

# ERGODIC PROPERTIES OF VISIBLE LATTICE POINTS

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ABSTRACT. Recently, the dynamical and spectral properties of square-free integers, visible lattice points and various generalisations have received increased attention. One reason is the connection of one-dimensional examples such as  $\mathcal{B}$ -free numbers with Sarnak's conjecture on the 'randomness' of the Möbius function, another the explicit computability of correlation functions as well as eigenfunctions for these systems together with intrinsic ergodicity properties. Here, we summarise some of the results, with focus on spectral and dynamical aspects, and expand a little on the implications for mathematical diffraction theory.

Dedicated to Nikolai P. Dolbilin on the occasion of his 70th birthday

## 1. INTRODUCTION

Delone sets are important mathematical descriptions of atomic arrangements, and Meyer sets are special cases with a rich spectral structure [3, 16, 17]. Particularly well-studied are cut and project sets or *model* sets, which underly the structure of perfect quasicrystals [36, 39, 3]. It is fair to say that such structures are rather well understood. This is much less so if one keeps uniform discreteness but relaxes relative denseness. In fact, one might expect to leave the realm of pure point spectrum, at least as soon as one has entropy [6]. However, there are interesting examples such as  $k$ -free numbers or visible lattice points that have positive topological entropy but nevertheless pure point dynamical and diffraction spectrum. They are examples of *weak* model sets, and deserve a better understanding. Here, we summarise some of the known results and add to the structure of their topological and spectral properties. The paper is organised as follows.

In Section 2, we use the visible points of  $\mathbb{Z}^2$  as a paradigm to formulate the results for this case in a geometrically oriented manner and to develop our notation and methods while we go along. Section 3 extends the findings to  $k$ -free points of  $n$ -dimensional lattices, while Section 4 looks into the setting of  $\mathcal{B}$ -free systems as introduced in [18, 25]. Finally, in Section 5, we analyse an example from the number field generalisation of [14] in our more geometric setting of diffraction analysis. For convenience and better readability, the more technical issues are presented in two appendices.

## 2. VISIBLE SQUARE LATTICE POINTS

Two classic examples for the structure we are after are provided by the *square-free integers* (the elements of  $\mathbb{Z}$  that are not divisible by any nontrivial square) and the *visible points* of a lattice (the points with coprime coordinates in a lattice basis). Since our focus is on higher-dimensional cases, let us take a closer look at the visible points of  $\mathbb{Z}^2$ ,

$$V = V_{\mathbb{Z}^2} = \mathbb{Z}^2 \setminus \bigcup_{p \text{ prime}} p\mathbb{Z}^2 = \{x \in \mathbb{Z}^2 \mid \gcd(x) = 1\},$$

where  $\gcd(x) = \gcd(x_1, x_2)$  for  $x = (x_1, x_2)$ . Note that, throughout the text,  $p$  will denote a prime number. The set is illustrated in Fig. 1, and can also be found in many textbooks including [2], where it is shown on the cover, and [3, Sec. 10.4]. The following result is standard; see [3, Prop. 10.4] and references therein for details.

**Proposition 1.** *The set  $V$  has the following properties.*

- (a) *The set  $V$  is uniformly discrete, but not relatively dense. In particular,  $V$  contains holes of arbitrary size that repeat lattice-periodically. More precisely, given an inradius  $\rho > 0$ , there is a sublattice of  $\mathbb{Z}^2$  depending on  $\rho$  such that a suitable translate of this sublattice consists of centres of holes of inradius at least  $\rho$ .*
- (b) *The group  $\mathrm{GL}(2, \mathbb{Z})$  acts transitively on  $V$ , and one has the partition  $\mathbb{Z}^2 = \dot{\bigcup}_{m \in \mathbb{N}_0} mV$  of  $\mathbb{Z}^2$  into  $\mathrm{GL}(2, \mathbb{Z})$ -invariant sets.*
- (c) *The difference set is  $V - V = \mathbb{Z}^2$ .*
- (d) *The natural density of  $V$  exists and is given by  $\mathrm{dens}(V) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$ , where  $\zeta$  denotes Riemann's zeta function.  $\square$*

Note that big holes are rare, but important; see [3, Rem. 10.6] for some examples. The interesting fact is that they do not destroy the existence of patch frequencies, though the latter clearly cannot exist uniformly. For the natural pair correlation (or autocorrelation) coefficients,

$$\eta(x) := \lim_{R \rightarrow \infty} \frac{1}{\pi R^2} |V \cap (-x + V) \cap B_R(0)|,$$

one finds the following result; compare [3, Lemma 10.6], [10, Thm. 2], as well as [31, Thm. 7].

**Lemma 1.** *For each  $x \in \mathbb{Z}^2$ , the natural autocorrelation coefficient  $\eta(x)$  of  $V$  exists, and is given by*

$$\eta(x) = \xi \prod_{p|\gcd(x)} \left(1 + \frac{1}{p^2 - 2}\right),$$

where  $\xi = \prod_p (1 - 2p^{-2}) \approx 0.3226$ . In particular, with  $\gcd(0) = 0$ , this also gives the density  $\eta(0) = \prod_p (1 - p^{-2}) = 1/\zeta(2)$ .  $\square$

The autocorrelation measure of  $V$ ,

$$\gamma = \sum_{x \in \mathbb{Z}^2} \eta(x) \delta_x,$$

is thus well-defined, and a translation bounded, positive definite measure on  $\mathbb{R}^2$  by construction. Its Fourier transform  $\hat{\gamma}$  exists by general arguments, compare [11, Ch. I.4] or [3, Rem. 8.7 and Prop. 8.6], and leads to the following result; see [3, Thm. 10.5] and [10, Thm. 3].

**Theorem 1.** *The natural diffraction measure  $\hat{\gamma}$  of the visible points  $V$  of the square lattice  $\mathbb{Z}^2$  exists. It is a positive pure point measure which is translation bounded and supported on the points of  $\mathbb{Q}^2$  with square-free denominator, the Fourier-Bohr spectrum of  $\gamma$ , so*

$$\hat{\gamma} = \sum_{\substack{k \in \mathbb{Q}^2 \\ \mathrm{den}(k) \text{ square-free}}} I(k) \delta_k,$$

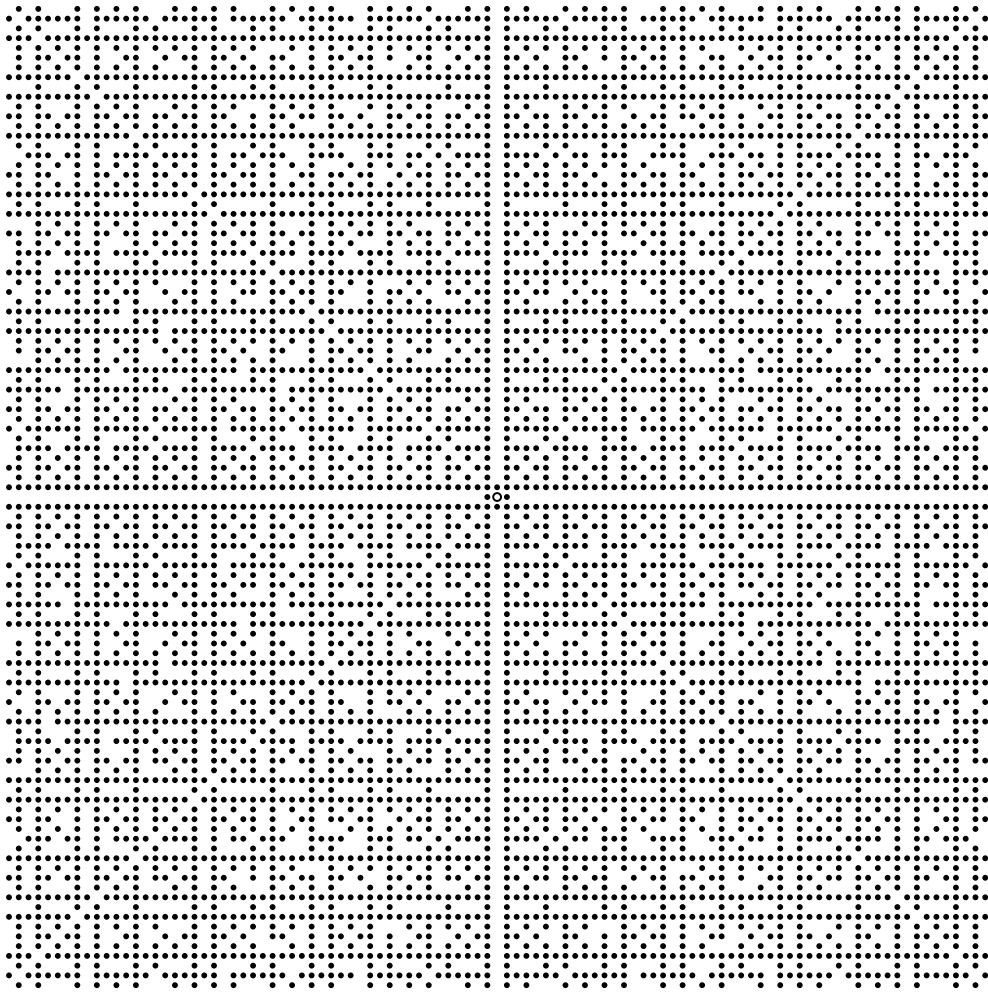


FIGURE 1. A central patch of the visible points  $V$  of the square lattice  $\mathbb{Z}^2$ . Note the invariance of  $V$  with respect to  $\text{GL}(2, \mathbb{Z})$ .

where  $\text{den}(k) := \gcd\{n \in \mathbb{N} \mid nk \in \mathbb{Z}^2\}$ . In particular,  $I(0) = (1/\zeta(2))^2 = 36/\pi^4$ , and when  $0 \neq k \in \mathbb{Q}^2$  has square-free denominator  $\text{den}(k)$ , the corresponding intensity is given by

$$I(k) = \left( \frac{6}{\pi^2} \prod_{p \mid \text{den}(k)} \frac{1}{p^2 - 1} \right)^2.$$

□

The essence of this result is the pure point nature of  $\hat{\gamma}$  together with its explicit computability via an intensity formula in the form of a *finite* product for any given  $k \in \mathbb{Q}^2$  with square-free denominator. Fig. 2 illustrates the diffraction measure. Note that  $\hat{\gamma}$  has the symmetry group  $\mathbb{Z}^2 \rtimes \text{GL}(2, \mathbb{Z})$ .

