A chaotic decomposition for generalized stochastic processes with independent values

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Abstract

We extend the result of Nualart and Schoutens on chaotic decomposition of the L^2-space of a Lévy process to the case of a generalized stochastic processes with independent values.

1 Introduction

Among all stochastic processes with independent increments, essentially only Brownian motion and Poisson process have a chaotic representation property. The latter property means that, by using multiple stochastic integrals with respect to the centered stochastic process, one can construct a unitary isomorphism between the L^2-space of the process and a symmetric Fock space. In the case of a Lévy process, several approaches have been proposed in order to construct a Fock space-type realization of the corresponding L^2-space. In this paper, we will be concerned with the approach of Nualart and Schoutens [9], who constructed a representation of every square integrable functional of a Lévy process in terms of orthogonalized Teugels martingales. Recall that, for a given Lévy process $(X_t)_{t\geq 0}$, its k-th order Teugels martingale is defined by centering the power jump process

$$X_t^{(k)} := \sum_{0<s\leq t} (\Delta X_s)^k, \quad k \in \mathbb{N}.$$

For numerous applications of this result, see e.g. [7, 10]. We also refer to [6] for an extension of this result to the case of a Lévy process taking values in $\mathbb{R}^d$, and to [1, 3] for a Nualart–Schotens-type decomposition for noncommutative (in particular, free) Lévy processes.

The aim of this note is to extend the Nualart–Schoutens decomposition to the case of a generalized stochastic process with independent values. Consider a standard triple $\mathcal{D} \subset L^2(\mathbb{R}^d, dx) \subset \mathcal{D}'$, where $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ is the nuclear space of all smooth, compactly supported functions on $\mathbb{R}^d$, and $\mathcal{D}'$ is the dual space of $\mathcal{D}$ with respect to the center space $L^2(\mathbb{R}^d, dx)$, see e.g. [2] for detail. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, we denote by $\langle\omega, \varphi\rangle$ the dual pairing of ω and φ. Denote by $\mathcal{C}(\mathcal{D}')$ the cylinder σ-algebra on $\mathcal{D}'$. A generalized stochastic process is a probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$. Thus, a generalized stochastic process is a random generalized function $\omega \in \mathcal{D}'$. One says that a generalized stochastic process has independent values, if for any $\varphi_1, \dots, \varphi_n \in \mathcal{D}$ which have mutually disjoint support, the random variables $\langle\omega, \varphi_1\rangle, \dots, \langle\omega, \varphi_n\rangle$ are independent. So, heuristically, we have that, for any $x_1, \dots, x_n \in \mathbb{R}^d$, the random variables $\omega(x_1), \dots, \omega(x_n)$ are independent. In the case where $d = 1$, one can (at least heuristically) interpret $\omega(t)$ as the time t derivative of a classical stochastic process $X = (X(t))_{t\in\mathbb{R}}$ with independent increments, so that, for $t \geq 0$, $X(t) = \int_0^t \omega(s)\, ds$.

If a generalized stochastic process with independent values, μ, has the property that the measure μ remains invariant under each transformation $x \mapsto x + a$ ($a \in \mathbb{R}^d$) of the underlying space, then one calls μ a Lévy process (which is, for $d = 1$, the time derivative of a classical Lévy process.) So, below, for a certain class of generalized stochastic processes with independent values, we will construct an orthogonal decomposition of the space $L^2(\mathcal{D}', \mu)$, which, in the case of a classical Lévy process, will be exactly the Nualart–Schotens decomposition from [9]. This paper will also extend the results of [8] for generalized stochastic processes being Lévy processes.

2 Preliminaries

We start by briefly recalling some results from [5]. Assume that for each $x \in \mathbb{R}^d$, $\sigma(x, ds)$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We also assume that for each $\Delta \in \mathcal{B}(\mathbb{R})$, $\mathbb{R}^d \ni x \mapsto \sigma(x, \Delta)$ is a measurable mapping. Hence, we can define a σ-finite measure $dx\, \sigma(x, ds)$ on $(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}))$. Let $\mathcal{B}_0(\mathbb{R}^d)$ denote the collection of all sets $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ which are bounded. We will additionally assume that, for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, there exists $C_\Lambda > 0$ such that

$$\int_{\mathbb{R}} |s|^n \sigma(x, ds) \leq C_\Lambda^n n! \quad n \in \mathbb{N}, \tag{2.1}$$

for all $x \in \Lambda$. We fix the Hilbert space $H = L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds))$. We denote by $\mathcal{F}(H) = \bigoplus_{n=0}^\infty H^{\odot n} n!$ the symmetric Fock space over H. Here $\odot$ denotes symmetric tensor product. We denote by $\mathfrak{D}$ the subset of $\mathcal{F}(H)$ which consists of all finite vectors $f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$ where each $f^{(k)}$ is a symmetric function on $(\mathbb{R}^d \times \mathbb{R})^k$ which is obtained as the symmetrization of a finite sum of functions of the form

$$g^{(k)}(x_1, s_1, \dots, x_k, s_k) = \phi(x_1, \dots, x_k) s_1^{i_1} \cdots s_k^{i_k},$$

where $\phi \in \mathcal{D}^{\otimes k} = C_0^\infty((\mathbb{R}^d)^k)$ and $i_1, \dots, i_k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For each $\varphi \in \mathcal{D}$, we define an operator $A(\varphi)$ in $\mathcal{F}(H)$ with domain $\mathfrak{D}$ by

$$A(\varphi) := a^+(\varphi \otimes m_0) + a^-(\varphi \otimes m_0) + a^0(\varphi \otimes m_1). \tag{2.2}$$

Here and below, for $i \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$,

$$(\varphi \otimes m_i)(x, s) := \varphi(x)s^i,$$

$a^+(\varphi \otimes m_i)$ is the creation operator corresponding to $\varphi \otimes m_i$:

$$a^+(\varphi \otimes m_i)f^{(k)} = f^{(k)} \odot (\varphi \otimes m_i), \quad f^{(k)} \in H^{\odot k},$$

$a^-(\varphi \otimes m_i)$ is the corresponding annihilation operator:

$$a^-(\varphi \otimes m_i)f^{(k)} = k \int_{\mathbb{R}^d \times \mathbb{R}} dy\, \sigma(y, du)\varphi(y)u^i f^{(k)}(y, u, \cdot),$$

and $a^0(\varphi \otimes m_i)$ is the neutral operator corresponding to $\varphi \otimes m_i$:

$$\begin{aligned}\big(a^0(\varphi \otimes m_i)f^{(k)}\big)&(x_1, s_1, \dots, x_k, s_k)\\ &= \big(\varphi(x_1)s_1^i + \dots + \varphi(x_k)s_k^i\big) f^{(k)}(x_1, s_1, \dots, x_k, s_k).\end{aligned}$$

Note that $A(\varphi)$ maps $\mathfrak{D}$ into itself, and it is a symmetric operator in $\mathcal{F}(H)$.

