LONG-TIME BEHAVIOR, INVARIANT MEASURES AND REGULARIZING EFFECTS FOR STOCHASTIC SCALAR CONSERVATION LAWS

BENJAMIN GESS AND PANAGIOTIS E. SOUGANIDIS

Abstract. We study the long-time behavior and regularity of the pathwise entropy solutions to stochastic scalar conservation laws with random in time spatially homogeneous fluxes and periodic initial data. We prove that the solutions converge to their spatial average, which is the unique invariant measure of the associated random dynamical system, and provide a rate of convergence, the latter being new even in the deterministic case for dimensions higher than two. The main tool is a new regularization result in the spirit of averaging lemmata for scalar conservation laws, which, in particular, implies a regularization by noise-type result for pathwise quasi-solutions.

1. Introduction and main results

1.1. **An overview.** We are interested in the long-time behavior, the existence and uniqueness of invariant measures and the regularity of pathwise entropy solutions of the spatially homogeneous stochastic scalar conservation laws (SSCL for short)

$$\begin{cases} du + \displaystyle\sum_{i=1}^{N} \partial_{x_i} A^i(u) \circ d\beta^i_t = 0 \ \text{ in } \ \mathbb{T}^N \times (0,\infty), \\ u = u_0 \ \text{ on } \ \mathbb{T}^N \times \{0\}, \end{cases} \tag{1.1}$$

where $\mathbb{T}^N$ is the N-dimensional torus, $\circ$ denotes the Stratonovich differential, the flux $\mathbf{A}$ is smooth with polynomial growth, that is, setting

$$\mathbf{A}''(\xi) := \partial^2_\xi \mathbf{A}(\xi) \ \text{ and } \ \mathbf{a}(\xi) := \mathbf{A}'(\xi) := \partial_\xi \mathbf{A}(\xi),$$

we assume that, for some $C > 0, m \in \mathbb{N}$ and all $\xi \in \mathbb{R}$,

$$\mathbf{A} = (A^1, \ldots, A^N) \in C^2(\mathbb{R}; \mathbb{R}^N) \ \text{ and } \ |\mathbf{A}''(\xi)| \le C(1 + |\xi|^m), \tag{1.2}$$

$$\boldsymbol{\beta} = (\beta^1, \ldots, \beta^N) \ \text{ is a standard two-sided Brownian motion,} \tag{1.3}$$

and

$$u_0 \in L^1(\mathbb{T}^N). \tag{1.4}$$

In order to have both nontrivial asymptotic behavior as well as to observe regularizing effects, we need to exclude the linear case by assuming that the flux is genuinely nonlinear. Since our estimates are based on a new stochastic averaging-type lemma (Theorem 1.3 below), it is convenient to quantify this property in a measure theoretic way.

Date: March 26, 2016.

2000 *Mathematics Subject Classification.* H6015, 35R60, 35L65.

Key words and phrases. Stochastic scalar conservation laws, averaging lemma, invariant measure, random dynamical systems, random attractor, regularity, regularization by noise.

Benjamin Gess is supported by the DFG through a research scholarship. Panagiotis Souganidis is supported by the NSF grant DMS-1266383.

We assume that there exist $\theta \in (0,1]$ and $C>0$ such that, for all $\sigma \in S^{N-1}$, $z \in \mathbb{R}^N$ and $\varepsilon > 0$,

$$|\{\xi \in \mathbb{R} : |\mathbf{a}(\xi)\sigma - z| \le \varepsilon\}| \le C\varepsilon^{\theta}, \tag{1.5}$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$ and, for $x=(x_1,\ldots,x_N), y=(y_1,\ldots,y_N) \in \mathbb{R}^N$, we set $xy := (x_1y_1,\ldots,x_Ny_N)$.

The genuine nonlinearity condition assumed typically in the deterministic setting, that is when $\boldsymbol{\beta} = (t,\ldots,t)$, is that there exist $\theta \in (0,1]$ and $C>0$ such that, for all $\sigma \in S^{N-1}$, $z \in \mathbb{R}$ and $\varepsilon>0$,

$$|\{\xi \in \mathbb{R} : |\mathbf{a}(\xi)\cdot\sigma - z| \le \varepsilon\}| \le C\varepsilon^{\theta}, \tag{1.6}$$

where $x \cdot y$ denotes the inner product of the vectors $x, y \in \mathbb{R}^N$.

Since, for some constant $C>0$,

$$|\mathbf{a}(\xi)\sigma - z| \ge C|\mathbf{a}(\xi)\cdot\sigma - \sum_{i=1}^{N} z_i|,$$

it is immediate that (1.6) implies (1.5).

The following example shows, however, that (1.5) is strictly weaker than (1.6) in dimensions larger than one. We fix $N=2$ and some $l \in \mathbb{N}$ and consider the flux $\mathbf{A} : \mathbb{R} \to \mathbb{R}^2$ given by

$$\mathbf{A}(\xi) = (\frac{1}{l+1}\xi^{l+1}, \frac{1}{l+1}\xi^{l+1}).$$

Then

$$\mathbf{a}(\xi) = (\xi^l, \xi^l).$$

In this case, if $\sigma = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $z=0$, we have $\mathbf{a}(\xi)\cdot\sigma - z = 0$, and, hence,

$$|\{\xi \in \mathbb{R} : |\mathbf{a}(\xi)\cdot\sigma - z| \le \varepsilon\}| = \infty$$

and (1.6) cannot be satisfied.

On the other hand, if $\sigma = (\sigma_1, \sigma_2) \in S^1$ and $z = (z_1, z_2)$,

$$|\mathbf{a}(\xi)\sigma - z|^2 = |\xi^l\sigma_1 - z_1|^2 + |\xi^l\sigma_2 - z_2|^2$$

Since $|\sigma| = 1$ yields $\sigma_1 > \frac{1}{\sqrt{2}}$ or $\sigma_2 > \frac{1}{\sqrt{2}}$, without loss of generality, next we assume that $\sigma_1 > \frac{1}{\sqrt{2}}$. Then

$$\begin{aligned}|\{\xi \in \mathbb{R} : |\mathbf{a}(\xi)\sigma - z| \le \varepsilon\}| &= |\{\xi \in \mathbb{R} : |\xi^l\sigma_1 - z_1|^2 + |\xi^l\sigma_2 - z_2|^2 \le \varepsilon^2\}| \\ &\le |\{\xi \in \mathbb{R} : |\xi^l\frac{\sigma_1}{|\sigma_1|} - \frac{z_1}{|\sigma_1|}| \le \frac{\varepsilon}{|\sigma_1|}\}| \\ &\le \frac{C}{|\sigma_1|^{\frac{1}{l}}}\varepsilon^{\frac{1}{l}} \le\ 2^{l/2}C\varepsilon^{\frac{1}{l}},\end{aligned}$$

where the first inequality follows from condition (1.6) for the one dimensional flux $a_1(\xi) = \xi^l$. Hence, (1.5) is satisfied.

The exponent θ in (1.6) depends on the dimension N. Indeed, it was shown in Berthelin and Junka [6] that, if $\mathbf{A}$ is smooth, then necessarily $\theta \in (0, \frac{1}{N}]$.

In contrast, this is not true for (1.5). Indeed, as we have seen above, for all $j = 1,\ldots,N$, in general we have,

$$\begin{aligned}|\{\xi \in \mathbb{R} : |\mathbf{a}(\xi)\sigma - z| \le \varepsilon\}| &= |\{\xi \in \mathbb{R} : \sum_{i=1}^{N} |a_i(\xi)\sigma_i - z_i|^2 \le \varepsilon^2\}| \\ &\le |\{\xi \in \mathbb{R} : |a_j(\xi)\sigma_j - z_j| \le \varepsilon\}|.\end{aligned}$$

Hence, if the one dimensional fluxes a_j satisfiy (1.5) with exponent θ_j, then $\mathbf{a}$ satisfies (1.5) with $\theta = \min(\theta_1, \dots, \theta_N)$. For example, in the particular case that $\mathbf{a}(\xi) = (\xi, \dots, \xi)$ which satisfies (1.5) with $\theta = 1$, we obtain $\theta = 1$ in (1.5) and there is no dependence on the dimension.

The main results of the paper are (i) the quantitative convergence, as $t \to \infty$, of solutions to (1.1) to the spatial average of the initial data, which turns out to be the unique invariant measure of the associated random dynamical system, (ii) a new regularizing property (averaging lemma) for (1.1), (iii) a rate for the convergence, as $t \to \infty$, of the deterministic entropy solutions to the their mean, and (iv) a "regularization by noise"-type result for pathwise quasi-solutions to (1.1).

The convergence to the spatial average, the rate of convergence and the regularizing effect are new results and, to the best of our knowledge, the only available for nonlinear problems with random fluxes. Providing a rate of convergence for deterministic scalar conservation laws with $N > 2$ solves a long-standing open problem. Concerning the regularization by noise, we show that pathwise quasi-solutions to the (stochastic) Burgers' equation are more regular than in the deterministic case. Again, to the best of our knowledge, this is the first such result for nonlinear random fluxes.

1.2. **The general setting.**

Without loss of generality in the following we work with the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ with the canonical realization on $\Omega = C_0(\mathbb{R}; \mathbb{R}^N) := \{\mathbf{b} \in C(\mathbb{R}; \mathbb{R}^N) \text{ and } \mathbf{b}(0) = 0\}$, and $\mathbb{P}$, $\mathbb{E}$, $\mathcal{F}_t$ and $\bar{\mathcal{F}}_t$ being respectively the two-sided standard Gaussian measure on Ω, the expectation with respect to $\mathbb{P}$, the canonical, uncompleted filtration and its completion.

Lions, Perthame and Souganidis introduced in [40] the notion of pathwise entropy solutions to (1.1) (actually [40] considered general continuous paths $\boldsymbol{\beta}$) and showed that, for each $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$, each path $t \mapsto \boldsymbol{\beta}_t(\omega)$ and all $T > 0$, there exists a unique pathwise entropy solution $u = u(\cdot; \boldsymbol{\beta}, u_0) = u(\cdot; \omega, u_0) \in C\big([0,\infty); L^1(\mathbb{R}^N)\big) \cap L^\infty(\mathbb{R}^N \times (0,T))$ to (1.1) and the solution operator is an L^1-contraction and, hence, is defined on $L^1(\mathbb{R}^N)$.
A straightforward modification of the arguments in [40] yields that the theory extends to (1.1) and is well posed in $L^\infty(\mathbb{T}^N)$, that is, for each $u_0 \in L^\infty(\mathbb{T}^N)$, each path $t \mapsto \boldsymbol{\beta}_t(\omega)$ and all $T > 0$, there exists a unique pathwise entropy solution $u = u(\cdot; \boldsymbol{\beta}, u_0) = u(\cdot; \omega, u_0) \in C\big([0,\infty); L^1(\mathbb{T}^N)\big) \cap L^\infty(\mathbb{T}^N \times [0,T])$, and the solution operator is an L^1-contraction and, hence, is defined on $L^1(\mathbb{T}^N)$.

Since the entropy solution to (1.1) is constructed in a pathwise manner, for each $u_0 \in L^1(\mathbb{T}^N)$ and $t \geq 0$, the map

$$\varphi(t,\omega)u_0 := u(\cdot, t; \omega, u_0) \tag{1.7}$$

defines a continuous random dynamical system (RDS for short) on $L^1(\mathbb{T}^N)$; we refer to Appendix A for some background on RDS.

The associated Markovian semigroup $(P_t)_{t \geq 0}$ is given, for each bounded measurable function f on $L^1(\mathbb{T}^N)$, $u_0 \in L^1(\mathbb{T}^N)$ and $t \geq 0$, by

$$P_t f(u_0) := \mathbb{E} f(u(\cdot, t; \cdot, u_0)) = \mathbb{E} f(\varphi(t, \cdot) u_0).$$

By duality we may consider the action of $(P_t)_{t \geq 0}$ on the space $\mathcal{M}_1$ of probability measures on $L^1(\mathbb{T}^N)$, that is, for $\mu \in \mathcal{M}_1$, we set

$$P_t^* \mu(f) := \int_{L^1} P_t f(x) d\mu(x).$$

A probability measure μ is an invariant measure for $(P_t)_{t \geq 0}$ if $P_t^* \mu = \mu$ for all $t \geq 0$. Moreover μ is said to be strongly mixing if, for each $\nu \in \mathcal{M}_1$, $P_t^* \nu \rightharpoonup \mu$ weak $\star$ in $\mathcal{M}_1$ as $t \to \infty$.

1.3. **The results.** We prove here (see Theorem 1.1 below) that, as $t \to \infty$ and $\mathbb{P}$-almost surely (a.s. for short),

$$u(\cdot, t; \omega, u_0) \to \bar{u}_0 := \int_{\mathbb{T}^N} u_0(x)dx,$$

provide a convergence rate and show that $\delta_{\bar{u}_0}$ is the unique invariant measure and $\bar{u}_0$ the random attractor. Here δ_c denotes the "Dirac mass" measure in $L^1(\mathbb{T}^N)$ charging the constant function with value $c \in \mathbb{R}$ and we set $L^1_c(\mathbb{T}^N)$ to be the space of all $L^1(\mathbb{T}^N)$ functions with spatial average c.

The first result is stated in the next theorem; for some of the terms in the statement we refer to Appendix A.

Theorem 1.1. *Assume* (1.2), (1.3), (1.4) *and* (1.5)*. Then, as* $t \to \infty$,

$$u(\cdot, t; \cdot, u_0) \to \bar{u}_0 \quad \text{in } L^1(\Omega; L^1(\mathbb{T}^N)) \text{ and } \mathbb{P}\text{-a.s. in } L^1(\mathbb{T}^N);$$

in particular, $\delta_{\bar{u}_0}$ *is the unique invariant measure for* $(P_t)_{t\geq 0}$ *on* $L^1_{\bar{u}_0}(\mathbb{T}^N)$ *and is strongly mixing, and, restricted to* $L^1_{\bar{u}_0}(\mathbb{T}^N)$*, the RDS* φ *has* $\bar{u}_0$ *as forward and pullback random attractor. Moreover, for* $t \geq 1$ *and* $u_0 \in L^{2+m}(\mathbb{T}^N)$,

$$\mathbb{E}\|u(\cdot, t; \cdot, u_0) - \bar{u}_0\|_1 \leq \left(\|u_0\|_{2+m}^{2+m} + 1\right) t^{-\frac{\theta}{3+\theta}},$$

and, for all $p \in (1, \infty)$,

$$\mathbb{E}\|u(\cdot, t; \cdot, u_0) - \bar{u}_0\|_p \leq 2\|u_0\|_\infty \left(\|u_0\|_{2+m}^{2+m} + 1\right) t^{-\frac{\theta}{3+\theta}}.$$

Following the strategy of the proof of Theorem 1.1, we also obtain a rate for the asymptotic behavior of the entropy solutions to the deterministic conservation law

$$\begin{cases} \partial_t u + \sum_{i=1}^N \partial_{x_i} A^i(u) = 0 \ \text{ in } \ \mathbb{T}^N \times (0, \infty), \\ u = u_0 \ \text{ on } \ \mathbb{T}^N \times \{0\}. \end{cases} \tag{1.8}$$

The result is stated next.

