An extended anyon Fock space and noncommutative Meixner-type orthogonal polynomials in infinite dimensions

Dedicated to Professor Anatoly Moiseevich Vershik on the occasion of his 80th birthday

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Abstract

Let ν be a finite measure on $\mathbb{R}$ whose Laplace transform is analytic in a neighborhood of zero. An anyon Lévy white noise on $(\mathbb{R}^d, dx)$ is a certain family of noncommuting operators $\langle\omega,\varphi\rangle$ in the anyon Fock space over $L^2(\mathbb{R}^d\times\mathbb{R}, dx\otimes\nu)$. Here $\varphi=\varphi(x)$ runs over a space of test functions on $\mathbb{R}^d$, while $\omega=\omega(x)$ is interpreted as an operator-valued distribution on $\mathbb{R}^d$. Let $L^2(\tau)$ be the noncommutative L^2-space generated by the algebra of polynomials in variables $\langle\omega,\varphi\rangle$, where τ is the vacuum expectation state. We construct noncommutative orthogonal polynomials in $L^2(\tau)$ of the form $\langle P_n(\omega), f^{(n)}\rangle$, where $f^{(n)}$ is a test function on $(\mathbb{R}^d)^n$. Using these orthogonal polynomials, we derive a unitary isomorphism U between $L^2(\tau)$ and an extended anyon Fock space over $L^2(\mathbb{R}^d, dx)$, denoted by $\mathbf{F}(L^2(\mathbb{R}^d, dx))$. The usual anyon Fock space over $L^2(\mathbb{R}^d, dx)$, denoted by $\mathcal{F}(L^2(\mathbb{R}^d, dx))$, is a subspace of $\mathbf{F}(L^2(\mathbb{R}^d, dx))$. Furthermore, we have the equality $\mathbf{F}(L^2(\mathbb{R}^d, dx))=\mathcal{F}(L^2(\mathbb{R}^d, dx))$ if and only if the measure ν is concentrated at one point, i.e., in the Gaussian/Poisson case. Using the unitary isomorphism U, we realize the operators $\langle\omega,\varphi\rangle$ as a Jacobi (i.e., tridiagonal) field in $\mathbf{F}(L^2(\mathbb{R}^d, dx))$. We derive a Meixner-type class of anyon Lévy white noise for which the respective Jacobi field in $\mathbf{F}(L^2(\mathbb{R}^d, dx))$ has a relatively simple structure. Each anyon Lévy white noise of the Meixner type is characterized by two parameters: $\lambda\in\mathbb{R}$ and $\eta\ge 0$. Furthermore, we get the representation $\omega(x)=\partial_x^\dagger+\lambda\partial_x^\dagger\partial_x+\eta\partial_x^\dagger\partial_x\partial_x+\partial_x$. Here ∂_x and $\partial_x^\dagger$ are annihilation and creation operators at point x.

1 Meixner polynomials in infinite dimensions

1.1 Meixner class of orthogonal polynomials

In 1934, Meixner [44] studied the following problem. Consider complex-valued functions $u(z)$ and $\Phi(z)$ which can be expanded into a power series of $z \in \mathbb{C}$ in a neighborhood of zero and suppose that $u(0) = 1$, $\Phi(0) = 0$, and $\Phi'(0) = 1$. Then the function

$$G(x,z) = \exp\big[x\Phi(z)\big]u(z) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!}\, z^n \tag{1.1}$$

generates a system of monic polynomials $P_n(x)$. Find all such polynomials which are orthogonal with respect to a probability measure μ on $\mathbb{R}$. Such polynomials are sometimes called orthogonal polynomials with generating function of exponential type.

Meixner [44] proved that a system of polynomials $P_n(x)$ belongs to this class if and only if it satisfies the recurrence relation

$$xP_n(x) = P_{n+1}(x) + (l + n\lambda)P_n(x) + n(k + \eta(n-1))P_{n-1}(x), \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $l \in \mathbb{R}$, $k > 0$, $\lambda \in \mathbb{R}$, $\eta \geq 0$. For each choice of the parameters, the corresponding measure of orthogonality, μ, is infinitely divisible. If $l = 0$, μ becomes centered, whereas $l \neq 0$ corresponds to the shift of μ by l. For $l = 0$ and $k \neq 1$, the measure μ is the k-th convolution power of the corresponding measure μ for $k = 1$.

One distinguishes five classes of polynomials satisfying (1.2) (see [23, 44]):

(i) For $\lambda = \eta = 0$, μ is a Gaussian measure, $(P_n)_{n=0}^{\infty}$ is a system of Hermite polynomials.

(ii) For $\lambda \neq 0$ and $\eta = 0$, μ is similar to a Poisson distribution (μ being a real Poisson distribution when $\lambda = 1$ and $l = 1$), $(P_n)_{n=0}^{\infty}$ is a system of Charlier polynomials.

(iii) For $|\lambda| = 2$ and $\eta \neq 0$, μ is a gamma distribution, $(P_n)_{n=0}^{\infty}$ is a system of Laguerre polynomials.

(iv) For $|\lambda| < 2$ and $\eta \neq 0$, μ is a Pascal (negative binomial) distribution, $(P_n)_{n=0}^{\infty}$ is a system of Meixner polynomials of the first kind.

(v) For $|\lambda| > 2$ and $\eta \neq 0$, μ is a Meixner distribution, $(P_n)_{n=0}^{\infty}$ is a system of Meixner polynomials of the second kind, or Meixner–Polaczek polynomials.

Note that, in each case, for z from a neighborhood of zero in $\mathbb{C}$,

$$G(x,z) = \exp\big[x\Phi(z) - \mathcal{C}(\Phi(z))\big], \tag{1.3}$$

where $\mathcal{C}(z) := \log\big(\int_{\mathbb{R}} e^{xz}\,\mu(dx)\big)$ is the cumulant transform of μ. We refer to [23,44] for explicit formulas of $\Phi(z)$ and $\mathcal{C}(z)$. If one introduces complex parameters $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta = -\lambda$ and $\alpha\beta = \eta$, using Taylor's expansion, one can write down explicit formulas for $\Phi(z)$ and $\mathcal{C}(z)$ in a unique form for all parameters α and β, see [47].

The two observations below will be crucial for our considerations. First, setting $l = 0$ and $k = 1$, we can rewrite formula (1.2) as follows

$$x = \partial^\dagger + \lambda\partial^\dagger\partial + \partial + \eta\partial^\dagger\partial\partial. \tag{1.4}$$

Here (with an abuse of notation) x denotes the operator of multiplication by the variable x in $L^2(\mathbb{R}, \mu)$, $\partial^\dagger$ is a creation (raising) operator: $\partial^\dagger P_n(x) = P_{n+1}(x)$, and ∂ is an annihilation (lowering) operator: $\partial P_n(x) = nP_{n-1}(x)$.

Second, Kolmogorov's representation of the Fourier transform of the infinitely divisible measure μ (with $l = 0$) has the form [48, 49]

$$\int_{\mathbb{R}} e^{iux}\,\mu(dx) = \exp\left[k\int_{\mathbb{R}}(e^{ius} - 1 - ius)s^{-2}\,\nu(ds)\right], \quad u \in \mathbb{R},$$

see also [30]. Here, for $\eta = 0$ (Gaussian and Poisson cases), $\nu = \delta_\lambda$, the Dirac measure with mass at λ, whereas for $\eta \neq 0$ (cases (iii)–(v)) ν is the probability measure on $\mathbb{R}$, whose system of monic orthogonal polynomials, $(p_n)_{n=0}^\infty$, satisfies the recurrence formula

$$sp_n(s) = p_{n+1}(s) + \lambda(n+1)s + \eta n(n+1)p_{n-1}(s). \tag{1.5}$$

In particular, $(p_n)_{n=0}^\infty$ is again a system of orthogonal polynomials from the Meixner class.

1.2 An infinite dimensional extension

It appears that the Meixner class of orthogonal polynomials is fundamental for infinite dimensional analysis, in particular, for the theory of Lévy white noise, see e.g. [1, 41, 42, 48, 52] and the references therein. Let $X := \mathbb{R}^d$ and let

$$\mathscr{D}(X) \subset L^2(X, dx) \subset \mathscr{D}'(X)$$

be a standard triple of spaces in which $\mathscr{D}(X)$ is the nuclear space of smooth, compactly supported functions on X and $\mathscr{D}'(X)$ is the dual space of $\mathscr{D}(X)$ with respect to zero space $L^2(X, dx)$. For $\omega \in \mathscr{D}'(X)$ and $\varphi \in \mathscr{D}(X)$, we denote by $\langle\omega, \varphi\rangle$ the dual pairing between ω and φ. Let μ be a probability measure on $\mathscr{D}'(X)$, and assume that μ is a generalized stochastic process with independent values, in the sense of [27], or using another terminology, a Lévy white noise measure [25]. We will assume that μ is centered and its Fourier transform has Kolmogorov's representation

$$\int_{\mathscr{D}'(X)} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) = \exp\left[\int_X\int_{\mathbb{R}}\left(e^{is\varphi(x)} - 1 - is\varphi(x)\right)s^{-2}\nu(ds)\,dx\right], \quad \varphi \in \mathscr{D}(X), \tag{1.6}$$

where ν is a probability measure on $\mathbb{R}$ which satisfies:

$$\int_{\mathbb{R}} e^{\varepsilon|s|}\,\nu(ds)<\infty \quad \text{for some } \varepsilon>0. \tag{1.7}$$

Note that the measure $s^{-2}\nu(ds)$ on $\mathbb{R}\setminus\{0\}$ is called the Lévy measure of μ, while $\nu(\{0\})$ describes the Gaussian part of μ (for $s=0$, the function under the integral sign in (1.6) is equal to $-(1/2)\varphi^2(x)$).

In the case $d=1$, for each $t\ge 0$, one can define by approximation in $L^2(\mathscr{D}'(X),\mu)$ a random variable $L_t(\omega)=\langle\omega,\chi_{[0,t]}\rangle$. Here $\chi_{[0,t]}$ denotes the indicator function of $[0,t]$. Then $(L_t)_{t\ge0}$ is a (version of a) Lévy process with Kolmogorov measure ν:

$$\int_{\mathscr{D}'(X)} e^{iuL_t(\omega)}\,\mu(d\omega)=\exp\left[t\int_{\mathbb{R}}(e^{ius}-1-ius)s^{-2}\nu(ds)\right].$$

Thus, the measure μ is indeed a Lévy white noise.

Denote by $\mathscr{C}\mathscr{P}$ the set of all continuous polynomials on $\mathscr{D}'(X)$, i.e., functions on $\mathscr{D}'(X)$ of the form

$$f^{(0)}+\sum_{i=1}^{n}\langle\omega^{\otimes i},f^{(i)}\rangle,\quad \omega\in\mathscr{D}'(X),\ f^{(0)}\in\mathbb{R},\ f^{(i)}\in\mathscr{D}(X)^{\otimes i},\ i=1,\dots,n,\ n\in\mathbb{N}. \tag{1.8}$$

If $f^{(n)}\ne 0$, one says that the polynomial in (1.8) has order n. The set $\mathscr{C}\mathscr{P}$ is dense in $L^2(\mathscr{D}'(X),\mu)$. So using the approach proposed by Skorohod [50], we may orthogonalize these polynomials. More precisely, we denote by $\mathscr{C}\mathscr{P}_n$ the linear space of all continuous polynomials on $\mathscr{D}'(X)$ of order $\le n$. Let $\mathscr{M}\mathscr{P}_n$ denote the closure of $\mathscr{C}\mathscr{P}_n$ in $L^2(\mathscr{D}'(X),\mu)$ (the set of measurable polynomials of order $\le n$). Let $\mathscr{O}\mathscr{P}_n:=\mathscr{M}\mathscr{P}_n\ominus\mathscr{M}\mathscr{P}_{n-1}$, the set of orthogonalized polynomials on $\mathscr{D}'(X)$ of order n. We clearly have

$$L^2(\mathscr{D}'(X),\mu)=\bigoplus_{n=0}^{\infty}\mathscr{O}\mathscr{P}_n. \tag{1.9}$$

Remark 1.1. An alternative orthogonal decomposition of the L^2-space of a Lévy process was derived by Vershik and Tsilevich in [57].

For each $f^{(n)}\in\mathscr{D}(X)^{\otimes n}$, we denote by $\langle P_n(\omega),f^{(n)}\rangle$ the orthogonal projection of the continuous monomial $\langle\omega^{\otimes n},f^{(n)}\rangle$ onto $\mathscr{O}\mathscr{P}_n$. We denote by $\mathscr{O}\mathscr{C}\mathscr{P}$ the linear space of orthogonalized continuous polynomials, i.e., the space of finite sums of functions of the form $\langle P_n(\omega),f^{(n)}\rangle$ and constants. It should be stressed that the function $\langle P_n(\omega),f^{(n)}\rangle$ does not necessarily belong to $\mathscr{C}\mathscr{P}$.

Theorem 1.2. *We have $\mathscr{C}\mathscr{P}=\mathscr{O}\mathscr{C}\mathscr{P}$ if and only if the Kolmogorov measure ν of μ satisfies: either $\nu=\delta_\lambda$ with $\lambda\in\mathbb{R}$ (in which case we set $\eta=0$) or the system of monic polynomials $(p_n)_{n=0}^{\infty}$ which are orthogonal with respect ν satisfies recurrence formula (1.5) with $\lambda\in\mathbb{R}$ and $\eta>0$.*

This theorem can be derived from the main result of [9]. It will also be a corollary of Theorem 3.5 below.

We define the generating function of the orthogonal polynomials by

$$G_\mu(\omega,\varphi) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n(\omega), \varphi^{\otimes n}\rangle,$$

and the cumulant transform of the measure μ by

$$\mathcal{C}_\mu(\varphi) := \log\left(\int_{\mathscr{D}'(X)} e^{\langle \omega,\varphi\rangle} \mu(d\omega)\right).$$

The following theorem shows, in particular, that formula (1.3) admits an extension to infinite dimensions, see [41] for a proof.

Theorem 1.3. *Fix any $\lambda \in \mathbb{R}$ and $\eta \geq 0$. Let μ be the corresponding probability measure μ on $\mathscr{D}'(X)$ from Theorem 1.2. Let $\mathcal{C}(\cdot)$ and $\Phi(\cdot)$ be the functions as in* (1.3) *for parameters $l = 0$, $k = 1$ and λ and η as above. Then*

$$\mathcal{C}_\mu(\varphi) = \int_X \mathcal{C}(\varphi(x))\, dx,$$

$$G_\mu(\omega,\varphi) = \exp\left[\langle \omega, \Phi(\varphi)\rangle - \int_X \mathcal{C}(\Phi(\varphi(x)))\, dx \right],$$

the formulas hold for φ from (at least) a neighborhood of zero in $\mathscr{D}(X)$.

In the case $\lambda = 0$, $\eta = 0$, μ is a Gaussian white noise measure. We refer to e.g. [8, 25, 33] for Gaussian white noise analysis.

In the case $\lambda \neq 0$ and $\eta = 0$, μ is a Poisson random measure (or point process), see e.g. [36]. We refer to [54] for a discussion of representations of the group of diffeomorphisms in the Poisson space, to [35] for Poisson white noise analysis, and to [2] for Poisson analysis on the configuration space.

For $\eta \neq 0$, the most important case of μ is when $\lambda = 2$ and $\eta = 1$. Then μ is the centered gamma measure. The gamma measure is concentrated on discrete Radon measure on X, $\sum_i s_i \delta_{x_i}$, such that the configuration of atoms, $\{x_i\}$, is a dense subset of X. A very important property of the gamma measure is that it is quasi-invariant with respect to a natural group of transformations of the weights, s_i, see [52] and the references therein. Furthermore, as shown in [52], the gamma measure is the unique law of a measure-valued Lévy process which has an equivalent σ-finite measure which is projective invariant with respect to the action of the group acting on the weights, s_i. This σ-finite measure is called in [52] the infinite dimensional Lebesgue measure, see also [53]. We also note that, in papers [26, 52, 55, 56], the gamma measure was used in the representation theory of the group $SL(2, F)$, where F is an algebra of functions on a manifold. White noise analysis related to the gamma measure was initiated in [37], and further developed in [38]. Gibbs perturbations of the gamma measure were constructed in [32]. A Laplace operator associated with the gamma measure was constructed and studied in [31]. Finally, infinite dimensional analysis related to the case of a general $\eta \neq 0$ was studied in [40, 41].

It is well known that, in the Gaussian and Poisson cases ($\eta = 0$), the decomposition of $L^2(\mathscr{D}'(X), \mu)$ in orthogonal polynomials yields the Wiener–Itô–Segal isomorphism between $L^2(\mathscr{D}'(X), \mu)$ and the symmetric Fock space over $L^2(X, dx)$. (An alternative derivation of this result is achieved by using multiple stochastic integrals, see e.g. [51] for the Poisson case.) This result admits the following extension, see [37, 38, 41].

Theorem 1.4. (i) *Let $\lambda \in \mathbb{R}$ and $\eta \geq 0$, and let μ be the corresponding probability measure on $\mathscr{D}'(X)$ from Theorem 1.2. For each $n \in \mathbb{N}$, there exists a measure $m_\nu^{(n)}$ on X^n which satisfies*

$$\int_{\mathscr{D}'(X)} \langle P_n(\omega), f^{(n)} \rangle^2 \, \mu(d\omega) = \int_{X^n} (\mathrm{Sym}_n \, f^{(n)})^2 \, dm_\nu^{(n)}, \quad f^{(n)} \in \mathscr{D}(X)^{\otimes n}. \tag{1.10}$$

Here $\mathrm{Sym}_n \, f^{(n)}$ *denotes the usual symmetrization of a function $f^{(n)}$. For $\eta = 0$, $m_\nu^{(n)} = \frac{1}{n!} \, dx_1 \cdots dx_n$, for $\eta \neq 0$ see subsec. 3.1 below for the explicit construction of $m_\nu^{(n)}$.*

(ii) *We define a Hilbert space*

$$\mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu) := \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}(X^n, m_\nu^{(n)}), \tag{1.11}$$

where $L^2_{\mathrm{sym}}(X^n, m_\nu^{(n)})$ is the subspace of $L^2(X^n, m_\nu^{(n)})$ consisting of all symmetric functions from this space. For $\eta = 0$, $\mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu)$ is the symmetric Fock space over $L^2(X, dx)$. For $\eta \neq 0$, $\mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu)$ contains the symmetric Fock space as a proper subspace. We then call $\mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu)$ an extended symmetric Fock space. The mapping

$$f^{(0)} + \sum_{i=i}^{n} \langle P_i(\omega), f^{(i)} \rangle \mapsto (f^{(0)}, \, \mathrm{Sym}_1 \, f^{(1)}, \ldots, \, \mathrm{Sym}_n \, f^{(n)}, 0, 0 \ldots) \in \mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu) \tag{1.12}$$

extends by continuity to a unitary operator $U : L^2(\mathscr{D}'(X), \mu) \mapsto \mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu)$.