An alternative view is possible by means of the Herglotz–Bochner theorem as follows. The autocorrelation measure  $\gamma$  is positive definite on  $\mathbb{R}^2$  if and only if the function  $\eta: \mathbb{Z}^2 \rightarrow \mathbb{R}$  is positive definite on  $\mathbb{Z}^2$ ; see [3, Lemma 8.4]. The latter property is equivalent to the existence

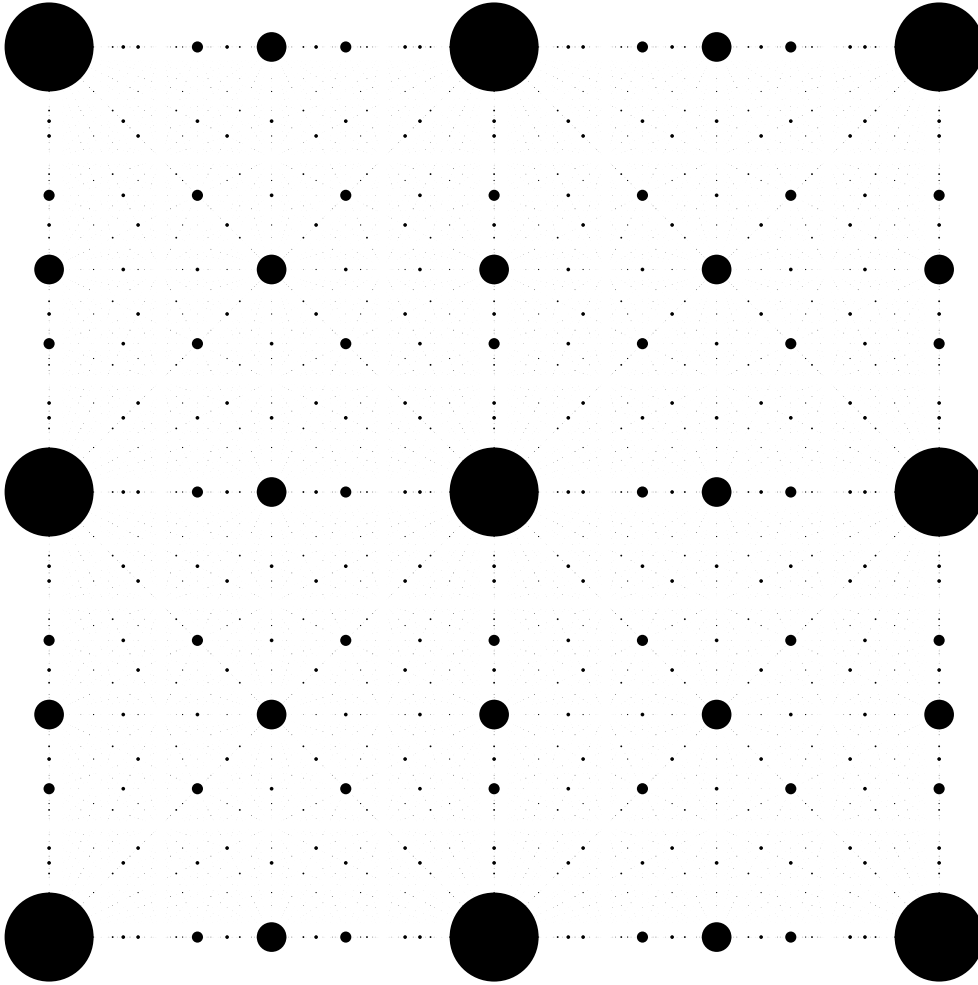


FIGURE 2. Diffraction  $\hat{\gamma}$  of the visible points of  $\mathbb{Z}^2$ . A point measure at  $k$  with intensity  $I(k)$  is shown as a disk centred at  $k$  with area proportional to  $I(k)$ . Shown are the intensities with  $I(k)/I(0) \geq 10^{-6}$  and  $k \in [0, 2]^2$ . Its lattice of periods is  $\mathbb{Z}^2$ , and  $\hat{\gamma}$  turns out to be  $\text{GL}(2, \mathbb{Z})$ -invariant.

of a positive measure  $\varrho$  on  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \simeq [0, 1]^2$  such that

$$\eta(x) = \int_{\mathbb{T}^2} e^{2\pi ixy} d\varrho(y),$$

where the connection to  $\hat{\gamma}$  is established by  $\varrho = \gamma|_{[0,1]^2}$ , so that  $\hat{\gamma} = \varrho * \delta_{\mathbb{Z}^2}$ . The finite positive measure  $\varrho$  is a spectral measure in the sense of dynamical system theory. It is related to the diffraction measure by convolution; for background, we refer to [5, 8] and references therein. We shall return to the dynamical point of view shortly.

Let us pause to comment on the history and the development of this problem. The arithmetic properties of  $V$  are classic and can be found in many places, including [2, 23]. The investigation of spectral aspects was advertised by Schroeder in [34], see also [35], by means of a numerical approach via FFT techniques. These results suffered from insufficient resolution

in the numerical treatment, and seemed to point towards continuous diffraction components, perhaps in line with the idea that the distribution of primes is sufficiently ‘random’.

Ten years later, on the basis of a formal Möbius inversion calculation for the amplitudes, Mosseri argued in [29] that the diffraction should be pure point rather than continuous, thus contradicting the earlier numerical findings of Schroeder. This was corroborated in [4] with further calculations on the diffraction intensities (still without proof), which gave the formula of Thm. 1 above. Also, a rather convincing comparison with an optical diffraction experiment was shown, which clearly indicated the correctness of the formal calculation. The first complete proof, with a detailed convergence result with precise error estimates, appeared in [BMP], and was recently improved and extended in [31], on the basis of number-theoretic results due to Mirsky [26, 27].

Simultaneously, due to the renewed interest in the square-free integers in connection with Sarnak’s conjecture, the dynamical systems point of view became more important, as is obvious from [13, 14, 31, 24]. Here, the focus is more on the dynamical spectrum, which is closely related to the diffraction measure as indicated above, and explained in detail in [5, 8].

To explain this, let us define the (discrete) hull of  $V$  as

$$\mathbb{X}_V = \overline{\{t + V \mid t \in \mathbb{Z}^2\}},$$

where the closure is taken in the product topology induced on  $\{0, 1\}^{\mathbb{Z}^2}$  by the discrete topology on  $\{0, 1\}$ . This topology is metric [38, 31] and is also called the *local topology*, because two elements of  $\{0, 1\}^{\mathbb{Z}^2}$  are close if they agree on a large ball around the origin. Clearly,  $\mathbb{X}_V$  is then compact, where here and below we simultaneously view subsets of  $\mathbb{Z}^2$  as configurations. In particular, the empty set is identified with the configuration  $\underline{0}$ , and  $\mathbb{Z}^2$  with  $\underline{1}$  this way. Since  $V$  contains holes of arbitrary size, the empty set is an element of  $\mathbb{X}_V$ .

For a natural number  $m$ , let  $\cdot_m: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2/m\mathbb{Z}^2$  denote the canonical projection  $x \mapsto [x]_m$ , where  $[x]_m = x + m\mathbb{Z}^2$ . For a subset  $X \subset \mathbb{Z}^2$ , we denote by  $X_m$  its image under this projection map. It should however be born in mind that, for elements  $x \in \mathbb{Z}^2$ , their images under the above map will always be written as  $[x]_m$  rather than  $x_m$ . This is due to the occasional need of regarding  $m$  in the last expression as an index. Let us recall the following result from [10].

**Proposition 2** (Chinese Remainder Theorem). [10, Prop. 2] *For pairwise coprime positive integers  $m_1, m_2, \dots, m_r$ , the natural group homomorphism*

$$(\mathbb{Z}^2)_{m_1 m_2 \dots m_r} \longrightarrow \prod_{i=1}^r (\mathbb{Z}^2)_{m_i}$$

*is an isomorphism. In particular, for  $x_1, x_2, \dots, x_r \in \mathbb{Z}^2$ , the simultaneous solutions  $t \in \mathbb{Z}^2$  of*

$$[t]_{m_i} = [x_i]_{m_i}, \quad 1 \leq i \leq r,$$

*comprise precisely one coset of  $m_1 m_2 \dots m_r \mathbb{Z}^2$  in  $\mathbb{Z}^2$ .* □

Next, let us come to a characterisation of  $\mathbb{X}_V$ . Let  $\mathbb{A}$  denote the set of *admissible* subsets  $A$  of  $\mathbb{Z}^2$ , i.e. subsets  $A \subset \mathbb{Z}^2$  with the property that, for every prime  $p$ ,  $A$  does *not* contain a full set of representatives modulo  $p\mathbb{Z}^2$ . In other words,  $A$  is admissible if and only if

$$|A_p| < p^2$$

for any prime  $p$ . Since  $V \in \mathbb{A}$  (otherwise some point of  $V$  were in  $p\mathbb{Z}^2$  for some prime  $p$ , a contradiction) and since  $\mathbb{A}$  is a  $\mathbb{Z}^2$ -invariant and closed subset of  $\{0, 1\}^{\mathbb{Z}^2}$ , it is clear that  $\mathbb{X}_V$

is a subset of  $\mathbb{A}$ . By [31, Thm. 2], the other inclusion is also true. This was first shown by Herzog and Stewart [22] for visible lattice points and by Sarnak [33] for the analogous case of the square-free integers. In fact, similar statements hold true for various generalisations discussed below; cf. [31, Thm. 6] for the case of  $k$ -free lattice points.