Theorem 1.2. *Assume* (1.2), (1.4) *and* (1.6)*. Then, as* $t \to \infty$,

$$u(\cdot, t; \cdot, u_0) \to \bar{u}_0 \quad \text{in } L^1(\mathbb{T}^N).$$

Moreover, for $t \geq 1$ *and* $u_0 \in L^{2+m}(\mathbb{T}^N)$,

$$\|u(\cdot, t; \cdot, u_0) - \bar{u}_0\|_1 \leq \left(\|u_0\|_{2+m}^{2+m} + 1\right) t^{-\frac{\theta}{2+\theta}}.$$

The rate in Theorem 1.2 is a new result for $N > 2$. We give some details on the existing literature for the (1.8) in the next subsection.

We do not make any claim of optimality for the rates obtained in Theorem 1.1 and Theorem 1.2. Moreover, although the rate for the stochastic problem is better than the one for the deterministic one, it remains an open question how the optimal rates compare.

The arguments and estimates leading to Theorem 1.1 are a special case of a more general new regularity result, which extends the ones obtained by Lions, Perthame and Souganidis in [41].

Theorem 1.3. *Assume* (1.2), (1.3), (1.4), (1.5) *and let* $u_0 \in L^{2+m}(\mathbb{T}^N)$*. Then, for all* $\lambda \in (0, \frac{4\theta}{2\theta+3})$ *and* $T > 0$*, there is a* $C > 0$ *such that*

$$\mathbb{E}\int_0^T \|u(t)\|_{W^{\lambda,1}} dt \leq C(1 + \|u_0\|_{2+m}^{2+m}),$$

and, for all $\delta > 0$,

$$\sup_{t\geq\delta} \mathbb{E}\|u(t)\|_{W^{\lambda,1}} < \infty.$$

We note that Theorem 1.3 yields higher regularity than in the corresponding deterministic result. Indeed, Jabin and Perthame proved in [36] that, if the genuine nonlinearity condition (1.6) holds for $\theta = 1$, then, for $t > 0$, entropy solutions to (1.8) satisfy $u(t) \in W^{\lambda,1}$ for each $\lambda \in (0, \frac{1}{3})$. In contrast, Theorem 1.3 yields that, for $t > 0$, $u(t) \in W^{\lambda,1}$ for each $\lambda \in (0, \frac{4}{5})$. As before, we do not know if the the regularity obtained in Theorem 1.3 is optimal.

Theorem 1.3 also implies a regularization by noise-type result in the following sense. In analogy to the deterministic theory, we introduce in Section 6 the class of quasi-solutions to the stochastic Burgers equation

$$\begin{cases} du + \partial_x u^2 \circ d\beta_t = 0 \quad \text{in} \quad \mathbb{T} \times (0,T), \\ u = u_0 \quad \text{on} \quad \mathbb{T} \times \{0\}, \end{cases} \tag{1.9}$$

and show that they are more regular than the ones to

$$\begin{cases} \partial_t u + \partial_x u^2 = 0 \quad \text{in} \quad \mathbb{T} \times (0,T), \\ u = u_0 \quad \text{on} \quad \mathbb{T} \times \{0\}. \end{cases} \tag{1.10}$$

The reason we work with such solutions is that the regularity implied by averaging techniques for (1.10) is essentially sharp for quasi-solutions, a fact which is not true for entropy solutions; see, for example, De Lellis and Westdickenberg [16] and De Lellis, Otto and Westdickenberg [15]. Indeed following the proof in [36] it easy to conclude that, for $t > 0$, quasi-solutions to (1.10) satisfy $u(t) \in W^{\lambda,1}$ for every $\lambda \in (0, \frac{1}{3})$; actually [36] works with entropy solutions, but a careful look at the proof yields that the result also holds for quasi-solutions. Optimal regularity for quasi-solutions was proven by Golse and Perthame in [32]. As it shown in [16], however, there are quasi-solutions such that $u(t) \notin W^{\lambda,1}$ for every $\lambda > \frac{1}{3}$.

In contrast, we show here that quasi-solutions to (1.9) satisfy $u(t) \in W^{\lambda,1}$ for every $\lambda \in (0, \frac{4}{5})$. In this sense, we see that the noise included in (1.9) has a regularizing effect.

Theorem 1.4. *Let* u *be a pathwise quasi-solution to* (1.9) *with* $u_0 \in L^2(\mathbb{T})$. *Then, for all* $\lambda \in (0, \frac{4}{5})$ *and* $T > 0$, *there is* $C > 0$ *such that*

$$\mathbb{E}\int_0^T \|u(t)\|_{W^{\lambda,1}} dt \leq C(1 + \|u_0\|_2^2 + \mathbb{E}|m|([0,T] \times \mathbb{T} \times \mathbb{R})).$$

and, for all $\delta > 0$,

$$\sup_{t\geq\delta} \mathbb{E}\|u(t)\|_{W^{\lambda,1}} < \infty.$$

We expect, and this is the subject of a subsequent work, that methods similar to the ones developed in this paper, in combination with arguments from [20], may be used to prove the existence and uniqueness of an invariant measure in the case of additive noise, that is for

$$du + \sum_{i=1}^N \partial_{x_i} A^i(u) \circ d\beta_t^i = dW_t,$$

where W is an infinite dimensional Wiener process independent of β.

1.4. **Brief review of the existing literature.** We discuss next briefly results about scalar conservation laws with some type of random dependence and their long-time behavior, we recall the basic literature for averaging lemmata and touch upon the issue of regularization by noise.

The effect of noise entering scalar conservation laws via randomness in the initial condition has been studied, for example, by Avellaneda and E [2], Burgers [9], Ryan [50] and Sinai [52].

For stochastic scalar conservation laws driven by additive noise, also including boundary value problems, we refer to the works by E, Khanin, Mazel and Sinai [22], Kim [37], Nakazawa [45], Saussereau and Stoica [51], Vallet and Wittbold [53] and the references therein.

The case of multiplicative semilinear noise dependence in Itô's form, that is stochastic PDE (SPDE for short) of the form

$$du + \mathrm{div} A(u)dt = g(u)dW_t, \tag{1.11}$$

where W denotes a possibly infinite dimensional Wiener process, has attracted considerable interest in recent years; see for example, Bauzet, Vallet and Wittbold [4], Chen, Ding and Karlsen, [10]. Debussche and Vovelle [19], Debussche, Hofmanova and Vovelle [17], Feng and Nualart [25], Hofmanova [33], Holden and Risebro [34], etc..

The long-time behavior of solutions to (1.11), the existence and uniqueness of an invariant measure and a singleton random point attractor was first studied in [22] for the stochastic Burgers equation with additive noise, that is

$$du + \partial_x u^2 dt = dW_t, \tag{1.12}$$

on the one-dimensional torus $\mathbb{T} = [0, 2\pi]$; for extensions of these results see Bakhtin [3], Boritchev [7], Iturriaga and Khanin [35], Saussereau and Stoica [51], etc..

More recently, these results were extended by Debussche and Vovelle [20] to general SSCL on $\mathbb{T}^N$ with additive noise, that is to

$$du + \mathrm{div} A(u)dt = \Phi dW_t, \tag{1.13}$$

where Φ is an appropriate Hilbert-Schmidt operator. In addition, it was shown in [20] that, under appropriate non-degeneracy conditions on the flux $\mathbf{A}$, (1.13) has a unique invariant measure.

We also refer to Dirr and Souganidis [21], who studied similar questions for Hamilton-Jacobi equations perturbed by additive noise in any dimension.

All of the above mentioned works consider semilinear SSCL in the sense that the noise is applied to functions of the solution and not its derivatives.

In contrast, [40] and its subsequent extensions by Lions, Perthame and Souganidis [42] and the authors [30] consider SSCL like

$$du + \sum_{i=1}^{N} \partial_{x_i} A_i(x, u) \circ d\beta_t^i = 0,$$

with inhomogeneous spatially dependent fluxes and single and multiple rough time dependence respectively, in the sense that the noise is applied to the flux and, hence, the derivatives of the solution.

The well-posedness of solutions to such SPDE was proven in [30, 40, 41] using a kinetic formulation. The case of joint transport noise and linear multiplicative noise has been treated by Friz and Gess in [29] by entropy and rough paths methods.

We next discuss some of the available literature about the long-time behavior of deterministic scalar conservation laws (SCL) with periodic initial data. Since there are many references, here we restrict to the ones that seem most relevant for our purpose. The one-dimensional setting is well understood, since Lax in [38] proved, using the Lax-Oleinik formula, asymptotic linear decay for strictly convex or concave fluxes. The first asymptotic result with a rate for $N=2$ is due to Engquist and E in [23]. The higher dimensional case was first studied by Chen and Frid in [11] and was subsequently generalized by Chen and Perthame [12] and Debussche and Vovelle [18]. These last three references assumed genuine non-linearity-type conditions on the flux and proved the convergence of the solutions to their spatial average without any rate. More recently and independently, Dafermos [14] and Panov [47] proved the same result under weaker genuine non-linearity conditions, which are also necessary. We note that with the exception of $N=1,2$ no rate was known before.

Many of the above mentioned results about the long-time behavior of deterministic SCL rely on averaging lemmata of the type introduced by Golse, Lions, Perthame, Sentis [31]. Averaging lemmata are one of the most important tools in the theory of (deterministic) conservation laws and were used very effectively for conservation laws by, among others, Perthame and Tadmor [49], Lions, Perthame and Tadmor [43], Lions, Perthame and Souganidis [39], Perthame and Souganidis [48] etc.. Some stochastic averaging lemmata, which lead to higher regularizing results than in the corresponding deterministic case, were obtained by Lions, Perthame and Souganidis [41].

Regularization by noise phenomena have been observed in the linear spatially inhomogeneous case, that is for transport equations of the form

$$du + b(x)\cdot Du\ dt = 0. \tag{1.14}$$

Indeed, it has been shown by Flandoli, Gubinelli and Priola [27] that adding perturbations of the type

$$du + b(x)\cdot Du\ dt = \partial_x u \circ d\beta_t, \tag{1.15}$$

regularizes the dynamics in the sense that well-posedness can be obtained for (1.15) in cases for which (1.14) is ill-posed. Moreover, it has been proven (see, for example, Fedrizzi and Flandoli [24], Flandoli, Gubinelli and Priola [28] and Mohammed, Nilssen and Proske [44]), that solutions to (1.15) are more regular than their deterministic analogues. For example, [24] shows that, if $u_0 \in \bigcap_{\lambda\geq 1} W^{\lambda,1}$ and $b\in L^p(\mathbb{R})$ with $p\geq 2$, then $u(t)\in \bigcap_{\lambda\geq 1} W^{\lambda,1}_{loc}$ a.s. and for every $t\in[0,T]$.

All the above results and techniques depend strongly on the linear structure of (1.15). In fact, Flandoli [26] showed that analogous regularizing effects are wrong in the nonlinear case, for example, the stochastic Burgers' equation

$$du + \partial_x u^2 dt = \partial_x u \circ d\beta_t,$$

where the inclusion of noise does not prevent characteristics to collide and thus BV-regularity is the best one can get even in the stochastic case.

1.5. **Organization of the paper and some notation.**

The paper is organized as follows: In Section 2 we recall the notion of stochastic pathwise entropy as well as some of the basic estimates that we need to prove the main results. Theorem 1.1, Theorem 1.2 and Theorem 1.3 are proved respectively in Section 3, Section 4 and Section 5. In Section 6 we recall the notion of quasi-solutions to (1.10), introduce the definition of pathwise quasi-solutions to (1.9) and prove Theorem 1.4. Appendix A states some of the basic properties of RDS, in Appendix B we present the proof of a very useful and basic analysis lemma which in the proofs, and finally, in Appendix C we give the rigorous justification of a formula which is at the core of the proofs.

In what follows, we will say pathwise instead of stochastic pathwise entropy solution, we will omit, when it does not cause any confusion, the dependence in ω and we occasionally write $m(x,\xi,s)dxd\xi ds$ instead of $dm(x,\xi,s)$. For notational convenience, we write $A\lesssim B$ if $A\leq CB$ for some $C>0$. If $A\lesssim B$ and $B\lesssim A$, we write $A\sim B$. We work with the homogeneous Bessel potential spaces $W^{\lambda,p}$ for $\lambda>0$, $p\in[1,\infty)$, that is

$$W^{\lambda,p} := \{f\in L^p(\mathbb{T}^N):\ (|n|^\lambda \hat{f}(n))^\vee \in L^p(\mathbb{T}^N)\},$$

where $\hat{f}$ denotes the discrete Fourier transform of f on $\mathbb{T}^N$ and $f^\vee$ its inverse. We note that the homogeneous Bessel potential spaces coincide with the domains of the fractional Laplace operators $(-\Delta)^{\frac{\lambda}{2}}$ on $L^p(\mathbb{T}^N)$. For notational simplicity we also set $H^\lambda := W^{\lambda,2}$.

2. PATHWISE ENTROPY SOLUTIONS AND SOME BASIC PROPERTIES

2.1. **The definition.**

We begin with the derivation of the notion of the pathwise solution to (1.1) and, for the moment, we assume that $\boldsymbol{\beta}$ is smooth in which case $\circ$ reduces to multiplication. Then (1.1) is a standard scalar conservation law with time dependent flux which is best studied using kinetic solutions.