(iii) *For each $\varphi \in \mathscr{D}(X)$, we keep the notation $\langle \omega, \varphi \rangle$ for the image of the operator of multiplication by the monomial $\langle \omega, \varphi \rangle$ in $L^2(\mathscr{D}'(X), \mu)$ under the unitary operator U. Then, analogously to (1.4), we have the following representation of the operator $\langle \omega, \varphi \rangle$ realized in the (extended) symmetric Fock space $\mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu)$:*

$$\langle \omega, \varphi \rangle = \int_X dx \, \varphi(x) (\partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x + \eta \partial_x^\dagger \partial_x \partial_x). \tag{1.13}$$

Here ∂_x is the annihilation operator at point x:

$$(\partial_x f^{(n)})(x_1, \dots, x_{n-1}) := n f^{(n)}(x, x_1, \dots, x_{n-1}), \tag{1.14}$$

and $\partial_x^\dagger$ is the creation operator at point x, satisfying

$$\int_X dx\, \varphi(x) \partial_x^\dagger f^{(n)} := \mathrm{Sym}_{n+1}(\varphi \otimes f^{(n)}), \tag{1.15}$$

see [41] for further details.

Note that, in view of formula (1.13), we may heuristically write

$$\omega(x) = \partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x + \eta \partial_x^\dagger \partial_x \partial_x. \tag{1.16}$$

As follows from Theorem 1.4, (iii), the operators $\langle \omega, \varphi \rangle$ realized in $\mathbf{F}_{\mathrm{sym}}(L^2(X, dx), \nu)$ form a Jacobi field, i.e., they have a tridiagonal structure; compare with e.g. [7, 9, 20, 21, 40].

1.3 A noncommutative extension for anyons — an introduction

The above discussed results have noncommutative analogs in the framework of free probability [16, 17], see also [4, 5, 10, 12, 13] and the references therein. See also [6, 14] for further connections between the classical distributions from the Meixner class and free probability.

However, in this paper, we will be interested in a noncommutative extension of Meixner polynomials for a so-called anyon statistics [28, 29, 39], see also [11]. The latter statistics, indexed by a complex number q of modulus one, forms a continuous bridge between the boson statistics ($q = 1$) and the fermi statistics ($q = -1$). One of the main aims of the present paper is to show that, in the anyon setting, one naturally arrives at noncommutative Meixner-type polynomials which have a representation like in (1.13).

In fact, one could think that it was hopeless to expect a counterpart of formula (1.13) in the fermion setting. Indeed, if the operators ∂_x and ∂_y anticommute, i.e., $\partial_x \partial_y = -\partial_y \partial_x$, then $\partial_x \partial_x = 0$, so that the term $\eta \partial_x^\dagger \partial_x \partial_x$ must be equal to zero. However, we do show that, even in the fermion setting, the integral $\int_X dx\, \varphi(x) \partial_x^\dagger \partial_x \partial_x$ leads to a well-defined, nontrivial operator in an extended antisymmetric Fock space $\mathbf{F}_{\mathrm{as}}(L^2(X, dx), \nu)$. The latter space contains the usual antisymmetric (fermion) Fock space $\mathcal{F}_{\mathrm{as}}(L^2(X, dx))$ as a subspace. On the space $\mathcal{F}_{\mathrm{as}}(L^2(X, dx))$, the operators ∂_x and ∂_y indeed anticommute. However, this anticommutaion fails on the whole space $\mathbf{F}_{\mathrm{as}}(L^2(X, dx), \nu)$. As a result, the extended antisymmetric Fock space leads to a proper renormalization (rather a nontrivial extension) of the operators ∂_x and $\partial_x^\dagger$.

Our discussion of this noncommutative extension is organized as follows. In Section 2, following [18, 28, 39], we briefly recall the construction of the anyon Fock space, standard operators on them, and the anyon commutation relations. We also recall the construction of a Lévy white noise for anyon statistics as a family of noncommutative self-adjoint operators $\langle\omega,\varphi\rangle$ in the anyon Fock space over $L^2(X\times\mathbb{R}, dx\,\nu(ds))$, see [18] for details. Note that, in this section, we do not explain why the 'increments' of this process can be understood as being 'anyon independent.' For this, we refer the reader to [18]. We only note that in the commutative, boson setting ($q=1$), we indeed recover a classical Lévy white noise, being realized as a family of commuting self-adjoint operators in the symmetric Fock space over $L^2(X\times\mathbb{R}, dx\,\nu(ds))$.

In Section 3, we formulate the main results of the paper. In particular, starting with a space $\mathscr{CP}$ of noncommutative continuous polynomials of anyon white noise, we construct a space $\mathscr{OCP}$ of orthogonalized continuous polynomials. By analogy with (1.10), for each $n\in\mathbb{N}$, we construct a measure $m_\nu^{(n)}$ on X^n and find the corresponding symmetrization operator Sym_n. This symmetry extends the anyon symmetry (in particular, the fermion symmetry) in a non-trivial way. By analogy with (1.11), we define an extended anyon Fock space, and then by analogy with (1.12), we construct a unitary operator U between the noncommutative L^2-space and the extended anyon Fock space. Under the unitary U, each operator $\langle\omega,\varphi\rangle$ takes a Jacobi form in the extended anyon Fock space. We show that this Jacobi field has the simplest form (in a sense) when ν is the same measure as in Theorem 1.2, i.e., ν is Kolmogorov's measure of a white noise measure μ from the Meixner class. Furthermore, in this case, analogs of formulas (1.13)–(1.15) hold.

Finally, Section 4 is devoted to the proofs of the main results.

Among numerous open problems regarding the anyon Meixner-type white noise, let us mention only two:

(i) In both the classical and free cases, the generating functions of the Meixner-type orthogonal polynomials are explicitly known and play an important role in the studies of these polynomials. In the anyon case, the form of the generating function is not yet known, even in the Gaussian case. The main difficulty lies in the fact that both the classical and free Meixner-type polynomials have corresponding systems of orthogonal polynomials on the real line. However, the anyon case is purely infinite dimensional and has no related one-dimensional theory.

(ii) As shown in [1], in the classical case, the Lévy processes from the Meixner class with $\eta>0$ are related to the renormalized squares of boson white noise. Is it possible to interpret anyon Meixner-type white noises as those related to renormalized squares of anyon white noise?

2 Noncommutative Lévy white noise for anyon statistics

2.1 Anyon Fock space and anyon commutation relations

Let $\mathcal{B}(X)$ denote the Borel σ-algebra on X, and let $\mathcal{B}_0(X)$ denote the family of all sets from $\mathcal{B}(X)$ which have compact closure. Let $m = m(dx) = dx$ denote the Lebesgue measure on $(X, \mathcal{B}(X))$.

For each $n \geq 2$, we define

$$X^{(n)} := \big\{(x_1, \dots, x_n) \in X^n \mid \forall 1 \leq i < j \leq n : \; x_i \neq x_j\big\}. \tag{2.1}$$

Since the measure m is non-atomic,

$$m^{\otimes n}(X \setminus X^{(n)}) = 0. \tag{2.2}$$

We introduce a strict total order on X as follows, for any $x = (x^1, \dots, x^d), y = (y^1, \dots, y^d) \in X$, $x \neq y$, we set $x < y$ if for some $j \in \{1, \dots, d\}$, we have $x^1 = y^1, \dots, x^{j-1} = y^{j-1}$ and $x^j < y^j$.

We fix a number $q \in \mathbb{C}$ with $|q| = 1$, and define a function $Q : X^{(2)} \to \mathbb{C}$ as follows:

$$Q(x, y) = \begin{cases} q, & \text{if } x < y, \\ \bar{q}, & \text{if } y < x. \end{cases}$$

Note that the function Q is Hermitian:

$$Q(x, y) = \overline{Q(y, x)}, \quad (x, y) \in X^{(2)}.$$

A function $f^{(n)} : X^{(n)} \to \mathbb{C}$ $(n \geq 2)$ is called Q-symmetric if, for each $j = 1, \dots, n-1$,

$$f^{(n)}(x_1, \dots, x_n) = Q(x_j, x_{j+1}) f^{(n)}(x_1, \dots, x_{j-1}, x_{j+1}, x_j, x_{j+2}, \dots, x_n). \tag{2.3}$$

Let $\mathcal{H} := L^2(X, m)$ be the Hilbert space of all complex-valued, square-integrable functions on X. Thus, for each $n \in \mathbb{N}$, $\mathcal{H}^{\otimes n} = L^2(X^n, m^{\otimes n})$. In view of (2.2), we have $\mathcal{H}^{\otimes n} = L^2(X^{(n)}, m^{\otimes n})$. We define a complex Hilbert space $\mathcal{H}^{\circledast n}$ as the (closed) subspace of $\mathcal{H}^{\otimes n}$ consisting of all ($m^{\otimes n}$-versions of) Q-symmetric functions in $\mathcal{H}^{\otimes n}$. Let Sym_n denote the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{H}^{\circledast n}$. This operator has the following explicit form: for each $f^{(n)} \in \mathcal{H}^{\otimes n}$,

$$(\operatorname{Sym}_n f^{(n)})(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} Q_\pi(x_1, \dots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}), \quad (x_1, \dots, x_n) \in X^{(n)}. \tag{2.4}$$

Here $\mathfrak{S}_n$ denotes the group of all permutations of $1, \dots, n$ and

$$Q_\pi(x_1, \dots, x_n) := \prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} Q(x_i, x_j), \quad (x_1, \dots, x_n) \in X^{(n)}. \tag{2.5}$$

We can now define a Q-symmetric tensor product $\circledast$. For any $m, n \in \mathbb{N}$ and any $f^{(m)} \in \mathcal{H}^{\circledast m}$ and $g^{(n)} \in \mathcal{H}^{\circledast n}$, we set $f^{(m)} \circledast g^{(n)} := \mathrm{Sym}_{m+n}(f^{(m)} \otimes g^{(n)})$. Note that this tensor product is associative. Note also that, for $q = 1$, $\circledast$ is the usual symmetric tensor product, while for $q = -1$, $\circledast$ is the usual antisymmetric tensor product.

We define an anyon Fock space by

$$\mathcal{F}^Q(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledast n} n! \,.$$

Thus, $\mathcal{F}^Q(\mathcal{H})$ is the Hilbert space which consists of all sequences $F = (f^{(0)}, f^{(1)}, f^{(2)}, \dots)$ with $f^{(n)} \in \mathcal{H}^{\circledast n}$ ($\mathcal{H}^{\circledast 0} := \mathbb{C}$) satisfying

$$\|F\|^2_{\mathcal{F}^Q(\mathcal{H})} := \sum_{n=0}^{\infty} \|f^{(n)}\|^2_{\mathcal{H}^{\circledast n}} n! < \infty.$$

(The inner product in $\mathcal{F}^Q(\mathcal{H})$ is induced by the norm in this space.) The vector $\Omega := (1, 0, 0, \dots) \in \mathcal{F}^Q(\mathcal{H})$ is called the vacuum. We denote by $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})$ the subspace of $\mathcal{F}^Q(\mathcal{H})$ consisting of all finite sequences

$$F = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$$

in which $f^{(i)} \in \mathcal{H}^{\circledast i}$ for $i = 0, 1, \dots, n$, $n \in \mathbb{N}$. This space can be endowed with the topology of the topological direct sum of the $\mathcal{H}^{\circledast n}$ spaces. Thus, convergence in $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})$ means uniform finiteness of non-zero components and coordinate-wise convergence in $\mathcal{H}^{\circledast n}$.

For each $h \in \mathcal{H}$, we define a creation operator $a^+(h)$ and an annihilation operator $a^-(h)$ as the linear operators acting on $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})$ given by

$$a^+(h) f^{(n)} := h \circledast f^{(n)}, \quad f^{(n)} \in \mathcal{H}^{\circledast n}, \quad a^-(h) := a^+(h)^* \restriction_{\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})} .$$

Both $a^+(h)$ and $a^-(h)$ act continuously on $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})$. In fact, for any $h \in \mathcal{H}$ and $f^{(n)} \in \mathcal{H}^{\circledast n}$, we have

$$(a^+(h) f^{(n)})(x_1, \dots, x_{n+1}) = \frac{1}{n+1} \Big[h(x_1) f^{(n)}(x_2, \dots, x_{n+1})$$
$$+ \sum_{k=2}^{n+1} Q(x_1, x_k) Q(x_2, x_k) \cdots Q(x_{k-1}, x_k) h(x_k) f^{(n)}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}) \Big],$$

$$(a^-(h)f^{(n)})(x_1,\dots,x_{n-1}) = n\int_X \overline{h(y)}\, f^{(n)}(y,x_1,\dots,x_{n-1})\,dy. \tag{2.6}$$

The action of the annihilation operator can also be written in the following form: for any $h\in\mathcal{H}$ and $f^{(n)}\in\mathcal{H}^{\otimes n}$,

$$(a^-(h)\operatorname{Sym}_n f^{(n)})(x_1,\dots,x_{n-1}) = \operatorname{Sym}_{n-1}\bigg(\int_X \overline{h(y)}\bigg[\sum_{k=1}^n Q(y,x_1)Q(y,x_2) \\ \times\dots\times Q(y,x_{k-1})f^{(n)}(x_1,x_2,\dots,x_{k-1},y,x_k,\dots,x_{n-1})\bigg]dy\bigg). \tag{2.7}$$

Let us now discuss the creation and annihilation operators at points of the space X. At least informally, for each $x\in X$, we may consider a delta function at x, denoted by δ_x. Then we can heuristically define $\partial_x^\dagger := a^+(\delta_x)$ and $\partial_x := a^-(\delta_x)$, so that

$$\partial_x^\dagger f^{(n)} = \delta_x \circledast f^{(n)}, \quad \partial_x f^{(n)} := nf^{(n)}(x,\cdot). \tag{2.8}$$

Thus,

$$a^+(h) := \int_X dx\, h(x)\partial_x^\dagger, \quad a^-(h) = \int_X dx\, \overline{h(x)}\,\partial_x\,. \tag{2.9}$$

Note that the second formula in (2.8) is a rigorous definition of ∂_x (for m-a.a. $x\in X$), while the first formula in (2.9) is the rigorous definition of the integral $\int_X dx\, h(x)\partial_x^\dagger$.

Let $B_0(X^n)$ denote the space of all complex-valued bounded measurable functions on X^n with compact support. Fix any sequence of $+$ and $-$ of length $n\geq 2$, and denote it by $(\sharp_1,\dots,\sharp_n)$. It is easy to see that, for any $g^{(n)}\in B_0(X^n)$, the expression

$$\int_{X^n} dx_1\cdots dx_n\, g^{(n)}(x_1,\dots,x_n)\partial_{x_1}^{\sharp_1}\cdots\partial_{x_n}^{\sharp_n}$$

identifies a linear continuous operator on $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H})$. Here we used the notation $\partial_x^+ := \partial_x^\dagger$, $\partial_x^- := \partial_x$.

The creation and annihilation operators satisfy the anyon commutation relations:

$$\partial_x\partial_y^\dagger = \delta(x,y) + Q(x,y)\partial_y^\dagger\partial_x, \tag{2.10}$$

$$\partial_x\partial_y = Q(y,x)\partial_y\partial_x, \tag{2.11}$$

$$\partial_x^\dagger\partial_y^\dagger = Q(y,x)\partial_y^\dagger\partial_x^\dagger. \tag{2.12}$$

Here $\delta(x,y)$ is understood as:

$$\int_{X^2} dx\,dy\, g^{(2)}(x,y)\delta(x,y) := \int_X dx\, g^{(2)}(x,x).$$

Formulas (2.10)–(2.12) make rigorous sense after smearing with functions $g^{(2)}\in B_0(X^2)$. Note that, for $q=1$, equations (2.10)–(2.12) become the canonical commutation relations, while for $q=-1$ they become the canonical anticommutation relations.

Remark 2.1. Let $D := \{(x,x) \mid x \in X\}$. Note that, for each $g^{(2)} \in B_0(X^2)$ which has support in D, the operator $\int_{X^2} dx\, dy\, g^{(2)}(x,y)\partial_y^\dagger \partial_x$ is equal to zero. Hence, it does not influence (2.10) that we have not identified the function Q on D.

For a bounded linear operator A in $\mathcal{H}$, we define the differential second quantization of A, denoted by $d\Gamma(A)$, as a linear continuous operator on $\mathcal{F}_{\text{fin}}^Q(\mathcal{H})$ given by $d\Gamma(A)\Omega := 0$ and

$$d\Gamma(A) \restriction \mathcal{H}^{\circledast n} := \operatorname{Sym}_n(A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes A)$$

for each $n \in \mathbb{N}$. For each a.e. bounded function $h \in L^\infty(X,m)$, we define a neutral operator

$$a^0(h) := \int_X dx\, h(x)\partial_x^\dagger \partial_x. \tag{2.13}$$

According to formulas (2.8) and (2.9), we have

$$\begin{aligned}\big(a^0(h)f^{(n)}\big)(x_1,\dots,x_n) &= \left(\int_X dx\, h(x)\partial_x^\dagger f^{(n)}(x,\cdot)\right)(x_1,\dots,x_n)\\ &= n \operatorname{Sym}_n \big(h(x_1)f^{(n)}(x_1,x_2,\dots,x_n)\big).\end{aligned} \tag{2.14}$$

From here one easily gets

$$(a^0(h)f^{(n)})(x_1,\dots,x_n) = \big(h(x_1)+\cdots+h(x_n)\big)f^{(n)}(x_1,\dots,x_n). \tag{2.15}$$

Hence, $a^0(h) = d\Gamma(M_h)$, where M_h is the operator of multiplication by h.

2.2 Anyon Lévy white noise and noncommutative orthogonal polynomials

Let us now recall the construction of a Lévy white noise over X for anyon statistics, see [18]. Let ν be a probability measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. (In fact, we can instead assume that ν is a finite measure. The results below will then require a trivial modification.) We denote by $\mathscr{P}(\mathbb{R})$ the linear space of polynomials on $\mathbb{R}$. We assume that $\mathscr{P}(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R},\nu)$. Note that the latter assumption is satisfied if, for example, (1.7) holds.