**Theorem 2.** *One has  $\mathbb{X}_V = \mathbb{A}$ .* □

It follows that  $\mathbb{X}_V$  is *hereditary*, i.e.

$$\forall X \in \mathbb{A} : (Y \subset X \Rightarrow Y \in \mathbb{A}),$$

and in particular contains *all* subsets of  $V$ . In other words,  $V$  is an *interpolating set* for  $\mathbb{X}_V$  in the sense of [42], which means that

$$\mathbb{X}_V|_V := \{X \cap V \mid X \in \mathbb{X}_V\} = \{0, 1\}^V.$$

Given a radius  $\rho > 0$  and a point  $t \in \mathbb{Z}^2$ , the  $\rho$ -*patch* of  $V$  at  $t$  is

$$(V - t) \cap B_\rho(0),$$

the translation to the origin of the part of  $V$  within a distance  $\rho$  of  $t$ . We denote by  $\mathcal{A}(\rho)$  the (finite) set of all  $\rho$ -patches of  $V$ , and by  $N(\rho) = |\mathcal{A}(\rho)|$  the number of distinct  $\rho$ -patches of  $V$ . For a  $\rho$ -patch  $\mathcal{P}$  of  $V$ , denote by  $C_{\mathcal{P}}$  the set of elements of  $\mathbb{X}_V$  whose  $\rho$ -patch at 0 is  $\mathcal{P}$ , the so-called *cylinder set* defined by the  $\rho$ -patch  $\mathcal{P}$ ; compare [15]. Note that these cylinder sets form a basis of the topology of  $\mathbb{X}_V$ .

The *patch counting entropy* of  $V$  is defined as

$$h_{\text{pc}}(V) := \lim_{\rho \rightarrow \infty} \frac{\log N(\rho)}{\pi \rho^2}.$$

Note that this differs from the definition in [31, 24], where, in view of the binary configuration space interpretation, we used the base 2 logarithm. It can be shown by a classic subadditivity argument that this limit exists. Since  $\mathbb{X}_V$  is hereditary, it follows that  $V$  has patch counting entropy  $h_{\text{pc}}(V)$  at least  $\text{dens}(V) \log(2) = (6/\pi^2) \log(2)$ . In fact, one has more.

**Theorem 3.** [31, Thm. 3] *One has  $h_{\text{pc}}(V) = (6/\pi^2) \log(2)$ .* □

The natural translational action of the group  $\mathbb{Z}^2$  on  $\mathbb{X}_V$  is continuous and  $(\mathbb{X}_V, \mathbb{Z}^2)$  thus is a *topological dynamical system*. By construction,  $(\mathbb{X}_V, \mathbb{Z}^2)$  is topologically transitive [1, 20, 42], as it is the orbit closure of one of its elements (namely  $V$ ). Equivalently, for any two non-empty open subsets  $U$  and  $W$  of  $\mathbb{X}_V$ , there is an element  $t \in \mathbb{Z}^2$  such that

$$U \cap (W + t) \neq \emptyset.$$

In accordance with Sarnak's findings [33] for square-free integers, one has the following result.

**Theorem 4.** *The topological dynamical system  $(\mathbb{X}_V, \mathbb{Z}^2)$  has the following properties.*

- (a)  $(\mathbb{X}_V, \mathbb{Z}^2)$  is topologically ergodic with positive topological entropy equal to  $(6/\pi^2) \log(2)$ .
- (b)  $(\mathbb{X}_V, \mathbb{Z}^2)$  is proximal, and  $\{\emptyset\}$  is the unique  $\mathbb{Z}^2$ -minimal subset of  $\mathbb{X}_V$ .
- (c)  $(\mathbb{X}_V, \mathbb{Z}^2)$  has no non-trivial topological Kronecker factor (i.e., minimal equicontinuous factor). In particular,  $(\mathbb{X}_V, \mathbb{Z}^2)$  has trivial topological point spectrum.
- (d)  $(\mathbb{X}_V, \mathbb{Z}^2)$  has a non-trivial joining with the Kronecker system given by  $K = (G, \mathbb{Z}^2)$ , where  $G$  is the compact Abelian group  $\prod_p (\mathbb{Z}^2)_p$  and  $\mathbb{Z}^2$  acts on  $G$  via addition of  $\iota(x) = ([x]_p)$ , i.e.  $g \mapsto g + \iota(x)$ , with  $g \in G$  and  $x \in \mathbb{Z}^2$ . In particular,  $(\mathbb{X}_V, \mathbb{Z}^2)$  fails to be topologically weakly mixing.

*Proof.* The topological entropy of the dynamical system  $(\mathbb{X}_V, \mathbb{Z}^2)$  is just  $h_{\text{pc}}(V)$ , so the assertion follows from Theorem 3; cf. [6, Thm. 1]. The topological ergodicity [1, 20] will follow from the existence of an ergodic full (non-empty open subsets have positive measure)  $\mathbb{Z}^2$ -invariant Borel measure on  $\mathbb{X}_V$ ; see Theorems 5 and 6(b) below.

For part (b), recall from Theorem 2 that the hull contains many more elements than the translates of  $V$ . Nevertheless, one can derive from Proposition 1(a) that every element of  $\mathbb{X}_V$  contains holes of arbitrary size that repeat lattice-periodically. This follows by standard compactness arguments from considering a sequence of the form  $(t_n + V)_{n \in \mathbb{N}}$  that converges in the local topology, via selecting suitable subsequences. In particular, let  $X, Y \in \mathbb{X}_V$  and a radius  $\rho$  be fixed. Let  $t_\rho + \Gamma$  be positions of holes of inradius  $\rho$  in  $X$ . Choose  $\rho'$  large enough such that  $B_{\rho'}(0)$  covers  $B_\rho(0) + F$ , where  $F$  is a fundamental domain of  $\Gamma$ . Then, any  $\rho'$ -hole of  $Y$  (which exists) contains a  $\rho$ -hole of  $X$ . Hence, for any  $\rho > 0$  and any two elements  $X, Y \in \mathbb{X}_V$ , there is a translation  $t \in \mathbb{Z}^2$  such that

$$(X + t) \cap B_\rho(0) = (Y + t) \cap B_\rho(0) = \emptyset,$$

meaning that both  $X$  and  $Y$  have the empty  $\rho$ -patch at  $-t$ . In terms of the metric  $d$  on  $\mathbb{X}_V$  [38, 31, 24] this means that  $d(X+t, Y+t) \leq 1/\rho$  and the proximality of the system follows. Similarly, the assertion on the unique  $\mathbb{Z}^2$ -minimal subset  $\{\emptyset\}$  follows from the fact that any element of  $\mathbb{X}_V$  contains arbitrarily large holes and thus any non-empty closed subsystem contains  $\emptyset$ .

Since Kronecker systems are distal, the first assertion of part (c) is an immediate consequence of the proximality of  $(\mathbb{X}_V, \mathbb{Z}^2)$ . This also implies that  $(\mathbb{X}_V, \mathbb{Z}^2)$  has trivial topological point spectrum; see [24] for an alternative argument that the non-zero constant function is the only continuous eigenfunction of the translation action.

For part (d), one can verify that a non-trivial (topological) joining [20] of  $(\mathbb{X}_V, \mathbb{Z}^2)$  with the Kronecker system  $K$  is given by

$$W := \bigcup_{X \in \mathbb{X}_V} \left( \{X\} \times \prod_p (\mathbb{Z}^2 \setminus X)_p \right).$$

Since the Kronecker system  $K$  is minimal and distal, a well-known disjointness theorem by Furstenberg [19, Thm. II.3] implies that  $(\mathbb{X}_V, \mathbb{Z}^2)$  fails to be topologically weakly mixing.  $\square$

Following [10, 31], the natural *frequency*  $\nu(\mathcal{P})$  of a  $\rho$ -patch  $\mathcal{P}$  of  $V$  is defined as

$$(1) \quad \nu(\mathcal{P}) := \text{dens}(\{t \in \mathbb{Z}^2 \mid (V - t) \cap B_\rho(0) = \mathcal{P}\}),$$

which can indeed be seen to exist.

**Theorem 5.** [31, Thms. 1 and 2] *Any  $\rho$ -patch  $\mathcal{P}$  of  $V$  occurs with positive frequency, which is given by*

$$\nu(\mathcal{P}) = \sum_{\mathcal{F} \subset (\mathbb{Z}^2 \cap B_\rho(0)) \setminus \mathcal{P}} (-1)^{|\mathcal{F}|} \prod_p \left( 1 - \frac{|(\mathcal{P} \cup \mathcal{F})_p|}{p^2} \right).$$

$\square$

The frequency function  $\nu$  from (1), regarded as a function on the cylinder sets by setting  $\nu(C_{\mathcal{P}}) := \nu(\mathcal{P})$ , is finitely additive on the cylinder sets with

$$\sum_{\mathcal{P} \in \mathcal{A}(\rho)} \nu(C_{\mathcal{P}}) = \frac{1}{|\det(\mathbb{Z}^2)|} = 1.$$

Since the family of cylinder sets is a (countable) semi-algebra that generates the Borel  $\sigma$ -algebra on  $\mathbb{X}_V$  (i.e. the smallest  $\sigma$ -algebra on  $\mathbb{X}_V$  which contains the open subsets of  $\mathbb{X}_V$ ),  $\nu$  extends uniquely to a probability measure on  $\mathbb{X}_V$ ; compare [15, Prop. 8.2] and references given there. Moreover, this probability measure is  $\mathbb{Z}^2$ -invariant by construction wherefore we have a measure-theoretic dynamical system  $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$ . For part (b) of the following claim, note that the Fourier–Bohr spectrum of  $V$  is itself a group and compare [8, Prop. 17].