Theorem 2.1. *For each $\varphi \in \mathcal{D}$, the operator $A(\varphi)$ is essentially self-adjoint on $\mathfrak{D}$. Furthermore, there exists a unique probability measure μ on $\mathcal{D}'$ such that the linear operator $I : \mathcal{F}(H) \to L^2(\mathcal{D}', \mu)$ given through $I\Omega = 1$, Ω being the vacuum vector $(1, 0, 0, \dots)$, and*

$$I(A(\varphi_1) \cdots A(\varphi_n)\Omega) = \langle \omega, \varphi_1 \rangle \cdots \langle \omega, \varphi_n \rangle,$$

is a unitary operator. The Fourier transform of the measure μ is given by

$$\begin{aligned}\int_{\mathcal{D}'} e^{i\langle \varphi, \omega \rangle} \mu(d\omega) = \exp\bigg[&-\frac{1}{2} \int_{\mathbb{R}^d} dx\, \sigma(x, \{0\})\varphi(x)^2 \\ &+ \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^*} \sigma(x, ds) \frac{1}{s^2}(e^{i\varphi(x)s} - i\varphi(x)s - 1) \bigg],\end{aligned} \tag{2.3}$$

where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. In particular, μ is a generalized stochastic process with independent values.

Note that, if the measure $\sigma(ds) = \sigma(x, ds)$ is the same for all $x \in \mathbb{R}^d$, then μ is a Lévy process.

3 An orthogonal decomposition of a Fock space

We will now discuss an orthogonal decomposition of a general symmetric Fock space. This decomposition generalizes the well-known basis of occupation numbers in the Fock space, see e.g. [2].

In this section, we will denote by H any real separable Hilbert space. Let $(H_k)_{k=0}^{\infty}$ be a sequence of closed subspaces of H such that $H = \bigoplus_{k=0}^{\infty} H_k$. Let $n \geq 2$. Then clearly

$$\begin{aligned} H^{\otimes n} &= \left(\bigoplus_{k_1=0}^{\infty} H_{k_1} \right) \otimes \left(\bigoplus_{k_2=0}^{\infty} H_{k_2} \right) \otimes \cdots \otimes \left(\bigoplus_{k_n=0}^{\infty} H_{k_n} \right) \\ &= \bigoplus_{(k_1,k_2,\dots,k_n)\in\mathbb{Z}_+^n} H_{k_1} \otimes H_{k_2} \otimes \cdots \otimes H_{k_n}. \end{aligned} \tag{3.1}$$

Denote by Sym_n the orthogonal projection of $H^{\otimes n}$ onto $H^{\odot n}$. Recall that, for any $f_1, f_2, \dots, f_n \in H$

$$f_1 \odot \cdots \odot f_n = \mathrm{Sym}_n\ f_1 \otimes \cdots \otimes f_n = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}. \tag{3.2}$$

(Here, S_n denotes the symmetric group of order n.) For each $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$, let $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$ denote the Hilbert space $\mathrm{Sym}_n(H_{k_1} \otimes H_{k_2} \otimes \cdots \otimes H_{k_n})$, i.e., the space of all Sym_n-projections of elements of $H_{k_1} \otimes H_{k_2} \otimes \cdots \otimes H_{k_n}$.

Assume that $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$, $(l_1, l_2, \dots, l_n) \in \mathbb{Z}_+^n$ are such that there exists a permutation $\sigma \in S_n$ such that

$$(k_1, k_2, \dots, k_n) = (l_{\sigma(1)}, l_{\sigma(2)}, \dots, l_{\sigma(n)}). \tag{3.3}$$

Then

$$H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n} = H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}. \tag{3.4}$$

Indeed, take any $f_1 \in H_{l_1}, f_2 \in H_{l_2}, \dots, f_n \in H_{l_n}$. Then

$$f_1 \odot f_2 \odot \cdots \odot f_n = f_{\sigma(1)} \odot f_{\sigma(2)} \odot \cdots \odot f_{\sigma(n)}. \tag{3.5}$$

We have $f_{\sigma(i)} \in H_{l_{\sigma(i)}} = H_{k_i}$. Therefore, the vector in (3.5) belongs to $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$. Since the set of all vectors of the form $f_1 \odot f_2 \odot \cdots \odot f_n$ with $f_i \in H_{l_i}$ is total in $H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}$, we therefore conclude that

$$H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n} \subset H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$$

By inverting the argument, we obtain the inverse conclusion, and so formula (3.4) holds.

If no permutation $\sigma \in S_n$ exists which satisfies (3.3), then

$$H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n} \perp H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}. \tag{3.6}$$

Indeed, take any $f_i \in H_{k_i}$, $g_i \in H_{l_i}$, $i = 1, 2, \dots, n$. Then, since Sym_n is an orthogonal projection,

$$\begin{aligned}
&\big(f_1 \odot f_2 \odot \cdots \odot f_n, g_1 \odot g_2 \odot \cdots \odot g_n\big)_{H^{\odot n}} \\
&\quad = \Big(\mathrm{Sym}_n \big(f_1 \otimes f_2 \otimes \cdots \otimes f_n\big), g_1 \otimes g_2 \otimes \cdots \otimes g_n \Big)_{H^{\otimes n}} \\
&\quad = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} (f_{\sigma(i)}, g_i)_H = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} (f_i, g_{\sigma(i)})_H = 0.
\end{aligned}$$

Since the vectors of the form $f_1 \odot f_2 \odot \cdots \odot f_n$ with $f_i \in H_{k_i}$ and $g_1 \odot g_2 \odot \cdots \odot g_n$ with $g_i \in H_{l_i}$ form a total set in $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$ and $H_{l_1} \odot H_{l_2} \odot \cdots \odot H_{l_n}$, respectively, we get (3.6).