We extend the function Q by setting

$$Q(x_1,s_1,x_2,s_2) := Q(x_1,x_2), \quad (x_1,x_2) \in X^{(2)},\ (s_1,s_2) \in \mathbb{R}^2.$$

Thus, the value of the function Q does not depend on s_1 and s_2. Analogously to (2.3), we define the notion of a Q-symmetric function $f^{(n)}$ defined on the set

$$\left\{(x_1,s_1,\dots,x_n,s_n) \in (X\times\mathbb{R})^n \mid (x_1,\dots,x_n) \in X^{(n)}\right\}.$$

For example, for $n = 2$, the Q-symmetry means:

$$f^{(2)}(x_1, s_1, x_2, s_2) = Q(x_1, x_2) f^{(2)}(x_2, s_2, x_1, s_1).$$

We next set

$$\mathcal{G} := L^2(X \times \mathbb{R}, m \otimes \nu) = \mathcal{H} \otimes L^2(\mathbb{R}, \nu),$$

and consider the corresponding Q-symmetric Fock space $\mathcal{F}^Q(\mathcal{G})$, which is constructed by analogy with $\mathcal{F}^Q(\mathcal{H})$. Let $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$ denote the linear subspace of $\mathcal{F}^Q(\mathcal{G})$ which consists of all finite sequences

$$F = (F^{(0)}, F^{(1)}, \dots, F^{(n)}, 0, 0, \dots), \quad n \in \mathbb{N}_0,$$

such that each $F^{(k)}$ with $k \neq 0$ has the form

$$F^{(k)}(x_1, s_1, \dots, x_k, s_k) = \mathrm{Sym}_k \left[\sum_{(i_1, i_2, \dots, i_k) \in \{0, 1, \dots, N\}^k} f_{(i_1, i_2, \dots, i_k)}(x_1, x_2, \dots, x_k) s_1^{i_1} s_2^{i_2} \cdots s_k^{i_k} \right],$$

where $f_{(i_1, i_2, \dots, i_k)} \in \mathcal{H}^{\otimes k}$ and $N \in \mathbb{N}$. Clearly, $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$ is dense in $\mathcal{F}^Q(\mathcal{G})$.

We denote $1(s) := 1$ and $\mathrm{id}(s) := s$ for $s \in \mathbb{R}$. Thus, $1, \mathrm{id} \in \mathscr{P}(\mathbb{R})$. We denote by $C_0(X \mapsto \mathbb{R})$ the space of all real-valued continuous functions on X with compact support. For each $f \in C_0(X \mapsto \mathbb{R})$, we define an operator

$$\langle \omega, f \rangle := a^+(f \otimes 1) + a^0(f \otimes \mathrm{id}) + a^-(f \otimes 1) \tag{2.16}$$

in $\mathcal{F}^Q(\mathcal{G})$ with domain $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$. Clearly, each operator $\langle \omega, f \rangle$ maps $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$ into itself. In fact, under assumption (1.7), each $F \in \mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$ is an analytic vector for each operator $\langle \omega, f \rangle$ with $f \in C_0(X \mapsto \mathbb{R})$, which implies that the operators $\langle \omega, f \rangle$ are essentially self-adjoint on $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$ (see e.g. [46, Sec. X.2]).

Remark 2.2. Let us keep the notation $\langle \omega, f \rangle$ for the closure of this operator in $\mathcal{F}^Q(\mathcal{G})$. Thus the operators $\langle \omega, f \rangle$ are self-adjoint. In the boson case, $q = 1$, these operators also commute in the sense of commutation of their resolutions of the identity. By using e.g. the projection spectral theorem [8], one shows [24] that there exists a unitary isomorphism between the symmetric Fock space $\mathcal{F}^Q(\mathcal{G})$ and the space $L^2(\mathscr{D}'(X), \mu)$, where μ is the Lévy white noise measure with Fourier transform (1.6). Under this unitary isomorphism, the vacuum vector Ω becomes the constant function 1, and each operator $\langle \omega, f \rangle$ becomes the operator of multiplication by the random variable $\langle \omega, f \rangle$ in $L^2(\mathscr{D}'(X), \mu)$. In other words, μ is the spectral measure of the family of commuting self-adjoint operators $\big(\langle \omega, f \rangle\big)_{f \in C_0(X \mapsto \mathbb{R})}$. In particular, the operators $\big(\langle \omega, f \rangle\big)_{f \in C_0(X \mapsto \mathbb{R})}$ in the symmetric Fock space $\mathcal{F}^Q(\mathcal{G})$ can indeed be thought of as a Lévy white noise. Let us also note that the unitary operator between $\mathcal{F}^Q(\mathcal{G})$ and $L^2(\mathscr{D}'(X), \mu)$ was originally derived by Itô, by using multiple stochastic integrals, see [34].

Remark 2.3. Note that, if the measure ν is concentrated at one point, $\lambda \in \mathbb{R}$, then $\mathcal{G} = \mathcal{H}$ and each operator $\langle \omega, f \rangle$ has the following form in $\mathcal{F}^Q(\mathcal{H})$:

$$\langle \omega, f \rangle := a^+(f) + a^-(f) + \lambda a^0(f). \tag{2.17}$$

The choice $\lambda = 0$ corresponds to an anyon Gaussian white noise, while $\lambda \neq 0$ corresponds to an anyon centered white noise. If we denote

$$\omega(x) := \partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x, \quad x \in X, \tag{2.18}$$

then, by (2.9), (2.13), (2.17), and (2.18), we get

$$\langle \omega, f \rangle = \int_X dx\, \omega(x) f(x), \quad f \in C_0(X \mapsto \mathbb{R}),$$

which justifies the notation $\langle \omega, f \rangle$. Thus, $(\omega(x))_{x \in X}$ is the anyon Gaussian/Poisson white noise. Note that $\omega(x)$ is informally treated as an operator-valued distribution.

We further denote by $C_0(X)$ the space of all complex-valued, continuous functions on X with compact support. For $f \in C_0(X)$, we set $\langle \omega, f \rangle := \langle \omega, \Re f \rangle + i \langle \omega, \Im f \rangle$.

Let $\mathscr{P}$ denote the complex unital $*$-algebra generated by $(\langle \omega, f \rangle)_{f \in C_0(X)}$, i.e., the algebra of noncommutative polynomials in variables $\langle \omega, f \rangle$. In particular, elements of $\mathscr{P}$ are linear operators acting on $\mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$, and for each $p \in \mathscr{P}$, p^* is the adjoint operator of p in $\mathcal{F}^Q(\mathcal{G})$, restricted to $\mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$.

We define a vacuum state on $\mathscr{P}$ by

$$\tau(p) := (p\Omega, \Omega)_{\mathcal{F}^Q(\mathcal{G})}, \quad p \in \mathscr{P}.$$

We introduce a scalar product on $\mathscr{P}$ by

$$(p_1, p_2)_{L^2(\tau)} := \tau(p_2^* p_1) = (p_1\Omega, p_2\Omega)_{\mathcal{F}^Q(\mathcal{G})}, \quad p_1, p_2 \in \mathscr{P}.$$

Let

$$\widetilde{\mathscr{P}} := \{p \in \mathscr{P} \mid (p, p)_{L^2(\tau)} = 0\}, \tag{2.19}$$

and define the noncommutative L^2-space $L^2(\tau)$ as the completion of the quotient space $\mathscr{P}/\widetilde{\mathscr{P}}$ with respect to the norm generated by the scalar product $(\cdot, \cdot)_{L^2(\tau)}$. Elements $p \in \mathscr{P}$ are treated as representatives of the equivalence classes from $\mathscr{P}/\widetilde{\mathscr{P}}$, and so $\mathscr{P}$ becomes a dense subspace of $L^2(\tau)$. As shown in [18], the vacuum vector Ω is cyclic for the family of operators $(\langle \omega, f \rangle)_{f \in C_0(X \mapsto \mathbb{R})}$. Consider a linear mapping $I : \mathscr{P} \to \mathcal{F}^Q(\mathcal{G})$ defined by

$$Ip := p\Omega \quad \text{for } p \in \mathscr{P}.$$

Then $Ip_1 = Ip_2$ if $p_1, p_2 \in \mathscr{P}$ are such that $p_1 - p_2 \in \widetilde{\mathscr{P}}$, and I extends to a unitary operator $I : L^2(\tau) \to \mathcal{F}^Q(\mathcal{G})$.

Note that, for each $p \in \mathscr{P}$ and $f \in C_0(X)$,

$$I\big(\langle \omega, f\rangle p\big) = \langle \omega, f\rangle (Ip), \tag{2.20}$$

i.e., under the unitary I, the operator of left multiplication by $\langle \omega, f\rangle$ in $L^2(\tau)$ becomes the operator $\langle \omega, f\rangle$ in $\mathcal{F}^Q(\mathcal{G})$.

Let us consider the topology on $C_0(X)$ which yields the following notion of convergence: $f_n \to f$ as $n \to \infty$ means that there exists a set $\Delta \in \mathcal{B}_0(X)$ such that $\operatorname{supp}(f_n) \subset \Delta$ for all $n \in \mathbb{N}$ and

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty. \tag{2.21}$$

By linearity and continuity we can extend the mapping

$$C_0(X)^n \ni (f_1, \dots, f_n) \mapsto \langle \omega^{\otimes n}, f_1 \otimes \cdots \otimes f_n\rangle = \langle \omega, f_1\rangle \cdots \langle \omega, f_n\rangle \in \mathscr{P}$$

to a mapping

$$C_0(X^n) \ni f^{(n)} \mapsto \langle \omega^{\otimes n}, f^{(n)}\rangle \in L^2(\tau),$$

and $\langle \omega^{\otimes n}, f^{(n)}\rangle$ can be thought of as a linear operator acting in $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H} \otimes \mathscr{P}(\mathbb{R}))$. We will think of $\langle \omega^{\otimes n}, f^{(n)}\rangle$ as a continuous monomial of order n. Sums of such operators and (complex) constants form the set $\mathscr{C}\mathscr{P}$ of continuous polynomials (of ω). Evidently, $\mathscr{P} \subset \mathscr{C}\mathscr{P}$.

Completely analogously to (1.9), we derive the orthogonal decomposition

$$L^2(\tau) = \bigoplus_{n=0}^{\infty} \mathscr{O}\mathscr{P}_n \tag{2.22}$$

(we used obvious notations). For any $f^{(n)} \in C_0(X^n)$, we denote by $\langle P_n(\omega), f^{(n)}\rangle$ the orthogonal projection of $\langle \omega^{\otimes n}, f^{(n)}\rangle$ onto $\mathscr{O}\mathscr{P}_n$. The set of finite linear sums of $\langle P_n(\omega), f^{(n)}\rangle$ and (complex) constants is denoted by $\mathscr{O}\mathscr{C}\mathscr{P}$ (orthogonalized continuous polynomials).

Remark 2.4. Note that $\langle P_1(\omega), f\rangle = \langle \omega, f\rangle$.

Remark 2.5. Note that, in subsec. 1.2, we used functions $f^{(n)} \in \mathscr{D}(X)^{\otimes n}$ when defining $\mathscr{C}\mathscr{P}$ and $\mathscr{O}\mathscr{C}\mathscr{P}$, while now we are using $f^{(n)} \in C_0(X^n)$ to define $\mathscr{C}\mathscr{P}$ and $\mathscr{O}\mathscr{C}\mathscr{P}$. The reason is that, in the noncommutative setting, there is no need for $f^{(n)}$ to be smooth, while in the classical case, $q = 1$, Theorem 1.2 still holds for the sets $\mathscr{C}\mathscr{P}$ and $\mathscr{O}\mathscr{C}\mathscr{P}$ as defined in this section.

3 Main results

3.1 The measures $m_\nu^{(n)}$

Let $(p_k)_{k=0}^\infty$ denote the system of monic orthogonal polynomials in $L^2(\mathbb{R},\nu)$. (If the support of ν is finite and consists of N points, we set $p_k := 0$ for $k \ge N$.) Hence, $(p_k)_{k=0}^\infty$ satisfy the recursion formula

$$sp_k(s) = p_{k+1}(s) + b_k p_k(s) + a_k p_{k-1}(s), \quad k \in \mathbb{N}_0, \tag{3.1}$$

with $p_{-1}(s) := 0$, $a_k > 0$, and $b_k \in \mathbb{R}$. (If the support of ν has N points, $a_k = 0$ for $k \ge N$.)

We define

$$c_k := a_0 a_1 \cdots a_{k-1}, \quad k \in \mathbb{N}, \tag{3.2}$$

where $a_0 := 1$ and the a_k's for $k \in \mathbb{N}$ are the coefficients from formula (3.1). We equivalently have:

$$c_k = \int_{\mathbb{R}} p_{k-1}(s)^2 \, \nu(ds), \quad k \in \mathbb{N}, \tag{3.3}$$

which is a well known fact of the theory of orthogonal polynomials. Note that $c_1 = 1$ and $c_k = 0$ for $k \ge 2$ if and only if the measure ν is concentrated at one point.

We denote by $\Pi(n)$ the set of all (unordered) partitions of the set $\{1, \dots, n\}$. For each partition $\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)$, we set $|\theta| := l$. For each $\theta \in \Pi(n)$, we denote by $X_\theta^{(n)}$ the subset of X^n which consists of all $(x_1, \dots, x_n) \in X^n$ such that, for all $1 \le i < j \le n$, $x_i = x_j$ if and only if i and j belong to the same element of the partition θ. Note that the sets $X_\theta^{(n)}$ with $\theta \in \Pi(n)$ form a partition of X^n. Note also that, by (2.1), $X^{(n)} = X_\theta^{(n)}$ for the minimal partition $\theta = \{\{1\}, \{2\}, \dots, \{n\}\}$.

Let us fix $n \in \mathbb{N}$, a permutation $\pi \in \mathfrak{S}_n$, and a partition $\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)$ satisfying

$$\max \theta_1 < \max \theta_2 < \cdots < \max \theta_l. \tag{3.4}$$

We define a measure $m_{\nu,\theta}^{(n)}$ on $X_\theta^{(n)}$ as the push-forward of the measure

$$\big(c_{|\theta_1|} \cdots c_{|\theta_l|}\big) n! \big(|\theta_1|! \cdots |\theta_l|!\big)^{-1} m^{\otimes l}$$

on $X^{(l)}$ under the mapping

$$X^{(l)} \ni y = (y_1, \dots, y_l) \mapsto (R_\theta^1 y, \dots, R_\theta^n y) \in X_\theta^{(n)},$$

where $R_\theta^i y = y_j$ for $i \in \theta_j$. Here $|\theta_i|$ denotes the number of elements of the set θ_i. Recalling that the sets $X_\theta^{(n)}$ with $\theta \in \Pi(n)$ form a partition of X^n, we define a measure $m_\nu^{(n)}$ on X^n such that the restriction of $m_\nu^{(n)}$ to each $X_\theta^{(n)}$ is equal to $m_{\nu,\theta}^{(n)}$. Note that the restriction of $m_\nu^{(n)}$ to $X^{(n)}$ is equal to $n! \, m^{\otimes n}$.

For example, for $n = 2$, we get

$$\int_{X^2} f^{(2)}(x_1, x_2)\, m_\nu^{(2)}(dx_1 \times dx_2) = \int_{\{x_1 \neq x_2\}} f^{(2)}(x_1, x_2)\, dx_1\, dx_2\, 2 + \int_X f^{(2)}(x, x)\, dx\, c_2$$
$$= \int_{X^2} f^{(2)}(x_1, x_2)\, dx_1\, dx_2\, 2 + \int_X f^{(2)}(x, x)\, dx\, c_2.$$

3.2 An extended anyon Fock space

Let us recall that, in subsec. 2.1, see in particular (2.3), we defined the notion of a Q-symmetric function $f^{(n)} : X^{(n)} \to \mathbb{C}$. Our next aim is to extend this notion to a complex-valued function defined on the whole X^n.

Let us fix a permutation $\pi \in \mathfrak{S}_n$ and a partition $\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)$ satisfying (3.4). The permutation π maps the partition θ into a new partition

$$\{\pi\theta_1, \dots, \pi\theta_l\} \in \Pi(n).$$

We call this new partition $\beta = \{\beta_1, \dots, \beta_l\}$, where the elements of the partition β are enumerated in such a way that

$$\max \beta_1 < \max \beta_2 < \cdots < \max \beta_l. \tag{3.5}$$

Thus, the permutation $\pi \in \mathfrak{S}_n$ identifies a permutation $\widehat{\pi} \in \mathfrak{S}_l$ (dependent on θ) such that

$$\pi\theta_i = \beta_{\widehat{\pi}(i)}, \quad i = 1, \dots, l. \tag{3.6}$$

Recall the complex-valued function $Q_\pi(x_1, \dots, x_n)$ on $X^{(n)}$ defined by (2.5). We will now extend this function to the whole set X^n as follows. We fix any $\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)$ satisfying (3.4) and any $(x_1, \dots, x_n) \in X_\theta^{(n)}$. We denote by $x_{\theta_1}, x_{\theta_2}, \dots, x_{\theta_l}$ the elements $x_{i_1}, x_{i_2}, \dots, x_{i_l}$ with $i_1 \in \theta_1, i_2 \in \theta_2, \dots, i_l \in \theta_l$, respectively. We set

$$Q_\pi(x_1, \dots, x_n) := \prod_{\substack{1 \le i < j \le l \\ \widehat{\pi}(i) > \widehat{\pi}(j)}} Q(x_{\theta_i}, x_{\theta_j}), \tag{3.7}$$

where the permutation $\widehat{\pi} \in \mathfrak{S}_l$ is as above. Note that, for the partition

$$\theta = \big\{\{1\}, \{2\}, \dots, \{n\}\big\},$$

the restriction of the function Q_π to the set $X_\theta^{(n)} = X^{(n)}$ is indeed equal to the function Q_π defined by (2.5).

We will say that a function $f^{(n)} : X^n \to \mathbb{C}$ is Q-symmetric if, for each permutation $\pi \in \mathfrak{S}_n$,

$$f^{(n)}(x_1, \dots, x_n) = Q_\pi(x_1, \dots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}), \quad (x_1, \dots, x_n) \in X^n. \tag{3.8}$$

In particular, the restriction of such a function to $X^{(n)}$ is then Q-symmetric according to our definition in subsec. 2.1, i.e., it satisfies (2.3).