**Theorem 6.** *The measure-theoretic dynamical system  $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$  has the following properties.*

- (a) *The  $\mathbb{Z}^2$ -orbit of  $V$  in  $\mathbb{X}_V$  is  $\nu$ -equidistributed, which means that for any function  $f \in C(\mathbb{X}_V)$ , one has*

$$\lim_{R \rightarrow \infty} \frac{1}{\pi R^2} \sum_{x \in \mathbb{Z}^2 \cap B_R(0)} f(V + x) = \int_{\mathbb{X}_V} f(X) \, d\nu(X).$$

*In other words,  $V$  is  $\nu$ -generic.*

- (b)  *$(\mathbb{X}_V, \mathbb{Z}^2, \nu)$  is ergodic, deterministic (i.e., it is of zero measure entropy) and has pure point dynamical spectrum. The latter is given by the Fourier–Bohr spectrum of the autocorrelation  $\gamma$ , as described in Theorem 1.*
- (c)  *$(\mathbb{X}_V, \mathbb{Z}^2, \nu)$  is metrically isomorphic to the Kronecker system  $K_\mu = (G, \mathbb{Z}^2, \mu)$ , where  $G$  is the compact Abelian group  $\prod_p (\mathbb{Z}^2)_p$ , the lattice  $\mathbb{Z}^2$  acts on  $G$  as above and  $\mu$  is the normalised Haar measure on  $G$ .*

*Proof.* For part (a), it suffices to show this for the characteristic functions of cylinder sets of finite patches, as their span is dense in  $C(\mathbb{X}_V)$ . But for such functions, the claim is clear as the left hand side is the patch frequency as used for the definition of the measure  $\nu$ .

For the ergodicity of  $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$ , one has to show that

$$\lim_{R \rightarrow \infty} \frac{1}{\pi R^2} \sum_{x \in \mathbb{Z}^2 \cap B_R(0)} \nu((C_{\mathcal{P}} + x) \cap C_{\mathcal{Q}}) = \nu(C_{\mathcal{P}})\nu(C_{\mathcal{Q}})$$

for arbitrary cylinder sets  $C_{\mathcal{P}}$  and  $C_{\mathcal{Q}}$ ; compare [41, Thm. 1.17]. The latter in turn follows from a straightforward (but lengthy) calculation using Theorem 5 and the definition of the measure  $\nu$  together with the Chinese Remainder Theorem. In fact, for technical reasons, it is better to work with a different semi-algebra that also generates the Borel  $\sigma$ -algebra on  $\mathbb{X}_V$ ; see Appendix A for the details.

Vanishing measure-theoretical entropy, i.e.

$$h_{\text{meas}}(V) = \lim_{\rho \rightarrow \infty} \frac{1}{\pi \rho^2} \sum_{\mathcal{P} \in \mathcal{A}(\rho)} -\nu(C_{\mathcal{P}}) \log \nu(C_{\mathcal{P}}) = 0,$$

was shown in [31, Thm. 4], which is in line with the results of [6]. Alternatively, it is an immediate consequence of part (c) above. As a consequence of part (a), the individual diffraction measure of  $V$  according to Theorem 1 coincides with the diffraction measure of the system  $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$  in the sense of [5]. Then, pure point diffraction means pure point dynamical spectrum [5, Thm. 7], and the latter is the group generated by the Fourier–Bohr spectrum; compare [5, Thm. 8] and [8, Prop. 17]. Since the intensity formula of Theorem 1 shows that there are no extinctions, the Fourier–Bohr spectrum here is itself a group, which completes part (b).

It is well known that  $K_\mu = (G, \mathbb{Z}^2, \mu)$  has the same pure point dynamical spectrum as  $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$ ; compare [13] for the details in the case of the square-free integers. In particular, the subgroup of  $\mathbb{T}^2$  given by the points of  $\mathbb{Q}^2 \cap [0, 1)^2$  with square-free denominator can easily be seen to be isomorphic to the direct sum  $\bigoplus_p \mathbb{Z}^2/p\mathbb{Z}^2$ , wherefore it is the Pontryagin dual of the direct product  $G = \prod_p \mathbb{Z}^2/p\mathbb{Z}^2$ ; cf. [32, Ch. 2.2]. By a theorem of von Neumann [40], two ergodic measure-preserving transformations with pure point dynamical spectrum are isomorphic if and only if they have the same dynamical spectrum. This proves part (c), which is a particular instance of the Halmos–von Neumann theorem; cf. [21].

Alternatively, the Kronecker system can be read off from the model set description, which also provides the compact Abelian group. The general formalism is developed in [7], though the torus parametrisation does not immediately apply. Some extra work is required here to establish the precise properties of the measure-theoretic homomorphism onto the compact Abelian group. Diagrammatically, the construction looks like this:

$$\begin{array}{ccccc}
 \mathbb{Z}^2 & \longleftarrow & \mathbb{Z}^2 \times \prod_p (\mathbb{Z}^2)_p & \longrightarrow & \prod_p (\mathbb{Z}^2)_p \\
 \cup & & \cup & & \cup \\
 M & & L & & W \\
 \uparrow & & \uparrow & & \uparrow \\
 x & \longleftrightarrow & (x, \iota(x)) & \longrightarrow & \iota(x)
 \end{array}$$

Here

$$L := \{(x, \iota(x)) \mid x \in \mathbb{Z}^2\} = \{(x, ([x]_p)) \mid x \in \mathbb{Z}^2\}$$

is the natural (diagonal) embedding of  $\mathbb{Z}^2$  into  $\mathbb{Z}^2 \times \prod_p (\mathbb{Z}^2)_p$  and

$$W := \prod_p (\mathbb{Z}^2)_p \setminus \{[0]_p\}$$

satisfies  $W = \partial W$  and has measure  $\mu(W) = \prod_p (1 - \frac{1}{p^2}) = 1/\zeta(2)$  with respect to the normalised Haar measure  $\mu$  on the compact group  $\prod_p (\mathbb{Z}^2)_p$ . Clearly, one has

$$M := M(W) := \{x \in \mathbb{Z}^2 \mid \iota(x) \in W\} = V$$

The above diagram is in fact a *cut and project scheme*:  $L$  is a lattice (a discrete co-compact subgroup) in  $\mathbb{Z}^2 \times \prod_p (\mathbb{Z}^2)_p$  with one-to-one projection onto the first factor and dense projection onto the second factor. This means that  $V$  is a *weak model set* [3]. The corresponding ‘torus’ is

$$\mathbb{T} := (\mathbb{Z}^2 \times \prod_p (\mathbb{Z}^2)_p) / L \simeq \prod_p (\mathbb{Z}^2)_p,$$

with the  $\mathbb{Z}^2$ -action given by addition of  $\iota(x) = ([x]_p)$ . A similar construction with translations from the group  $\mathbb{R}^2$  in mind is given in [10]; see also [37, Ch. 5a].

The so-called *torus parametrisation* [28] is the Borel map

$$\varphi: \mathbb{T} \rightarrow \mathbb{X}_V,$$

given by

$$([y_p]_p) \mapsto M((y_p]_p) + W).$$

Clearly,  $\varphi$  intertwines the  $\mathbb{Z}^2$ -actions. Note that, since  $\mathbb{X}_V = \mathbb{A}$ , one indeed has

$$M((y_p]_p) + W = \mathbb{Z}^2 \setminus \bigcup_p (y_p + p\mathbb{Z}^2) \in \mathbb{X}_V.$$

The map  $\varphi$  fails to be injective. For example, the fibre  $\varphi^{-1}(\emptyset)$  over the empty set  $\emptyset$  is easily seen to be uncountable. Furthermore,  $\varphi$  is not continuous, since, e.g.,  $\varphi(\left[(0,0)_p\right]) = V$  but

$$V \ni (1,0) \notin \varphi(\left[(0,0)_{p_1}, \dots, (0,0)_{p_{n-1}}, (1,0)_{p_n}, (0,0)_{p_{n+1}}, \dots\right]).$$

Nevertheless, employing the ergodicity of the measure  $\nu$ , one can show that  $\varphi$  is in fact a measure-theoretic isomorphism between the two systems; see Appendix B for the details.  $\square$

Let us mention that our approach is complementary to that of [13]. There, ergodicity and pure point spectrum are consequences of determining all eigenfunctions, then concluding via 1 being a simple eigenvalue and via the basis property of the eigenfunctions. Here, we establish the  $\nu$ -genericity of  $V$  and the ergodicity of the measure  $\nu$  and afterwards use the equivalence theorem between pure point dynamical and diffraction spectrum [5, Thm. 7], hence employing the diffraction measure of  $V$  calculated in [10, 31].