By (3.1), the closed linear span of the spaces $H_{k_1} \odot H_{k_2} \odot \cdots \odot H_{k_n}$ with $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$ coincides with $H^{\odot n}$. Hence, by (3.4) and (3.6), we get the orthogonal decomposition

$$H^{\odot n} = \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^{\infty},\, |\alpha| = n} H_0^{\odot \alpha_0} \odot H_1^{\odot \alpha_1} \odot H_2^{\odot \alpha_2} \odot \cdots . \tag{3.7}$$

Here $\mathbb{Z}_{+,0}^{\infty}$ denotes the set of indices $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ such that all $\alpha_i \in \mathbb{Z}_+$ and $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2 + \cdots < \infty$. Hence, by (3.7), we get the following

Lemma 3.1. *We have the orthogonal decomposition of the symmetric Fock space* $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H^{\odot n} n!$ *:*

$$\mathcal{F}(H) = \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^{\infty}} \big(H_0^{\odot \alpha_0} \odot H_1^{\odot \alpha_1} \odot H_2^{\odot \alpha_2} \cdots \big) |\alpha|! \,. \tag{3.8}$$

Next, we have:

Lemma 3.2. *Let* $\alpha \in \mathbb{Z}_{+,0}^{\infty}$. *Then*

$$\begin{aligned}
&\mathrm{Sym}_{|\alpha|} : \big(H_0^{\odot \alpha_0} \otimes H_1^{\odot \alpha_1} \otimes H_2^{\odot \alpha_2} \otimes \cdots \big) \alpha_0! \alpha_1! \alpha_2! \cdots \\
&\quad \to \big(H_0^{\odot \alpha_0} \odot H_1^{\odot \alpha_1} \odot H_2^{\odot \alpha_2} \odot \cdots \big) \, |\alpha|!
\end{aligned} \tag{3.9}$$

is a unitary operator.

Proof. We start the proof with the following well-known observation. Let $k, l \geq 1$, $n := k + l$. Then $\mathrm{Sym}_n = \mathrm{Sym}_n(\mathrm{Sym}_k \otimes \mathrm{Sym}_l)$. Hence, for any $\alpha \in \mathbb{Z}_{+,0}^{\infty}$, $|\alpha| = n$, we get $\mathrm{Sym}_n = \mathrm{Sym}_n(\mathrm{Sym}_{\alpha_0} \otimes \mathrm{Sym}_{\alpha_1} \otimes \mathrm{Sym}_{\alpha_2} \otimes \cdots)$. Therefore, we have the following equality of subspaces of $H^{\otimes n}$:

$$H_0^{\odot \alpha_0} \odot H_1^{\odot \alpha_1} \odot H_2^{\odot \alpha_2} \odot \cdots$$

$$= \mathrm{Sym}_n \big(H_0^{\otimes\alpha_0} \otimes H_1^{\otimes\alpha_1} \otimes H_2^{\otimes\alpha_2} \otimes \cdots\big)$$
$$= \mathrm{Sym}_n \big(\mathrm{Sym}_{\alpha_0} \otimes \mathrm{Sym}_{\alpha_1} \otimes \mathrm{Sym}_{\alpha_2} \otimes \cdots\big)\big(H_0^{\otimes\alpha_0} \otimes H_1^{\otimes\alpha_1} \otimes H_2^{\otimes\alpha_2} \otimes \cdots\big)$$
$$= \mathrm{Sym}_n \big(H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes H_2^{\odot\alpha_2} \otimes \cdots\big).$$

This shows that the image of the operator Sym_n in (3.9) is the whole space $H_0^{\odot\alpha_0} \odot H_1^{\odot\alpha_1} \odot H_2^{\odot\alpha_2} \odot \cdots n!$. Hence, we only need to prove that this operator is an isometry.

Fix any $f_i, g_i \in H_i$ with $i \in \mathbb{Z}_+$ and any $\alpha \in \mathbb{Z}_{+,0}^\infty$. Then, by (3.2)

$$\big(\mathrm{Sym}_n \big(f_0^{\otimes\alpha_0} \otimes f_1^{\otimes\alpha_1} \otimes f_2^{\otimes\alpha_2} \otimes \cdots\big), \mathrm{Sym}_n \big(g_0^{\otimes\alpha_0} \otimes g_1^{\otimes\alpha_1} \otimes g_2^{\otimes\alpha_2} \otimes \cdots\big)\big)_{H^{\odot n}}$$
$$= \big(\mathrm{Sym}_n \big(f_0^{\otimes\alpha_0} \otimes f_1^{\otimes\alpha_1} \otimes f_2^{\otimes\alpha_2} \otimes \cdots\big), g_0^{\otimes\alpha_0} \otimes g_1^{\otimes\alpha_1} \otimes g_2^{\otimes\alpha_2} \otimes \cdots\big)_{H^{\otimes n}}$$
$$= \frac{1}{n!} \sum_{\sigma_0 \in S_{\alpha_0}} (f_0, g_0)_{H_0}^{\alpha_0} \cdot \sum_{\sigma_1 \in S_{\alpha_1}} (f_1, g_1)_{H_1}^{\alpha_1} \cdots$$
$$= \frac{1}{n!} \big(f_0^{\otimes\alpha_0}, g_0^{\otimes\alpha_0}\big)_{H_0^{\odot\alpha_0}} \alpha_0! \big(f_1^{\otimes\alpha_1}, g_1^{\otimes\alpha_1}\big)_{H_1^{\odot\alpha_1}} \alpha_1! \cdots$$
$$= \frac{1}{n!} \big(f_0^{\otimes\alpha_0} \otimes f_1^{\otimes\alpha_1} \otimes \cdots, \ g_0^{\otimes\alpha_0} \otimes g_1^{\otimes\alpha_1} \otimes \cdots\big)_{H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes \cdots} \alpha_0! \alpha_1! \cdots .$$