Next, for a function $f^{(n)}: X^n \to \mathbb{C}$, we define

$$(\mathrm{Sym}_n\, f^{(n)})(x_1,\dots,x_n) = \frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n} Q_\pi(x_1,\dots,x_n) f^{(n)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)}),\quad (x_1,\dots,x_n)\in X^n. \quad (3.9)$$

Clearly, the restriction of the function $\mathrm{Sym}_n\, f^{(n)}$ to the set $X^{(n)}$ is still given by (2.4).

We denote by $\mathbf{F}_n^Q(\mathcal{H},\nu)$ the subspace of the complex L^2-space $L^2(X^n, m_\nu^{(n)})$ which consists of ($m_\nu^{(n)}$-versions of) Q-symmetric functions.

Proposition 3.1. *For each $n\in\mathbb{N}$, Sym_n is the orthogonal projection of $L^2(X^n, m_\nu^{(n)})$ onto $\mathbf{F}_n^Q(\mathcal{H},\nu)$.*

We also set $\mathbf{F}_0^Q(\mathcal{H},\nu)=\{c\Omega \mid c\in\mathbb{C}\}$, where Ω is the vacuum vector. We define an extended anyon Fock space

$$\mathbf{F}^Q(\mathcal{H},\nu) := \bigoplus_{n=0}^{\infty} \mathbf{F}_n^Q(\mathcal{H},\nu).$$

If the measure ν is concentrated at one point (and so $c_1=1$ and $c_k=0$ for $k\ge 2$), we get $\mathbf{F}^Q(\mathcal{H},\nu)=\mathcal{F}^Q(\mathcal{H})$, i.e., $\mathbf{F}^Q(\mathcal{H},\nu)$ is the usual anyon Fock space. Otherwise, $\mathcal{F}^Q(\mathcal{H})$ is a proper subspace of $\mathbf{F}^{\mathbf{Q}}(\mathcal{H},\nu)$. Indeed, recalling formula (2.2), we may embed $\mathcal{F}^Q(\mathcal{H})$ into $\mathbf{F}^Q(\mathcal{H},\nu)$ by identifying each function $f^{(n)}\in\mathcal{H}^{\circledast n}$ with the function from $\mathbf{F}_n^Q(\mathcal{H},\nu)$ which is equal to $f^{(n)}$ on $X^{(n)}$, and to 0 otherwise. Evidently, the orthogonal compliment to $\mathcal{F}^Q(\mathcal{H})$ in $\mathbf{F}^{\mathbf{Q}}(\mathcal{H},\nu)$ is a non-zero space in this case.

Using the orthogonal decomposition (2.22), we will now construct a unitary isomorphism between $L^2(\tau)$ and the extended anyon Fock space $\mathbf{F}^{\mathbf{Q}}(\mathcal{H},\nu)$.

Theorem 3.2. *Let $f^{(n)}, g^{(n)}\in C_0(X^n)$. Then*

$$\big(\langle P_n(\omega), f^{(n)}\rangle, \langle P_n(\omega), g^{(n)}\rangle\big)_{L^2(\tau)} = (\mathrm{Sym}_n\, f^{(n)}, \mathrm{Sym}_n\, g^{(n)})_{\mathbf{F}_n^Q(\mathcal{H},\nu)}. \quad (3.10)$$

Since the set $C_0(X^n)$ is dense in $L^2(X^n, m_\nu^{(n)})$, Theorem 3.2 implies that we can extended the mapping

$$C_0(X^n)\ni f^{(n)}\mapsto \langle P_n(\omega), f^{(n)}\rangle \in L^2(\tau)$$

to a linear continuous operator

$$L^2(X^n, m_\nu^{(n)})\ni f^{(n)}\mapsto \langle P_n(\omega), f^{(n)}\rangle \in L^2(\tau).$$

Note that, by Theorem 3.2, for each $f^{(n)}\in L^2(X^n, m_\nu^{(n)})$,

$$\langle P_n(\omega), f^{(n)}\rangle = \langle P_n(\omega), \mathrm{Sym}_n\, f^{(n)}\rangle.$$

Thus, Theorem 3.2 immediately implies

Corollary 3.3. *We have a unitary isomorphism*

$$\mathbf{F}^Q(\mathcal{H},\nu) \ni (f^{(n)})_{n=0}^\infty \mapsto f^{(0)} + \sum_{n=1}^\infty \langle P_n(\omega), f^{(n)}\rangle \in L^2(\tau). \tag{3.11}$$

We denote the inverse of the unitary operator in (3.11) by U. Thus, $U : L^2(\tau) \to \mathbf{F}^Q(\mathcal{H},\nu)$ is a unitary operator; compare with Theorem 1.4 (ii) in the boson case, $q = 1$.

3.3 Anyon Lévy white noise as a Jacobi field

In view of subsec. 2.2 and Corollary 3.3, we have the following chain of unitary operators:

$$\mathbf{F}^Q(\mathcal{H},\nu) \xleftarrow{U} L^2(\tau) \xrightarrow{I} \mathcal{F}^Q(\mathcal{G}).$$

We also define a unitary operator

$$\mathbf{U} : \mathbf{F}^Q(\mathcal{H},\nu) \to \mathcal{F}^Q(\mathcal{G}), \quad \mathbf{U} := IU^{-1}.$$

Let $h \in C_0(X)$. Recall formula (2.20), which says that, under I^{-1}, the operator $\langle\omega,h\rangle$ in $\mathcal{F}^Q(\mathcal{G})$ becomes the operator of left multiplication by $\langle\omega,h\rangle$ in $L^2(\tau)$. We denote

$$\mathbf{J}(h) := \mathbf{U}^{-1}\langle\omega,h\rangle\mathbf{U}. \tag{3.12}$$

Obviously, the operators $\mathbf{J}(h)$ form a Jacobi field in the extended anyon Fock space $\mathbf{F}^Q(\mathcal{H},c)$, i.e., each operator $\mathbf{J}(h)$ has a representation

$$\mathbf{J}(h) = \mathbf{J}^+(h) + \mathbf{J}^0(f) + \mathbf{J}^-(h), \tag{3.13}$$

where $\mathbf{J}^+(h)$ is a creation operator, $\mathbf{J}^0(h)$ is a neutral operator, and $\mathbf{J}^-(h)$ is an annihilation operator. Equivalently, we have

$$\langle\omega,h\rangle\langle P_n(\omega), f^{(n)}\rangle = \langle P_{n+1}(\omega), \mathbf{J}^+(h)f^{(n)}\rangle + \langle P_n(\omega), \mathbf{J}^0(h)f^{(n)}\rangle + \langle P_{n-1}(\omega), \mathbf{J}^-(h)f^{(n)}\rangle.$$

Our next aim is to explicitly calculate the operators $\mathbf{J}^\sharp(h)$, $\sharp = +, 0, -$.

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We define a linear space $\mathcal{F}_{\text{fin}}(B_0(X))$ of all finite vectors $(f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$, where $f^{(0)} \in \mathbb{C}$, $f^{(i)} \in B_0(X^i)$, $i \geq 1$. Evidently, the vacuum vector, Ω, belongs to $\mathcal{F}_{\text{fin}}(B_0(X))$.

For each $h \in C_0(X)$, we define a neutral operator $\mathscr{J}^0(h)$ and an annihilation operator $\mathscr{J}_1^-(h)$ acting on $\mathcal{F}_{\text{fin}}(B_0(X))$ as follows. We first set

$$\mathscr{J}^0(h)\Omega = \mathscr{J}_1^-(h)\Omega := 0. \tag{3.14}$$

Next,

$$(\mathscr{J}^0(h)f^{(n)})(x_1,\dots,x_n) := \sum_{i=1}^{n} h(x_i)f^{(n)}(x_1,\dots,x_n)R_i^{(n)}(x_1,\dots,x_n). \tag{3.15}$$

Here, for each $\theta = \{\theta_1,\dots,\theta_l\} \in \Pi(n)$, the restriction of the function $R_i^{(n)} : X^n \to \mathbb{R}$ to the set $X_\theta^{(n)}$ is given by

$$R_i^{(n)} \restriction X_\theta^{(n)} := b_{\gamma(i,\theta)-1}/\gamma(i,\theta) \tag{3.16}$$

In formula (3.16), $\gamma(i,\theta) := |\theta_u|$ with $\theta_u \in \theta$ being chosen so that $i \in \theta_u$, and $(b_k)_{k=0}^\infty$ are the coefficients from (3.1). Finally,

$$\begin{aligned}(\mathscr{J}_1^-(h)f^{(n)})&(x_1,\dots,x_{n-1})\\ :=& \sum_{1\le i<j\le n} h(x_{j-1})f^{(n)}(x_1,\dots,x_{i-1},\underbrace{x_{j-1}}_{i\text{-th place}},x_i,x_{i+1},\dots,\underbrace{x_{j-1}}_{j\text{-th place}},\dots,x_{n-1})\\ &\times S_{j-1}^{(n)}(x_1,\dots,x_{n-1}),\end{aligned} \tag{3.17}$$

where for any $\theta \in \Pi(n-1)$

$$S_{j-1}^{(n)} \restriction X_\theta^{(n-1)} := \frac{2a_{\gamma(j-1,\theta)}}{\gamma(j-1,\theta)(\gamma(j-1,\theta)+1)}. \tag{3.18}$$

Here $(a_k)_{k=1}^\infty$ are also the coefficients from (3.1).

We define

$$\mathbf{F}_{\mathrm{fin}}^Q(B_0(X)) := \mathrm{Sym}\ \mathcal{F}_{\mathrm{fin}}(B_0(X)),$$

where Sym is the linear operator on $\mathcal{F}_{\mathrm{fin}}(B_0(X))$ satisfying $\mathrm{Sym}\, f^{(n)} := \mathrm{Sym}_n\, f^{(n)}$ for $f^{(n)} \in B_0(X^n)$. We also denote $\mathbf{B}_0^Q(X^n) := \mathrm{Sym}_n\, B_0(X^n)$.

On $\mathbf{F}_{\mathrm{fin}}^Q(B_0(X))$, we define a Q-symmetric tensor product by setting, for any $f^{(m)} \in \mathbf{B}_0^Q(X^m)$, $g^{(m)} \in \mathbf{B}_0^Q(X^n)$,

$$f^{(m)} \circledast g^{(n)} := \mathrm{Sym}_{m+n}(f^{(m)} \otimes g^{(n)}), \tag{3.19}$$

and extending it by linearity. Here $f^{(m)} \otimes g^{(n)} \in B_0(X^{m+n})$ is given by

$$(f^{(m)} \otimes g^{(n)})(x_1,\dots,x_{m+n}) = f^{(m)}(x_1,\dots,x_m)g^{(n)}(x_{m+1},\dots,x_{m+n}).$$

We will prove below that the tensor product $\circledast$ is associative. Furthermore, the restriction of $f^{(m)} \circledast g^{(n)}$ to $X^{(m+m)}$ evidently coincides with $f^{(m)} \circledast g^{(n)}$ as defined in subsec. 2.1.

Theorem 3.4. *For each $h \in C_0(X)$, $\mathbf{J}(h)$ is a linear operator on $\mathbf{F}_{\mathrm{fin}}^Q(B_0(X))$ which has representation* (3.13). *For each $F \in \mathbf{F}_{\mathrm{fin}}^Q(B_0(X))$, we have*

$$\mathbf{J}^+(h)F = h \circledast F,$$
$$\mathbf{J}^0(h)F = \mathrm{Sym}(\mathscr{J}^0(h)F),$$

and

$$\mathbf{J}^-(h) = \mathbf{J}_1^-(h) + \mathbf{J}_2^-(h). \tag{3.20}$$

Here,

$$\mathbf{J}_1^-(h)F = \mathrm{Sym}(\mathscr{J}_1^-(h)F)$$

and for each $f^{(n)} \in \mathbf{B}_0^Q(X^n)$

$$(\mathbf{J}_2^-(h)f^{(n)})(x_1, \dots, x_{n-1}) = n \int_X dy\, h(y) f^{(n)}(y, x_1, \dots, x_{n-1}). \tag{3.21}$$

3.4 A characterization of Meixner-type polynomials

Recall that the operators $\mathscr{J}^0(h)$ and $\mathscr{J}^-(h)$ were defined by using the coefficients of the recursion relation (3.1) (i.e., by the measure ν), and these operators do not depend on the type of anyon statistics, i.e., they are independent of Q.

Recall the set of orthogonalized continuous polynomials, $\mathscr{OCP}$, defined in subsec. 2.2. Let us consider the following condition.

(C) For each $h \in C_0(X \mapsto \mathbb{R})$, the linear operators $\mathbf{J}^0(h)$ and $\mathbf{J}_1^-(h)$ map the set $\mathscr{OCP}$ into itself.

Theorem 3.5. *Assume that either $q \neq -1$ or $q = -1$ and the support of the measure ν does not consist of exactly two points. Then condition* (C) *is satisfied if and only if there exist constants $\lambda \in \mathbb{R}$ and $\eta \geq 0$ such that the coefficients a_k, b_k in the recursion formula* (3.1) *are given by*

$$a_k = \eta k(k+1) \quad (k \in \mathbb{N}), \qquad b_k = \lambda(k+1) \quad (k \in \mathbb{N}_0). \tag{3.22}$$

In the latter case, for any $h, f_1, \dots, f_n \in C_0(X)$, we have

$$\begin{aligned}
\mathbf{J}(h) f_1 \circledast \dots \circledast f_n &= h \circledast f_1 \circledast \dots \circledast f_n \\
&+ \lambda \sum_{i=1}^n f_1 \circledast \dots \circledast f_{i-1} \circledast (hf_i) \circledast f_{i+1} \circledast \dots \diamond f_n \\
&+ 2\eta \sum_{1 \leq i < j \leq n} f_1 \circledast \dots \circledast f_{i-1} \circledast f_{i+1} \circledast \dots \circledast f_{j-1} \circledast (hf_if_j) \circledast f_{j+1} \circledast \dots \circledast f_n \\
&+ n \int_X dy\, h(y)(f_1 \circledast \dots \circledast f_n)(y, \cdot).
\end{aligned} \tag{3.23}$$

We see that, in the classical case, $q = 1$, Theorem 3.5 gives exactly the Meixner class of infinite dimensional polynomials, discussed in subsec. 1.2. Note that the obtained class of the measures ν is independent on q. So, for such a choice of ν, we call $\big(\langle P_n(\omega), f^{(n)}\rangle\big)$ a Meixner-type system of orthogonal (noncommutative) polynomials for anyon statistics.

Remark 3.6. In the fermion case ($q = -1$), if the support of the measure ν consists of exactly two points, we could not prove that condition (C) always fails, but we conjecture this indeed to be the case.

The following result can be easily proven.

Proposition 3.7. *For each $q \in \mathbb{C}$, $|q| = 1$ we have equality $\mathscr{CP} = \mathscr{OCP}$ in the anyon Gaussian/Poisson case, i.e., when formula* (3.22) *holds with $\lambda \in \mathbb{R}$ and $\eta = 0$.*

However, due to the form of the operator $\mathbf{J}_2^-(h)$, see (3.21), equality $\mathscr{CP} = \mathscr{OCP}$ fails if $q \neq 1$ and the the measure ν is not concentrated at one point. Still, in the classical case, $q = 1$, Theorem 3.5 implies Theorem 1.2.

3.5 Anyon Meixner-type white noise

We will assume in this subsection that (3.22) holds. We may, at least informally, define,

$$\omega(x) = \langle \omega, \delta_x \rangle, \quad x \in X,$$

so that for $h \in C_0(X)$,

$$\langle \omega, h \rangle = \int_X dx\, \omega(x) h(x). \tag{3.24}$$

Hence, we may think of $(\omega(x))_{x\in X}$ as an anyon Meixner-type white noise.

For $x \in X$, we define an annihilation operator ∂_x as the linear operator acting on $\mathbf{F}^Q_{\mathrm{fin}}(B_0(X))$ by the formula:

$$(\partial_x f^{(n)})(x_1, \dots, x_{n-1}) := n f^{(n)}(x, x_1, \dots, x_{n-1}), \quad (x_1, \dots, x_{n-1}) \in X^{n-1}, \tag{3.25}$$

for $f^{(n)} \in \mathbf{B}_0^Q(X^n)$. Then, by (3.21), for $h \in C_0(X)$, we may interpret the operator $\mathbf{J}_2^-(h)$ as the integral

$$\mathbf{J}_2^-(h) = \int_X dx\, h(x) \partial_x\,. \tag{3.26}$$

Next, we introduce an 'operator-valued distribution' $X \ni x \mapsto \partial_x^\dagger$ so that, for any $h \in C_0(X)$ and $f^{(n)} \in \mathbf{B}_0^Q(X^n)$,

$$\int_X dx\, h(x) \partial_x^\dagger f^{(n)} := h \circledast f^{(n)}. \tag{3.27}$$

In other words, we may think $\partial_x^\dagger f^{(n)} = \delta_x \circledast f^{(n)}$. Thus,

$$\mathbf{J}^+(h) = \int_X dx\, h(x)\partial_x^\dagger\,. \tag{3.28}$$

For $h \in C_0(X)$, we will now need operators

$$\int_X dx\, h(x)\partial_x^\dagger\partial_x, \qquad \int_X dx\, h(x)\partial_x^\dagger\partial_x\partial_x$$

acting on $\mathbf{F}_{\mathrm{fin}}^Q(B_0(X))$. In view of (3.25) and (3.27), we have, for each $f^{(n)} \in \mathbf{B}_0^Q(X^n)$,

$$\begin{aligned}\left(\int_X dx\, h(x)\partial_x^\dagger\partial_x\, f^{(n)}\right)(x_1,\dots,x_n) &= n\left(\int_X dx\, h(x)\partial_x^\dagger f^{(n)}(x,\cdot)\right)(x_1,\dots,x_n)\\ &= n\ \mathrm{Sym}_n\left(h(x_1)f^{(n)}(x_1,x_2,\dots,x_n)\right)\end{aligned}$$

(compare with (2.14)), and

$$\begin{aligned}&\left(\int_X dx\, h(x)\partial_x^\dagger\partial_x\partial_x\, f^{(n)}\right)(x_1,\dots,x_{n-1})\\ &\quad= n(n-1)\left(\int_X dx\, h(x)\partial_x^\dagger f^{(n)}(x,x,\cdot)\right)(x_1,\dots,x_{n-1})\\ &\quad= n(n-1)\ \mathrm{Sym}_{n-1}\left(h(x_1)f^{(n)}(x_1,x_1,x_2,x_3,\dots,x_{n-1})\right).\end{aligned} \tag{3.29}$$

Theorem 3.8. *Assume* (3.22) *holds. Then for* $h \in C_0(X)$*,*

$$\mathbf{J}^0(h) = \int_X dx\, h(x)\lambda\partial_x^\dagger\partial_x, \tag{3.30}$$

$$\mathbf{J}_1^-(h) = \int_X dx\, h(x)\eta\partial_x^\dagger\partial_x\partial_x\,. \tag{3.31}$$

Thus,

$$\mathbf{J}(h) = \int_X dx\, h(x)(\partial_x^\dagger + \lambda\partial_x^\dagger\partial_x + \eta\partial_x^\dagger\partial_x\partial_x + \partial_x). \tag{3.32}$$

In view of formula (3.12), the operator $\mathbf{J}(h)$ is a realization of the operator $\langle\omega, h\rangle$ in the extended anyon Fock space $\mathbf{F}^Q(\mathcal{H},\nu)$. So, with an abuse of notation, we may denote $\mathbf{J}(h)$ by $\langle\omega, h\rangle$. Then, by (3.24) and (3.32), we get the following representation of the anyon Meixner-type white noise (realized in the extended anyon Fock space $\mathbf{F}^Q(\mathcal{H},\nu)$):

$$\omega(x) = \partial_x^\dagger + \lambda\partial_x^\dagger\partial_x + \eta\partial_x^\dagger\partial_x\partial_x + \partial_x.$$

Remark 3.9. We note that, for q-commutation relations with q being real from either the interval $(-1,0)$ or the interval $(0,1)$ [3, 15, 19], there is no analog of a q-Lévy process which would have a representation like in (3.32). Nevertheless, as shown in [22], there exist classical Markov processes whose transition probabilities are measures of orthogonality for q-Meixner (orthogonal) polynomials on the real line.