### 3. $k$ -FREE LATTICE POINTS

The square-free integers and the visible points of the square lattice are particular cases of the following natural generalisation. Let  $\Lambda \subset \mathbb{R}^n$  be a lattice. The  $k$ -free points  $V = V(\Lambda, k)$  of  $\Lambda$  are then defined by

$$V = \Lambda \setminus \bigcup_{p \text{ prime}} p^k \Lambda.$$

They are the points with the property that the greatest common divisor of their coordinates (in an arbitrary lattice basis) is not divisible by any non-trivial  $k$ th power of an integer. Without restriction, we shall assume that  $\Lambda$  is *unimodular*, i.e.  $|\det(\Lambda)| = 1$ . Moreover, we exclude the trivial case  $n = k = 1$ , where  $V$  consists of just the two points of  $\Lambda$  closest to 0 on either side. On the basis of the results in [10, 31], one can then show analogous versions of any of the above findings. In particular, one has [31, Cor. 1]

$$\text{dens}(V) = \frac{1}{\zeta(nk)}$$

and the result for the diffraction measure  $\hat{\gamma}$  of  $V$  looks as follows. Recall that the *dual* or *reciprocal lattice*  $\Lambda^*$  of  $\Lambda$  is

$$\Lambda^* := \{y \in \mathbb{R}^n \mid y \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

Further, the *denominator* of a point  $\ell$  in the  $\mathbb{Q}$ -span  $\mathbb{Q}\Lambda^*$  of  $\Lambda^*$  is defined as

$$\text{den}(\ell) := \gcd\{m \in \mathbb{N} \mid m\ell \in \Lambda^*\}.$$

**Theorem 7.** [10, Thms. 3 and 5] [31, Thm. 8] *The natural diffraction measure  $\hat{\gamma}$  of the autocorrelation  $\gamma$  of  $V$  exists. It is a positive pure point measure which is translation bounded and supported on the set of points in  $\mathbb{Q}\Lambda^*$  with  $(k+1)$ -free denominator, so*

$$\hat{\gamma} = \sum_{\substack{\ell \in \mathbb{Q}\Lambda^* \\ \text{den}(\ell) \text{ } (k+1)\text{-free}}} I(\ell) \delta_\ell.$$

*In particular,  $I(0) = (1/\zeta(nk))^2$  and when  $0 \neq \ell \in \mathbb{Q}\Lambda^*$  has  $(k+1)$ -free denominator  $\text{den}(\ell)$ , the corresponding intensity is given by*

$$I(\ell) = \left( \frac{1}{\zeta(nk)} \prod_{p \mid \text{den}(\ell)} \frac{1}{p^{nk} - 1} \right)^2.$$

□

Again, the hull

$$\mathbb{X}_V = \overline{\{t + V \mid t \in \Lambda\}}$$

of  $V$  turns out to be just the set of *admissible* subsets of  $\Lambda$ , i.e. subsets  $A$  of  $\Lambda$  with

$$|A_p| < p^{nk}$$

for any prime  $p$ , where  $A_p$  denotes the reduction of  $A$  modulo  $p^k \Lambda$ ; see [31, Thm. 6]. The natural topological dynamical system  $(\mathbb{X}_V, \Lambda)$  has the analogous properties as in the special case discussed above in Theorem 4. In particular, it has positive topological entropy equal to the patch counting entropy of  $V$ , i.e.

$$h_{\text{pc}}(V) = \frac{\log(2)}{\zeta(nk)}$$

by [31, Thm. 3]. For the patch frequencies, one has the following result.

**Theorem 8.** [31, Thms. 1 and 2] *Any  $\rho$ -patch  $\mathcal{P}$  of  $V$  occurs with positive frequency, which is given by*

$$\nu(\mathcal{P}) = \sum_{\mathcal{F} \subset (\Lambda \cap B_\rho(0)) \setminus \mathcal{P}} (-1)^{|\mathcal{F}|} \prod_p \left( 1 - \frac{|(\mathcal{P} \cup \mathcal{F})_p|}{p^{nk}} \right).$$

□

This gives rise to a measure-theoretic dynamical system  $(\mathbb{X}_V, \Lambda, \nu)$  which can be seen, as above, to be ergodic and metrically isomorphic to  $(G, \Lambda, \mu)$ , where  $G$  is the compact Abelian group

$$G = \prod_p \Lambda_p = \prod_p \Lambda/p^k \Lambda$$

on which the lattice  $\Lambda$  acts via addition of  $\iota(x) = ([x]_p)$ . As before,  $\mu$  is the normalised Haar measure on  $G$ . It follows that  $(\mathbb{X}_V, \Lambda, \nu)$  has zero measure entropy; see also [31, Thm. 4]. Again,  $V$  turns out to be  $\nu$ -generic. We thus get the analogous result to Theorem 6 also in this more general setting.

#### 4. $\mathcal{B}$ -FREE LATTICE POINTS

One further generalisation step seems possible as follows. In [18], Lemańczyk et al. studied the dynamical properties of  $\mathcal{B}$ -free systems, i.e.  $\mathcal{B} \subset \{2, 3, \dots\}$  consists of pairwise coprime integers satisfying

$$\sum_{b \in \mathcal{B}} \frac{1}{b} < \infty$$

and the hull  $\mathbb{X}_{\mathcal{B}} = \overline{\{t + V \mid t \in \mathbb{Z}\}}$  is the orbit closure of the set

$$V = \mathbb{Z} \setminus \bigcup_{b \in \mathcal{B}} b\mathbb{Z}$$

of  $\mathcal{B}$ -free numbers (integers with no factor from  $\mathcal{B}$ ). Substituting the one-dimensional lattice  $\mathbb{Z} \subset \mathbb{R}$  by other unimodular lattices  $\Lambda \subset \mathbb{R}^n$  in the above definitions and requiring that  $\sum_{b \in \mathcal{B}} \frac{1}{b^n} < \infty$ , one arrives at  $\mathcal{B}$ -free lattice points  $V$  and the associated topological dynamical systems  $(\mathbb{X}_V, \Lambda)$ . The  $k$ -free lattice points from the previous section then arise from the particular choice  $\mathcal{B} = \{p^k \mid p \text{ prime}\}$ . Since the proofs in [10, 31] do not use special properties

of  $k$ th powers of prime numbers except their pairwise coprimality, the above results carry over to the case of  $\mathcal{B}$ -free lattice points with almost identical proofs. In particular, the density and topological (patch counting) entropy of  $V$  are given by

$$\text{dens}(V) = \prod_{b \in \mathcal{B}} \left(1 - \frac{1}{b^n}\right)$$

and  $h_{\text{pc}}(V) = \log(2) \text{dens}(V)$ , respectively. Again,  $\mathbb{X}_V$  contains the *admissible* subsets  $A$  of  $\Lambda$ , i.e.

$$|A_b| < b^n$$

for any  $b \in \mathcal{B}$ , where  $A_b$  denotes the reduction of  $A$  modulo  $b\Lambda$ . Moreover, the diffraction measure  $\widehat{\gamma}$  of  $V$  exists. It is a pure point measure that is supported on the set of points  $\ell \in \mathbb{Q}\Lambda^*$  with the property that the denominator  $\text{den}(\ell)$  divides a finite product of distinct  $b$ 's, the intensity at such a point being given by

$$I(\ell) = \left( \text{dens}(V) \prod_{\substack{b \in \mathcal{B} \\ \text{gcd}(\text{den}(\ell), b) \neq 1}} \frac{1}{b^n - 1} \right)^2.$$

Both the pure point nature and the intensity formula can be shown by writing  $\widehat{\gamma}$  as a vague limit of diffraction measures of approximating crystallographic systems. A Weierstraß M-test, as in [10], implies that one also has the stronger norm convergence, which preserves the pure point nature of approximating measures in the limit via [3, Thm. 8.4].

Further, the frequency of a  $\rho$ -patch  $\mathcal{P}$  of  $V$  is positive and given by the expression

$$\nu(\mathcal{P}) = \sum_{\mathcal{F} \subset (\Lambda \cap B_\rho(0)) \setminus \mathcal{P}} (-1)^{|\mathcal{F}|} \prod_{b \in \mathcal{B}} \left(1 - \frac{|\mathcal{P} \cup \mathcal{F}|_b}{b^n}\right).$$

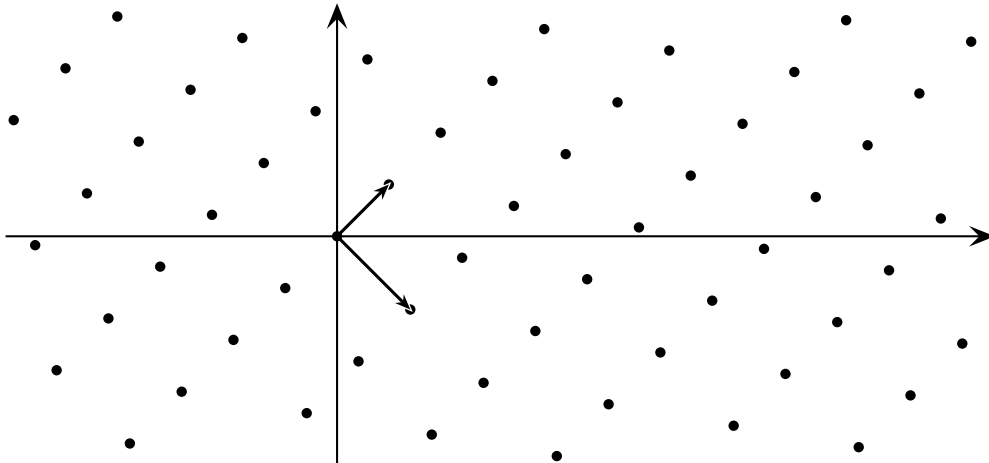
The associated measure-theoretic dynamical system  $(\mathbb{X}_V, \Lambda, \nu)$  can be seen, as above, to be ergodic and metrically isomorphic to  $(G, \Lambda, \mu)$ , where  $G$  is the compact Abelian group

$$G = \prod_{b \in \mathcal{B}} \Lambda_b = \prod_{b \in \mathcal{B}} \Lambda/b\Lambda$$

on which the lattice  $\Lambda$  acts via addition of  $\iota(x) = ([x]_b)$  and  $\mu$  is the normalised Haar measure on  $G$ . Again,  $V$  turns out to be  $\nu$ -generic. Due to the similarity of the structures, we leave all details to the reader.