Since the set of all vectors of the form $f_i^{\otimes\alpha_i}$ with $f_i \in H_i$ is a total subset of $H_i^{\odot\alpha_i}$, we conclude that the operator in (3.9) is indeed an isometry. □

We define the symmetrization operator

$$\mathrm{Sym} : \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^\infty} \big(H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes H_2^{\odot\alpha_2} \otimes \cdots\big) \alpha_0! \alpha_1! \alpha_2! \cdots \to \mathcal{F}(H) \tag{3.10}$$

so that the restriction of Sym to each space

$$\big(H_0^{\odot\alpha_0} \otimes H_1^{\odot\alpha_1} \otimes H_2^{\odot\alpha_2} \otimes \cdots\big) \alpha_0! \alpha_1! \alpha_2! \cdots$$

is equal to $\mathrm{Sym}_{|\alpha|}$. By Lemmas 3.1 and 3.2, we get

Lemma 3.3. *The symmetrization operator* Sym *is a unitary operator.*

Remark 3.4. Let us assume that each Hilbert space H_k is one-dimensional and in each H_k we fix a vector $e_k \in H_k$ such that $||e_k|| = 1$. Thus, $(e_k)_{k=0}^\infty$ is an orthonormal basis of H. By Lemma 3.3, the set of the vectors

$$\Big(\big(\alpha_0! \alpha_1! \alpha_2! \cdots\big)^{-\frac{1}{2}} e_0^{\otimes\alpha_0} \odot e_1^{\otimes\alpha_1} \odot e_2^{\otimes\alpha_2} \odot \cdots\Big)_{\alpha \in \mathbb{Z}_{+,0}^\infty}$$

is an orthonormal basis of $\mathcal{F}(H)$. This basis is called a basis of occupation numbers.

4 An orthogonal decomposition of $L^2(\mathcal{D}', \mu)$

We want to apply the general result about the orthogonal decomposition of the Fock space to the case of $\mathcal{F}(H)$, where $H = L^2(\mathbb{R}^d \times \mathbb{R}, dx\, \sigma(x, ds))$. We note that, by (2.1), for each $x \in \mathbb{R}^d$, the set of polynomials is dense in $L^2(\mathbb{R}, \sigma(x, ds))$. We denote by $(q^{(n)}(x, s))_{n \geq 0}$ the sequence of monic polynomials which are orthogonal with respect to the measure $\sigma(x, ds)$. These polynomials satisfy the following recursive formula:

$$\begin{aligned} sq^{(n)}(x, s) &= q^{(n+1)}(x, s) + b_n(x) q^{(n)}(x, s) + a_n(x) q^{(n-1)}(x, s), \quad n \geq 1, \\ sq^{(0)}(x, s) &= q^{(1)}(x, s) + b_0(x) \end{aligned} \tag{4.1}$$

with some $b_n(x) \in \mathbb{R}$ and $a_n(x) > 0$. [Note that if the support of $\sigma(x, ds)$ consists of $k < \infty$ points, then, for $n \geq k$, we set $q^{(n)}(x, s) = 0$, $a_n(x) = 0$ with $b_n(x) \in \mathbb{R}$ being arbitrary.]

From now on, we will assume that the following condition is satisfied:

(A) For each $n \in \mathbb{N}$, the function $a_n(x)$ from (4.1) is locally bounded on $\mathbb{R}^d$, i.e., for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, $\sup_{x \in \Lambda} a_n(x) < \infty$.

Denote by $\mathfrak{L}$ the linear space of all functions on $\mathbb{R}^d \times \mathbb{R}$ which have the form

$$f(x, s) = \sum_{k=0}^{n} a_k(x) q^{(k)}(x, s), \tag{4.2}$$

where $n \in \mathbb{N}$, $a_k \in \mathcal{D}$, $k = 0, 1, \dots, n$.

Lemma 4.1. *The space $\mathfrak{L}$ is densely embedded into H.*

Proof. Let $f(x, s) = a(x) q^{(k)}(x, s)$, where $a \in \mathcal{D}$. Let us show that $f \in H$. Denote $\Lambda := \operatorname{supp}(a)$. We have, for some $C > 0$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}} dx\, \sigma(x, ds) f(x, s)^2 \leq C \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds)\, q^{(k)}(x, s)^2. \tag{4.3}$$

If $k = 0$, then $q^{(0)}(x, s) = 1$, and the right hand side of (4.3) is evidently finite. By the theory of orthogonal polynomials (see e.g. [4])

$$\int_{\mathbb{R}} \sigma(x, ds)\, q^{(k)}(x, s)^2 = a_1(x) a_2(x) \cdots a_k(x), \quad k \geq 1. \tag{4.4}$$

Hence we continue (4.3)

$$\leq C \int_{\Lambda} dx\, a_1(x) a_2(x) \cdots a_k(x) < \infty$$

by (A). Thus, $\mathfrak{L} \subset H$.