4 Proofs

4.1 Proof of Proposition

3.1

Note that $\mathrm{Sym}_1 = \mathbf{1}$, so we need to prove the statement for $n \geq 2$. Following [18], let us first briefly recall how one shows that the operator Sym_n given by (2.4) is an orthogonal projection in the space $L^2(X^n, m^{\otimes n})$. For each $\pi \in \mathfrak{S}_n$, we define

$$(\Psi_\pi f^{(n)})(x_1, \dots, x_n) = Q_\pi(x_1, \dots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}) \tag{4.1}$$

for $(x_1, \dots, x_n) \in X^{(n)}$. Thus, $\mathrm{Sym}_n = \frac{1}{n!}\sum_{\pi \in \mathfrak{S}_n} \Psi_\pi$. We then have $\Psi_\pi^* = \Psi_{\pi^{-1}}$, which implies $\mathrm{Sym}_n^* = \mathrm{Sym}_n$. Furthermore, for each permutation $\varkappa \in \mathfrak{S}_n$, we have

$$\Psi_\pi \Psi_\varkappa = \Psi_{\varkappa\pi}. \tag{4.2}$$

Therefore, on $X^{(n)}$,

$$\mathrm{Sym}_n^2 = \frac{1}{(n!)^2} \sum_{\pi, \varkappa \in \mathfrak{S}_n} \Psi_\pi \Psi_\varkappa = \frac{1}{(n!)^2} \sum_{\pi \in \mathfrak{S}_n} \sum_{\varkappa \in \mathfrak{S}_n} \Psi_{\pi\varkappa} = \frac{1}{n!} \sum_{\varkappa \in \mathfrak{S}_n} \Psi_\varkappa = \mathrm{Sym}_n\,. \tag{4.3}$$

Thus, Sym_n is an orthogonal projection. Note that formula (4.2) implies that, for $\varkappa, \pi \in \mathfrak{S}_n$,

$$\left(\prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} Q(x_i, x_j) \right) \left(\prod_{\substack{1 \leq k < l \leq n \\ \varkappa(k) > \varkappa(l)}} Q(x_{\pi^{-1}(k)}, x_{\pi^{-1}(l)}) \right)$$
$$= \prod_{\substack{1 \leq i < j \leq n \\ (\varkappa\pi)(i) > (\varkappa\pi)(j)}} Q(x_i, x_j), \quad (x_1, \dots, x_n) \in X^{(n)}. \tag{4.4}$$

Now let us consider the linear, bounded operator Sym_n in $L^2(X^n, m_\nu^{(n)})$. We represent the operator Sym_n as

$$\mathrm{Sym}_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \Psi_\pi, \tag{4.5}$$

with $\Psi_\pi f^{(n)}$ being defined on the whole X^n by the right hand side of formula (4.1) in which the function $Q_\pi(x_1, \dots, x_n)$ on X^n is defined in subsec. 3.2.

We fix a permutation $\pi \in \mathfrak{S}_n$ and a partition $\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)$ satisfying (3.4), and let (3.5), (3.6) hold. Further, let $\varkappa \in \mathfrak{S}_n$ and let $\zeta = \{\zeta_1, \dots, \zeta_l\} \in \Pi(n)$ be such that

$$\max \zeta_1 < \max \zeta_2 < \dots < \max \zeta_l, \tag{4.6}$$

and

$$\varkappa\beta_i = \zeta_{\widehat{\varkappa}(i)}, \quad i = 1, \dots, l,$$

where $\widehat{\varkappa} \in \mathfrak{S}_l$.

Then, for each function $f^{(n)} : X^n \to \mathbb{C}$ and $(x_1, \dots, x_n) \in X^n$, we have

$$(\Psi_\pi \Psi_\varkappa f^{(n)})(x_1, \dots, x_n) = \left(\prod_{\substack{1 \le i < j \le l \\ \widehat{\pi}(i) > \widehat{\pi}(j)}} Q(x_{\theta_i}, x_{\theta_j}) \right) (\Psi_\varkappa f^{(n)})(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}). \quad (4.7)$$

Denote $y_i = x_{\pi^{-1}(i)}$, or equivalently $y_{\pi(i)} = x_i$ for $i = 1, \dots, n$. Thus, $y_{\pi(i)} = y_{\pi(j)}$ if and only if i and j belong to the same element of the partition θ. Equivalently, $y_i = y_j$ if and only if i and j belong to the same element of the partition β. Therefore,

$$(\Psi_\varkappa f^{(n)})(y_1, \dots, y_n) = \left(\prod_{\substack{1 \le u < v \le l \\ \widehat{\varkappa}(u) > \widehat{\varkappa}(v)}} Q(y_{\beta_u}, y_{\beta_v}) \right) f^{(n)}(y_{\varkappa^{-1}(1)}, \dots, y_{\varkappa^{-1}(n)}).$$

Hence,

$$\begin{aligned} &(\Psi_\varkappa f^{(n)})(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}) \\ &\quad = \left(\prod_{\substack{1 \le u < v \le l \\ \widehat{\varkappa}(u) > \widehat{\varkappa}(v)}} Q(x_{\pi^{-1}\beta_u}, x_{\pi^{-1}\beta_v}) \right) f^{(n)}(x_{\pi^{-1}\varkappa^{-1}(1)}, \dots, x_{\pi^{-1}\varkappa^{-1}(n)}) \\ &\quad = \left(\prod_{\substack{1 \le u < v \le l \\ \widehat{\varkappa}(u) > \widehat{\varkappa}(v)}} Q(x_{\theta_{\widehat{\pi}^{-1}(u)}}, x_{\theta_{\widehat{\pi}^{-1}(v)}}) \right) f^{(n)}(x_{(\varkappa\pi)^{-1}(1)}, \dots, x_{(\varkappa\pi)^{-1}(n)}), \end{aligned} \quad (4.8)$$

where we used the observation that, for each $u = 1, \dots, l$,

$$\pi^{-1}\beta_u = \theta_{\widehat{\pi}^{-1}(u)}.$$

Using (4.4), we get

$$\left(\prod_{\substack{1 \le i < j \le l \\ \widehat{\pi}(i) > \widehat{\pi}(j)}} Q(x_{\theta_i}, x_{\theta_j}) \right) \left(\prod_{\substack{1 \le u < v \le l \\ \widehat{\varkappa}(u) > \widehat{\varkappa}(v)}} Q(x_{\theta_{\widehat{\pi}^{-1}(u)}}, x_{\theta_{\widehat{\pi}^{-1}(v)}}) \right) = \left(\prod_{\substack{1 \le i < j \le l \\ \widehat{\varkappa\pi}(i) > \widehat{\varkappa\pi}(j)}} Q(x_{\theta_i}, x_{\theta_j}) \right). \quad (4.9)$$

Here $\widehat{\varkappa\pi}$ is the permutation from $\mathfrak{S}_l$ induced by the permutation $\varkappa\pi \in \mathfrak{S}_n$ and the partition θ. In (4.9), we used the observation that $\widehat{\varkappa\pi} = \widehat{\varkappa}\,\widehat{\pi}$. Now, substituting (4.8) into (4.7) and using (4.9), we conclude that

$$\Psi_\pi \Psi_\varkappa = \Psi_{\varkappa\pi}, \tag{4.10}$$

and hence analogously to (4.3), we get $\mathrm{Sym}_n^2 = \mathrm{Sym}_n$.

Next, we note that the measure $m_\nu^{(n)}$ remains invariant under the transformation

$$X^n \ni (x_1, \dots, x_n) \mapsto (x_{\pi^{-1}(1)}, \dots x_{\pi^{-1}(n)}) \in X^n.$$

Furthermore, as easily seen, the equality

$$\overline{Q_{\pi^{-1}}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)})} = Q_\pi(x_1, \dots, x_n)$$

holds for each $(x_1, \dots, x_n) \in X^n$. Hence, for each $\pi \in \mathfrak{S}_n$, $\Psi_\pi^* = \Psi_{\pi^{-1}}$, which implies $\mathrm{Sym}_n^* = \mathrm{Sym}_n$.

Thus, Sym_n is an orthogonal projection in $L^2(X^n, m_\nu^{(n)})$. Analogously to [18, Proposition 2.5], we easily conclude that the image of $\mathrm{Sym}_n$is indeed $\mathbf{F}_n^Q(\mathcal{H}, \nu)$. Thus, Proposition 3.1 is proven.

Recall the tensor product $\circledast$ defined on $\mathbf{F}_{\mathrm{fin}}^Q(B_0(X))$ by formula (3.19). Using (4.5) and (4.10), it is easy to show that, for any $f^{(m)} \in B_0(X^m)$ and $g^{(n)} \in B_0(X^n)$, we have

$$\begin{aligned}(\mathrm{Sym}_m\, f^{(m)}) \circledast (\mathrm{Sym}_n\, g^{(n)}) &= \mathrm{Sym}_{m+n}((\mathrm{Sym}_m\, f^{(m)}) \otimes (\mathrm{Sym}_n\, g^{(n)})) \\ &= \mathrm{Sym}_{m+n}(f^{(m)} \otimes g^{(n)}).\end{aligned}$$

Therefore, the tensor product $\circledast$ is associative on $\mathbf{F}_{\mathrm{fin}}^Q(B_0(X))$.

4.2 Proof of Theorem 3.2

Recall the unitary operator $I : L^2(\tau) \to \mathcal{F}^Q(\mathcal{G})$. Our next aim is to obtain an explicit form of the subspace $I(\mathscr{OP}_n)$ of $\mathcal{F}^Q(\mathcal{G})$.

Denote by $\mathbb{N}_{0,\mathrm{fin}}^\infty$ the set of all infinite sequences $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots) \in \mathbb{N}_0^\infty$ such that only a finite number of α_j's are not equal to zero. Let $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2 + \cdots$. For each $\alpha \in \mathbb{N}_{0,\mathrm{fin}}^\infty$ with $|\alpha| \geq 1$, we denote by $\mathcal{F}_\alpha$ the subspace of the Fock space $\mathcal{F}^Q(\mathcal{G})$ which consists of all elements of the form

$$\mathrm{Sym}_{|\alpha|}\left(f^{(|\alpha|)}(x_1, \dots, x_{|\alpha|}) p_0(s_1) \cdots p_0(s_{\alpha_0}) p_1(s_{\alpha_0+1}) \cdots p_1(s_{\alpha_0+\alpha_1}) p_2(s_{\alpha_0+\alpha_1+1}) \cdots\right),$$

where $f^{(|\alpha|)} \in \mathcal{H}^{\otimes|\alpha|}$. For $\alpha \in \mathbb{N}_{0,\mathrm{fin}}^\infty$ with $|\alpha| = 0$, we set $\mathcal{F}_\alpha := \{c\Omega \mid c \in \mathbb{C}\}$. The following proposition is proven in [18, Section 7]. This result is a counterpart of the Nualart–Schoutens decomposition of the L^2-space of a classical Lévy process [45], see also [48].

Proposition 4.1. *We have*

$$\mathcal{F}^Q(\mathcal{G}) = \bigoplus_{\alpha\in\mathbb{N}_{0,\,\mathrm{fin}}^\infty} \mathcal{F}_\alpha. \tag{4.11}$$

For each $n \in \mathbb{N}_0$, we define

$$\mathbb{F}_n := \bigoplus_{\substack{\alpha\in\mathbb{N}_{0,\,\mathrm{fin}}^\infty \\ \alpha_0+2\alpha_1+3\alpha_2+\cdots=n}} \mathcal{F}_\alpha.$$

Note that, by (4.11),

$$\mathcal{F}^Q(\mathcal{G}) = \bigoplus_{n=0}^{\infty} \mathbb{F}_n.$$

Proposition 4.2. *For each $n \in \mathbb{Z}_+$,*

$$I\mathscr{O}\mathscr{P}_n = \mathbb{F}_n.$$

Proof. It suffices to prove that, for each $n \in \mathbb{N}$,

$$I\mathscr{M}\mathscr{P}_n = \bigoplus_{\substack{\alpha\in\mathbb{N}_{0,\,\mathrm{fin}}^\infty \\ \alpha_0+2\alpha_1+3\alpha_2+\cdots\le n}} \mathcal{F}_\alpha =: \mathbb{M}_n. \tag{4.12}$$

Lemma 4.3. *The space $\mathbb{M}_n$ consists of all finite sums of elements of the form*

$$\mathrm{Sym}_k \big(f^{(k)}(x_1,\dots,x_k) s_1^{i_1} s_2^{i_2}\cdots s_k^{i_k}\big), \tag{4.13}$$

where $f^{(k)} \in \mathcal{H}^{\otimes k}$ and $i_1 + i_2 + \cdots + i_k + k \le n$.

Proof. For each $\pi \in \mathfrak{S}_k$, we define a unitary operator Ψ_π on $(\mathcal{H}\otimes L^2(\mathbb{R},\nu))^{\otimes k}$ by

$$(\Psi_\pi g^{(k)})(x_1, s_1,\dots,x_k,s_k) = Q_\pi(x_1,\dots,x_k) g^{(k)}(x_{\pi^{-1}(1)}, s_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(k)}, s_{\pi^{-1}(k)}).$$

Here the function Q_π is defined by (2.5). Then, by [18], the operators Ψ_π form a unitary representation of the symmetric group $\mathfrak{S}_k$, and for each $\pi \in \mathfrak{S}_k$ we have $\mathrm{Sym}_k = \mathrm{Sym}_k \Psi_\pi$. Hence, for any permutation $\pi \in \mathfrak{S}_k$, $u^{(k)} \in \mathcal{H}^{\otimes k}$, and any polynomial $r^{(k)}(s_1,\dots,s_k)$ in the $s_1,\dots,s_k$ variables,

$$\mathrm{Sym}_k \big(f^{(k)}(x_1,\dots,x_k) r^{(k)}(s_1,\dots,s_k)\big) = \mathrm{Sym}_k \big(u^{(k)}(x_1,\dots,x_k) r^{(k)}(s_{\pi^{-1}(1)},\dots,s_{\pi^{-1}(k)})\big),$$

where

$$u^{(k)}(x_1,\dots,x_k) = Q_\pi(x_1,\dots,x_k) f^{(k)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(k)}).$$

In particular, $u^{(k)} \in \mathcal{H}^{\otimes k}$.