For the latter we introduce the auxiliary nonlinear function

$$\chi(x,\xi,t) = \chi(u(x,t),\xi) = \begin{cases} +1 & \text{if } \; 0 \le \xi \le u(x,t), \\ -1 & \text{if } \; u(x,t) \le \xi \le 0, \\ \;0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The kinetic formulation of (1.1) is the assertion that the χ given by (2.1) is a solution, in the sense of distributions, to

$$\begin{cases} \partial_t \chi + \sum_{i=1}^N a^i(\xi)\partial_{x_i}\chi\dot{\beta}^i(t) = \partial_\xi m & \text{in } \; \mathbb{T}^N \times \mathbb{R} \times (0,T), \\ \chi = \chi_0(\cdot,\cdot) := \chi(u_0(\cdot),\cdot) & \text{on } \mathbb{T}^N \times \mathbb{R} \times \{0\}, \end{cases} \tag{2.2}$$

where

$$m \;\text{ is a nonnegative bounded measure on } \mathbb{T}^N \times \mathbb{R} \times (0,T) \text{ for each } T > 0.$$

To introduce the pathwise entropy solutions we use the transport equation

$$\begin{cases} \partial_t \varrho + \sum_{i=1}^N a^i(\xi)\partial_{x_i}\varrho\dot{\beta}^i(t) = 0 & \text{in } \; \mathbb{T}^N \times \mathbb{R} \times (0,\infty), \\ \varrho = \varrho_0 & \text{on } \mathbb{T}^N \times \mathbb{R} \times \{0\}. \end{cases} \tag{2.3}$$

For each $y \in \mathbb{T}^N$ and $\rho_0 \in C^\infty(\mathbb{T}^N)$, the solution $\varrho = \varrho(x,y,\xi,t)$ to (2.3) with initial condition $\varrho_0(\cdot - y)$ is given by

$$\varrho(x,y,\xi,t) = \varrho_0\left(x - y - \sum_{i=1}^N a^i(\xi)\beta^i(t)\right). \tag{2.4}$$

We define the "convolution along characteristics" $\varrho * \chi : \mathbb{T}^N \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ by

$$\varrho * \chi(y,\xi,t) := \int_{\mathbb{T}^N} \varrho(x,y,\xi,t)\chi(x,\xi,t)dx.$$

It follows that, in the sense of distributions in ξ and t,

$$\partial_t(\varrho * \chi)(y,\xi,t) = \int_{\mathbb{T}^N} \varrho(x,y,\xi,t)\partial_\xi m(x,\xi,t)dx \;\text{ in } \mathbb{T}^N \times \mathbb{R} \times (0,T), \tag{2.5}$$

and, after integrating in t, again in the the sense of distributions in ξ and for all $t \in [0,T]$,

$$(\varrho * \chi)(y,\xi,t) - (\varrho * \xi)(y,\xi,0) = \int_0^t \int_{\mathbb{T}^N} \varrho(x,y,\xi,r)\partial_\xi m(x,\xi,r)dxdr.$$

Note that this last identity is actually equivalent to the kinetic formulation for conservation laws with smooth time dependence. Moreover, the dependence on the derivatives of the paths has disappeared. These two observations are the idea behind the notion of pathwise entropy solutions, that is, to use (2.5) as the definition of a solution. To state the latter we need to introduce one more notion.

We say that a measurable map $m : \Omega \to \mathcal{M}$, the space of nonnegative bounded measures on $\mathbb{T}^N \times \mathbb{R} \times [0,T]$, is a kinetic measure, if the process $t \mapsto m(\cdot,[0,t])$ with values in the space of nonnegative bounded measures on $\mathbb{T}^N \times \mathbb{R}$ is $\mathcal{F}_t$-adapted.

Definition 2.1. A map $u : \mathbb{T}^N \times [0,\infty) \times \Omega \to \mathbb{R}$ such that, for all $T > 0$, $u \in (L^1 \cap L^\infty)(\mathbb{T}^N \times [0,T] \times \Omega)$, and $\mathbb{P}$-a.s. in ω, $u(\cdot,\omega) \in C\big([0,T];L^1(\mathbb{T}^N)\big) \cap L^\infty(\mathbb{T}^N \times [0,T])$, and $t \mapsto u(t,\cdot)$ is an $\mathcal{F}_t$-adapted process in $L^1(\mathbb{T}^N)$, is a (stochastic) pathwise entropy solution to (1.1), if there exists a kinetic measure m such that, for all $y \in \mathbb{T}^N$ and all test functions ϱ given by (2.4) with $\varrho_0 \in C^\infty(\mathbb{T}^N)$ and all $\varphi \in C_c^\infty([0,T)), \psi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \partial_t\varphi(r)\psi(\xi)(\varrho * \chi)(y,\xi,r)drd\xi + \int_{\mathbb{R}} \varphi(0)\psi(\xi)(\varrho * \chi)(y,\xi,0)d\xi \\
&= \int_0^T \int_{\mathbb{T}^N \times \mathbb{R}} \varphi(r)\partial_\xi\left(\psi(\xi)\varrho(x,y,\xi,r)\right)dm(x,\xi,r).
\end{aligned}$$

2.2. **The basic properties.** The next proposition summarizes the key properties of the pathwise entropy solution and is obtained by a straightforward modification of the proof of the analogous result in [40].

Proposition 2.2. *Assume* (1.2), (1.3) *and* (1.4). *For each* $u_0 \in L^\infty(\mathbb{T}^N)$ *and each path* $t \mapsto \boldsymbol{\beta}_t(\omega)$, *there exists a unique stochastic pathwise entropy solution* $u = u(\cdot,\omega;u_0)$. *Moreover, for all* $t \geq 0$, $p \in [1,\infty]$ *and* $\mathbb{P}$-*a.s.,*

$$\int u(x,t,\omega;u_0)dx = \int u_0(x)dx, \tag{2.6}$$

$$\|u(\cdot,t,\omega;u_0)\|_p \leq \|u_0\|_p, \tag{2.7}$$

and

$$\|u(\cdot,t,\omega;u_0)\|_{BV} \leq \|u_0\|_{BV\cdot}, \tag{2.8}$$

and, if $u^1(\cdot,\cdot;\omega), u^2(\cdot,\cdot;\omega)$ *are two pathwise entropy solutions, then, for all* $t, s \geq 0$ *with* $s \leq t$ *and* $\mathbb{P}$-*a.s.,*

$$\|u^1(\cdot,t,\omega) - u^2(\cdot,t,\omega)\|_{L^1(\mathbb{T}^N)} \leq \|u^1(\cdot,s,\omega) - u^2(\cdot,s,\omega)\|_{L^1(\mathbb{T}^N)}. \tag{2.9}$$

Note that Proposition 2.2 and, in particular, (2.9), yield that the solutions $u = u(\cdot,\omega;u_0)$ are well-defined for all $u_0 \in L^1(\mathbb{T}^N)$. It is then easy to see that the map defined in (1.7) is indeed a continuous RDS on $L^1(\mathbb{T}^N)$.

For future use we remark that, in view the definition of χ, for each $u \in \mathbb{R}$, it holds $\int_{\mathbb{R}} |\chi(u,\xi)|d\xi = |u|$ and, thus, for any $u \in L^1(\mathbb{T}^N)$ and $p \geq 1$, we have

$$\begin{aligned}
\|\chi(u(\cdot),\cdot)\|^p_{L^p(\mathbb{T}^N \times \mathbb{R})} \leq \|\chi(u(\cdot),\cdot)\|_{L^1(\mathbb{T}^N \times \mathbb{R})} &= \int_{\mathbb{T}^N \times \mathbb{R}} |\chi(u(x),\xi)|d\xi dx \\
&= \int_{\mathbb{T}^N} |u(x)|dx = \|u\|_{L^1(\mathbb{T}^N)}.
\end{aligned}$$

In the sequel we will also need the following bound on the mass of the entropy defect measure m, which a small extension of the analogous bounds in [30, 40, 42], where we refer to for the proofs.

Lemma 2.3. *Assume* (1.2), (1.3) *and* (1.4). *For* $u_0 \in L^\infty(\mathbb{T}^N)$, $m \in \mathbb{N}$, $t \geq 0$ *and* $\mathbb{P}$-*a.s.,*

$$\|u(\cdot,t,\omega,u_0)\|^{m+2}_{m+2} + (m+2)(m+1)\int_0^t \int_{\mathbb{T}^N \times \mathbb{R}} |\xi|^m dm(x,\xi,s) = \|u_0\|^{m+2}_{m+2},$$

and

$$(m+2)(m+1)\mathbb{E}\int_0^\infty\int_{\mathbb{T}^N\times\mathbb{R}}|\xi|^m dm(x,\xi,s)\le \|u_0\|_{m+2}^{m+2}.$$

3. The quantitative asymptotic behavior.

The key step here is to introduce a regularization in the spirit of the classical averaging lemmata, which allows to obtain estimates that control the long time behavior of the solution to (1.1). The typical proofs of such averaging lemmata employ space time Fourier transforms, which is not possible in our context because of the rough time dependence of the flux. Instead, here we use only Fourier transforms with respect to the space variable x, a technique developed in Bouchut and Desvillettes [8] and used in Debussche and Vovelle [20] as well as in [41]. Although we follow the arguments of [20], here we need to deal with the new difficulties arising because of the stochastic nature of the flux in (1.1). An important tool is a technical result (Lemma 3.1), which uses the genuine nonlinearity condition (1.5). Since the proof is rather long, we divide it in several subsections.

3.1. **Averaging lemmata, a regularization result and a "new formula" for the solution.**

In view of (2.9) and the density of $L^\infty(\mathbb{T}^N)$ in $L^1(\mathbb{T}^N)$, it is clear that it suffices to consider $u_0\in L^\infty(\mathbb{T}^N)$. Moreover, to keep the presentation simple, throughout this section we restrict the presentation to the zero average case, that is, henceforth, we assume that

$$\int u_0(x)dx=0 \tag{3.1}$$

and leave it up to the reader to fill in the details for the general case, which is easily reduced to (3.1).

The main idea is that averaging over the "kinetic variable" ξ of the solution χ to (2.3) yields more regularity and provides new estimates for the solutions to the conservation laws. The difficulty, of course, comes from the very low regularity of the right hand side $\partial_\xi m$ in (2.2). Typically this problem is dealt with by adding a regularizing (parabolic)-type term in the kinetic equation and taking advantage of the smoothing it generates.

In this paper we use the fractional Laplacian operator as a regularizing term (see [8] and [20] for similar types of arguments)

$$B:=(-\Delta)^\alpha+Id\ \text{ with } \alpha\in(0,1], \tag{3.2}$$

and, for $\gamma>0$, we rewrite (2.2) as

$$\begin{cases}\partial_t\chi+\sum_i a^i(\xi)\partial_{x_i}\chi\circ d\beta^i(t)+\gamma B\chi=\gamma B\chi+\partial_\xi m \ \text{ in } \ \mathbb{T}^N\times\mathbb{R}\times(0,T),\\ \chi=\chi(u_0(\cdot),\cdot) \text{ on } \mathbb{T}^N\times\mathbb{R}\times\{0\}.\end{cases} \tag{3.3}$$

Note that in what follows we make strong use of the properties of the Gaussian paths and our approach and results do not extend to general continuous driving signals replacing $\boldsymbol{\beta}$.

Let $S_{A_\gamma(\xi)}(s,t)$ denote the solution operator (group) of

$$\partial_t v+\sum_i a^i(\xi)\dot\beta^i(t)\partial_{x_i}v+\gamma Bv=0 \text{ in } \mathbb{T}^N\times\mathbb{R}\times(s,\infty). \tag{3.4}$$

It is immediate that, for all f in the appropriate function space,

$$S_{A_\gamma(\xi)}(s,t)f(x)=\left(S_{\gamma B}(t-s)f\right)\left(x-\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s))\right), \tag{3.5}$$

where $S_{\gamma B}(t)$ is the solution semigroup to

$$\partial_t v+\gamma Bv=0\ \text{ in } \mathbb{R}^N\times(0,\infty), \tag{3.6}$$

and

$$\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)) = \big(a^1(\xi)(\beta^1(t)-\beta^1(s)),\dots,a^N(\xi)(\beta^N(t)-\beta^N(s))\big).$$

Note that, for each $n \in \mathbb{Z}^N$, the Fourier transform of $S_{A_\gamma(\xi)}$ corresponds to multiplication by

$$\exp(-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s))\cdot n - \gamma(|n|^{2\alpha}+1)(t-s)).$$

It follows (see Appendix C for a rigorous justification) from the variation of parameters formula that the solution χ to (3.2) is given, in the sense of distributions in x and ξ, by

$$\chi(x,\xi,t) = S_{A_\gamma(\xi)}(0,t)\chi_0(x,\xi) + \int_0^t S_{A_\gamma(\xi)}(s,t)(\gamma B\chi(s) + m_\xi(x,\xi,s))ds. \tag{3.7}$$

After integrating in ξ, we find that, for $t \in [0,T]$ and $\varphi \in C^\infty(\mathbb{T}^N)$,

$$\begin{aligned}\int_{\mathbb{T}^N}\varphi(x)u(x,t)dx &= \int_{\mathbb{T}^N\times\mathbb{R}}\varphi(x)\chi(x,\xi,t)dxd\xi\\ &= \int_{\mathbb{T}^N\times\mathbb{R}}\varphi(x)S_{A_\gamma(\xi)}(0,t)\chi_0(x,\xi)dxd\xi\\ &\quad+\int_0^t\int_{\mathbb{T}^N\times\mathbb{R}}\gamma B(S^*_{A_\gamma(\xi)}(s,t)\varphi)(x)\chi(x,\xi,s)dxd\xi ds\\ &\quad-\int_0^t\int_{\mathbb{T}^N\times\mathbb{R}}\partial_\xi(S^*_{A_\gamma(\xi)}(s,t)\varphi)(x)dm(x,\xi,s),\end{aligned} \tag{3.8}$$

where $S^*_{A_\gamma(\xi)}$ denotes the adjoint semigroup to $S_{A_\gamma(\xi)}$.

Accordingly, in the sense of distributions in x, we write (recall that, a.s. in ω, $u \in C([0,\infty);L^1(\mathbb{T}^N))$)

$$u(t) = u^0(t) + u^1(t) + Q(t), \tag{3.9}$$

where, for $\varphi \in C^\infty(\mathbb{T}^N)$,

$$\langle u^0(t),\varphi\rangle := \int_{\mathbb{T}^N\times\mathbb{R}}\varphi(x)S_{A_\gamma(\xi)}(0,t)\chi_0(x,\xi)dxd\xi, \tag{3.10}$$

$$\langle u^1(t),\varphi\rangle := \int_0^t\int_{\mathbb{T}^N\times\mathbb{R}}\gamma(B(S^*_{A_\gamma(\xi)}(s,t)\varphi)(x)\chi(x,\xi,s)dxd\xi ds \tag{3.11}$$

and

$$\langle Q(t),\varphi\rangle := -\int_0^t\int_{\mathbb{T}^N\times\mathbb{R}}\partial_\xi(S^*_{A_\gamma(\xi)}(s,t)\varphi)(x)m(x,\xi,s)dxd\xi ds. \tag{3.12}$$

Next, we estimate each of (3.10), (3.11) and (3.12) separately using averaging techniques.