We now have to show that $\mathfrak{L}$ is a dense subset of H. Let $g \in H$ be such that $(g, f)_H = 0$ for all $f \in \mathfrak{L}$. Hence for any $a \in \mathcal{D}$ and $k \geq 0$

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, a(x)\, q^{(k)}(x, s) = 0.$$

Fix any compact set Λ in $\mathbb{R}^d$ and let $a \in \mathcal{D}$ be such that the support of a is a subset of Λ. Then,

$$\int_{\mathbb{R}^d} dx\, a(x) \left(\int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\ q^{(k)}(x, s) \right) = 0.$$

Hence

$$\int_{\Lambda} dx\, a(x) \left(\int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s) \right) = 0. \tag{4.5}$$

We state that the function

$$\Lambda \ni x \mapsto \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s)$$

belongs to $L^2(\Lambda,\, dx)$. Indeed, if $k = 0$, then $q^{(0)}(x, s) = 1$ and this statement evidently follows from Cauchy's inequality. Assume that $k \geq 1$. Then by Cauchy's inequality, (4.3), and condition (A),

$$\begin{aligned}
&\int_{\Lambda} dx \left(\int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s) \right)^2 \\
&\qquad \leq \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds_1)\, g(x, s_1)^2 \int_{\mathbb{R}} \sigma(x, ds_2)\, q^{(k)}(x, s_2)^2 \\
&= \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)^2\, a_1(x) a_2(x) \cdots a_k(x) \\
&\leq \left(\prod_{i=1}^{k} \sup_{x \in \Lambda} a_i(x) \right) \int_{\Lambda} dx \int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)^2 < \infty.
\end{aligned}$$

Since the set of all functions $a \in \mathcal{D}$ with support in Λ is dense in $L^2(\Lambda, dx)$, we therefore conclude from (4.5) that, for dx-a.a $x \in \Lambda$,

$$\int_{\mathbb{R}} \sigma(x, ds)\, g(x, s)\, q^{(k)}(x, s) = 0, \quad \forall k \geq 0. \tag{4.6}$$

Since $g \in H$, we get that, for dx-a.a. $x \in \mathbb{R}^d$, $g(x, \cdot) \in L^2(\mathbb{R}, \sigma(x, ds))$. Since $\{q^{(k)}(x, \cdot)\}_{k=0}^{\infty}$ form an orthogonal basis in $L^2(\mathbb{R}, \sigma(x, ds))$, we conclude from (4.6) that for dx-a.a. $x \in \mathbb{R}^d$ $g(x, s) = 0$ for $\sigma(x, ds)$-a.a. $s \in \mathbb{R}$. From here, we easily conclude that $g = 0$ as an element of H. Hence $\mathfrak{L}$ is indeed dense in H. $\square$

For each $n \in \mathbb{Z}_+$, we define

$$\mathfrak{L}_n := \big\{ g_n(x,s) = f(x)\, q^{(n)}(x,s) \mid f \in \mathcal{D} \big\}.$$

We have $\mathfrak{L}_n \subset \mathfrak{L}$, and the linear span of the $\mathfrak{L}_n$ spaces coincides with $\mathfrak{L}$. For any $g_n(x,s) = f_n(x)\, q^{(n)}(x,s) \in \mathfrak{L}_n$ and $g_m(x,s) = f_m(x)\, q^{(m)}(x,s) \in \mathfrak{L}_m$, $n, m \in \mathbb{Z}_+$, we have

$$\begin{aligned}(g_n, g_m)_H &= \int_{\mathbb{R}^d \times \mathbb{R}} g_n(x,s)\, g_m(x,s) dx\, \sigma(x, ds) \\ &= \int_{\mathbb{R}^d} f_n(x)\, f_m(x) \bigg(\int_{\mathbb{R}} q^{(n)}(x,s)\, q^{(m)}(x,s)\, \sigma(x,ds) \bigg) dx.\end{aligned} \tag{4.7}$$

Hence, if $n \neq m$, then

$$(g_n, g_m)_H = 0,$$

which implies that the linear spaces $\{\mathfrak{L}_n\}_{n=0}^{\infty}$ are mutually orthogonal in H. Denote by H_n the closure of $\mathfrak{L}_n$ in H. Then by Lemma 4.1, $H = \bigoplus_{n=0}^{\infty} H_n$.

By (4.7), setting $n = m$, we get

$$\|g_n\|_{H_n}^2 = \int_{\mathbb{R}^d} f_n^2(x) \bigg(\int_{\mathbb{R}} q^{(n)}(x,s)^2\, \sigma(x,ds) \bigg) dx = \int_{\mathbb{R}^d} f_n^2(x) \rho_n(dx), \tag{4.8}$$

where

$$\rho_n(dx) = \bigg(\int_{\mathbb{R}} q^{(n)}(x,s)^2\, \sigma(x,ds) \bigg) dx$$

is a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Consider a linear operator

$$\mathcal{D} \ni f_n \mapsto \big(J_n f_n\big)(x,s) := f_n(x) q^{(n)}(x,s) \in \mathfrak{L}_n.$$

The image of J_n is clearly the whole $\mathfrak{L}_n$. Now, $\mathfrak{L}_n$ is dense in H_n, while $\mathcal{D}$ is evidently dense in $L^2(\mathbb{R}^d, \rho_n(dx))$. By (4.8), for each $f_n \in \mathcal{D}$,

$$\|J_n f_n\|_{H_n} = \|f_n\|_{L^2(\mathbb{R}^d,\, \rho_n(dx))}.$$

Therefore, we can extend the operator J_n by continuity to a unitary operator

$$J_n : L^2(\mathbb{R}^d, \rho_n(dx)) \to H_n. \tag{4.9}$$

In particular,

$$H_n = \left\{ f_n(x)\, q^{(n)}(x,s) \mid f_n \in L^2(\mathbb{R}^d, \rho_n(dx)) \right\}.$$

Therefore, for each $k \geq 2$

$$\begin{aligned} H_n^{\otimes k} = \bigg\{ & f_n^{(k)}(x_1, \dots, x_k)\, q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k) \mid \\ & f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k} = L^2\big((\mathbb{R}^d)^k, \rho_n(dx_1) \cdots \rho_n(dx_k)\big) \bigg\}. \end{aligned}$$

Since the operator J_n in (4.9) is unitary, we get that the operator

$$J_n^{\otimes k} : L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k} \to H_n^{\otimes k}$$

is also unitary. The restriction of $J_n^{\otimes k}$ to $L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$ is a unitary operator

$$J_n^{\otimes k} : L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k} \to H_n^{\odot k}. \tag{4.10}$$

Indeed, take any $f_n \in L^2(\mathbb{R}^d, \rho_n(dx))$. Then $f_n^{\otimes k} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$ and the set of all such vectors is total in $L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$. Now, by the definition of $J_n^{\otimes k}$, we get

$$J_n^{\otimes k} f_n^{\otimes k} = (J_n\, f_n)^{\otimes k} \in H_n^{\odot k},$$

and furthermore the set of all vectors of the form $(J_n\, f_n)^{\otimes k}$ is total in $H_n^{\odot k}$. Hence, the statement follows.