Noting the evident representations

$$p_l(s) = \sum_{i=0}^{l} \alpha_{il}\, s^i, \quad s^l = \sum_{i=0}^{l} \beta_{il}\, p_i(s),$$

we easily conclude the lemma. □

We now finish the proof of (4.12). Let $\mathcal{F}_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$ be the linear subspace of the full Fock space over $\mathcal{H}\otimes L^2(\mathbb{R},\nu)$ which consists of finite sums of $c\Omega$ ($c\in\mathbb{C}$) and elements of the form

$$f^{(k)}(x_1,\dots,x_k)s_1^{i_1}s_2^{i_2}\cdots s_k^{i_k} \tag{4.14}$$

with $f^{(k)}\in\mathcal{H}^{\otimes k}$, $i_1,i_2,\dots,i_k\in\mathbb{Z}_+$, $k\in\mathbb{N}$. We set

$$\mathrm{Sym}:=\mathbf{1}\oplus\mathrm{Sym}_1\oplus\mathrm{Sym}_2\oplus\mathrm{Sym}_3\oplus\cdots. \tag{4.15}$$

This operator projects $\mathcal{F}_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$ onto $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$. We have, for each $h\in C_0(X)$ and $F\in\mathcal{F}_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$,

$$a^+(h\otimes 1)\,\mathrm{Sym}\,F=\mathrm{Sym}\,J^+(h\otimes 1)F,\quad a^-(h\otimes 1)\,\mathrm{Sym}\,F=\mathrm{Sym}\,J^-(h\otimes 1)F, \tag{4.16}$$

$$a^0(h\otimes\mathrm{id})\,\mathrm{Sym}\,F=\mathrm{Sym}\,J^0(h\otimes\mathrm{id})F. \tag{4.17}$$

Here, for each F as in (4.14),

$$(J^+(h\otimes 1)F)(x_1,s_1,\dots,x_{k+1},s_{k+1})=h(x_1)1(s_1)f^{(k)}(x_2,\dots,x_{k+1})s_2^{i_1}s_3^{i_2}\cdots s_{k+1}^{i_k},$$
$$(J^0(h\otimes\mathrm{id})F)(x_1,s_1,\dots,x_k,s_k)=(h(x_1)s_1+\dots+h(x_k)s_k)f^{(k)}(x_1,\dots,x_k)s_1^{i_1}s_2^{i_2}\cdots s_k^{i_k},$$
$$(J^-(h\otimes 1)F)(x_1,s_1,\dots,x_{k-1},s_{k-1})=\sum_{j=1}^k\int_X dy\int_{\mathbb{R}}\nu(dt)\,h(y)Q(y,x_1)\cdots Q(y,x_{j-1})$$
$$\times f^{(k)}(x_1,\dots,x_{j-1},y,x_j,\dots,x_{k-1})s_1^{i_1}\cdots s_{j-1}^{i_{j-1}}t^{i_j}s_j^{i_{j+1}}\cdots s_{k-1}^{i_{j_k}}. \tag{4.18}$$

Hence, it follows by induction from Lemma 4.3 and (4.16)–(4.18) that

$$\langle\omega,h_1\rangle\cdots\langle\omega,h_n\rangle\Omega\subset\mathbb{M}_n$$

for any $h_1,\dots,h_n\in C_0(X)$, $n\in\mathbb{N}$. Since $\mathbb{M}_n$ is a closed subspace of $\mathcal{F}^Q(\mathcal{G})$, we therefore get the inclusion $I\mathscr{M}\mathscr{P}_n\subset\mathbb{M}_n$. On the other hand, it directly follows from the proof of [18, Proposition 6.7] that each element of $\mathbb{M}_n$ which has form (4.13) belongs to $I\mathscr{M}\mathscr{P}_n$. Hence, we get the inverse inclusion $\mathbb{M}_n\subset I\mathscr{M}\mathscr{P}_n$. □

Note that, for each $h\in C_0(X)$,

$$a^0(h\otimes\mathrm{id})=d\Gamma(M_{h\otimes\mathrm{id}})=d\Gamma(M_h\otimes M_{\mathrm{id}}), \tag{4.19}$$

where M_h is the operator of multiplication by the function $h(x)$ in $\mathcal{H}$ and M_{id} is the (restriction to $\mathscr{P}(\mathbb{R})$ of the) operator of multiplication by the monomial $\mathrm{id}(s)=s$ in $L^2(\mathbb{R},\nu)$. (Note that the operator M_{id} is unbounded in $L^2(\mathbb{R},\nu)$ if the support of measure ν is unbounded, and the second quantization operator has domain $\mathcal{F}^Q_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$.) In view of the recursion formula (3.1), we get the representation

$$M_{\mathrm{id}}=A^++A^0+A^-,$$

where A^+, A^0, and A^- are the linear operators on $\mathscr{P}(\mathbb{R})$ given by

$$A^+p_k := p_{k+1}, \quad A^0p_k = b_kp_k, \quad A^-p_k = a_kp_{k-1}. \tag{4.20}$$

By (4.19) and (4.20),

$$a^0(h\otimes \mathrm{id}) = d\Gamma(M_h\otimes A^+) + d\Gamma(M_h\otimes A^0) + d\Gamma(M_h\otimes A^-). \tag{4.21}$$

By (2.16) and (4.21), we get, for each $h\in C_0(X)$,

$$\langle \omega, h\rangle = \mathcal{A}^+(h) + \mathcal{A}^0(h) + \mathcal{A}^-(h), \tag{4.22}$$

where

$$\begin{aligned} \mathcal{A}^+(h) &:= a^+(h\otimes 1) + d\Gamma(M_h\otimes A^+),\\ \mathcal{A}^0(h) &:= d\Gamma(M_h\otimes A^0),\\ \mathcal{A}^-(h) &:= a^-(h\otimes 1) + d\Gamma(M_h\otimes A^-). \end{aligned} \tag{4.23}$$

Proposition 4.4. *For each $h\in C_0(X)$, we have $\mathcal{A}^+(h):\mathbb{F}_n\to\mathbb{F}_{n+1}$, $\mathcal{A}^0(h):\mathbb{F}_n\to\mathbb{F}_n$, $\mathcal{A}^-(h):\mathbb{F}_n\to\mathbb{F}_{n-1}$.*

Proof. Let $\sharp = +, 0, -$. For each $h\in C_0(X)$, we define an operator $N(M_h\otimes A^\sharp)$ on $\mathcal{F}_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$ by setting $N(M_h\otimes A^\sharp)\Omega := 0$ and for each $n\in\mathbb{N}$,

$$\begin{aligned} &N(M_h\otimes A^\sharp)\restriction\big(\mathcal{F}_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))\cap\mathcal{G}^{\otimes n}\big)\\ &:= (M_h\otimes A^\sharp)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}+\mathbf{1}\otimes(M_h\otimes A^\sharp)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}+\cdots+\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes(M_h\otimes A^\sharp). \end{aligned}$$

Lemma 4.5. *Let $\sharp = +, 0, -$. For any $h\in C_0(X\mapsto\mathbb{R})$ and $F\in\mathcal{F}_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$, we have*

$$d\Gamma(M_h\otimes A^\sharp)\,\mathrm{Sym}\,F = \mathrm{Sym}\,N(M_h\otimes A^\sharp)F.$$

Proof. Fix any $F\in\mathcal{F}_{\mathrm{fin}}(\mathcal{H}\otimes\mathscr{P}(\mathbb{R}))$ of the form

$$F(x_1,s_1,\dots,x_n,s_n) = f^{(n)}(x_1,\dots,x_n)p_{i_1}(s_1)\cdots p_{i_n}(s_n).$$

By (2.4),

$$\begin{aligned} &(\mathrm{Sym}_n F)(x_1,s_1,\dots,x_k,s_k)\\ &\quad= \frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n} Q_\pi(x_1,\dots,x_n)f^{(n)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})p_{i_1}(s_{\pi(1)})\cdots p_{i_n}(s_{\pi(n)}). \end{aligned} \tag{4.24}$$

Note that

$$d\Gamma(M_h\otimes A^+) = \mathrm{Sym}\,N(M_h\otimes A^+). \tag{4.25}$$

By (4.24),

$$(N(M_h \otimes A^+) \operatorname{Sym}_n F)(x_1, s_1, \dots, x_n, s_n)$$
$$= \frac{1}{n!} \sum_{j=1}^{n} \sum_{\pi \in \mathfrak{S}_n} Q_\pi(x_1, \dots, x_n) h(x_{\pi^{-1}(j)}) f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)})$$
$$\times p_{i_1}(s_{\pi^{-1}(1)}) \cdots p_{i_j+1}(s_{\pi^{-1}(j)}) \cdots p_{i_n}(s_{\pi^{-1}(n)})$$
$$= \frac{1}{n!} \sum_{j=1}^{n} \sum_{\pi \in \mathfrak{S}_n} Q_\pi(x_1, \dots, x_n) g_j^{(n)}(x_{\pi^{-1}(1)}, s_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}, s_{\pi^{-1}(n)}). \tag{4.26}$$

Here, for $j = 1, \dots, n$,

$$g_j^{(n)}(x_1, s_1, \dots, x_n, s_n) := h(x_j) f^{(n)}(x_1, \dots, x_n) p_{i_1}(s_1) \cdots p_{i_j+1}(s_j) \cdots p_{i_n}(s_n).$$

Then, by (4.25) and (4.26),

$$(d\Gamma(M_h \otimes A^+) \operatorname{Sym}_n F)(x_1, s_1, \dots, x_n, s_n)$$
$$= \frac{1}{(n!)^2} \sum_{j=1}^{n} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\pi \in \mathfrak{S}_n} Q_\sigma(x_1, \dots, x_n) Q_\pi(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$
$$\times g_j^{(n)}(x_{\sigma^{-1}(\pi^{-1}(1))}, s_{\sigma^{-1}(\pi^{-1}(1))}, \dots x_{\sigma^{-1}(\pi^{-1}(n))}, s_{\sigma^{-1}(\pi^{-1}(n))}).$$

Hence,

$$d\Gamma(M_h \otimes A^+) \operatorname{Sym}_n F = \sum_{j=1}^{n} \operatorname{Sym}_n^2 g_j^{(n)} = \sum_{j=1}^{n} \operatorname{Sym}_n g_j^{(n)}$$
$$= \operatorname{Sym}_n \left(\sum_{j=1}^{n} g_j^{(n)} \right) = \operatorname{Sym}_n N(M_h \otimes A^+) F.$$

The proof for A^0 and A^- is analogous. □

Now, the proposition follows directly from the definition of the spaces $\mathbb{F}^{(n)}$, formula (4.16), and Lemma 4.5. □

Proposition 4.6. *For any $h_1, \dots, h_n \in C_0(X)$, we have*

$$I\langle P_n(\omega), h_1 \otimes \cdots \otimes h_n \rangle = \mathcal{A}^+(h_1) \cdots \mathcal{A}^+(h_n)\Omega.$$

Proof. Recall that $\langle P_n(\omega), h_1 \otimes \cdots \otimes h_n \rangle$ is the orthogonal projection of the monomial

$$\langle h_1, \omega \rangle \cdots \langle h_n, \omega \rangle = \langle h_1 \otimes \cdots \otimes h_n, \omega^{\otimes n} \rangle$$

onto $\mathscr{O}\mathscr{P}_n$. The statement follows from Propositions 4.2 and 4.4 if we note that

$$I\langle P_n(\omega), h_1 \otimes \cdots \otimes h_n \rangle$$

is equal to the orthogonal projection of

$$\langle \omega, h_1 \rangle \cdots \langle \omega, h_n \rangle \Omega = (\mathcal{A}^+(h_1) + \mathcal{A}^0(h_1) + \mathcal{A}^-(h_1)) \cdots (\mathcal{A}^+(h_n) + \mathcal{A}^0(h_n) + \mathcal{A}^-(h_n))\Omega$$

onto $\mathbb{F}_n$. $\square$

We will now explicitly calculate the vector $I\langle P_n(\omega), h_1 \otimes \cdots \otimes h_n \rangle$. We introduce a topology on $B_0(X^n)$ which yields the following notion of convergence: $f_n \to f$ as $n \to \infty$ means that there exists a set $\Delta \in \mathcal{B}_0(X)$ such that $\mathrm{supp}(f_n) \subset \Delta$ for all $n \in \mathbb{N}$ and (2.21) holds. Note that $C_0(X^n)$ is a topological subspace of $B_0(X^n)$.

For each $\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n)$ with $\theta_1, \ldots, \theta_l$ satisfying (3.4), we define, for $f^{(n)} \in B_0(X^n)$, $(x_1, \ldots, x_l) \in X^{(l)}$, and $(s_1, \ldots, s_l) \in \mathbb{R}^l$,

$$(\mathcal{E}_\theta f^{(n)})(x_1, s_1, \ldots, x_l, s_l) := f_\theta^{(n)}(x_1, \ldots, x_l) p_{|\theta_1|-1}(s_1) p_{|\theta_2|-1}(s_2) \cdots p_{|\theta_l|-1}(s_l). \quad (4.27)$$

Here the function $f_\theta^{(n)}(x_1, \ldots, x_l)$ is obtained from the function $f^{(n)}(y_1, \ldots, y_n)$ by replacing y_{i_1} with x_1 for all $i_1 \in \theta_1$, y_{i_2} with x_2 for all $i_2 \in \theta_2$, and so on. Note that the function $f_\theta^{(n)} : X^{(l)} \to \mathbb{C}$ is completely identified by the restriction of the function $f^{(n)} : X^n \to \mathbb{C}$ to the set $X_\theta^{(n)}$.

For example, let $n = 6$ and let $\theta = \{\theta_1, \theta_2, \theta_3\} \in \Pi(6)$ be of the form

$$\theta_1 = \{1, 3\}, \quad \theta_2 = \{2, 4, 6\}, \quad \theta_3 = \{5\}.$$

Then, for each $(x_1, x_2, x_3) \in X^{(3)}$ and $(s_1, s_2, s_3) \in \mathbb{R}^3$,

$$(\mathcal{E}_\theta f^{(6)})(x_1, s_1, x_2, s_2, x_3, s_3) = f^{(6)}(x_1, x_2, x_1, x_2, x_3, x_2) p_1(s_1) p_2(s_2) p_0(s_3).$$

Proposition 4.7. *For each $n \in \mathbb{N}$, the mapping*

$$C_0(X)^n \ni (h_1, \ldots, h_n) \mapsto \langle P_n(\omega), h_1 \otimes \cdots \otimes h_n \rangle \in L^2(\tau)$$

may be extended by linearity and continuity to a mapping

$$B_0(X^n) \ni f^{(n)} \to \langle P_n(\omega), f^{(n)} \rangle \in L^2(\tau).$$

Furthermore, for each $f^{(n)} \in B_0(X^n)$, we have

$$I\langle P_n(\omega), f^{(n)} \rangle = \mathrm{Sym}\left(\sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \right). \quad (4.28)$$

Proof. Fix any $h_1, \dots, h_n \in C_0(X)$ and set $f^{(n)}(x_1, \dots, x_n) = h_1(x_1) \cdots h_n(x_n)$. Then, by Proposition 4.6, formula (4.28) is equivalent to

$$\big(a^+(h_1 \otimes 1) + d\Gamma(M_{h_1} \otimes A^+)\big) \cdots \big(a^+(h_n \otimes 1) + d\Gamma(M_{h_n} \otimes A^+)\big)\Omega = \operatorname{Sym}\left(\sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)}\right). \tag{4.29}$$

By (4.16) and Lemmma 4.5, formula (4.29) would follow from

$$\big(J^+(h_1 \otimes 1) + N(M_{h_1} \otimes A^+)\big) \cdots \big(J^+(h_n \otimes 1) + N(M_{h_n} \otimes A^+)\big)\Omega = \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)}. \tag{4.30}$$

Let $\beta = \{\beta_1, \dots, \beta_k\}$ be an (unordered) partition of $\{i+1, i+2, \dots, n\}$. Then

$$J^+(h_i \otimes 1)\mathcal{E}_\beta(h_{i+1} \otimes h_{i+2} \otimes \cdots \otimes h_n) = \mathcal{E}_{\beta^+}(h_i \otimes h_{i+1} \otimes \cdots \otimes h_n), \tag{4.31}$$

where $\beta^+ := \{\{i\}, \beta_1, \dots, \beta_k\}$ is a partition of $\{i, i+1, \dots, n\}$. Furthermore,

$$N(M_{h_i} \otimes A^+)\mathcal{E}_\beta(h_{i+1} \otimes h_{i+2} \otimes \cdots \otimes h_n) = \sum_{j=1}^{k} \mathcal{E}_{\beta_j^0}(h_i \otimes h_{i+1} \otimes \cdots \otimes h_n), \tag{4.32}$$

where β_j^0 is the partition of $\{i, i+1, \dots, n\}$ obtained from β by adding i to the set β_j, i.e.,

$$\beta_j^0 := \{\beta_1, \dots, \beta_j \cup \{i\}, \dots \beta_k\}.$$

By (4.31) and (4.32), formula (4.30) follows by induction. Finally, the extension of formula (4.28) to the case of a general $f^{(n)} \in B_0(X^n)$ follows by linearity and approximation. $\square$

We will now prove Theorem 3.2. Even, a bit more generally, we will prove that formula (3.10) holds for any $f^{(n)}, g^{(n)} \in B_0(X^n)$.

We first note that it suffices to prove formula (3.10) in the case where $f^{(n)} = g^{(n)} = h_1 \otimes \cdots \otimes h_n$ with $h_1, \dots, h_n \in B_0(X)$. By Proposition 4.7,

$$\begin{aligned}
&\big(\langle P_n(\omega), f^{(n)}\rangle,\, \langle P_n(\omega), f^{(n)}\rangle\big)_{L^2(\tau)} \\
&\quad = \left(\sum_{\theta \in \Pi(n)} \operatorname{Sym}_{|\theta|}(\mathcal{E}_\theta f^{(n)}),\, \sum_{\zeta \in \Pi(n)} \operatorname{Sym}_{|\zeta|}(\mathcal{E}_\zeta f^{(n)})\right)_{\mathcal{F}^Q(\mathcal{G})} \\
&\quad = \sum_{l=1}^{n} \sum_{\substack{\theta, \zeta \in \Pi(n) \\ |\theta| = |\zeta| = l}} \big(\operatorname{Sym}_l(\mathcal{E}_\theta f^{(n)}), \mathcal{E}_\zeta f^{(n)}\big)_{L^2((X \times \mathbb{R})^l, (m \otimes \nu)^{\otimes l})}\, l!\,.
\end{aligned} \tag{4.33}$$

Note that, by Proposition 3.1,

$$(\mathrm{Sym}_n\ f^{(n)}, \mathrm{Sym}_n\ f^{(n)})_{\mathbf{F}_n^Q(\mathcal{H},\nu)} = \int_{X^n} (\mathrm{Sym}_n\, f^{(n)}) f^{(n)}\, dm_\nu^{(n)}$$
$$= \sum_{\zeta\in\Pi(n)} \int_{X_\zeta^{(n)}} (\mathrm{Sym}_n\ f^{(n)}) f^{(n)}\, dm_{\nu,\zeta}^{(n)}. \tag{4.34}$$

By (4.33) and (4.34), formula (3.10) will follow if we show that, for a fixed $\zeta \in \Pi(n)$ with $|\zeta| = l$,

$$\sum_{\theta\in\Pi(n),\, |\theta|=l} \big(\mathrm{Sym}_l(\mathcal{E}_\theta f^{(n)}), \mathcal{E}_\zeta f^{(n)}\big)_{L^2((X\times\mathbb{R})^l,(m\otimes\nu)^{\otimes l})}\, l! = \int_{X_\zeta^{(n)}} (\mathrm{Sym}_n\ f^{(n)}) f^{(n)}\, dm_{\nu,\zeta}^{(n)}. \tag{4.35}$$

So, let us fix a partition $\zeta = \{\zeta_1, \dots, \zeta_l\} \in \Pi(n)$ and assume that (4.6) holds. Denote $k_i := |\zeta_i|$, $i = 1, \dots, l$. We have, by the definition of $\mathcal{E}_\zeta f^{(n)}$:

$$(\mathcal{E}_\zeta f^{(n)}) = \left(\prod_{i_1\in\zeta_1} h_{i_1}\right) \otimes p_{k_1-1} \otimes \cdots \otimes \left(\prod_{i_l\in\zeta_l} h_{i_l}\right) \otimes p_{k_l-1}. \tag{4.36}$$

Let $\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)$ and assume that (3.4) holds. Let $r_i := |\theta_i|$, $i = 1, \dots, l$. We may assume that there exists a permutation $\widehat{\pi} \in \mathfrak{S}_l$ such that

$$r_i = k_{\widehat{\pi}(i)}, \quad i = 1, \dots, l. \tag{4.37}$$

Indeed, otherwise the corresponding term in the sum on the left hand side of formula (4.35) vanishes. Analogously to (4.36), we have

$$l!\ \mathrm{Sym}_l(\mathcal{E}_\theta f^{(n)})(y_1, s_1, \dots, y_l, s_l) = \sum_{\varkappa\in S_l} Q_\varkappa(y_1, \dots, y_l)$$
$$\times \left(\left(\prod_{j_1\in\theta_{\varkappa(1)}} h_{j_1}\right) \otimes p_{r_{\varkappa(1)}-1} \otimes \cdots \otimes \left(\prod_{j_l\in\theta_{\varkappa(l)}} h_{j_l}\right) \otimes p_{r_{\varkappa(l)}-1}\right)(y_1, s_1, \dots, y_l, s_l).$$