## 5. $k$ -FREE INTEGERS IN NUMBER FIELDS

Another possible extension is given by the number theoretic setting that was studied by Cellarosi and Vinogradov in [14]. Let  $K$  be an algebraic number field, i.e. a finite extension of  $\mathbb{Q}$ , say of degree  $[K : \mathbb{Q}] = d$ ; see [12, 30] for background material on algebraic number theory. Further, consider the Dedekind domain  $\mathcal{O}_K$  of integers in  $K$  (in particular, any ideal  $0 \neq \mathfrak{a} \neq \mathcal{O}_K$  of  $\mathcal{O}_K$  factors uniquely as a product of prime ideals) and define, for  $k \geq 2$ , the set  $V$  of  $k$ -free integers of  $K$  as the nonzero elements  $a \in \mathcal{O}_K$  with the property that, if  $a$  is not a unit, then the prime ideal factorisation of the corresponding principal ideal  $0 \neq (a) \neq \mathcal{O}_K$  contains no  $k$ th powers of prime ideals, i.e.  $(a) \not\subseteq \mathfrak{p}^k$  for any prime ideal  $\mathfrak{p} \subset \mathcal{O}_K$ . In other


 FIGURE 3. The Minkowski embedding of  $\mathbb{Z}[\sqrt{2}]$  in  $\mathbb{R}^2$ .

words, one has

$$V = V(\mathcal{O}_K, k) = \mathcal{O}_K \setminus \bigcup_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p} \text{ prime ideal}}} \mathfrak{p}^k.$$

It is well known that  $\mathcal{O}_K$  is a free  $\mathbb{Z}$ -module of rank  $d$  and is thus isomorphic to the lattice  $\mathbb{Z}^d$  as a group. In particular, there is a natural isomorphism from  $\mathcal{O}_K$  to a lattice in  $\mathbb{R}^d$ , namely the *Minkowski embedding*; see [12, 30] and [3, Ch. 3.4]. In order to illustrate this, we prefer to discuss the specific real quadratic number field  $K = \mathbb{Q}(\sqrt{2})$  with  $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ . It is well known that  $K$  is a Galois extension and thus has precisely two field automorphisms, namely the identity and the non-trivial automorphism determined by  $\sqrt{2} \mapsto -\sqrt{2}$ . We denote the latter automorphism by  $x \mapsto x'$ . One can then easily check that  $\mathcal{O}_K$  corresponds under the *Minkowski embedding*  $j: K \rightarrow \mathbb{R}^2$ , given by

$$x \mapsto (x, x'),$$

to a non-unimodular lattice  $\mathcal{L}$  in  $\mathbb{R}^2$  with area  $|\det \mathcal{L}| = 2\sqrt{2} = \sqrt{|d_K|}$ , where  $d_K = 8$  is the discriminant of  $K$ . In fact, since  $\mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z}\sqrt{2}$ , one has  $\mathcal{L} = \mathbb{Z}(1, 1) \oplus \mathbb{Z}(\sqrt{2}, -\sqrt{2})$ ; see Figure 3.

Moreover, the image  $j(\mathfrak{a})$  of any ideal  $0 \neq \mathfrak{a} \subset \mathcal{O}_K$  is a lattice in  $\mathbb{R}^2$  with area

$$(2) \quad |\det j(\mathfrak{a})| = 2\sqrt{2}(\mathcal{O}_K : \mathfrak{a}),$$

where  $(\mathcal{O}_K : \mathfrak{a})$  denotes the (finite) subgroup index of  $\mathfrak{a}$  in  $\mathcal{O}_K$ , i.e. the absolute norm  $N(\mathfrak{a})$  of  $\mathfrak{a}$ . Note that the absolute norm is a totally multiplicative function on the set of non-zero ideals of  $\mathcal{O}_K$ . We are thus lead to consider the familiar object

$$j(V) = \mathcal{L} \setminus \bigcup_{\mathfrak{p}} j(\mathfrak{p}^k),$$

where here and below  $\mathfrak{p}$  runs through the prime ideals of  $\mathcal{O}_K$ ; see Fig. 4 for an illustration. Again, the proofs in [10, 31] can be adjusted to obtain similar results as above.

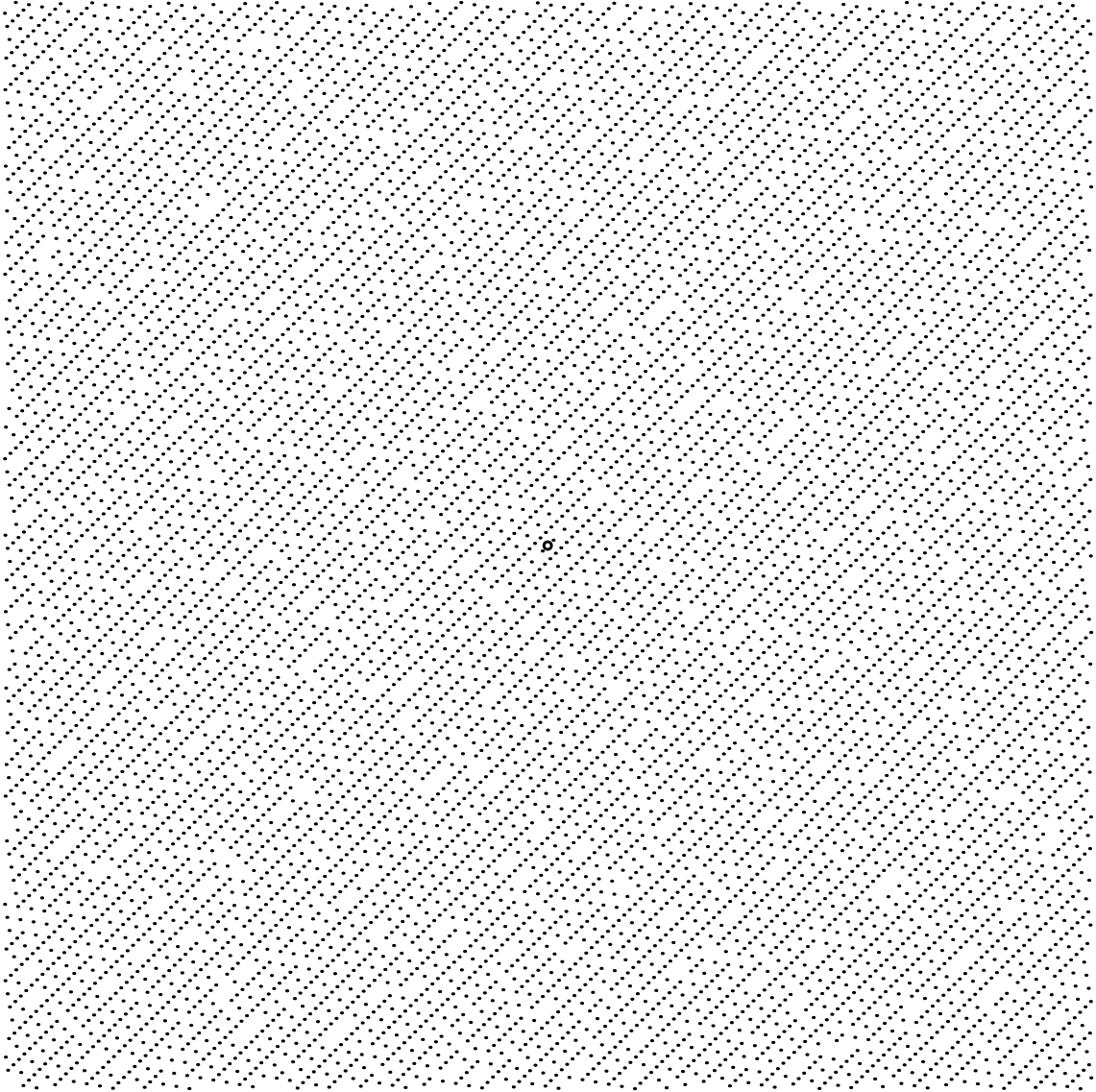


FIGURE 4. A central patch of the Minkowski embedding of the square-free integers  $V$  of  $\mathbb{Q}(\sqrt{2})$ .

Denote by

$$\zeta_K(s) = \zeta_{\mathbb{Q}(\sqrt{2})}(s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ \mathfrak{a} \neq 0 \text{ ideal}}} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

the *Dedekind  $\zeta$ -function* of  $K$ , which converges for all  $s$  with  $\operatorname{Re}(s) > 1$ . Employing Eq. (2), a similar reasoning as in the previous sections now shows that the density and topological

(patch counting) entropy of  $j(V)$  are given by

$$\text{dens}(j(V)) = \frac{1}{2\sqrt{2}} \frac{1}{\zeta_K(k)} = \frac{1}{2\sqrt{2}} \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^k}\right)$$

and  $\log(2) \text{dens}(j(V))$ , respectively. Note that the Chinese Remainder Theorem in its general form says that, given pairwise coprime ideals  $\mathfrak{a}_1, \dots, \mathfrak{a}_r$  in a ring  $\mathcal{O}$  ( $\mathfrak{a}_s + \mathfrak{a}_t = \mathcal{O}$  for  $s \neq t$ ), one has  $\prod_{i=1}^r \mathfrak{a}_i = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_r$  and

$$\mathcal{O} / \prod_{i=1}^r \mathfrak{a}_i \simeq \prod_{i=1}^r \mathcal{O} / \mathfrak{a}_i.$$