In the analysis we need a basic integral estimate which is proved in Appendix B. For its statement it is convenient to introduce, for each $b:\mathbb{R}\to\mathbb{R}^N$ and $f \in L^2$, the function $\phi(\cdot;b,f):\mathbb{R}^N\to\mathbb{R}$ given by

$$\phi(w;b,f) := e^{-\frac{|w|^2}{a}}\int_{\mathbb{R}} e^{ib(\xi)\cdot w}f(\xi)d\xi. \tag{3.13}$$

Lemma 3.1. *Let $b:\mathbb{R}\to\mathbb{R}^N$ be such that, for all $\varepsilon>0, z\in\mathbb{R}^N$ and some nondecreasing $\iota:[0,\infty)\to[0,\infty)$ with $\lim_{\varepsilon\to 0} i(\varepsilon)=0$,*

$$|\{\xi\in\mathbb{R} : |b(\xi)-z|\le\varepsilon\}| \le \iota(\varepsilon).$$

Then, for all $a>0$ and $f\in L^2(\mathbb{R})$,

$$\|\phi(\cdot;b,f)\|^2_{L^2} \le 2\sqrt{a\pi}\int_0^\infty \tau e^{-\tau^2}\iota(\frac{2\tau}{\sqrt{a}})d\tau\|f\|_2^2.$$

3.2. **The estimate of u^0.** Taking Fourier transforms in (3.10) yields, for each $n \in \mathbb{Z}^N$,

$$\hat{u}^0(n,t) = \int e^{-i\mathbf{a}(\xi)\boldsymbol{\beta}(t)\cdot n-\gamma(|n|^{2\alpha}+1)t}\hat{\chi}_0(n,\xi)d\xi.$$

It is immediate, in view of (3.1), that

$$\hat{u}^0(0,t) = \int e^{-\gamma t}\hat{\chi}_0(0,\xi)d\xi = \int\int e^{-\gamma t}\chi_0(x,\xi)d\xi dx = \int e^{-\gamma t}u^0(x)dx = 0.$$

When $n \in \mathbb{Z}^N \setminus \{0\}$, integrating in time, taking expectations and using the scaling properties of the Brownian paths, we find

$$\begin{aligned}
\mathbb{E}\int_0^T |\hat{u}^0(n,t)|^2dt &= \mathbb{E}\int_0^T \Big|\int e^{-i\mathbf{a}(\xi)\boldsymbol{\beta}(t)\cdot n-\gamma(|n|^{2\alpha}+1)t}\hat{\chi}_0(n,\xi)d\xi\Big|^2dt\\
&= \int_0^T e^{-2\gamma(|n|^{2\alpha}+1)t}\mathbb{E}\Big|\int e^{-i\mathbf{a}(\xi)\boldsymbol{\beta}(t|n|^2)\cdot\frac{n}{|n|}}\hat{\chi}_0(n,\xi)d\xi\Big|^2dt\\
&= \int_0^T e^{-2\gamma(|n|^{2\alpha}+1)t}\int\Big|\int e^{-i\mathbf{a}(\xi)w\cdot\frac{n}{|n|}}\hat{\chi}_0(n,\xi)d\xi\Big|^2\frac{e^{-\frac{|w|^2}{2|n|^2t}}}{\sqrt{2\pi|n|^2t}}dwdt\\
&= \int_0^T \frac{e^{-2\gamma(|n|^{2\alpha}+1)t}}{\sqrt{2\pi|n|^2t}}\int\Big|e^{-\frac{|w|^2}{4|n|^2t}}\int e^{-i\mathbf{a}(\xi)\frac{n}{|n|}\cdot w}\hat{\chi}_0(n,\xi)d\xi\Big|^2dwdt.
\end{aligned}$$

Using Lemma 3.1 with $a = 4|n|^2t$, $b(\xi) = \mathbf{a}(\xi)\cdot\frac{n}{|n|}$ and $i(\varepsilon) \sim \varepsilon^\theta$ we get

$$\begin{aligned}
\mathbb{E}\int_0^T |\hat{u}^0(n,t)|^2dt &\le 2\sqrt{a\pi}\int_0^T \frac{e^{-2\gamma(|n|^{2\alpha}+1)t}}{\sqrt{2\pi|n|^2t}}\int_0^\infty \tau e^{-\tau^2}\iota(\frac{2\tau}{\sqrt{a}})d\tau dt\|\hat{\chi}_0(n,\cdot)\|_2^2\\
&\lesssim a^{\frac{1-\theta}{2}}\int_0^T \frac{e^{-2\gamma(|n|^{2\alpha}+1)t}}{\sqrt{2\pi|n|^2t}}\int_0^\infty \tau^{1+\theta}e^{-\tau^2}d\tau dt\|\hat{\chi}_0(n,\cdot)\|_2^2\\
&\lesssim \int_0^T e^{-2\gamma(|n|^{2\alpha}+1)t}(|n|^2t)^{-\frac{\theta}{2}}dt\|\hat{\chi}_0(n,\cdot)\|_2^2\\
&\lesssim |n|^{-\theta}\int_0^T e^{-2\gamma(|n|^{2\alpha}+1)t}t^{-\frac{\theta}{2}}dt\|\hat{\chi}_0(n,\cdot)\|_2^2\\
&\lesssim |n|^{-\theta}\gamma^{-\frac{2-\theta}{2}}(|n|^{2\alpha}+1)^{-\frac{2-\theta}{2}}\int_0^\infty e^{-t}t^{-\frac{\theta}{2}}dt\|\hat{\chi}_0(\cdot,\cdot)\|_2^2,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\mathbb{E}\int_0^T |\hat{u}^0(n,t)|^2dt &\lesssim |n|^{-\theta}\gamma^{-\frac{2-\theta}{2}}(|n|^{2\alpha}+1)^{-\frac{2-\theta}{2}}\|\hat{\chi}_0(n,\cdot)\|_2^2\\
&\lesssim \gamma^{-\frac{2-\theta}{2}}\|\hat{\chi}_0(n,\cdot)\|_2^2.
\end{aligned}\tag{3.14}$$

Combining the previous estimates, after summing over n, we obtain

$$\begin{aligned}
\mathbb{E}\int_0^T \|u^0(t)\|_2^2dt = \mathbb{E}\int_0^T \|\hat{u}(t)\|^2dt &\lesssim \gamma^{-\frac{2-\theta}{2}}\|\chi_0(\cdot,\cdot)\|_2^2\\
&\lesssim \gamma^{-\frac{2-\theta}{2}}\|\chi_0(\cdot,\cdot)\|_1 = \gamma^{-\frac{2-\theta}{2}}\|u_0\|_1.
\end{aligned}\tag{3.15}$$

3.3. **The estimate of u^1.** Let $\bar{\omega}_n := \gamma(|n|^{2\alpha}+1)$. For each $n \in \mathbb{Z}^N$, the Fourier transform $\hat{u}^1(n,t)$ of $u^1(t)$ in x is given by

$$\begin{aligned}\hat{u}^1(n,t) &= \bar{\omega}_n \int_0^t \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s))\cdot n - \gamma(|n|^{2\alpha}+1)(t-s)} \hat{\chi}(n,\xi,s) d\xi ds \\ &= \bar{\omega}_n \int_0^t e^{-\bar{\omega}_n(t-s)} \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s))\cdot n} \hat{\chi}(n,\xi,s) d\xi ds.\end{aligned}$$

Integrating in t, taking expectation and using that $\int_0^t \bar{\omega}_n e^{-\bar{\omega}_n r} dr \leq 1$, we find

$$\begin{aligned}\mathbb{E}\int_0^T |\hat{u}^1|^2(n,t)dt =& \mathbb{E}\int_0^T \Big| \int_0^t \bar{\omega}_n e^{-\bar{\omega}_n(t-s)} \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s))\cdot n} \hat{\chi}(n,\xi,s) d\xi ds\Big|^2 dt \\ =& \mathbb{E}\int_0^T \Big| \int_0^t \bar{\omega}_n e^{-\bar{\omega}_n r} \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r))\cdot n} \hat{\chi}(n,\xi,t-r) d\xi dr\Big|^2 dt \\ \leq& \mathbb{E}\int_0^T \int_0^t \bar{\omega}_n e^{-\bar{\omega}_n r} \Big| \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r))\cdot n} \hat{\chi}(n,\xi,t-r) d\xi\Big|^2 dr dt.\end{aligned}$$

In view of the facts that $\hat{\chi}$ is $\mathcal{F}_t$-adapted and the increments $\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r)$ are independent of $\mathcal{F}_{t-r}$, using the scaling properties of the Brownian motion we find

$$\begin{aligned}&\mathbb{E}\Big| \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r))\cdot n} \hat{\chi}(n,\xi,t-r) d\xi\Big|^2 \\ &= \mathbb{E}[\mathbb{E}\Big| \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r))\cdot n} \hat{\chi}(n,\xi,t-r) d\xi\Big|^2 | \mathcal{F}_{t-r}] \\ &= \mathbb{E}\tilde{\mathbb{E}}\Big| \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r))(\tilde{\omega})\cdot n} \hat{\chi}(n,\xi,t-r)(\omega) d\xi\Big|^2 \\ &= \mathbb{E}\tilde{\mathbb{E}}\Big| \int e^{-i\mathbf{a}(\xi)\boldsymbol{\beta}(|n|^2 r)(\tilde{\omega})\cdot\frac{n}{|n|}} \hat{\chi}(n,\xi,t-r)(\omega) d\xi\Big|^2 \\ &= \mathbb{E}\int \Big| \int e^{-i\mathbf{a}(\xi) w\cdot\frac{n}{|n|}} \hat{\chi}(n,\xi,t-r) d\xi\Big|^2 \frac{e^{-\frac{|w|^2}{2|n|^2 r}}}{\sqrt{2\pi|n|^2 r}} dw \\ &= \frac{1}{\sqrt{2\pi|n|^2 r}} \mathbb{E}\int \Big| e^{-\frac{|w|^2}{4|n|^2 r}} \int e^{-i\mathbf{a}(\xi) w\cdot\frac{n}{|n|}} \hat{\chi}(n,\xi,t-r) d\xi\Big|^2 dw,\end{aligned}$$

where $\tilde{\mathbb{E}}$ denotes the expectation with respect to $\tilde{\omega}$.

Employing again Lemma 3.1 with $a = 4|n|^2 r$ and $b(\xi) = \mathbf{a}(\xi)\cdot\frac{n}{|n|}$ we get

$$\begin{aligned}&\int \Big| e^{-\frac{|w|^2}{4|n|^2 r}} \int e^{-i\mathbf{a}(\xi) w\cdot\frac{n}{|n|}} \hat{\chi}(n,\xi,t-r) d\xi\Big|^2 dw \\ &\lesssim \sqrt{a}\int_0^\infty \tau e^{-\tau^2} \iota(\frac{2\tau}{\sqrt{a}}) d\tau \|\hat{\chi}(n,\cdot,t-r)\|_2^2 \\ &\lesssim a^{\frac{1-\theta}{2}} \int_0^\infty \tau^{\theta+1} e^{-\tau^2} d\tau \|\hat{\chi}(n,\cdot,t-r)\|_2^2 \lesssim (|n|^2 r)^{\frac{1-\theta}{2}} \|\hat{\chi}(n,\cdot,t-r)\|_2^2,\end{aligned}$$

and, hence,

$$\mathbb{E}\Big| \int e^{-i\mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r))\cdot n} \hat{\chi}(n,t-r) d\xi\Big|^2 \lesssim (|n|^2 r)^{-\frac{\theta}{2}} \mathbb{E}\|\hat{\chi}(n,\cdot,t-r)\|_2^2.$$

Combining all the above estimates we find

$$\mathbb{E}\int_0^T |\hat{u}^1|^2(n,t)dt \lesssim \int_0^T \int_0^t \bar{\omega}_n e^{-\bar{\omega}_n r}(|n|^2 r)^{-\frac{\theta}{2}} \mathbb{E}\|\hat{\chi}(n,\cdot,t-r)\|_2^2 dr dt.$$

Young's inequality then yields

$$\mathbb{E}\int_0^T |\hat{u}^1|^2(n,t)dt \lesssim \int_0^T \bar{\omega}_n e^{-\bar{\omega}_n r}(|n|^2 r)^{-\frac{\theta}{2}} dr \int_0^T \mathbb{E}\|\hat{\chi}(n,\cdot,r)\|_2^2 dr,$$

and, in view of the fact that, for $\theta \in [0,1]$,

$$\int_0^T \bar{\omega}_n e^{-\bar{\omega}_n r} r^{-\frac{\theta}{2}} dr \le \bar{\omega}_n^{1+\frac{\theta}{2}} \int_{\mathbb{R}} e^{-\bar{\omega}_n r}(\bar{\omega}_n r)^{-\frac{\theta}{2}} dr = \bar{\omega}_n^{\frac{\theta}{2}} \int_{\mathbb{R}} e^{-r} r^{-\frac{\theta}{2}} dr < \infty,$$

we conclude that, for $n \in \mathbb{Z}^N \setminus \{0\}$,

$$\mathbb{E}\int_0^T |\hat{u}^1|^2(n,t)dt \lesssim \gamma^{\frac{\theta}{2}}(|n|^{2\alpha-2} + |n|^{-2})^{\frac{\theta}{2}} \mathbb{E}\|\hat{\chi}(n,\cdot,\cdot)\|^2_{L^2(\mathbb{R}\times[0,T])}, \tag{3.16}$$

while, in view of (2.6),

$$\begin{aligned}\hat{u}^1(0,t) &= \int_{\mathbb{T}^N} u^1(t,x)dx = \int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} \gamma e^{-\gamma(t-s)} \hat{\chi}(0,\xi,s) d\xi dx ds \\ &= \int_0^t \gamma e^{-\gamma(t-s)} \int_{\mathbb{T}^N} u(x,s) dx ds = 0.\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}\|\hat{\chi}\|^2_{L^2(\mathbb{Z}^N\times\mathbb{R}\times[0,T])} &= \sum_n \mathbb{E}\int_0^T \int_{\mathbb{R}} |\hat{\chi}(n,\xi,t)|^2 d\xi dt = \mathbb{E}\|\chi\|^2_{L^2(\mathbb{T}^N\times\mathbb{R}\times[0,T])} \\ &\le \mathbb{E}\|\chi\|_{L^1(\mathbb{T}^N\times\mathbb{R}\times[0,T])} = \mathbb{E}\int_0^T \|u(t)\|_1 dt,\end{aligned}$$

we conclude that

$$\mathbb{E}\int_0^T \|u^1(t)\|_2^2 dt \lesssim \gamma^{\frac{\theta}{2}} \mathbb{E}\int_0^T \|u(t)\|_1 dt. \tag{3.17}$$

3.4. **The estimate of Q.** For $\lambda \ge 0$ and $\varphi \in L^\infty(\Omega \times [0,T]; C^\infty(\mathbb{T}^N))$ (in what follows, unless necessary, we do not display the dependence of φ in ω), let

$$\begin{aligned}< (-\Delta)^{\frac{\lambda}{2}} Q(t), \varphi(t) > &:= \int_0^t \int \partial_\xi (S_{A_\gamma(\xi)}(s,t)((-\Delta)^{\frac{\lambda}{2}}\varphi(t))(x) dm(x,\xi,s) \\ &= \int_0^t \int \boldsymbol{a}'(\xi)(\boldsymbol{\beta}_t - \boldsymbol{\beta}_s) \cdot DS^*_{A_\gamma(\xi)}(s,t)((-\Delta)^{\frac{\lambda}{2}}\varphi(t))(x) dm(x,\xi,s),\end{aligned}$$

where the second equality is immediate from the definitions of $S_{A_\gamma(\xi)}$ and $S^*_{A_\gamma(\xi)}$.