Hence, by (3.3),

$$\big(\mathrm{Sym}_l(\mathcal{E}_\theta f^{(n)}), \mathcal{E}_\zeta f^{(n)}\big)_{L^2((X\times\mathbb{R})^l,(m\otimes\nu)^{\otimes l})}\, l!$$
$$= \sum_{\widehat{\pi}} \int_{X^l} Q_{\widehat{\pi}}(y_1, \dots, y_l) \left(\prod_{j_1\in\theta_{\widehat{\pi}(1)}} h_{j_1}(y_1)\right) \left(\prod_{i_1\in\zeta_1} h_{i_1}(y_1)\right)$$
$$\times \cdots \times \left(\prod_{j_l\in\theta_{\widehat{\pi}(l)}} h_{j_l}(y_l)\right) \left(\prod_{i_l\in\zeta_l} h_{i_l}(y_l)\right) dy_1 \cdots dy_l\, c_{k_1} \cdots c_{k_l}, \tag{4.38}$$

where the summation is over all permutations $\widehat{\pi} \in S_l$ which satisfy (4.37). Let us fix such a permutation $\widehat{\pi}$. Then, there exist

$$r_1! \cdots r_l! = k_1! \cdots k_l!$$

permutations $\pi \in \mathfrak{S}_n$ which satisfy

$$\pi\zeta_i = \theta_{\widehat{\pi}(i)}, \quad i = 1, \dots, l. \tag{4.39}$$

Note that, for each permutation π satisfying (4.39) and for $(x_1, \dots, x_n) \in X_\zeta^{(n)}$,

$$\begin{aligned} & f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}) \\ & \quad = (h_1 \otimes \cdots \otimes h_n)(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}) \\ & \quad = (h_{\pi(1)} \otimes \cdots \otimes h_{\pi(n)})(x_1, \dots, x_n) \\ & \quad = \left(\prod_{j_1 \in \pi\zeta_1} h_{j_1} \right)(y_1) \cdots \left(\prod_{j_l \in \pi\zeta_l} h_{j_l} \right)(y_l) \\ & \quad = \left(\prod_{j_1 \in \theta_{\widehat{\pi}(1)}} h_{j_1} \right)(y_1) \cdots \left(\prod_{j_l \in \theta_{\widehat{\pi}(l)}} h_{j_l} \right)(y_l), \end{aligned} \tag{4.40}$$

where $y_1 = x_{i_1}$ for $i_1 \in \zeta_1$,..., $y_l = x_{i_l}$ for $i_l \in \zeta_l$.

Let $\zeta, \theta \in \Pi(n)$ be such that condition (4.37) is satisfied by some permutation $\widehat{\pi} \in \mathfrak{S}_l$. That is, the corresponding sequences $(k_1, \dots, k_l)$ and $(r_1, \dots, r_l)$ coincide up to a permutation. Denote by $\mathfrak{S}_n[\zeta, \theta]$ the set of all permutations $\pi \in \mathfrak{S}_n$ which satisfy (4.39) with some permutation $\widehat{\pi} \in \mathfrak{S}_l$. (Note that the permutation $\widehat{\pi}$ is then completely identified by π, ζ and θ and automatically satisfies (4.39).) Clearly, if θ and θ' are from $\Pi(n)$ with $|\theta| = |\theta'| = l$, both satisfying (4.39), and $\theta \neq \theta'$, then

$$\mathfrak{S}_n[\zeta, \theta] \cap \mathfrak{S}_n[\zeta, \theta'] = \varnothing. \tag{4.41}$$

Furthermore,

$$\bigcup_{\substack{\theta \in \Pi(n),\, |\theta| = l \\ \theta \text{ satisfying (4.39)}}} \mathfrak{S}_n[\zeta, \theta] = \mathfrak{S}_n. \tag{4.42}$$

Therefore, by the definition of the measure $m_{c,\zeta}^{(n)}$ and formulas (3.7), (4.38), (4.40)–(4.42),

$$\begin{aligned} & \left(\mathrm{Sym}_l(\mathcal{E}_\theta f^{(n)}), \mathcal{E}_\zeta f^{(n)} \right)_{L^2((X \times \mathbb{R})^l, (m \otimes \nu)^{\otimes l})} l! \\ & \quad = \frac{1}{n!} \sum_{\pi \in S_n[\zeta, \theta]} \int_{X_\zeta^{(n)}} \mathbf{Q}_\pi(x_1, \dots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}) \end{aligned}$$

$$\times f^{(n)}(x_1,\dots,x_n)\, m^{(n)}_{\nu,\zeta}(dx_1\times\dots\times dx_n).$$

Hence

$$\sum_{\theta\in\Pi(n),\,|\theta|=l}\big(\mathrm{Sym}_l(\mathcal{E}_\theta f^{(n)}),\mathcal{E}_\zeta f^{(n)}\big)_{L^2((X\times\mathbb{R})^l,(m\otimes\nu)^{\otimes l})}\,l!$$
$$=\sum_{\substack{\theta\in\Pi(n),\,|\theta|=l\\ \theta\text{ satisfying (4.39)}}}\big(\mathrm{Sym}_l(\mathcal{E}_\theta f^{(n)}),\mathcal{E}_\zeta f^{(n)}\big)_{L^2((X\times\mathbb{R})^l,(m\otimes\nu)^{\otimes l})}\,l!$$
$$=\frac{1}{n!}\sum_{\substack{\theta\in\Pi(n),\,|\theta|=l\\ \theta\text{ satisfying (4.39)}}}\ \sum_{\pi\in S_n[\zeta,\theta]}\int_{X^{(n)}_\zeta}Q_\pi(x_1,\dots,x_n)f^{(n)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})$$
$$\times f^{(n)}(x_1,\dots,x_n)\, m^{(n)}_{\nu,\zeta}(dx_1\times\dots\times dx_n)$$
$$=\frac{1}{n!}\sum_{\pi\in S_n}\int_{X^{(n)}_\zeta}Q_\pi(x_1,\dots,x_n)f^{(n)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})$$
$$\times f^{(n)}(x_1,\dots,x_n)\, m^{(n)}_{\nu,\zeta}(dx_1\times\dots\times dx_n)$$
$$=\int_{X^{(n)}_\zeta}(\mathrm{Sym}_n\ f^{(n)})f^{(n)}\,dm^{(n)}_{\nu,\zeta}\,.$$

Thus, Theorem 3.2 is proven.

4.3 Proof of Theorem 3.4

Let us first prove the following

Lemma 4.8. *Let $h\in C_0(X)$ and $f^n\in B_0(X^n)$, $n\in\mathbb{N}$. Then formulas* (3.13), (3.20) *hold with*

$$\mathbf{J}^+(h)\,\mathrm{Sym}_n f^{(n)}=\mathrm{Sym}_{n+1}(h\otimes f^{(n)}),$$
$$\mathbf{J}^0(h)\,\mathrm{Sym}_n f^{(n)}=\mathrm{Sym}_n\,\mathscr{J}^0(h)f^{(n)},$$
$$\mathbf{J}^-_1(h)\,\mathrm{Sym}_n f^{(n)}=\mathrm{Sym}_{n-1}\,\mathscr{J}^-_1(h)f^{(n)},$$
$$\mathbf{J}^-_2(h)\,\mathrm{Sym}_n f^{(n)}=\mathrm{Sym}_{n-1}\,\mathscr{J}^-_2(h)f^{(n)},$$

Here

$$(\mathscr{J}^-_2(h)f^{(n)})(x_1,\dots,x_{n-1})$$
$$:=\sum_{i=1}^n\int_X dy\,h(y)f^{(n)}(x_1,\dots,x_{i-1},y,x_i,\dots,x_{n-1})T_i(y,x_1,\dots,x_{n-1}),\quad(4.43)$$

where for any $\theta \in \Pi(n-1)$

$$T_i^{(n)} \restriction X \times X_\theta^{(n-1)} := \prod_{\theta_u \in \theta:\, \max \theta_u \le i-1} Q(y, x_{\theta_u}). \tag{4.44}$$

Proof. By (4.22) and (4.23), we have

$$\langle \omega, h \rangle = \mathcal{A}^+(h) + \mathcal{A}^0(h) + d\Gamma(M_h \otimes A^-) + a^-(h \otimes 1). \tag{4.45}$$

(i) ($\mathbf{J}^+(h)$ part) From the proof of Proposition 4.7 it follows that

$$\mathbf{U}^{-1}\mathcal{A}^+(h)\mathbf{U}\,\mathrm{Sym}_n\, f^{(n)} = \mathrm{Sym}_{n+1}(h \otimes f^{(n)}) = \mathbf{J}^+(h)\,\mathrm{Sym}_n\, f^{(n)}. \tag{4.46}$$

(ii) ($\mathbf{J}^0(h)$ *part*) By Lemma 4.5, Proposition 4.7, (4.15),(4.23), (3.15) and (3.16),

$$\begin{aligned}
\mathbf{U}^{-1}\mathcal{A}^0(h)\mathbf{U}\,\mathrm{Sym}_n\, f^{(n)} &= \mathbf{U}^{-1}\mathcal{A}^0(h)\,\mathrm{Sym} \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \\
&= \mathbf{U}^{-1}\,\mathrm{Sym}\, N(M_h \otimes A^0) \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \\
&= \mathbf{U}^{-1}\,\mathrm{Sym} \sum_{\theta \in \Pi(n)} \sum_{i=1}^{n} \mathcal{E}_\theta (h \times_i f^{(n)}) b_{\gamma(i,\theta)-1}\, \gamma(i,\theta)^{-1} \\
&= \mathrm{Sym}_n\, \mathscr{J}^0(h) f^{(n)} \\
&= \mathbf{J}^0(h)\,\mathrm{Sym}_n\, f^{(n)}.
\end{aligned} \tag{4.47}$$

Here,

$$(h \times_i f^{(n)})(x_1, \dots, x_n) := h(x_i) f^{(n)}(x_1, \dots, x_n).$$

(iii) ($\mathbf{J}_1^-(h)$ *part*) Analogously,

$$\begin{aligned}
&\mathbf{U}^{-1} d\Gamma(M_h \otimes A^-)\mathbf{U}\,\mathrm{Sym}_n\, f^{(n)} = \mathbf{U}^{-1}\,\mathrm{Sym}\, N(M_h \otimes A^-) \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \\
&= \mathbf{U}^{-1}\,\mathrm{Sym} \sum_{l=1}^{n} \sum_{\substack{\theta \in \Pi(n) \\ |\theta| = l}} \sum_{k=1}^{l} \mathbf{1}^{\otimes(k-1)} \otimes (M_h \otimes A^-) \otimes \mathbf{1}^{\otimes(l-k)} \mathcal{E}_\theta f^{(n)} \\
&= \mathbf{U}^{-1}\,\mathrm{Sym} \sum_{l=1}^{n-1} \sum_{\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)} \sum_{\substack{k=1,\dots,l \\ |\theta_k| \ge 2}} \mathbf{1}^{\otimes(k-1)} \otimes (M_h \otimes A^-) \otimes \mathbf{1}^{\otimes(l-k)} \mathcal{E}_\theta f^{(n)},
\end{aligned} \tag{4.48}$$

where (3.4) is supposed to hold. Note that, for $\theta = \{\theta_1, \dots, \theta_l\} \in \Pi(n)$ satisfying (3.4) and $k \in \{1, \dots, l\}$ with $|\theta_k| \ge 2$, we have

$$\big(\mathbf{1}^{\otimes(k-1)} \otimes (M_h \otimes A^-) \otimes \mathbf{1}^{\otimes(l-k)} \mathcal{E}_\theta f^{(n)}\big)(x_1, s_1, \dots, x_l, s_l)$$

$$
\begin{aligned}
&= a_{|\theta_k|-1} h(x_k) f_\theta^{(n)}(x_1,\dots,x_k,\dots,x_l) p_{|\theta_1|-1}(s_1)\cdots p_{|\theta_{k-1}|-1}(s_{k-1}) \\
&\quad \times p_{|\theta_k|-2}(s_k) p_{|\theta_{k+1}|-1}(s_{k+1})\cdots p_{|\theta_l|-1}(s_l).
\end{aligned} \tag{4.49}
$$

Let us fix any $i, j \in \{1,\dots,n\}$ with $i < j$. Consider the set

$$
L_i := \{1, 2, \dots, i-1, i+1, \dots, n\},
$$

which has $n-1$ elements. Then any partition $\zeta = \{\zeta_1,\dots,\zeta_l\} \in \Pi(n-1)$ identifies a partition $\tilde{\zeta} = \{\tilde{\zeta}_1,\dots,\tilde{\zeta}_l\}$ of L_i: $\tilde{\zeta}_u := K_i\zeta_u$, $u = 1,\dots,l$, where

$$
K_i v := \begin{cases} v, & \text{if } v \le i-1, \\ v+1, & \text{if } v \ge i. \end{cases}
$$

Let $\tilde{\zeta}_k$ be the element of $\tilde{\zeta}$ which contains j. Set

$$
\theta_u := \begin{cases} \tilde{\zeta}_u, & \text{if } u \neq k, \\ \tilde{\zeta}_k \cup \{i\}, & \text{if } u = k. \end{cases}
$$

Thus, we have constructed a partition $\theta = \{\theta_1,\dots,\theta_l\} \in \Pi(n)$ with $l \le n-1$. Next, consider an arbitrary partition $\theta = \{\theta_1,\dots,\theta_l\} \in \Pi(n)$ with $l \le n-1$. Choose any $k \in \{1,\dots,l\}$ such that $|\theta_k| \ge 2$. In how many ways can we obtain θ from i, j and $\zeta \in \Pi(n-1)$ as above? This number is evidently equal to the number of all choices of $i, j \in \{1,\dots,n\}$ with $i < j$ and $i, j \in \theta_k$, i.e.,

$$
|\theta_k|(|\theta_k|-1)/2 = (|\tilde{\zeta}_k|+1)|\tilde{\zeta}_k|/2 = (|\zeta_k|+1)|\zeta_k|/2,
$$

where $j \in \tilde{\zeta}_k$, or equivalently $j-1 \in \zeta_k$. Hence, by (3.4) (3.17), (3.18), (4.48), and (4.49), we get

$$
\mathbf{U}^{-1} d\Gamma(M_h \otimes A^-) \mathbf{U} \operatorname{Sym}_n f^{(n)} = \operatorname{Sym}_{n-1} \mathscr{J}_1^-(h) f^{(n)} = \mathbf{J}_1^-(h) \operatorname{Sym}_n f^{(n)}. \tag{4.50}
$$

(iv) ($\mathbf{J}_2^-(h)$ *part*). For each $\theta = \{\theta_1,\dots,\theta_l\} \in \Pi(n)$ satisfying (3.4), we have

$$
\begin{aligned}
&\big(a^-(h\otimes 1) \operatorname{Sym}_l \mathcal{E}_\theta f^{(n)}\big)(x_1, s_1, \dots, x_{l-1}, s_{l-1}) \\
&\quad = \operatorname{Sym}_{l-1} \Bigg(\int_X dy \sum_{\substack{i=1,\dots,l \\ |\theta_i|=1}} h(y) Q(y,x_1) Q(y,x_2) \cdots Q(y,x_{i-1}) \\
&\qquad \times f_\theta^{(n)}(x_1,\dots,x_{i-1},y,x_i,\dots,x_{l-1}) \\
&\qquad \times p_{|\theta_1|-1}(s_1)\cdots p_{|\theta_{i-1}|-1|}(s_{i-1}) p_{|\theta_{i+1}|-1}(s_i)\cdots p_{|\theta_l|-1}(s_{l-1}) \Bigg),
\end{aligned} \tag{4.51}
$$

where we used (2.7) and (4.27). Hence, by (4.43), (4.44), and (4.51),

$$
\mathbf{U}^{-1} a^-(h\otimes 1) \mathbf{U} \operatorname{Sym}_n f^{(n)} = \operatorname{Sym}_{n-1} \mathscr{J}_2^-(h) f^{(n)} = \mathbf{J}_2^-(h) \operatorname{Sym}_n f^{(n)}. \tag{4.52}
$$

□

Lemma 4.9. *For any $h \in C_0(X)$ and $f^{(n)} \in \mathbf{B}_0^Q(X^n)$, we have*

$$(\mathbf{J}_2^-(h)f^{(n)})(x_1,\dots,x_{n-1}) = (\mathscr{J}_2^-(h)f^{(n)})(x_1,\dots,x_{n-1})$$
$$= n\int_X dy\, h(y) f^{(n)}(y,x_1,\dots,x_{n-1}). \tag{4.53}$$

Proof. Fix any $n \geq 2$ and $i \in \{2,\dots,n\}$. Let a permutation $\pi \in \mathfrak{S}_n$ be given by $\pi(1) = i$, $\pi(j) = j-1$ for $j = 2,\dots,i$, and $\pi(j) = j$ for $j = i+1,\dots,n$. Recall the operator Ψ_π defined in subsec. 3.1. By (3.7) and (4.44), we have, for each $(x_1,\dots,x_n) \in X^n$ such that $x_1 \neq x_j$ for $j \in \{2,\dots,n\}$,

$$(\Psi_\pi f^{(n)})(x_1,\dots,x_n) = f^{(n)}(x_2,x_3,\dots,x_i,x_1,x_{i+1},\dots,x_n)T_i(x_1,x_2,\dots,x_n). \tag{4.54}$$

Since $f \in \mathbf{B}_0^Q(X^n)$, by (4.5) and (4.10),

$$\Psi_\pi f^{(n)} = \Psi_\pi \operatorname{Sym}_n f^{(n)} = \operatorname{Sym}_n f^{(n)} = f^{(n)}. \tag{4.55}$$

By (4.54) and (4.55), for each $(x_1,\dots,x_{n-1}) \in X^{n-1}$

$$\int_X dy\, h(y) f^{(n)}(x_1,\dots,x_{i-1},y,x_i,\dots,x_{n-1})T_i(y,x_1,\dots,x_{n-1})$$
$$= \int_{X\setminus\{x_1,\dots,x_{n-1}\}} dy\, h(y) f^{(n)}(x_1,\dots,x_{i-1},y,x_i,\dots,x_{n-1})T_i(y,x_1,\dots,x_{n-1})$$
$$= \int_X dy\, h(y) f^{(n)}(y,x_1,\dots,x_{n-1}).$$

Hence, by (4.43),

$$(\mathscr{J}_2^-(h)f^{(n)})(x_1,\dots,x_{n-1}) = n\int_X dy\, h(y) f^{(n)}(y,x_1,\dots,x_{n-1}) =: g^{(n-1)}(x_1,\dots,x_{n-1}). \tag{4.56}$$

Since $f^{(n)} \in \mathbf{B}_0^Q(X^n)$, formula (3.8) holds for each $\pi \in \mathfrak{S}_n$. Hence, for each $\pi \in \mathfrak{S}_{n-1}$,

$$g^{(n-1)}(x_1,\dots,x_{n-1}) = Q_\pi(x_1,\dots,x_{n-1}) g^{(n)}(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)}),$$

see (3.7). Therefore,

$$\operatorname{Sym} g^{(n-1)} = g^{(n-1)}. \tag{4.57}$$

So, the lemma follows from (4.56) and (4.57). □

Now, Theorem 3.4 follows from Lemmas 4.8, 4.9.