Recall that the *dual* or *reciprocal module*  $\mathcal{O}_K^*$  of  $\mathcal{O}_K$  is the fractional ideal

$$\mathcal{O}_K^* := \{y \in K \mid \text{Tr}_{K/\mathbb{Q}}(yx) \in \mathbb{Z} \text{ for all } x \in \mathcal{O}_K\}$$

of  $K$  containing  $\mathcal{O}_K$ , where  $\text{Tr}_{K/\mathbb{Q}}(yx) = yx + y'x'$  is the *trace* of  $yx$ . Then  $j(\mathcal{O}_K^*) = \mathcal{L}^*$  and hence  $j(\mathbb{Q}\mathcal{O}_K^*) = \mathbb{Q}\mathcal{L}^*$ . Here, one calculates that  $\mathcal{O}_K^* = \mathbb{Z}\frac{1}{2} \oplus \mathbb{Z}\frac{\sqrt{2}}{4}$  and thus  $\mathbb{Q}\mathcal{O}_K = \mathbb{Q}\mathcal{O}_K^* = K$  as well as  $\mathbb{Q}\mathcal{L}^* = \mathbb{Q}\mathcal{L}$ . Further, the *denominator* of a point  $\ell$  in  $\mathbb{Q}\mathcal{L}^*$  is defined as the non-zero ideal

$$\text{den}(\ell) := \{x \in \mathcal{O}_K \mid xj^{-1}(\ell) \in \mathcal{O}_K^*\} \subset \mathcal{O}_K.$$

Then, the diffraction measure  $\hat{\gamma}$  of  $j(V)$  is pure point (for the same reason as above) and is supported on the set of points  $\ell \in j(\mathbb{Q}\mathcal{O}_K^*) = \mathbb{Q}\mathcal{L}^*$  with  $(k+1)$ -free denominator  $\text{den}(\ell)$  (i.e., either  $\text{den}(\ell) = \mathcal{O}_K$  or the unique prime ideal factorization of  $\text{den}(\ell)$  contains no  $(k+1)$ th powers), the intensity at such a point being given by

$$I(\ell) = \left( \frac{1}{2\sqrt{2}} \frac{1}{\zeta_K(k)} \prod_{\substack{\mathfrak{p} \\ \text{den}(\ell) \subset \mathfrak{p}}} \frac{1}{N(\mathfrak{p})^k - 1} \right)^2.$$

See Fig. 5 for an illustration, where the restriction of  $\hat{\gamma}$  to a compact region is shown.

With a view to the general case of an algebraic number field  $K$ , the above did not make use of the additional fact that  $\mathbb{Z}[\sqrt{2}]$  is in fact a *Euclidean domain* and thus a *principal ideal domain* (in particular, a *unique factorisation domain*). Here, one has  $N(\mathfrak{a}) = |N_{K/\mathbb{Q}}(a)|$  for  $\mathfrak{a} = (a)$ , where  $N_{K/\mathbb{Q}}(a) = aa'$  is the *norm* of  $a$ . Note that  $\mathbb{Z}[\sqrt{2}]$  is a Euclidean domain with respect to the norm function  $x \mapsto |N_{K/\mathbb{Q}}(x)|$ , i.e.  $a + b\sqrt{2} \mapsto |a^2 - 2b^2|$ . Furthermore, the Dedekind  $\zeta$ -function of  $K$  can be written more explicitly in terms of the usual rational primes, i.e.

$$\zeta_K(s) = \frac{1}{1 - 2^{-s}} \prod_{p \equiv \pm 1(8)} \frac{1}{(1 - p^{-s})^2} \prod_{p \equiv \pm 3(8)} \frac{1}{1 - p^{-2s}};$$

see [9, Eq. (7)]. E.g., one has  $\zeta_K(2) = \frac{\pi^4}{48\sqrt{2}}$ ; see [9, Eq. (58)]. Henceforth, the intensity  $I(\ell)$  can be computed explicitly in terms of the prime elements of  $\mathcal{O}_K$  dividing any generator of the (principal) ideal  $\text{den}(\ell)$ .

Let us finally turn to the associated topological dynamical system  $(\mathbb{X}_V, \mathcal{O}_K) \simeq (\mathbb{X}_{j(V)}, \mathcal{L})$ , where  $\mathbb{X}_V$  resp.  $\mathbb{X}_{j(V)}$  are given as the translation orbit closure of  $V$  resp.  $j(V)$  with respect to

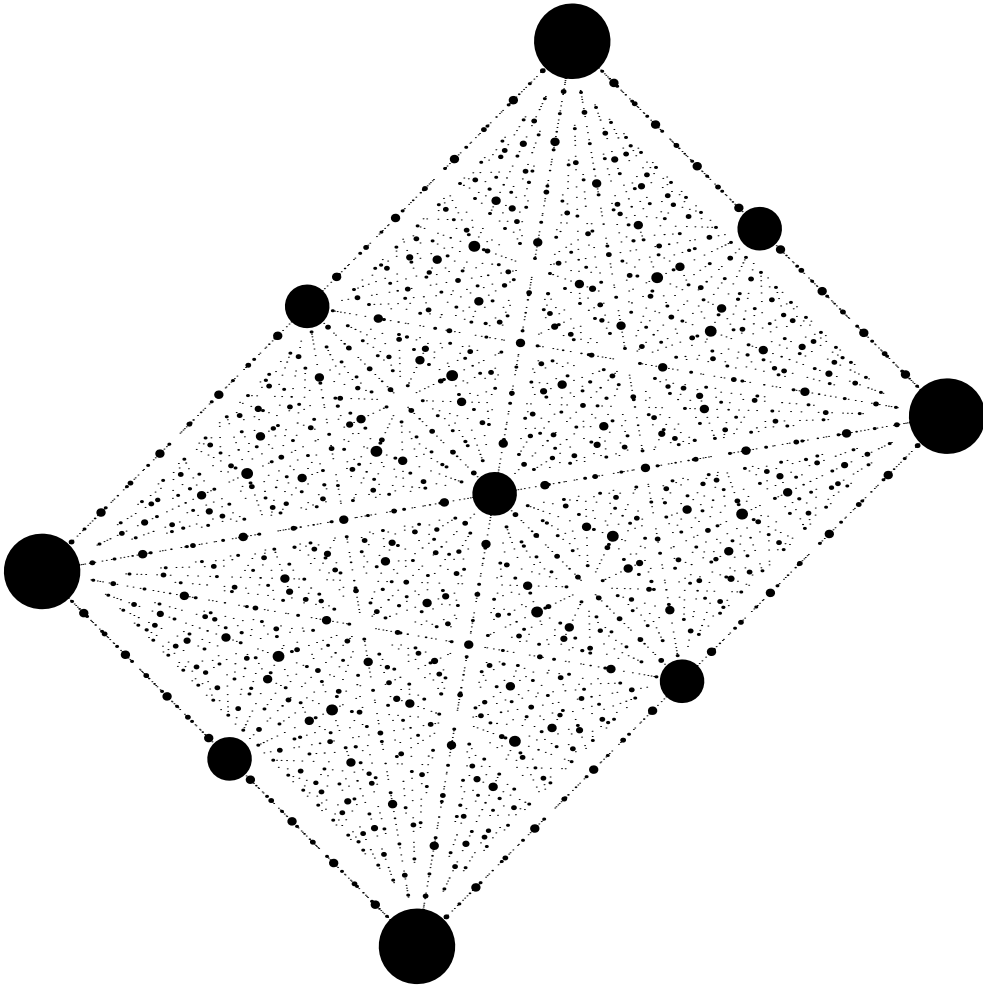


FIGURE 5. Diffraction  $\hat{\gamma}$  of the Minkowski embedding of  $V = V(\mathbb{Z}[\sqrt{2}], 2)$ , with the intensities rescaled by the function  $x \mapsto \sqrt[4]{x}/20$  for better visibility of small intensities. Its lattice of periods is  $\mathcal{L}^*$  as described in the text and the picture shows the intensities inside the closure of one fundamental domain.

the product topology on  $\{0, 1\}^{\mathcal{O}_K}$  resp.  $\{0, 1\}^{\mathcal{L}}$ . Again,  $\mathbb{X}_V$  resp.  $\mathbb{X}_{j(V)}$  can be characterised as the *admissible* subsets  $A$  of  $\mathcal{O}_K$  resp.  $\mathcal{L}$ , i.e.

$$|A_{\mathfrak{p}}| < N(\mathfrak{p})^k$$

for any prime ideal  $\mathfrak{p}$  of  $\mathcal{O}_K$ , where  $A_{\mathfrak{p}}$  denotes the reduction of  $A$  modulo  $\mathfrak{p}^k$  resp.  $j(\mathfrak{p}^k)$ . Further, the frequency of a  $\rho$ -patch  $\mathcal{P}$  of  $j(V)$  is positive and given by the expression

$$\nu(\mathcal{P}) = \sum_{\mathcal{F} \subset (\mathcal{L} \cap B_{\rho}(0)) \setminus \mathcal{P}} (-1)^{|\mathcal{F}|} \prod_{\mathfrak{p}} \left( 1 - \frac{|(\mathcal{P} \cup \mathcal{F})_{\mathfrak{p}}|}{N(\mathfrak{p})^k} \right).$$

The associated measure-theoretic dynamical system  $(\mathbb{X}_{j(V)}, \mathcal{L}, \nu) \simeq (\mathbb{X}_V, \mathcal{O}_K, \nu)$  can be seen, as above, to be ergodic and metrically isomorphic to  $(G, \mathcal{L}, \mu)$ , where  $G$  is the compact

Abelian group

$$G = \prod_{\mathfrak{p}} \mathcal{L}_{\mathfrak{p}} = \prod_{\mathfrak{p}} \mathcal{L}/j(\mathfrak{p}^k) \simeq \prod_{\mathfrak{p}} \mathcal{O}_K/\mathfrak{p}^k = \prod_{\mathfrak{p}} (\mathcal{O}_K)_{\mathfrak{p}}$$

on which the lattice  $\mathcal{L}$  (resp. the group  $\mathcal{O}_K$ ) acts via addition of  $\iota(j(x)) = ([j(x)]_{\mathfrak{p}})$  (resp.  $\iota(x) = ([x]_{\mathfrak{p}})$ ), where  $x \in \mathcal{O}_K$ , and  $\mu$  is the normalised Haar measure on  $G$ . As above,  $V$  turns out to be  $\nu$ -generic.