To conclude the ongoing proof it is enough to take $\lambda = 0$, while $\lambda > 0$ is needed for Theorem 1.3. Since the arguments are similar, we present here the general case.

We aim for a bound of the $L^1(\mathbb{T}^N \times \mathbb{R} \times [0,T] \times \Omega)$ norm of u and, hence, we need to estimate

$$\begin{aligned}&\mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} Q(t), \varphi(t)\rangle dt = \\ &\mathbb{E}\int_0^T \int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} \boldsymbol{a}'(\xi)(\boldsymbol{\beta}(t) - \boldsymbol{\beta}(s)) \cdot DS^*_{A_\gamma(\xi)}(s,t)((-\Delta)^{\frac{\lambda}{2}}\varphi(t))(x) dm(x,\xi,s) dt.\end{aligned}$$

In view of [20, Lemma 9], for any $\psi \in C^\infty(\mathbb{T}^N)$, we have

$$\begin{aligned}
&\|D(S^*_{A_\gamma(\xi)}(s,t)(-\Delta)^{\frac{\lambda}{2}}\psi)\|_\infty = \|D\left(e^{-\gamma(t-s)B}(-\Delta)^{\frac{\lambda}{2}}\psi(\cdot)\right)(\cdot - \mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)))\,\|_\infty \\
&= \|D\left(e^{-\gamma(t-s)B}(-\Delta)^{\frac{\lambda}{2}}\psi(\cdot)\right)(\cdot - \mathbf{a}(\xi)(\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)))\,\|_\infty \\
&\le e^{-\gamma(t-s)}\|De^{-\gamma(t-s)(-\Delta)^\alpha}(-\Delta)^{\frac{\lambda}{2}}\psi\|_\infty \lesssim e^{-\gamma(t-s)}(\gamma(t-s))^{-\frac{\lambda+1}{2\alpha}}\|\psi\|_\infty.
\end{aligned}$$

Let $\mu_{\alpha,\lambda} := \frac{\lambda+1}{2\alpha}$. It follows that, for all $\varphi \in L^\infty(\Omega\times[0,T];C^\infty(\mathbb{T}^N))$,

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}}Q(t),\varphi(t)\rangle dt \\
&= \mathbb{E}\int_0^T\int_0^t\int_{\mathbb{T}^N\times\mathbb{R}} \partial_\xi(S_{A_\gamma(\xi)}(s,t)(-\Delta)^{\frac{\lambda}{2}}\varphi(t))dm(x,\xi,s)dt \\
&\lesssim \|\varphi\|_\infty\mathbb{E}\int_0^T\int_0^t|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|e^{-\gamma(t-s)}(\gamma(t-s))^{-\mu_{\alpha,\lambda}}\int_{\mathbb{T}^N\times\mathbb{R}}(1+|\xi|^m)dm(x,\xi,s)dt.
\end{aligned}$$

At this point, as in the previous step, we take conditional expectations and use that $\boldsymbol{\beta}(t)-\boldsymbol{\beta}(t-r)$ is independent of $\mathcal{F}_{t-r}$. The argument is, however, more complicated due to the lack of regularity of m which requires the use of an additional regularization layer.

Let l be the random measure on $[0,T]$ defined by

$$l([s,t]) := \int_s^t\int_{\mathbb{T}^N\times\mathbb{R}}(1+|\xi|^m)dm(x,\xi,r).$$

Since m is a kinetic measure, we can choose $\mathcal{F}_t$-adapted approximations l^ε such that, $\mathbb{P}$-a.s., $l^\varepsilon \in C([0,T])$ and $l^\varepsilon \to l$ weak $\star$. Hence,

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^t|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|e^{-\gamma(t-s)}(\gamma(t-s))^{-\mu_{\alpha,\lambda}}\int_{\mathbb{T}^N\times\mathbb{R}}(1+|\xi|^m)dm(x,\xi,s)dt \\
&= \mathbb{E}\int_0^T\int_0^t|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|e^{-\gamma(t-s)}(\gamma(t-s))^{-\mu_{\alpha,\lambda}}dl(s)dt \\
&= \lim_{\varepsilon\to0}\mathbb{E}\int_0^T\int_0^t|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|e^{-\gamma(t-s)}(\gamma(t-s))^{-\mu_{\alpha,\lambda}}l^\varepsilon(s)dsdt.
\end{aligned}$$

Using the independence of $\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)$ from $\mathcal{F}_s$ and the $\mathcal{F}_s$-measurability of l^ε we find

$$\begin{aligned}
\mathbb{E}|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|l^\varepsilon(s) &= \mathbb{E}[\mathbb{E}|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|l^\varepsilon(s)|\mathcal{F}_s] = \mathbb{E}|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|\mathbb{E}l^\varepsilon(s) \\
&= (t-s)^{\frac{1}{2}}\mathbb{E}l^\varepsilon(s).
\end{aligned}$$

Employing again Young's inequality we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^t|\boldsymbol{\beta}(t)-\boldsymbol{\beta}(s)|e^{-\gamma(t-s)}(\gamma(t-s))^{-\mu_{\alpha,\lambda}}\int_{\mathbb{T}^N\times\mathbb{R}}(1+|\xi|^m)dm(x,\xi,s)dt \\
&= \lim_{\varepsilon\to0}\int_0^T\int_0^t(t-s)^{\frac{1}{2}}e^{-\gamma(t-s)}(\gamma(t-s))^{-\mu_{\alpha,\lambda}}\mathbb{E}l^\varepsilon(s)dsdt \\
&\le \lim_{\varepsilon\to0}\int_0^T t^{\frac{1}{2}}e^{-\gamma t}(\gamma t)^{-\mu_{\alpha,\lambda}}dt\int_0^T\mathbb{E}l^\varepsilon(t)dt \\
&\le \int_0^T t^{\frac{1}{2}}e^{-\gamma t}(\gamma t)^{-\mu_{\alpha,\lambda}}dt\,\mathbb{E}\int_0^T\int_{\mathbb{T}^N\times\mathbb{R}}(1+|\xi|^m)dm(x,\xi,t).
\end{aligned}$$

In conclusion,

$$\begin{aligned}&\mathbb{E}\int_0^T \langle (-\Delta)^{\frac{\lambda}{2}} Q(t), \varphi(t)\rangle dt\\ &\lesssim \|\varphi\|_\infty \gamma^{-\mu_{\alpha,\lambda}} \int_0^T e^{-\gamma t} t^{\frac{1}{2}-\mu_{\alpha,\lambda}} dt \mathbb{E}\int_0^T \int_{\mathbb{T}^N\times\mathbb{R}} (1+|\xi|^m) dm(x,\xi,s).\end{aligned}$$

Moreover note that, if $\mu_{\alpha,\lambda} < \frac{3}{2}$, then

$$\begin{aligned}\int_0^T t^{\frac{1}{2}-\mu_{\alpha,\lambda}} e^{-\gamma t} dt &= \gamma^{-\frac{1}{2}+\mu_{\alpha,\lambda}} \int_0^T (\gamma t)^{\frac{1}{2}-\mu_{\alpha,\lambda}} e^{-\gamma t} dt = \gamma^{-\frac{3}{2}+\mu_{\alpha,\lambda}} \int_{\mathbb{R}_+} t^{\frac{1}{2}-\mu_{\alpha,\lambda}} e^{-t} dt\\ &\le C\gamma^{-\frac{3}{2}+\mu_{\alpha,\lambda}}.\end{aligned}$$

We use next Lemma 2.3 and, in view of all the above, we get

$$\mathbb{E}\int_0^T \langle (-\Delta)^{\frac{\lambda}{2}} Q(t), \varphi(t)\rangle dt \lesssim \|\varphi\|_\infty \gamma^{-\frac{3}{2}} (\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}). \tag{3.18}$$

As mentioned earlier, for an estimate on the energy decay it is enough to consider $\lambda = 0$ in which case $\mu_{\alpha,0} = \frac{1}{2\alpha}$ and, assuming $\alpha > \frac{1}{3}$, we obtain the estimate

$$\mathbb{E}\int_0^T \langle Q(t), \varphi(t)\rangle dt \lesssim \gamma^{-\frac{3}{2}} \|\varphi\|_\infty (\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}). \tag{3.19}$$

3.5. **The proof of Theorem 1.1.** Since, for all $\varphi \in L^\infty(\Omega\times[0,T]; C^\infty(\mathbb{T}^N))$,

$$\langle u(t), \varphi\rangle = \langle u^0(t), \varphi\rangle + \langle u^1(t), \varphi\rangle + \langle Q(t), \varphi\rangle,$$

we find

$$\begin{aligned}&\mathbb{E}\int_0^T \langle u(t), \varphi(t)\rangle dt\\ &\le \|\varphi\|_\infty \mathbb{E}\int_0^T [\|u^0(t)\|_1 + \|u^1(t)\|_1] dt + \mathbb{E}\int_0^T \langle Q(t), \varphi(t)\rangle dt\\ &\le 2\|\varphi\|_\infty T^{\frac{1}{2}} \left(\mathbb{E}\int_0^T [\|u^0(t)\|_2^2 + \|u^1(t)\|_2^2] dt\right)^{\frac{1}{2}} + \mathbb{E}\int_0^T \langle Q(t), \varphi(t)\rangle dt.\end{aligned}$$

The inequality above as well as (3.15), (3.17) and (3.19) and the Cauchy-Schwartz inequality yield, for some $C > 0$, the following sequence of inequalities.

$$\begin{aligned}&\mathbb{E}\int_0^T \langle u(t), \varphi(t)\rangle dt\\ &\le C\|\varphi\|_\infty T^{\frac{1}{2}} (\gamma^{-\frac{2-\theta}{2}} \|u_0\|_1 + \gamma^{\frac{\theta}{2}} \mathbb{E}\int_0^T \|u(t)\|_1 dt)^{\frac{1}{2}} + \gamma^{-\frac{3}{2}} (\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2})\\ &\le C\|\varphi\|_\infty T^{\frac{1}{2}} (\gamma^{-\frac{2-\theta}{2}} \|u_0\|_1 + 4TC^2\gamma^\theta + \frac{1}{4TC^2} (\mathbb{E}\int_0^T \|u(t)\|_1 dt)^2)^{\frac{1}{2}}\\ &\qquad + \gamma^{-\frac{3}{2}} (\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2})\\ &\le \|\varphi\|_\infty [C(T^{\frac{1}{2}} \gamma^{-\frac{2-\theta}{4}} \|u_0\|_1^{\frac{1}{2}} + T\gamma^{\frac{\theta}{2}} + \gamma^{-\frac{3}{2}} (\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2})) + \frac{1}{2}\mathbb{E}\int_0^T \|u(t)\|_1 dt],\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}\int_0^T \|u(t)\|_1 dt\\&\le C(T^{\frac{1}{2}}\gamma^{-\frac{2-\theta}{4}}\|u_0\|_1^{\frac{1}{2}} + T\gamma^{\frac{\theta}{2}} + \gamma^{-\frac{3}{2}}(\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2})) + \frac{1}{2}\mathbb{E}\int_0^T \|u(t)\|_1 dt\end{aligned}$$

and, thus,

$$\mathbb{E}\frac{1}{T}\int_0^T \|u(t)\|_1 dt \lesssim T^{-\frac{1}{2}}\gamma^{-\frac{2-\theta}{4}}\|u_0\|_1^{\frac{1}{2}} + \gamma^{\frac{\theta}{2}} + \gamma^{-\frac{3}{2}}T^{-1}(\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}),$$

and, since $t \mapsto \|u(t)\|_1$ in non-increasing,

$$\begin{aligned}\mathbb{E}\|u(T)\|_1 &\le \mathbb{E}\frac{1}{T}\int_0^T \|u(t)\|_1 dt\\&\lesssim T^{-\frac{1}{2}}\gamma^{-\frac{2-\theta}{4}}\|u_0\|_1^{\frac{1}{2}} + \gamma^{\frac{\theta}{2}} + \gamma^{-\frac{3}{2}}T^{-1}(\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}).\end{aligned}$$

Choosing $\gamma = T^{-a}$ we get

$$\mathbb{E}\|u(T)\|_1 \lesssim T^{-\frac{1}{2}+a(\frac{2-\theta}{4})}\|u_0\|_1^{\frac{1}{2}} + 2T^{-\frac{a\theta}{2}} + T^{\frac{3a}{2}-1}(\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}).$$

Finally, we let $a = \frac{2}{3+\theta}$ and obtain, assuming that $T \ge 1$,

$$\mathbb{E}\|u(T)\|_1 \lesssim T^{-\frac{\theta}{3+\theta}}\left(\|u_0\|_1^{\frac{1}{2}} + 1 + \|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}\right);$$

note that the rate is independent of the choice of α.