4.4 Proof of Theorem 3.5

Assume that (3.22) holds. Then, by (3.16) and (3.18), we get $R_i^{(n)} \equiv \lambda$ and $S_{j-1}^{(n)} \equiv 2\eta$. Hence, by (3.15) and (3.17), for any $h \in C_0(X)$, the operators $\mathscr{J}^0(h)$ and $\mathscr{J}_1^-(h)$ map $\mathcal{F}_{\mathrm{fin}}(C_0(X))$ into itself. Hence, condition (C) is satisfied. Furthermore, equality (3.23) follows Theorem 3.4.

To show that (3.22) is necessary for condition (C) to hold, we proceed as follows. We first assume that the measure $\nu = \delta_\lambda$ for some $\lambda \in \mathbb{R}$ (Guassian/Poisson). Then $a_k = 0$ for all $k \in \mathbb{N}$, $b_0 = \lambda$, and the values of b_k for $k \in \mathbb{N}$ maybe chosen arbitrarily. Thus, (3.22) holds in this case with $\eta = 0$.

We next assume that the support of the measure ν contains an infinite number of points. Thus, $a_k > 0$ for all $k \in \mathbb{N}$.

Lemma 4.10. *Let $q \neq -1$. Let $a_k > 0$ for all $k \in \mathbb{N}$. Let $n \geq 2$ and let $f^{(n)} \in C_0(X^n)$ be such that* $\mathrm{Sym}_n\, f^{(n)} = 0$ *$m_\nu^{(n)}$-a.e. on the set $X_\theta^{(n)}$, where $\theta = \{\theta_1, \theta_2\} \in \Pi(n)$ with $\theta_1 = \{1\}$ and $\theta_2 = \{2, \dots, n\}$. Then $f^{(n)}(x, \dots, x) = 0$ for all $x \in X$.*

In the fermion case, $q = -1$, the above result remains true for $n \geq 3$.

Proof. Let $x_1, x_2 \in X$ be such that $x_1^1 < x_2^1$. (Recall that x^i denotes the i-th coordinate of $x = (x^1, \dots, x^d) \in X$.) In particular, $x_1 < x_2$. Then

$$(\mathrm{Sym}_n\, f^{(n)})(x_1, x_2, x_2, \dots, x_2) = \frac{1}{n}\big(f^{(n)}(x_1, x_2, x_2, \dots, x_2) + f^{(n)}(x_2, x_1, x_2, \dots, x_2) \\ + \dots + f^{(n)}(x_2, \dots, x_2, x_1, x_2) + q f^{(n)}(x_2, \dots, x_2, x_1)\big) = 0. \tag{4.58}$$

Since the function $f^{(n)}$ is continuous, equality (4.58) holds point-wise on the open set

$$\{(x_1, x_2) \in X^2 \mid x_1^1 < x_2^1\}.$$

Therefore, for all $x \in X$, we get $\frac{n-1+q}{n} f^{(n)}(x, \dots, x) = 0$. Thus, $f^{(n)}(x, \dots, x) = 0$ if either $q \neq -1$ and $n \geq 2$, or $q = -1$ and $n \geq 3$. □

We now set $\lambda := b_0$. Let us show that, if (C) holds, then $b_k = \lambda(k+1)$ for all $k \in \mathbb{Z}_+$. The proof below works for any anyon statisics, however, in the case where $q \neq -1$, this proof can be significantly simplified.

Let $\varepsilon \in \mathbb{R}$ be such that $b_1 = 2\lambda + \varepsilon$. We will now show by induction that

$$b_k = \lambda(k+1) + \varepsilon, \quad k \geq 1. \tag{4.59}$$

Assume that equality in (4.59) holds for $k = 1, \dots, n$. Fix any $h \in C_0(X)$ and $f^{(n+2)} \in C_0(X^{n+2})$. We define a function $g^{(n+2)} \in C_0(X^{n+2})$ by

$$g^{(n+2)}(x_1, \dots, x_{n+2}) := f^{(n+2)}(x_1, \dots, x_{n+2})\big(\lambda h(x_1) + h(x_2)(\lambda(n+1) + \varepsilon)\big). \tag{4.60}$$

Let $\theta = \{\theta_1, \theta_2\} \in \Pi(n+2)$ with $\theta_1 = \{1\}$, $\theta_2 = \{2, \dots, n+2\}$. By (3.15) and (3.16), we have $m_\nu^{(n+2)}$-a.e. on $X_\theta^{(n+2)}$:

$$\begin{aligned}(\mathscr{J}^0(h) f^{(n+2)})&(x_1, \dots, x_{n+2})\\ &= f^{(n+2)}(x_1, \dots, x_{n+2})\big(\lambda h(x_1) + (n+1)h(x_2)(\lambda(n+1) + \varepsilon)/(n+1)\big)\\ &= g^{(n+2)}(x_1, \dots, x_{n+2}).\end{aligned}$$

Since (C) holds, there exists a function $u^{(n+2)} \in C_0(X^{n+2})$ such that

$$\operatorname{Sym}_{n+2} \mathscr{J}^0(h) f^{(n+2)} = \operatorname{Sym}_{n+2} u^{(n+2)} \tag{4.61}$$

$m_\nu^{(n+2)}$-a.e. on X^{n+2}. Hence,

$$\mathbf{Sym}_{n+2}(g^{(n+2)} - u^{(n+2)})(x_1, \dots, x_{n+2}) = 0$$

for $m_c^{(n+2)}$-a.a. $(x_1, \dots, x_{n+2}) \in X_\theta^{(n+2)}$. Noting that $g^{(n+2)} - u^{(n+2)} \in C_0(X^{n+2})$, we conclude from Lemma 4.10 that

$$u^{(n+2)}(x, \dots, x) = g^{(n+2)}(x, \dots, x), \quad x \in X. \tag{4.62}$$

By (4.60)–(4.62),

$$(\mathscr{J}^0(h) f^{(n+2)})(x, \dots, x) = \big(\lambda(n+2) + \varepsilon\big) h(x) f^{(n+2)}(x, \dots, x) \tag{4.63}$$

for all $x \in X$. By (3.15), (3.16), and (4.63), we therefore get $b_{n+1} = \lambda(n+2) + \varepsilon$. Thus, (4.59) is proven.

Our next aim is to show that $\varepsilon = 0$. We first derive the following analog of Lemma 4.10.

Lemma 4.11. *Let $a_k > 0$ for all $k \in \mathbb{N}$. Let $f^{(5)} \in C_0(X^5)$ be such that* $\operatorname{Sym}_5 f^{(5)} = 0$ *$m_\nu^{(5)}$-a.e. on the set $X_\theta^{(5)}$, where $\theta = \{\theta_1, \theta_2\} \in \Pi(5)$ with $\theta_1 = \{1, 2\}$, $\theta_2 = \{3, 4, 5\}$. Then $f^{(5)}(x, \dots, x) = 0$ for all $x \in X$.*

Proof. The proof is similar to that of Lemma 4.10. In fact, from the condition of Lemma 4.11, we get $\frac{6+4q}{10} f^{(5)}(x, \dots, x) = 0$, which implies the statement. □

By (3.15), (3.16), and (4.59), we have, for $m_\nu^{(5)}$-a.e. $(x_1, \dots, x_5) \in X_\theta^{(5)}$ with $\theta \in \Pi(5)$ being as in Lemma 4.11,

$$(\mathscr{J}^0(h) f^{(5)})(x_1, \dots, x_5) = f^{(5)}(x_1, \dots, x_5)\big(h(x_1)(2\lambda + \varepsilon) + h(x_3)(3\lambda + \varepsilon)\big). \tag{4.64}$$

Analogously to derivation of formula (4.63), we conclude from condition (C), Lemma 4.11, and (4.64) that, for all $x \in X$,

$$(\mathscr{J}^0(h) f^{(5)})(x, \dots, x) = f^{(5)}(x, \dots, x) h(x)(5\lambda + 2\varepsilon). \tag{4.65}$$

On the other hand, by (3.15), (3.16), and (4.59), we have, for all $x \in X$

$$(\mathscr{J}^0(h)f^{(5)})(x,\dots,x) = f^{(5)}(x,\dots,x)h(x)(5\lambda+\varepsilon). \tag{4.66}$$

Comparing (4.65) and (4.66), we see that ε must be equal to zero.

The proof of the equality $a_k = \eta k(k+1)$ for $k \in \mathbb{N}$ is similar, so we only outline it. Denote $\eta := a_1/2$. Using Lemma 4.10 and formulas (3.17), (3.18), we get the recursive formula

$$a_{n+1} = 2\eta + \big((n+1)(n+2)-2\big)\frac{a_n}{n(n+1)} \tag{4.67}$$

for $n \geq 2$. Choose $\varepsilon \in \mathbb{R}$ so that $a_2 = 6\eta + \varepsilon$. Then, by (4.67),

$$a_3 = 12\eta + \frac{10}{6}\varepsilon, \quad a_4 = 20\eta + \frac{5}{2}\varepsilon, \quad a_5 = 30\eta + \frac{7}{2}\varepsilon. \tag{4.68}$$

On the other hand, by Lemma 4.11,

$$a_5 = a_2 + 2a_3. \tag{4.69}$$

From (4.68) and (4.69), we get $\varepsilon = 0$. Hence, the recursive formula (4.67) holds for all $n \geq 1$. From here the desired equality follows.

We finally consider the case where the support of the measure ν consists of l points with $l \geq 2$ being finite. In the case where $q = -1$, we will additionally assume that $l \geq 3$. Then $a_1 > 0$, $a_2 > 0, \dots, a_{l-1} > 0$, $a_i = 0$ for$i \geq l$. Furthermore, by (3.2), $c_1 > 0$, $c_2 > 0, \dots, c_k > 0$, $c_i = 0$ for $i \geq l+1$. Let condition (C) be satisfied. Then, in view of the construction of the measures $m_\nu^{(n)}$, analogously to the above, we conclude that formula (4.67) holds for $n = 1, 2, \dots, l-1$. In particular, we get

$$a_l = a_1 + \big(l(l+1)-2\big)\frac{a_{l-1}}{(l-1)l}\,.$$

Since $a_1 > 0$ and $a_{l-1} > 0$, we therefore get $a_l > 0$, which contradicts the fact that $a_l = 0$. Thus, (C) can not be satisfied. Theorem 3.5 is proven.

We leave the easy proof of Proposition 3.7 to the interested reader. Let us show, however, how Theorem 1.2 can now be easily derived.

Assume $q = 1$. Assume that $\mathscr{CP} = \mathscr{OCP}$. Then, for any $h \in C_0(X)$ and $f^{(n)} \in C_0(X^n)$, we have

$$\langle \omega, h\rangle \langle P_n(\omega), f^{(n)}\rangle \in \mathscr{OCP} \tag{4.70}$$

(we used that product of any polynomials from $\mathscr{CP}$ belongs to $\mathscr{CP}$). Since

$$\mathbf{J}_2^-(h)\langle f^{(n)}, P_n(\omega)\rangle = \langle \mathscr{J}_2^-(h)f^{(n)}, P_{n-1}(\omega)\rangle \in \mathscr{OCP},$$

we therefore conclude from Theorem 3.4 and (4.70) that (C) holds. Hence, by Theorem 3.5, (3.22) holds.

Let us now assume that (3.22) holds. Then, as follows from the proof of Theorem 3.5, $h \in C_0(X)$, the operators $\mathscr{J}^0(h)$ and $\mathscr{J}_1^-(h)$ map $\mathcal{F}_{\mathrm{fin}}(C_0(X))$ into itself. Hence, for any $f^{(n)} \in C_0(X^n)$, (4.70) holds. From here the equality $\mathscr{CP} = \mathscr{OCP}$ can be deduced analogously to the proof of [16, Theorem 4.1].

4.5 Proof of Theorem 3.8

We will only prove equality (3.31) as the proof of equality (3.30) is similar and simpler. Note also that formula (3.32) will follow from (3.26)–(3.31).

It suffices to prove that, for any $h \in C_0(X)$,

$$\mathbf{J}_1^-(h)g^{(n)} = \int_X dx\, h(x)\eta\partial_x^\dagger\partial_x\partial_x\, g^{(n)},$$

where $g^{(n)} \in \mathbf{B}_0^Q(X^n)$ is of the form $g^{(n)} = f_1 \circledast \cdots \circledast f_n$, with $f_1, \dots, f_n \in B_0(X)$. We have

$$g^{(n)}(x_1, \dots, x_n) := \frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n} Q_\pi(x_1, \dots, x_n) f_{\pi(1)}(x_1)\cdots f_{\pi(n)}(x_n).$$

Hence, by (3.29),

$$\begin{aligned}
&\left(\int_X dx\, h(x)\partial_x^\dagger\partial_x\partial_x g^{(n)}\right)(x_1, \dots, x_{n-1})\\
&= \operatorname{Sym}_{n-1}\Bigg(\frac{1}{(n-2)!}\sum_{\pi\in\mathfrak{S}_n} Q_\pi(x_1, x_1, x_2, \dots, x_{n-1})\\
&\qquad\qquad \times (hf_{\pi(1)}f_{\pi(2)})(x_1)f_{\pi(3)}(x_2)\cdots f_{\pi(n)}(x_{n-1})\Bigg)\\
&= \sum_{1\le i<j\le n}\frac{1}{(n-2)!}\sum_{\substack{\pi\in\mathfrak{S}_n\\ \pi\{1,2\}=\{i,j\}}} \operatorname{Sym}_{n-1}\big(Q_\pi(x_1, x_1, x_2, \dots, x_{n-1})\\
&\qquad\qquad\qquad \times (hf_if_j)(x_1)f_{\pi(3)}(x_2)\cdots f_{\pi(n)}(x_{n-1})\big). \qquad (4.71)
\end{aligned}$$

By (3.7), for any $\pi \in \mathfrak{S}_n$ satisfying $\pi\{1,2\} = \{i,j\}$ with $i < j$, and any $(x_1, x_2, \dots, x_{n-1}) \in X^{n-1}$, we have

$$Q_\pi(x_1, x_1, x_2, \dots, x_{n-1}) = Q_{\sigma_{ij}(\pi)}(x_1, x_2, \dots, x_{n-1}). \qquad (4.72)$$

Here the permutation $\sigma_{ij}(\pi) \in \mathfrak{S}_{n-1}$ is defined as follows:

$$\sigma_{ij}(\pi)(1) := j,$$

and for $k = 2, \dots, n-1$,

$$\sigma_{ij}(\pi)(k) := \begin{cases}\pi(k+1), & \text{if } \pi(k+1) < i,\\ \pi(k+1) - 1, & \text{if } \pi(k+1) > i.\end{cases}$$

By (4.72), for any $\pi \in \mathfrak{S}_n$ satisfying $\pi\{1,2\} = \{i,j\}$ with $i < j$,

$$Q_\pi(x_1, x_1, x_2, \dots, x_{n-1})(hf_if_j)(x_1)f_{\pi(3)}(x_2)\cdots f_{\pi(n)}(x_{n-1})$$

$$
\begin{aligned}
&= Q_{\sigma_{ij}(\pi)}(x_1, x_2, \ldots, x_{n-1})\big(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \\
&\quad \otimes \cdots \otimes f_{j-1} \otimes (hf_i f_j) \otimes f_{j+1} \otimes \cdots \otimes f_n\big)\big(x_{\sigma_{ij}(\pi)^{-1}(1)}, \ldots, x_{\sigma_{ij}(\pi)^{-1}(n-1)}\big) \\
&= \Psi_{\sigma_{ij}(\pi)}\big(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \\
&\quad \otimes \cdots \otimes f_{j-1} \otimes (hf_i f_j) \otimes f_{j+1} \otimes \cdots \otimes f_n\big)(x_1, \ldots, x_{n-1}).
\end{aligned}
$$

Hence, by (4.5) and (4.10),

$$
\begin{aligned}
&\mathrm{Sym}\,\big(Q_\pi(x_1, x_1, x_2, \ldots, x_{n-1})(hf_i f_j)(x_1) f_{\pi(3)}(x_2) \cdots f_{\pi(n)}(x_{n-1})\big) \\
&\quad = (f_1 \circledast \cdots \circledast f_{i-1} \circledast f_{i+1} \circledast \cdots \circledast f_{j-1} \circledast (hf_i f_j) \circledast f_{j+1} \circledast \cdots \circledast f_n)(x_1, \ldots, x_{n-1}).
\end{aligned} \tag{4.73}
$$

By (4.71) and (4.73), we thus get

$$
\begin{aligned}
&\int_X dx\, h(x) \partial_x^\dagger \partial_x \partial_x g^{(n)} \\
&\quad = 2 \sum_{1 \le i < j \le n} f_1 \circledast \cdots \circledast f_{i-1} \circledast f_{i+1} \circledast \cdots \circledast f_{j-1} \circledast (hf_i f_j) \circledast f_{j+1} \circledast \cdots \circledast f_n.
\end{aligned}
$$

From here equality (3.31) follows.

Acknowledgements

M.B. and E.L. acknowledge the financial support of the Polish National Science Center, grant no. Dec-2012/05/B/ST1/00626, and of the SFB 701 "Spectral structures and topological methods in mathematics", Bielefeld University. MB was partially supported by the MAESTRO grant DEC-2011/02/A/ST1/00119.

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