For any $f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\otimes k}$,

$$\big(J_n^{\otimes k} f_n^{(k)}\big)(x_1, s_1, \dots, x_k, s_k) = f_n^{(k)}(x_1, \dots, x_k) q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k).$$

Hence, the unitary operator (4.10) acts as follows

$$\begin{aligned} &L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k} \ni f_n^{(k)}(x_1, \dots, x_k) \\ &\quad \mapsto \big(J_n^{\otimes k} f_n^{(k)}\big)(x_1, s_1, \dots, x_k, s_k) = f_n^{(k)}(x_1, \dots, x_k) q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k). \end{aligned}$$

Thus, each function $g_n^{(k)} \in H_n^{\odot k}$ has a representation

$$g_n^{(k)}(x_1, s_1, \dots, x_k, s_k) = f_n^{(k)}(x_1, \dots, x_k) q^{(n)}(x_1, s_1) \cdots q^{(n)}(x_k, s_k),$$

where $f_n^{(k)} \in L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}$ and $\|g_n^{(k)}\|_{H_n^{\odot k}} = \|f_n^{(k)}\|_{L^2(\mathbb{R}^d, \rho_n(dx))^{\odot k}}$.

For each $\alpha \in \mathbb{Z}_{+,0}^\infty$, we consider the Hilbert space

$$L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big) := L^2(\mathbb{R}^d, \rho_0(dx))^{\odot \alpha_0} \otimes L^2(\mathbb{R}^d, \rho_1(dx))^{\odot \alpha_1} \otimes \cdots . \tag{4.11}$$

We now define a unitary operator

$$J_\alpha : L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big) \to H_0^{\odot \alpha_0} \otimes H_1^{\odot \alpha_1} \otimes \cdots ,$$

where

$$J_\alpha = J_0^{\otimes \alpha_0} \otimes J_1^{\otimes \alpha_1} \otimes \cdots .$$

We evidently have, for each $f_\alpha \in L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big)$,

$$\begin{aligned} &\big(J_\alpha\, f_\alpha\big)(x_1, s_1, x_2, s_2, \dots, x_{|\alpha|}, s_{|\alpha|}) \\ &\quad = f_\alpha(x_1, x_2, \dots, x_{|\alpha|}) q^{(0)}(x_1, s_1) \cdots q^{(0)}(x_{\alpha_0}, s_{\alpha_0}) \\ &\quad \times q^{(1)}(x_{\alpha_0+1}, s_{\alpha_0+1}) \cdots q^{(1)}(x_{\alpha_0+\alpha_1}, s_{\alpha_0+\alpha_1}) \cdots . \end{aligned}$$

For each $\alpha \in \mathbb{Z}_{+,0}^{\infty}$, we define a Hilbert space

$$\mathcal{G}_\alpha := L^2_\alpha\big((\mathbb{R}^d)^{|\alpha|}\big)\alpha_0!\alpha_1!\cdots .$$

The J_α is evidently a unitary operator

$$J_\alpha : \mathcal{G}_\alpha \to (H_0^{\odot \alpha_0} \otimes H_1^{\odot \alpha_1} \otimes \cdots)\alpha_0!\alpha_1!\cdots .$$

Denote $\mathcal{G} := \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^{\infty}} \mathcal{G}_\alpha$. Hence, we can construct a unitary operator

$$J : \mathcal{G} \to \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^{\infty}} (H_0^{\odot \alpha_0} \otimes H_1^{\odot \alpha_1} \otimes \cdots)\alpha_0!\alpha_1!\cdots$$

by setting $J := \bigoplus_{\alpha \in \mathbb{Z}_{+,0}^{\infty}} J_\alpha$. By Lemma 3.3, we get a unitary operator $\mathcal{R} : \mathcal{G} \to \mathcal{F}(H)$, by setting $\mathcal{R} := \operatorname{Sym} J$. Thus, by Theorem 2.1, we get

Theorem 4.2. *Let condition (A) be satisfied. We have a unitary isomorphism $\mathcal{K} : \mathcal{G} \to L^2(\mathcal{D}', \mu)$ given by $\mathcal{K} := I\mathcal{R}$, where the unitary operator $I : \mathcal{F}(H) \to L^2(\mathcal{D}', \mu)$ is from Theorem 2.1.*

5 The unitary isomorphism $\mathcal{K}$ through multiple stochastic integrals

We will now give an interpretation of the unitary isomorphism $\mathcal{K}$ in terms of multiple stochastic integrals. We will only present a sketch of the proof, omitting some technical details.

Let us recall the operators $A(\varphi)$ in $\mathcal{F}(H)$ defined by (2.2). Now, for each $k \in \mathbb{N}$, we define operators

$$A^{(k)}(\varphi) := a^+(\varphi \otimes m_{k-1}) + a^0(\varphi \otimes m_k) + a^-(\varphi \otimes m_{k-1}). \tag{5.1}$$

In particular, $A^{(1)}(\varphi) = A(\varphi)$. The operator $A^{(k)}(\varphi)$ being symmetric, we denote by $A^{(k)}(\varphi)^\sim$ the closure of $A^{(k)}(\varphi)$. For each $k \in \mathbb{N}$ and $\varphi \in \mathcal{D}$, we define $Y^{(k-1)}(\varphi) := I(\varphi \otimes m_{k-1})$. It can be shown that, for each $k \in \mathbb{N}$, $IA^{(k)}(\varphi)^\sim I^{-1}$ *is the operator of multiplication by the function* $Y^{(k-1)}$.