It follows that δ_0 is the unique invariant measure and, since $t \mapsto \|u(t)\|_1$ is pathwise non-increasing, we have that a.s.

$$\lim_{t\to\infty} \|u(t)\|_1 = 0. \tag{3.20}$$

3.6. **The random attractor.** Since $L^1(\Omega)$ is separable, there exist a countable set $M \subset L^1(\Omega) \cap L^2(\Omega)$, dense in $L^1(\Omega)$ and $\Omega_0 \subseteq \Omega$ of full $\mathbb{P}$-measure, such that, for all $u_0 \in M$ and all $\omega \in \Omega_0$, as $t \to \infty$,

$$\|\varphi(t,\omega)u_0\|_1 = \|u^{u_0}(t,\omega)\|_1 \to 0.$$

Fix $\varepsilon > 0$ and a compact $K \subset L^1(\Omega)$. There exist a positive integer $L = L(K,\varepsilon)$ and $\{x_i\}_{i=1}^L \subset M$ such that $K \subset \bigcup_{i=1}^{L} B(x_i, \frac{1}{2}\varepsilon)$. It follows that, for all $\omega \in \Omega_0$, there exists $t_0(\varepsilon) > 0$ such that, if $t \ge t_0(\varepsilon)$,

$$\|\varphi(t,\omega)x_i\|_1 \le \frac{\varepsilon}{2}.$$

The pathwise contraction property in $L^1(\Omega)$ then yields that $\varphi(t,\omega)(K)$ has a cover by balls of radius $\frac{\varepsilon}{2}$ centered at $\{\varphi(t,\omega)x_i\}_{i=1}^L$ and, thus, for all $\omega \in \Omega_0$ and $t \ge t_0(\varepsilon)$,

$$\sup_{x\in K} \|\varphi(t,\omega)x\|_1 \le \varepsilon.$$

Hence, 0 is a forward random attractor.

Moreover, in view of the $\mathbb{P}$-invariance of θ_t, $\sup_{x\in K}\|\varphi(t,\theta_{-t}\omega)x\|_1 \to 0$ in probability. Since $t \mapsto \|\varphi(t,\theta_{-t}\omega)x\|_1$ is pathwise non-increasing, the above observation also implies, that, $\mathbb{P}$-a.s.,

$$\sup_{x\in K} \|\varphi(t,\theta_{-t}\omega)x\|_1 \to 0$$

and, thus, 0 is also a pullback random attractor.

4. Rate of convergence for deterministic SCL.

The techniques used in the proof of Theorem 1.1 may also be employed in the deterministic setting. In fact, the arguments are somewhat simpler and the estimates can be taken from [20] since no difficulties arise when the flux is not stochastic.

Proof of Theorem 1.2. We follow the proof Theorem 1.1 noting that instead of applying Lemma 3.1 we use [8, Lemma 2.4] which yields the estimates:

$$\int_0^T \|u^0(t)\|_{H^{\alpha+(\frac{1}{2}-\alpha)\theta}} dt \lesssim \gamma^{\theta-1}\|u_0\|_{L^1} dt,$$

$$\int_0^T \|u^1(t)\|_{H^{(\frac{1}{2}-\alpha)\theta}} dt \lesssim \gamma^{\theta} \int_0^T \|u^1(t)\|_{L^1} dt,$$

and

$$\|Q\|_{L^1([0,T];W^{\lambda,1})} \lesssim \gamma^{-2}(\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}).$$

Proceeding as before we get

$$\|u(T)\|_1 \lesssim T^{-\frac{1}{2}}\gamma^{-\frac{1-\theta}{2}}\|u_0\|_1^{\frac{1}{2}} + \gamma^{\theta} + \gamma^{-2}T^{-1}(\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}).$$

Choosing $\gamma = T^{-a}$ we find

$$\|u(T)\|_1 \lesssim T^{-\frac{1}{2}+a(\frac{1-\theta}{2})}\|u_0\|_1^{\frac{1}{2}} + 2T^{-a\theta} + T^{2a-1}(\|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}),$$

which, for $a = \frac{1}{\theta+2}$ and for all $T \geq 1$, yields

$$\|u(T)\|_1 \lesssim T^{-\frac{\theta}{\theta+2}}(1 + \|u_0\|_{m+2}^{m+2}).$$

□

5. Regularity of pathwise entropy solutions.

In this section we present the proof of Theorem 1.3. Since the argument is long, we divide it into three steps. In the first step, we obtain a bound, which we use in the second step to bootstrap and, hence, to improve the estimate. The conclusion is then shown in the last step.

Proof of Theorem 1.3. Step 1: We assume that, for some $\tau \in [0,1]$, $\chi = \chi(u) \in L^2(\mathbb{R} \times [0,T] \times \Omega; H^{\tau}(\mathbb{R}^N))$, and note that (3.14) with $\gamma = 1$ yields

$$\mathbb{E}\int_0^T \|u^0(t)\|_{H^{\theta+\alpha(2-\theta)}}^2 dt \lesssim \|u_0\|_1.$$

Moreover, multiplying (3.16) by $|n|^{\tau}$ and taking $\gamma = 1$, we obtain

$$\mathbb{E}\int_0^T \|u^1(t)\|_{H^{\theta(1-\alpha)+\tau}}^2 dt \lesssim \|\chi\|_{L^2(\mathbb{R}\times[0,T]\times\Omega;H^{\tau}(\mathbb{R}^N))}^2.$$

As in the last section, for all $\varphi \in L^\infty([0,T]\times\Omega; C^\infty(\mathbb{T}^N))$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} u(t), \varphi(t)\rangle dt \\
&= \mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} u^0(t), \varphi(t)\rangle + \langle(-\Delta)^{\frac{\lambda}{2}} u^1(t), \varphi(t)\rangle dt + \mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} Q(t), \varphi(t)\rangle dt \\
&\le \|\varphi\|_\infty \mathbb{E}\int_0^T [\|u^0(t)\|_{W^{\lambda,1}} + \|u^1(t)\|_{W^{\lambda,1}}] dt + \mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} Q(t), \varphi(t)\rangle dt \\
&\le \|\varphi\|_\infty T^{\frac12}(\mathbb{E}\int_0^T [\|u^0(t)\|^2_{W^{\lambda,1}} + \|u^1(t)\|^2_{W^{\lambda,1}}] dt)^{\frac12} + \mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} Q(t), \varphi(t)\rangle dt.
\end{aligned}$$

We assume for the moment that, for $\lambda \in (0,1)$,

$$H^{\theta+\alpha(2-\theta)}, H^{\theta(1-\alpha)+\tau} \hookrightarrow W^{\lambda,1}, \tag{5.1}$$

where $\hookrightarrow$ denotes continuous embedding, and use that $t \to \|u(t)\|_1$ is non-increasing in t, the above estimates and (3.18) with $\gamma = 1$, to get

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle(-\Delta)^{\frac{\lambda}{2}} u(t), \varphi(t)\rangle dt \\
&\lesssim \|\varphi\|_\infty (T^{\frac12}(\|u_0\|_1 + \|\chi\|^2_{L^2(\mathbb{R}\times[0,T]\times\Omega; H^\tau(\mathbb{R}^N))})^{\frac12} + \|u_0\|_2^2 + \|u_0\|_{m+2}^{m+2}) \\
&\lesssim \|\varphi\|_\infty (T^{\frac12}(1+T^{\frac12})\|u_0\|_1^{\frac12} + \|\chi\|_{L^2(\mathbb{R}\times[0,T]\times\Omega; H^\tau(\mathbb{R}^N))} + \|u_0\|_{m+2}^{m+2}) \\
&\lesssim \|\varphi\|_\infty (1 + \|\chi\|^2_{L^2(\mathbb{R}\times[0,T]\times\Omega; H^\tau(\mathbb{R}^N))} + \|u_0\|_{2+m}^{2+m}),
\end{aligned} \tag{5.2}$$

as long as

$$\mu_{\alpha,\lambda} = \frac{\lambda+1}{2\alpha} < \frac{3}{2}. \tag{5.3}$$

Next we take $u_0 \in BV$ and recall that, in view of Proposition 2.2, $u(t) \in BV$ for all $t > 0$, Then (5.2) implies

$$\mathbb{E}\int_0^T \|u(t)\|_{W^{\lambda,1}} dt \le C(1 + \|\chi\|^2_{L^2(\mathbb{R}\times[0,T]\times\Omega; H^\tau(\mathbb{R}^N))} + \|u_0\|_{2+m}^{2+m}). \tag{5.4}$$

The general case $u_0 \in L^{2+m}(\mathbb{T}^N)$ follows easily using approximations and the continuity of the solutions with respect to the initial condition.
From (5.4) it follows

$$\|\chi\|_{L^1(\mathbb{R}\times[0,T]\times\Omega; W^{\lambda,1}(\mathbb{R}^N))} \le C(1 + \|\chi\|^2_{L^2(\mathbb{R}\times[0,T]\times\Omega; H^\tau(\mathbb{R}^N))} + \|u_0\|_{2+m}^{2+m}). \tag{5.5}$$

Interpolating (5.4) with the obvious bound $\|\chi\|_{L^\infty(\mathbb{R}\times[0,T]\times\Omega\times\mathbb{R}^N)} \le 1$ yields

$$\|\chi\|^2_{L^2(\mathbb{R}\times[0,T]\times\Omega; W^{\frac{\lambda}{2},2}(\mathbb{R}^N))} \le C(1 + \|\chi\|^2_{L^2(\mathbb{R}\times[0,T]\times\Omega; H^\tau(\mathbb{R}^N))} + \|u_0\|_{2+m}^{2+m}). \tag{5.6}$$

It remains to choose λ and α to justify the above calculations. To this end, we first note that (5.3) is equivalent to $\lambda < 3\alpha - 1$, and noting that $W^{r,p} \hookrightarrow W^{s,q}$ if $r \ge s$ and $\frac{r}{N} - \frac{1}{p} \ge \frac{s}{N} - \frac{1}{q}$, we further require that $\theta(1-\alpha) + \tau \ge \lambda$, which gives

$$\frac{\theta(1-\alpha)+\tau}{N} - \frac{1}{2} \ge \frac{\lambda}{N} - 1,$$

and, hence, the requirement that

$$\lambda < (3\alpha - 1) \wedge (\theta(1-\alpha) + \tau).$$

Maximizing the right hand side yields $\alpha = \frac{\theta+1+\tau}{\theta+3} \in (0,1)$ and we obtain

$$\lambda < 3\alpha - 1 = \frac{2\theta}{\theta+3} + \frac{3}{\theta+3}\tau. \tag{5.7}$$

We note that, since $\tau \le 1$, we have $2\alpha \ge \tau$, and, hence,

$$\frac{\theta + \alpha(2-\theta)}{N} - \frac{1}{2} \ge \frac{\theta(1-\alpha)+\tau}{N} - \frac{1}{2}.$$

It follows now that $H^{\theta+\alpha(2-\theta)} \hookrightarrow W^{\lambda,1}$.

Step 2: Bootstrapping
We bootstrap the arguments in the first step. Using (5.6) with $\tau = 0$ and noting that

$$\|\chi\|_{L^2(\mathbb{R}\times[0,T]\times\Omega;L^2(\mathbb{R}^N))} \le \|\chi\|_{L^1(\mathbb{R}\times[0,T]\times\Omega;L^1(\mathbb{R}^N))} \lesssim \|u_0\|_1$$

gives, for $\lambda < \lambda_0 := \frac{2\theta}{\theta+3}$,

$$\|\chi\|_{L^2(\mathbb{R}\times[0,T]\times\Omega;W^{\frac{\lambda}{2},2}(\mathbb{R}^N))} \lesssim (1 + \|u_0\|_{2+m}^{2+m}).$$

Next we iterate this argument and get, from (5.5), that for

$$\lambda < \lambda_n := \frac{2\theta}{\theta+3} + \frac{3}{\theta+3}\frac{\lambda_{n-1}}{2},$$

and some constant C_n,

$$\|\chi\|_{L^1(\mathbb{R}\times[0,T]\times\Omega;W^{\lambda,1}(\mathbb{R}^N))} \le C_n(1 + \|u_0\|_{2+m}^{2+m}).$$

Since, as $n \to \infty$,

$$\lambda_n \uparrow \lambda_* = \frac{4\theta}{2\theta+3},$$

using (5.4) we obtain that, for any

$$\lambda < \lambda_* = \frac{4\theta}{2\theta+3},$$

there is some $C > 0$ such that

$$\mathbb{E}\int_0^T \|u(t)\|_{W^{\lambda,1}} dt \le C(1 + \|u_0\|_{2+m}^{2+m}). \tag{5.8}$$

Step 3: Conclusion
In view of (5.8), for each $\delta > 0$, there is a $\delta_0 \in [0,\delta]$ such that $\mathbb{E}\|u(\delta_0)\|_{W^{\lambda,1}} < \infty$. The spatial homogeneity and the contraction property with respect to the initial condition imply that $t \mapsto \|u(t)\|_{W^{\lambda,1}}$ is non-increasing, and, hence,

$$\sup_{t\ge\delta} \mathbb{E}\|u(t)\|_{W^{\lambda,1}} < \infty.$$

□

6. Pathwise quasi-solutions and regularization by noise.

6.1. **Deterministic background.** We recall (see [15,47]) that $u \in C([0,T];L^1(\mathbb{T})) \cap L^\infty(\mathbb{T}\times[0,T])$ is a quasi-solution to the deterministic Burgers' equation (1.10) if, for every convex entropy-entropy flux pair (η,q), $-\mu_\eta := \partial_t\eta(u) + \partial_x q(u)$ is a Radon measure. Note that the difference between quasi- and entropy solutions is that for the latter μ_η is, in addition, non-negative.

Moreover (see, for example, Benilan and Kruzkov [5]), $u \in C([0,T];L^1(\mathbb{T})) \cap L^\infty(\mathbb{T}\times[0,T])$ is an entropy sub-solution (resp. super-solution) to (1.10), if, for every convex entropy-entropy flux pair (η,q) with η nondecreasing (resp. nonincreasing), $-\mu_\eta := \partial_t\eta(u) + \partial_x q(u)$ is a nonnegative Radon measure.

The next claim is about the existence of quasi-solutions to (1.10).

Lemma 6.1. *(i) Any entropy sub- or super-solution to* (1.10) *is also a quasi-solution.*
(ii) An entropy solution to $\partial_t u + \partial_x u^2 = g$, *with source* $g \in L^1_{loc}(\mathbb{T})$, *is a quasi-solution to* (1.10).

Proof. The second assertion is proved in [15,47]. For the first claim we consider here the case of u being an entropy sub-solution, since the argument can be easily modified for super-solutions.

Let (η,q) be a convex entropy-entropy flux pair. In order to use the sub-solution property of u, we show that it is possible to write (η,q) as the difference of two convex entropy-entropy flux pairs (η_1,q_1) and (η_2,q_2) with η_1 and η_2 nonincreasing. We also remark that, since $u \in L^\infty(\mathbb{T}\times[0,T])$, it is enough to obtain this decomposition in $[-\|u\|_\infty, \|u\|_\infty]$.

Fix $s_0 := -(\|u\|_\infty + 1)$. Since η is convex, there exists some $p_0 \in \mathbb{R}$ such that, for all $s \in \mathbb{R}$,

$$\eta(s) \geq \eta(s_0) + p_0(s - s_0).$$

Let $\eta_1, \eta_2 : \mathbb{R} \to \mathbb{R}$ be the convex and nondecreasing functions given by

$$\eta_1(s) := \begin{cases} \eta(s) + |p_0|(s-s_0) \text{ if } \ s \geq s_0, \\ \eta(s_0) \text{ if } s < s_0, \end{cases} \qquad \text{and} \quad \eta_2(s) := |p_0|(s-s_0).$$

It follows that

$$\eta = \eta_1 - \eta_2 \ \text{ and } q = q_1 - q_2 \ \text{ in } \ [s_0, \infty).$$

Then $-\mu_\eta := \partial_t\eta(u) + \partial_x q(u)$ is a Radon measure, since

$$\begin{aligned} -\mu_\eta &:= \partial_t\eta(u) + \partial_x q(u) = \partial_t\eta_1(u) + \partial_x q_1(u) - (\partial_t\eta_2(u) + \partial_x q_2(u)) \\ &= (-\mu_{\eta_1}) - (-\mu_{\eta_2}), \end{aligned}$$

where $-\mu_{\eta_1}, -\mu_{\eta_2}$ are the nonnegative Radon measures coming from the sub-solution property of u with entropies η_1, η_2 respectively. ☐

Although in general quasi-solutions need not be weak solutions, [16] provides an explicit example of a quasi-solution to (1.10) that is a weak but not an entropy solution. We remark that in [16], which is about the Cauchy problem in $\mathbb{R}$, is not required for quasi-solutions to be continuous in $L^1(\mathbb{T})$. It follows, however, from the explicit construction therein, that the same argument also works in the periodic setting and yields a quasi-solution $u \in C([0,T];L^1(\mathbb{T}))$ to (1.10) with the above properties.

The kinetic formulation of a quasi-solution u to (1.10) (see [15]) is that the χ defined as in (2.1) satisfies, in the sense of distributions,

$$\partial_t\chi + 2\xi\partial_x\chi = \partial_\xi m, \tag{6.1}$$

where m is a Radon measure on $\mathbb{T}\times\mathbb{R}\times[0,\infty)$, which is supported in $[-\|u\|_\infty, \|u\|_\infty]$ in the ξ-variable and has finite total variation $|m|$ in $\mathbb{T}\times\mathbb{R}\times[0,T]$, for each $T > 0$.

Once there is a kinetic formulation, it is immediate that quasi-solutions can also be described using the convolution along characteristics, as it was done in Section 2.1 for entropy solutions, that is to assume (2.5) with m as above.

Finally, it is straightforward that all the previous statements about quasi-, sub- and super-solutions extend to the time-dependent deterministic Burger's equation

$$\partial_u + \partial_x u^2 \dot{\beta} = 0 \text{ in } \mathbb{T} \times (0, \infty),$$

for any smooth path β.

6.2. **Pathwise quasi-solutions.** Using the convolution by characteristics formulation, we introduce now the notion of quasi-solutions to (1.10) to the stochastic Burger's equation (1.9). For the definition we consider kinetic Radon measures, which have the same measurability properties as kinetic measures used in the definition of pathwise entropy solutions but need not be nonnegative.

Definition 6.2. A map $u : \mathbb{T}^N \times [0, \infty) \times \Omega \to \mathbb{R}$ such that, for all $T > 0$, $u \in (L^1 \cap L^\infty)(\mathbb{T}^N \times [0, T] \times \Omega)$, and $\mathbb{P}$-a.s. in ω, $u(\cdot, \omega) \in C\big([0, T]; L^1(\mathbb{T}^N)\big) \cap L^\infty(\mathbb{T}^N \times [0, T])$, and $t \mapsto u(t, \cdot)$ is an $\mathcal{F}_t$-adapted process in $L^1(\mathbb{T}^N)$, is a pathwise quasi-solution to (1.9), if there exists a kinetic Radon measure m such that, for all $y \in \mathbb{T}^N$ and $\varrho(x, y, \xi, t) = \rho_0(x - y + \xi(\beta(t) - \beta(s)))$ with $\varrho_0 \in C^\infty(\mathbb{T}^N)$ and all $\varphi \in C_c^\infty([0, T)), \psi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \partial_t \varphi(r) \psi(\xi) (\varrho * \chi)(y, \xi, r) dr d\xi + \int_{\mathbb{R}} \varphi(0) \psi(\xi) (\varrho * \chi)(y, \xi, 0) d\xi \\ &= \int_0^T \int_{\mathbb{T}^N \times \mathbb{R}} \varphi(r) \partial_\xi \left(\psi(\xi) \varrho(x, y, \xi, r)\right) dm(x, \xi, r). \end{aligned} \tag{6.2}$$

Next we present an example of a pathwise quasi-solution which is not a pathwise entropy solution. Its construction is based on the observation in Lemma 6.1(i) that, in the deterministic setting, entropy sub- and super-solutions are quasi-solutions.

Fix a non-negative $u_0 \in L^\infty(\mathbb{T})$ and a family $(\beta^\varepsilon)_{\varepsilon > 0}$ of smooth paths approximating the Brownian motion β, that is, for $\varepsilon > 0$, $\beta^\varepsilon \in C^\infty([0, \infty))$ and, as $\varepsilon \to 0$, $\beta^\varepsilon \to \beta$ in $C([0, T])$ for all $T \geq 0$, and let u^ε be the nonnegative entropy solution to

$$\begin{cases} \partial_t u^\varepsilon + \partial_x (u^\varepsilon)^2 \dot{\beta}^\varepsilon = u^\varepsilon \text{ in } \mathbb{T} \times (0, \infty), \\ u^\varepsilon = u_0 \text{ on } \mathbb{T} \times \{0\}. \end{cases}$$

Since u^ε is an entropy super-solution to

$$\begin{cases} \partial_t u^\varepsilon + \partial_x (u^\varepsilon)^2 \dot{\beta}^\varepsilon = 0 \text{ in } \mathbb{T} \times (0, \infty), \\ u^\varepsilon = u_0 \text{ on } \mathbb{T} \times \{0\}, \end{cases}$$

in view of Lemma 6.1, u^ε is also a quasi-solution with dissipation measure m^ε having a non-vanishing negative part given by $\chi(u^\varepsilon, \xi) u^\varepsilon$.

It follows from the density arguments in [40], that the u^ε's converge in $C([0, T]; L^1(\mathbb{T}))$ to the pathwise entropy solution to

$$\begin{cases} du + \partial_x u^2 \circ d\beta = u dt \text{ in } \mathbb{T} \times (0, \infty), \\ u = u_0 \text{ on } \mathbb{T} \times \{0\}. \end{cases}$$

Moreover, since the dissipation measures m^ε's have finite total variation, we may extract subsequences $u^{\varepsilon_n} \to u$ and $m^{\varepsilon_n} \rightharpoonup^* m$. Writing (6.2) for u^{ε_n} and m^{ε_n} and then taking the limit yields that u is a quasi-solution to (1.9), with dissipation measure m having non-vanishing negative part $\chi(u, \xi) u$.

We remark that it can be shown that pathwise quasi-solutions do not satisfy, in general, the density property of the pathwise entropy solutions. Indeed quasi-solutions to equations with smooth paths do not converge, in general, to pathwise quasi-solutions.

6.3. **The proof of Theorem 1.4.** We present now the

Proof of Theorem 1.4. We first note that the derivation of (3.8) given in Appendix C does not rely on the assumption that m is a nonnegative measure. Hence, (3.8) also holds for pathwise quasi-solutions. The proof now is the same as the one for Theorem 1.3, with the exception that in the estimation of Q we cannot use Lemma 2.3, since it is not satisfied for quasi-solutions. Instead, we observe (note that here we have $A(u)=u^2$ and $\theta=1$) that

$$\mathbb{E}\int_0^T\langle(-\Delta)^{\frac{\lambda}{2}}Q(t),\varphi(t)\rangle dt\lesssim\|\varphi\|_\infty\gamma^{-\frac{3}{2}}\mathbb{E}\int_0^T\int_{\mathbb{T}^N\times\mathbb{R}}d|m|(x,\xi,s). \tag{6.3}$$

It then follows, as before, that

$$\mathbb{E}\int_0^T\langle(-\Delta)^{\frac{\lambda}{2}}u(t),\varphi(t)\rangle dt\lesssim\|\varphi\|_\infty\left(1+\|u_0\|_2^2+\mathbb{E}|m|([0,T]\times\mathbb{T}\times\mathbb{R})\right),$$

which implies the claim. □

Appendix A. Random dynamical systems and random attractors

We briefly recall some of the concepts used earlier in the proofs and we refer, for example, to Arnold [1], Crauel and Flandoli [13] and Ochs [46], for a comprehensive treatment.

Let (E,d) and $(\Omega,\mathcal{F},\mathbb{P},\theta)$ be respectively a complete, separable metric space and a metric dynamical system, where $(\Omega,\mathcal{F},\mathbb{P})$ is, a not necessarily complete, probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $\theta:=(\theta_t)_{t\in\mathbb{R}}$ a group of jointly measurable maps on $(\Omega,\mathcal{F},\mathbb{P})$ which leaves $\mathbb{P}$ invariant.

A random dynamical system is a measurable map $\varphi:[0,\infty)\times\Omega\times E\to E$ such that, for all $\omega\in\Omega, x\in E$ and $s,t\in[0,\infty)$,

$$\varphi(0,\omega)x=x\text{ and }\varphi(t+s,\omega)=\varphi(t,\theta_s\omega)\circ\varphi(s,\omega).$$

If $x\mapsto\varphi(t,\omega)x$ is continuous for all $t\in\mathbb{R}$, $\omega\in\Omega$, then φ is a continuous RDS.

A family $\{D(\omega)\}_{\omega\in\Omega}$ of non-empty subsets of E is called a random closed (resp. compact) set if it is $\mathbb{P}$-a.s. closed (resp. compact) and $\mathcal{F}$-measurable, that is, for each $x\in E$,

$$\omega\mapsto d(x,D(\omega)):=\inf_{y\in D(\omega)}d(x,y)\text{ is }\mathcal{F}\text{-measurable},$$

and is called φ-invariant, if for all $t\geq0$ and a.s. in $\omega\in\Omega$,

$$\varphi(t,\omega)D(\omega)=D(\theta_t\omega).$$

A random, compact set A is called a pullback random attractor of the RDS φ, if A is φ-invariant, and for every compact set B in E and a.s.,

$$\lim_{t\to\infty}\sup_{x\in B}d(\varphi(t,\theta_{-t}\omega)x,A(\omega))=0. \tag{A.1}$$

If (A.1) is replaced by

$$\lim_{t\to\infty}\sup_{x\in B}d(\varphi(t,\omega)x,A(\theta_t\omega))=0,$$

then A is said to be a forward random attractor.

Appendix B. The proof of Lemma 3.1

We present here the proof of Lemma 3.1.

Proof. We compute the Fourier transform $\hat{\phi}$ of ϕ. Using the elementary fact that $\int e^{-2\pi i z\cdot w}e^{-\frac{|w|^2}{a}}dw = \sqrt{a\pi}e^{-a\pi^2|z|^2}$, we find

$$\begin{aligned}\hat{\phi}(z) &= \int e^{-2\pi i z\cdot w}\phi(w)dw = \int e^{-2\pi i z\cdot w}e^{-\frac{|w|^2}{a}}\int e^{ib(\xi)\cdot w}f(\xi)d\xi dw\\ &= \int\int e^{-2\pi i(z-\frac{1}{2\pi}b(\xi))\cdot w}e^{-\frac{|w|^2}{a}}dwf(\xi)d\xi = \int \sqrt{a\pi}e^{-|\sqrt{a}\pi(\frac{1}{2\pi}b(\xi)-z)|^2}f(\xi)d\xi,\end{aligned}$$

and, hence,

$$\begin{aligned}|\hat{\phi}(z)|^2 &= \left|\int \sqrt{a\pi}e^{-|\sqrt{a}\pi(\frac{1}{2\pi}b(\xi)-z)|^2}f(\xi)d\xi\right|^2\\ &\le a\pi\int e^{-|\sqrt{a}\pi(\frac{1}{2\pi}b(\xi)-z)|^2}d\xi\int e^{-|\sqrt{a}\pi(\frac{1}{2\pi}b(\xi)-z)|^2}f^2(\xi)d\xi.\end{aligned}$$

Next we use the assumption on b which enters in the following straightforward estimate:

$$\begin{aligned}\int e^{-|\sqrt{a}\pi(\frac{1}{2\pi}b(\xi)-z)|^2}d\xi &= \int_0^\infty 2\tau e^{-\tau^2}|\{\xi:\sqrt{a}\pi|\frac{1}{2\pi}b(\xi)-z)|<\tau\}|d\tau\\ &= \int_0^\infty 2\tau e^{-\tau^2}|\{\xi:|b(\xi)-2\pi z|<\frac{2\tau}{\sqrt{a}}\}|d\tau\\ &\le \int_0^\infty 2\tau e^{-\tau^2}\iota(\frac{2\tau}{\sqrt{a}})d\tau.\end{aligned}$$

Hence

$$|\hat{\phi}(z)|^2 \le 2a\pi\int_0^\infty \tau e^{-\tau^2}\iota(\frac{2\tau}{\sqrt{a}})d\tau\int e^{-|\sqrt{a}\pi(\frac{1}{2\pi}b(\xi)-z)|^2}f^2(\xi)d\xi.$$

Integrating the above inequality in z and using that

$$\int e^{-|\sqrt{a}\pi(\frac{1}{2\pi}b(\xi)-z)|^2}dz = \int e^{-a\pi^2|z|^2}dz = \frac{1}{\sqrt{a\pi}}$$

yields

$$\int |\hat{\phi}(z)|^2dz \le 2\sqrt{a\pi}\int_0^\infty \tau e^{-\tau^2}\iota(\frac{2\tau}{\sqrt{a}})d\tau\|f\|_2^2.$$

□

Appendix C. The derivation of equation (3.8)

We give here the details about the derivation of (3.8). Since $u \in C([0,T];L^1(\mathbb{T}^N))$, the definition of pathwise entropy solutions yields that, for all $t \ge 0$, all ϱ given by (2.4) with $\varrho_0 \in C^\infty(\mathbb{T}^N)$ and all