Suppose, for a moment, that the measures $\sigma(x, ds)$ do not depend on $x \in \mathbb{R}^d$. For a fixed $\varphi \in \mathcal{D}$, let us orthogonalize in $L^2(\mathcal{D}', \mu)$ the functions $(Y^{(k)}(\varphi))_{k=0}^\infty$. This is of course equivalent to the orthogonalization of the monomials $(s^k)_{k=0}^\infty$ in $L^2(\mathbb{R}, \sigma)$. Denote by $(q^{(k)})_{k=0}^\infty$ the system of monic orthogonal polynomials with respect to the measure σ. Denote $(\varphi \otimes q^{(k)})(x, s) := \varphi(x)q^{(k)}(s)$. Thus, the random variables

$$Z^{(k)}(\varphi) := I(\varphi \otimes q^{(k)}), \quad k \in \mathbb{Z}_+,$$

appear as a result of the orthogonalization of $(Y^{(k)}(\varphi))_{k=0}^\infty$. Since $q^{(0)}(s)=1$, we have

$$Z^{(0)}(\varphi)=Y^{(0)}(\varphi)=\langle\cdot,\varphi\rangle.$$

For each $k\ge 1$, we have a representation of $q^{(k)}(s)$ as follows:

$$q^{(k)}(s)=\sum_{i=0}^k b_i^{(k)}\,s^i.$$

Thus,

$$Z^{(k)}(\varphi)=I(\varphi\otimes q^{(k)})=\sum_{i=0}^k b_i^{(k)} I(\varphi\otimes m_i)=\sum_{i=0}^k b_i^{(k)} Y^{(i)}(\varphi).$$

Hence, under I^{-1}, the image of the operator of multiplication by $Z^{(k)}(\varphi)$ is the operator

$$\begin{aligned}R^{(k)}(\varphi):&=\sum_{i=0}^k b_i^{(k)}(a^+(\varphi\otimes m_i)+a^-(\varphi\otimes m_i)+a^0(\varphi\otimes m_{i+1}))\\ &=a^+(\varphi\otimes q^{(k)})+a^-(\varphi\otimes q^{(k)})+a^0(\varphi\otimes\rho^{(k)}),\end{aligned}$$

where $\rho^{(k)}(s):=sq^{(k)}(s)$.

Let us now consider the general case, i.e., the case where the measure $\sigma(x,ds)$ does depend on $x\in\mathbb{R}^d$. We are using the monic polynomials $(q^{(k)}(x,\cdot))_{k=0}^\infty$ which are orthogonal with respect to the measure $\sigma(x,ds)$. We have

$$q^{(k)}(x,s)=\sum_{i=0}^k b_i^{(k)}(x)\,s^i.$$

We define

$$Z^{(k)}(\varphi):=I(\varphi q^{(k)})=\sum_{i=0}^k Y^{(i)}(\varphi b_i^{(k)}),$$

where $(\varphi q^{(k)})(x,s):=\varphi(x)q^{(k)}(x,s)$. Hence, under I^{-1}, the image of the operator of multiplication by $Z^{(k)}(\varphi)$ is the operator

$$\begin{aligned}R^{(k)}(\varphi):&=\sum_{i=0}^k \big(a^+((\varphi b_i^{(k)})\otimes m_i)+a^-((\varphi b_i^{(k)})\otimes m_i)+a^0((\varphi b_i^{(k)})\otimes m_{i+1})\big)\\ &=a^+\left(\left(\varphi\sum_{i=0}^k b_i^{(k)}\right)\otimes m_i\right)+a^-\left(\left(\varphi\sum_{i=0}^k b_i^{(k)}\right)\otimes m_i\right)\\ &\quad+a^0\left(\left(\varphi\sum_{i=0}^k b_i^{(k)}\right)\otimes m_{i+1}\right)\end{aligned}$$

$$= a^+(\varphi q^{(k)}) + a^-(\varphi q^{(k)}) + a^0(\varphi \rho^{(k)}),$$

where $\rho^{(k)}(x,s) := sq^{(k)}(x,s)$.

It is not hard to see that the above definitions and formulas can be easily extended to the case where the function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is just measurable, bounded, and has compact support. In particular, for each $\Delta \in \mathcal{B}_0(\mathbb{R}^d)$, we will use the operators $Z^{(k)}(\Delta) := Z^{(k)}(\chi_\Delta)$.

We will now introduce a multiple Wiener–Itô integral with respect to $Z^{(k)}$'s. So, we fix any $\alpha \in \mathbb{Z}_{+,0}^\infty$, $|\alpha| = n$, $n \in \mathbb{N}$. Take any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(\mathbb{R}^d)$, mutually disjoint. Then we define

$$\begin{aligned}
&\int_{\Delta_1\times\Delta_2\times\cdots\times\Delta_n} dZ^{(0)}(x_1)\cdots dZ^{(0)}(x_{\alpha_0}) dZ^{(1)}(x_{\alpha_0+1})\cdots dZ^{(1)}(x_{\alpha_0+\alpha_1})\\
&\quad \times dZ^{(2)}(x_{\alpha_0+\alpha_1+1})\cdots\\
&= \int_{(\mathbb{R}^d)^n} \chi_{\Delta_1}(x_1)\chi_{\Delta_2}(x_2)\cdots\chi_{\Delta_n}(x_n) dZ^{(0)}(x_1)\cdots dZ^{(0)}(x_{\alpha_0})\\
&\quad \times dZ^{(1)}(x_{\alpha_0+1})\cdots dZ^{(1)}(x_{\alpha_0+\alpha_1}) dZ^{(2)}(x_{\alpha_0+\alpha_1+1})\cdots\\
&:= Z^{(0)}(\Delta_1)\cdots Z^{(0)}(\Delta_{\alpha_0}) Z^{(1)}(\Delta_{\alpha_0+1})\cdots Z^{(1)}(\Delta_{\alpha_0+\alpha_1}) Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1})\cdots .
\end{aligned}$$