$\varphi \in C^1(\mathbb{T}^N \times [0,t])$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}\times\mathbb{T}^N} \partial_s\varphi(y,s)(\varrho * \chi)(y,\xi,s)dyd\xi ds + \int_{\mathbb{R}\times\mathbb{T}^N} \varphi(y,0)(\varrho * \chi)(y,\xi,0)dyd\xi \\
&= \int_{\mathbb{R}\times\mathbb{T}^N} \varphi(y,t)(\varrho * \chi)(y,\xi,t)dyd\xi \\
&\quad + \int_{\mathbb{T}^N}\int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} \partial_\xi\big(\varphi(y,s)\varrho(x,y,\xi,s)\big)dm(x,\xi,s)dy.
\end{aligned}$$

Replacing φ by $\varphi(y - \boldsymbol{a}(\xi)\cdot\boldsymbol{\beta}_t, s)$ implies

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}\times\mathbb{T}^N} \partial_s\varphi(y,s)\varrho_0(x-y-\boldsymbol{a}(\xi)\cdot(\boldsymbol{\beta}_s-\boldsymbol{\beta}_t))\chi(x,\xi,s)dxdyd\xi ds \\
&+ \int_{\mathbb{R}\times\mathbb{T}^N}\int_{\mathbb{T}^N} \varphi(y,0)\varrho_0(x-y-\boldsymbol{a}(\xi)\cdot(\boldsymbol{\beta}_0-\boldsymbol{\beta}_t))\chi(x,\xi,0)dxdyd\xi \\
&= \int_{\mathbb{R}\times\mathbb{T}^N}\int_{\mathbb{T}^N} \varphi(y,t)\varrho_0(x-y)\chi(x,\xi,t)dxdyd\xi \\
&+ \int_{\mathbb{T}^N}\int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} \partial_\xi\big(\varphi(y,s)\varrho_0(x-y-\boldsymbol{a}(\xi)\cdot(\boldsymbol{\beta}_s-\boldsymbol{\beta}_t))\big)dm(x,\xi,s)dy.
\end{aligned}$$

Given $\varphi \in C^1(\mathbb{T}^N)$ we use next $S^*_{\gamma B}(t-\cdot)\varphi$ as a test function in the formula above. Since

$$\partial_s(S^*_{\gamma B}(t-s))\varphi = \gamma B(S^*_{\gamma B}(t-s))\varphi,$$

we obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}\times\mathbb{T}^N} \gamma B(S^*_{\gamma B}(t-s)\varphi)(y)\varrho_0(x-y-\boldsymbol{a}(\xi)\cdot(\boldsymbol{\beta}_s-\boldsymbol{\beta}_t))\chi(x,\xi,s)dxdyd\xi ds \\
&+ \int_{\mathbb{R}\times\mathbb{T}^N}\int_{\mathbb{T}^N} (S^*_{\gamma B}(t)\varphi)(y)\varrho_0(x-y-\boldsymbol{a}(\xi)\cdot(\boldsymbol{\beta}_0-\boldsymbol{\beta}_t))\chi(x,\xi,0)dxdyd\xi \\
&= \int_{\mathbb{R}\times\mathbb{T}^N}\int_{\mathbb{T}^N} \varphi(y)\varrho_0(x-y)\chi(x,\xi,t)dxdyd\xi \\
&+ \int_{\mathbb{T}^N}\int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} \partial_\xi\left((S^*_{\gamma B}(t-r)\varphi)(y)\varrho_0(x-y-\boldsymbol{a}(\xi)\cdot(\boldsymbol{\beta}_s-\boldsymbol{\beta}_t))\right)dm(x,\xi,s)dy,
\end{aligned}$$

and, in view of (3.5),

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}\times\mathbb{T}^N} \gamma B(S^*_{A_\gamma(\xi)}(s,t)\varphi)(y)\varrho_0(x-y)\chi(x,\xi,s)dxdyd\xi ds \\
&+ \int_{\mathbb{R}\times\mathbb{T}^N}\int_{\mathbb{T}^N} (S^*_{A_\gamma(\xi)}(0,t)\varphi)(y)\varrho_0(x-y)\chi(x,\xi,0)dxdyd\xi \\
&= \int_{\mathbb{R}\times\mathbb{T}^N}\int_{\mathbb{T}^N} \varphi(y)\varrho_0(x-y)\chi(x,\xi,t)dxdyd\xi \\
&+ \int_{\mathbb{T}^N}\int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} \partial_\xi\left((S^*_{A_\gamma(\xi)}(s,t)\varphi)(y)\right)\varrho_0(x-y)dm(x,\xi,s)dy.
\end{aligned}$$

Let ϱ_0^ε be a approximation of the identity. Letting $\varepsilon \to 0$ in the above equation with ϱ_0 replaced by ϱ_0^ε yields

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}^N} \varphi(x)\chi(x,\xi,t)dxd\xi = &\int_{\mathbb{R}\times\mathbb{T}^N} (S^*_{A_\gamma(\xi)}(0,t)\varphi)(x)\chi(x,\xi,0)dxd\xi \\ &+ \int_0^t \int_{\mathbb{R}\times\mathbb{T}^N} \gamma B(S^*_{A_\gamma(\xi)}(s,t)\varphi)(x)\chi(x,\xi,s)dxd\xi ds \\ &- \int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} \partial_\xi \left((S^*_{A_\gamma(\xi)}(s,t)\varphi)(x) \right) dm(x,\xi,s),\end{aligned}$$

which finishes the proof.

References

[1] Arnold, L. *Random dynamical systems.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[2] Avellaneda, M., and E, W. Statistical properties of shocks in Burgers turbulence. *Comm. Math. Phys. 172*, 1 (1995), 13–38.

[3] Bakhtin, Y. The Burgers equation with Poisson random forcing. *Ann. Probab. 41*, 4 (2013), 2961–2989.

[4] Bauzet, C., Vallet, G., and Wittbold, P. The Cauchy problem for conservation laws with a multiplicative stochastic perturbation. *J. Hyperbolic Differ. Equ. 9*, 4 (2013), 661–709.

[5] Bénilan, P., and Kružkov, S. Conservation laws with continuous flux functions. *NoDEA Nonlinear Differential Equations Appl. 3*, 4 (1996), 395–419.

[6] Berthelin, F., and Junca, S. Averaging lemmas with a force term in the transport equation. *J. Math. Pures Appl. (9) 93*, 2 (2010), 113–131.

[7] Boritchev, A. Sharp estimates for turbulence in white-forced generalised Burgers equation. *Geom. Funct. Anal. 23*, 6 (2013), 1730–1771.

[8] Bouchut, F., and Desvillettes, L. Averaging lemmas without time Fourier transform and application to discretized kinetic equations. *Proc. Roy. Soc. Edinburgh Sect. A 129*, 1 (1999), 19–36.

[9] Burgers, J. M. *The Non-Linear Diffusion Equation: Asymptotic Solutions and Statistical Problems.* Lecture series. Springer, 1974.

[10] Chen, G.-Q., Ding, Q., and Karlsen, K. H. On nonlinear stochastic balance laws. *Arch. Ration. Mech. Anal. 204*, 3 (2012), 707–743.

[11] Chen, G.-Q., and Frid, H. Decay of entropy solutions of nonlinear conservation laws. *Arch. Ration. Mech. Anal. 146*, 2 (1999), 95–127.

[12] Chen, G.-Q., and Perthame, B. Large-time behavior of periodic entropy solutions to anisotropic degenerate parabolic-hyperbolic equations. *Proc. Amer. Math. Soc. 137*, 9 (2009), 3003–3011.

[13] Crauel, H., and Flandoli, F. Attractors for random dynamical systems. *Probab. Theory Related Fields 100*, 3 (1994), 365–393.

[14] Dafermos, C. M. Long time behavior of periodic solutions to scalar conservation laws in several space dimensions. *SIAM J. Math. Anal. 45*, 4 (2013), 2064–2070.

[15] De Lellis, C., Otto, F., and Westdickenberg, M. Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Ration. Mech. Anal. 170*, 2 (2003), 137–184.

[16] De Lellis, C., and Westdickenberg, M. On the optimality of velocity averaging lemmas. *Ann. Inst. H. Poincaré Anal. Non Linéaire 20*, 6 (2003), 1075–1085.

[17] Debussche, A., Hofmanová, M., and Vovelle, J. Degenerate parabolic stochastic partial differential equations: Quasilinear case. *preprint* (2013), 1–34.

[18] Debussche, A., and Vovelle, J. Long-time behavior in scalar conservation laws. *Differential Integral Equations 22*, 3-4 (2009), 225–238.

[19] Debussche, A., and Vovelle, J. Scalar conservation laws with stochastic forcing. *J. Funct. Anal. 259*, 4 (2010), 1014–1042.

[20] Debussche, A., and Vovelle, J. Invariant measure of scalar first-order conservation laws with stochastic forcing. *Probab. Theory Related Fields 163*, 3-4 (2015), 575–611.

[21] Dirr, N., and Souganidis, P. E. Large-time behavior for viscous and nonviscous Hamilton-Jacobi equations forced by additive noise. *SIAM J. Math. Anal. 37*, 3 (2005), 777–796 (electronic).

[22] E, W., Khanin, K. M., Mazel, A. E., and Sinaĭ, Y. G. Invariant measures for Burgers equation with stochastic forcing. *Ann. of Math. (2) 151*, 3 (2000), 877–960.

[23] Engquist, B., and E, W. Large time behavior and homogenization of solutions of two-dimensional conservation laws. *Comm. Pure Appl. Math. 46*, 1 (1993), 1–26.

[24] Fedrizzi, E., and Flandoli, F. Noise prevents singularities in linear transport equations. *J. Funct. Anal. 264*, 6 (2013), 1329–1354.

[25] Feng, J., and Nualart, D. Stochastic scalar conservation laws. *J. Funct. Anal. 255*, 2 (2008), 313–373.

[26] Flandoli, F. *Random perturbation of PDEs and fluid dynamic models*, vol. 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.

[27] Flandoli, F., Gubinelli, M., and Priola, E. Well-posedness of the transport equation by stochastic perturbation. *Invent. Math. 180*, 1 (2010), 1–53.

[28] Flandoli, F., Gubinelli, M., and Priola, E. Remarks on the stochastic transport equation with Hölder drift. *Rend. Semin. Mat. Univ. Politec. Torino 70*, 1 (2012), 53–73.

[29] Friz, P. K., and Gess, B. Stochastic scalar conservation laws driven by rough paths. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis* (2015). http://dx.doi.org/10.1016/j.anihpc.2015.01.009.

[30] Gess, B., and Souganidis, P. E. Scalar conservation laws with multiple rough fluxes. *Commun. Math. Sci. 13*, 6 (2015), 1569–1597.

[31] Golse, F., Lions, P.-L., Perthame, B., and Sentis, R. Regularity of the moments of the solution of a transport equation. *J. Funct. Anal. 76*, 1 (1988), 110–125.

[32] Golse, F., and Perthame, B. Optimal regularizing effect for scalar conservation laws. *Rev. Mat. Iberoam. 29*, 4 (2013), 1477–1504.

[33] Hofmanová, M. Degenerate parabolic stochastic partial differential equations. *Stochastic Process. Appl. 123*, 12 (2013), 4294–4336.

[34] Holden, H., and Risebro, N. H. Conservation laws with a random source. *Appl. Math. Optim. 36*, 2 (1997), 229–241.

[35] Iturriaga, R., and Khanin, K. M. Burgers turbulence and random Lagrangian systems. *Comm. Math. Phys. 232*, 3 (2003), 377–428.

[36] Jabin, P.-E., and Perthame, B. Regularity in kinetic formulations via averaging lemmas. *ESAIM Control Optim. Calc. Var. 8* (2002), 761–774 (electronic). A tribute to J. L. Lions.

[37] Kim, J. U. On a stochastic scalar conservation law. *Indiana Univ. Math. J. 52*, 1 (2003), 227–256.

[38] Lax, P. D. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math. 10* (1957), 537–566.

[39] Lions, P.-L., Perthame, B., and Souganidis, P. E. Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. *Comm. Pure Appl. Math. 49*, 6 (1996), 599–638.

[40] Lions, P.-L., Perthame, B., and Souganidis, P. E. Scalar conservation laws with rough (stochastic) fluxes. *Stochastic Partial Differential Equations: Analysis and Computations 1*, 4 (2013), 664–686.

[41] Lions, P.-L., Perthame, B., and Souganidis, P. E. Stochastic averaging lemmas for kinetic equations. *Séminaire Laurent Schwartz - EDP et applications* (2013), 1–17.

[42] Lions, P.-L., Perthame, B., and Souganidis, P. E. Scalar conservation laws with rough (stochastic) fluxes: the spatially dependent case. *Stoch. Partial Differ. Equ. Anal. Comput. 2*, 4 (2014), 517–538.

[43] Lions, P.-L., Perthame, B., and Tadmor, E. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc. 7*, 1 (1994), 169–191.

[44] Mohammed, S.-E. A., Nilssen, T. K., and Proske, F. N. Sobolev differentiable stochastic flows for SDEs with singular coefficients: applications to the transport equation. *Ann. Probab. 43*, 3 (2015), 1535–1576.

[45] Nakazawa, H. Stochastic Burgers' equation in the inviscid limit. *Adv. in Appl. Math. 3*, 1 (1982), 18–42.

[46] Ochs, G. Weak random attractors. *preprint* (1999), 1–18.

[47] Panov, E. Y. On decay of periodic entropy solutions to a scalar conservation law. *Ann. Inst. H. Poincaré Anal. Non Linéaire 30*, 6 (2013), 997–1007.

[48] Perthame, B., and Souganidis, P. E. A limiting case for velocity averaging. *Ann. Sci. École Norm. Sup. (4) 31*, 4 (1998), 591–598.

[49] Perthame, B., and Tadmor, E. A kinetic equation with kinetic entropy functions for scalar conservation laws. *Comm. Math. Phys. 136*, 3 (1991), 501–517.

[50] Ryan, R. Large-deviation analysis of Burgers turbulence with white-noise initial data. *Comm. Pure Appl. Math. 51*, 1 (1998), 47–75.

[51] Saussereau, B., and Stoica, I. L. Scalar conservation laws with fractional stochastic forcing: existence, uniqueness and invariant measure. *Stochastic Process. Appl. 122*, 4 (2012), 1456–1486.

[52] Sinaĭ, Y. G. Statistics of shocks in solutions of inviscid Burgers equation. *Comm. Math. Phys. 148*, 3 (1992), 601–621.

[53] Vallet, G., and Wittbold, P. On a stochastic first-order hyperbolic equation in a bounded domain. *Infin. Dimens. Anal. Quantum Probab. Relat. Top. 12*, 4 (2009), 613–651.

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
E-mail address: `gess@uchicago.edu`

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
E-mail address: `souganidis@math.uchicago.edu`