Using the fact that the sets $\Delta_1, \dots, \Delta_n$ are mutually disjoint,

$$\begin{aligned}
&I^{-1}(Z^{(0)}(\Delta_1)\cdots Z^{(0)}(\Delta_{\alpha_0}) Z^{(1)}(\Delta_{\alpha_0+1})\cdots Z^{(1)}(\Delta_{\alpha_0+\alpha_1}) Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1})\cdots)\\
&= R^{(0)}(\chi_{\Delta_1})\cdots R^{(0)}(\chi_{\Delta_{\alpha_0}}) R^{(1)}(\chi_{\Delta_{\alpha_0+1}})\cdots R^{(1)}(\chi_{\Delta_{\alpha_0+\alpha_1}}) R^{(2)}(\chi_{\Delta_{\alpha_0+\alpha_1+1}})\cdots\\
&= a^+(\chi_{\Delta_1} q^{(0)})\cdots a^+(\chi_{\Delta_{\alpha_0}} q^{(0)}) a^+(\chi_{\Delta_{\alpha_0+1}} q^{(1)})\cdots a^+(\chi_{\Delta_{\alpha_0+\alpha_1}} q^{(1)})\\
&\quad \times a^+(\chi_{\Delta_{\alpha_0+\alpha_1+1}} q^{(2)})\cdots\Omega\\
&= (\chi_{\Delta_1} q^{(0)})\odot\cdots\odot(\chi_{\Delta_{\alpha_0}} q^{(0)})\odot(\chi_{\Delta_{\alpha_0+1}} q^{(1)})\odot\cdots\odot(\chi_{\Delta_{\alpha_0+\alpha_1}} q^{(1)})\\
&\quad \odot(\chi_{\Delta_{\alpha_0+\alpha_1+1}} q^{(2)})\odot\cdots\\
&= \mathrm{Sym}_n\Bigg(\Big[(\chi_{\Delta_1} q^{(0)})\odot\cdots\odot(\chi_{\Delta_{\alpha_0}} q^{(0)})\Big]\otimes\Big[(\chi_{\Delta_{\alpha_0+1}} q^{(1)})\odot\cdots\\
&\quad \odot(\chi_{\Delta_{\alpha_0+\alpha_1}} q^{(1)})\Big]\otimes\cdots\Bigg)\\
&= \mathrm{Sym}_n\Bigg(\Big[(\chi_{\Delta_1}\odot\cdots\odot\chi_{\Delta_{\alpha_0}})(x_1,\dots,x_{\alpha_0}) q^{(0)}(x_1,s_1)\cdots q^{(0)}(x_{\alpha_0},s_{\alpha_0})\Big]\\
&\quad \otimes\Big[(\chi_{\Delta_{\alpha_0+1}}\odot\cdots\odot\chi_{\Delta_{\alpha_0+\alpha_1}})(x_{\alpha_0+1},\dots,x_{\alpha_0+\alpha_1})\, q^{(1)}(x_{\alpha_0+1},s_{\alpha_0+1})\\
&\quad \cdots q^{(1)}(x_{\alpha_0+\alpha_1},s_{\alpha_0+\alpha_1})\Big]\otimes\cdots\Bigg)\\
&= \mathcal{R}\Big((\chi_{\Delta_1}\odot\cdots\odot\chi_{\Delta_{\alpha_0}})\otimes(\chi_{\Delta_{\alpha_0+1}}\odot\cdots\odot\chi_{\Delta_{\alpha_0+\alpha_1}})\otimes\cdots\Big).
\end{aligned}$$

Hence

$$Z^{(0)}(\Delta_1)\cdots Z^{(0)}(\Delta_{\alpha_0})Z^{(1)}(\Delta_{\alpha_0+1})\cdots Z^{(1)}(\Delta_{\alpha_0+\alpha_1})Z^{(2)}(\Delta_{\alpha_0+\alpha_1+1})\cdots$$
$$=\mathcal{K}((\chi_{\Delta_1}\odot\cdots\odot\chi_{\Delta_{\alpha_0}})\otimes(\chi_{\Delta_{\alpha_0+1}}\odot\cdots\odot\chi_{\Delta_{\alpha_0+\alpha_1}})\otimes\cdots).$$

The set of all vectors of the form

$$((\chi_{\Delta_1}\odot\cdots\odot\chi_{\Delta_{\alpha_0}})\otimes(\chi_{\Delta_{\alpha_0+1}}\odot\cdots\odot\chi_{\Delta_{\alpha_0+\alpha_1}})\otimes\cdots)$$

is total in $\mathcal{G}_\alpha$. Therefore, by linearity and continuity, we can extend the definition of the multiple Winner–Itô integral to the whole space $\mathcal{G}_\alpha$. Thus, we get, for each $f_\alpha\in\mathcal{G}_\alpha$,

$$\int_{(\mathbb{R}^d)^{|\alpha|}} f_\alpha(x_1,\dots,x_{|\alpha|})dZ^{(0)}(x_1)\cdots dZ^{(0)}(x_{\alpha_0})dZ^{(1)}(x_{\alpha_0+1})\cdots dZ^{(1)}(x_{\alpha_0+\alpha_1})$$
$$\times\, dZ^{(2)}(x_{\alpha_0+\alpha_1+1})\cdots=\mathcal{K}f_\alpha.$$

Thus, we have the following theorem.

Theorem 5.1. *The unitary isomorphism $\mathcal{K}:\mathcal{G}\to L^2(\mathcal{D}',\mu)$ from Theorem 4.2 is given by*

$$\mathcal{G}=\bigoplus_{\alpha\in\mathbb{Z}^\infty_{+,0}}\mathcal{G}_\alpha\ni(f_\alpha)_{\alpha\in\mathbb{Z}^\infty_{+,0}}=f\mapsto\mathcal{K}f$$
$$=\sum_{\alpha\in\mathbb{Z}^\infty_{+,0}}\int_{(\mathbb{R}^d)^{|\alpha|}} f_\alpha(x_1,\dots,x_{|\alpha|})dZ^{(0)}(x_1)\cdots dZ^{(0)}(x_{\alpha_0})$$
$$\times\, dZ^{(1)}(x_{\alpha_0+1})\cdots dZ^{(1)}(x_{\alpha_0+\alpha_1})dZ^{(2)}(x_{\alpha_0+\alpha_1+1})\cdots\,.$$

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