Since none of the above uses special properties of the quadratic field  $\mathbb{Q}(\sqrt{2})$ , similar results hold for the general case of an arbitrary algebraic number field  $K$ . Moreover, even the extension to  $\mathcal{B}$ -free integers in  $K$ , i.e.

$$V = V_{\mathcal{O}_K} = \mathcal{O}_K \setminus \bigcup_{\mathfrak{b} \in \mathcal{B}} \mathfrak{b},$$

where  $\mathcal{B}$  is a set of pairwise coprime ideals  $\mathfrak{b} \subsetneq \mathcal{O}_K$  satisfying

$$\sum_{\mathfrak{b} \in \mathcal{B}} \frac{1}{N(\mathfrak{b})} < \infty,$$

should be possible.

#### APPENDIX A. ERGODICITY OF THE PATCH FREQUENCY MEASURE

Below, we shall only treat the paradigmatic case  $V = V_{\mathbb{Z}^2}$  from Section 2. For a  $\rho$ -patch  $\mathcal{P} \in \mathcal{A}(\rho)$  of  $V$  (i.e.  $P \subset B_{\rho}(0) \cap \mathbb{Z}^2$  with  $P \in \mathbb{A}$ ), denote by  $B_{\mathcal{P}}$  the set of elements of  $\mathbb{X}_V = \mathbb{A}$  whose  $\rho$ -patch at 0 contains  $\mathcal{P}$ . One readily checks that the sets of type  $B_{\mathcal{P}}$  form a semi-algebra that also generates the Borel  $\sigma$ -algebra on  $\mathbb{X}_V$ . In fact, one has

$$C_{\mathcal{P}} = B_{\mathcal{P}} \setminus \bigcup_{\substack{\mathcal{Q} \in \mathcal{A}(\rho) \\ \mathcal{P} \subsetneq \mathcal{Q}}} B_{\mathcal{Q}},$$

and

$$(A1) \quad B_{\mathcal{P}} = \bigcup_{\substack{\mathcal{Q} \in \mathcal{A}(\rho) \\ \mathcal{P} \subset \mathcal{Q}}} C_{\mathcal{Q}}.$$

**Corollary 1.** *For any  $\rho$ -patch  $\mathcal{P}$  of  $V$ , one has*

$$\nu(B_{\mathcal{P}}) = \prod_p \left( 1 - \frac{|\mathcal{P}_p|}{p^2} \right).$$

*Proof.* Let  $\mathcal{P}$  be a  $\rho$ -patch of  $V$ . By Theorem 5 and the definition of  $\nu$ , (A1) implies that

$$\begin{aligned}
\nu(B_{\mathcal{P}}) &= \sum_{\substack{\mathcal{Q} \in \mathcal{A}(\rho) \\ \mathcal{P} \subset \mathcal{Q}}} \nu(C_{\mathcal{Q}}) \\
&= \sum_{\substack{\mathcal{Q} \in \mathcal{A}(\rho) \\ \mathcal{P} \subset \mathcal{Q}}} \sum_{\mathcal{Q} \subset \mathcal{F} \subset \mathbb{Z}^2 \cap B_{\rho}(0)} (-1)^{|\mathcal{F} \setminus \mathcal{Q}|} \prod_p \left(1 - \frac{|\mathcal{F}_p|}{p^2}\right) \\
&= \sum_{\mathcal{P} \subset \mathcal{Q} \subset \mathcal{F} \subset \mathbb{Z}^2 \cap B_{\rho}(0)} (-1)^{|\mathcal{F} \setminus \mathcal{Q}|} \prod_p \left(1 - \frac{|\mathcal{F}_p|}{p^2}\right) \\
&= \prod_p \left(1 - \frac{|\mathcal{P}_p|}{p^2}\right) + \sum_{\substack{\mathcal{P} \subset \mathcal{Q} \subset \mathcal{F} \subset \mathbb{Z}^2 \cap B_{\rho}(0) \\ \mathcal{F} \neq \mathcal{P}}} (-1)^{|\mathcal{F} \setminus \mathcal{Q}|} \prod_p \left(1 - \frac{|\mathcal{F}_p|}{p^2}\right) \\
&= \prod_p \left(1 - \frac{|\mathcal{P}_p|}{p^2}\right),
\end{aligned}$$

since, for fixed  $\mathcal{F} \subset \mathbb{Z}^2 \cap B_{\rho}(0)$  with  $\mathcal{P} \subsetneq \mathcal{F}$ , one indeed has

$$\begin{aligned}
\sum_{\mathcal{P} \subset \mathcal{Q} \subset \mathcal{F}} (-1)^{|\mathcal{F} \setminus \mathcal{Q}|} \prod_p \left(1 - \frac{|\mathcal{F}_p|}{p^2}\right) &= \prod_p \left(1 - \frac{|\mathcal{F}_p|}{p^2}\right) \sum_{\mathcal{P} \subset \mathcal{Q} \subset \mathcal{F}} (-1)^{|\mathcal{F} \setminus \mathcal{Q}|} \\
&= \prod_p \left(1 - \frac{|\mathcal{F}_p|}{p^2}\right) \sum_{\mathcal{R} \subset \mathcal{F} \setminus \mathcal{P}} (-1)^{|\mathcal{R}|} \\
&= \prod_p \left(1 - \frac{|\mathcal{F}_p|}{p^2}\right) \sum_{i=0}^{|\mathcal{F} \setminus \mathcal{P}|} \binom{|\mathcal{F} \setminus \mathcal{P}|}{i} (-1)^i \\
&= 0,
\end{aligned}$$

where the last equality follows from the binomial theorem since  $\mathcal{F} \setminus \mathcal{P} \neq \emptyset$ .  $\square$

For a natural number  $m$ , finite subsets  $\mathcal{P}$  and  $\mathcal{Q}$  of  $\mathbb{Z}^2$  and  $S \subset \mathcal{P}_m$ , we set

$$\mathcal{Q}_{S, \mathcal{P}}^m = \left( \bigcap_{s \in S} \mathcal{Q}_m - s \right) \setminus \left( \bigcup_{s \in \mathcal{P}_m \setminus S} \mathcal{Q}_m - s \right),$$

i.e. the set of elements of  $(\mathbb{Z}^2)_m$  that lie in  $\mathcal{Q}_m - s$  precisely for those  $s \in \mathcal{P}_m$  with  $s \in S \subset \mathcal{P}_m$ , in particular  $\mathcal{Q}_{\emptyset, \mathcal{P}}^m = (\mathbb{Z}^2)_m \setminus (\mathcal{Q}_m - \mathcal{P}_m)$ . With  $q_{S, \mathcal{P}}^m = |\mathcal{Q}_{S, \mathcal{P}}^m|$ , one then has  $q_{\emptyset, \mathcal{P}}^m = m^2 - |\mathcal{Q}_m - \mathcal{P}_m|$  and, since the difference set  $\mathcal{Q}_m - \mathcal{P}_m$  is the disjoint union of the various  $\mathcal{Q}_{S, \mathcal{P}}^m$ , where one has  $\emptyset \neq S \subset \mathcal{P}_m$ ,

$$(\text{A2}) \quad \sum_{S \subset \mathcal{P}_m} q_{S, \mathcal{P}}^m = m^2.$$

Note that the following two lemmas also hold for any finite subsets  $\mathcal{P}, \mathcal{Q}$  of an arbitrary finite group  $G$  instead of  $G = (\mathbb{Z}^2)_m$ .

**Lemma 2.** *For any finite subsets  $\mathcal{P}$  and  $\mathcal{Q}$  of  $\mathbb{Z}^2$  and any natural number  $m$ , one has*

$$\sum_{S \subset \mathcal{P}_m} |S| q_{S, \mathcal{P}}^m = |\mathcal{P}_m| |\mathcal{Q}_m|.$$

*Proof.* By induction on  $|\mathcal{P}_m|$ . For the induction basis  $|\mathcal{P}_m| = 0$ , the assertion is trivially true. For the induction step, consider  $\mathcal{P}$  with  $|\mathcal{P}_m| > 0$  and fix an element, say  $*$ , of  $\mathcal{P}_m$ . It follows that

$$\begin{aligned} \sum_{S \subset \mathcal{P}_m} |S| q_{S, \mathcal{P}}^m &= \sum_{S \subset \mathcal{P}_m \setminus \{*\}} |S| q_{S, \mathcal{P}}^m + \sum_{S \subset \mathcal{P}_m \setminus \{*\}} (|S| + 1) q_{S \cup \{*\}, \mathcal{P}}^m \\ &= \sum_{S \subset \mathcal{P}_m \setminus \{*\}} |S| (q_{S, \mathcal{P}}^m + q_{S \cup \{*\}, \mathcal{P}}^m) + \sum_{S \subset \mathcal{P}_m \setminus \{*\}} q_{S \cup \{*\}, \mathcal{P}}^m \\ &= \sum_{S \subset \mathcal{P}_m \setminus \{*\}} |S| (q_{S, \mathcal{P}}^m + q_{S \cup \{*\}, \mathcal{P}}^m) + |\mathcal{Q}_m|, \end{aligned}$$

since

$$\begin{aligned} \sum_{S \subset \mathcal{P}_m \setminus \{*\}} q_{S \cup \{*\}, \mathcal{P}}^m &= \sum_{S \subset \mathcal{P}_m \setminus \{*\}} \left| \left( (\mathcal{Q}_m - \{*\}) \cap \left( \bigcap_{s \in S} \mathcal{Q}_m - s \right) \right) \setminus \left( \bigcup_{s \in \mathcal{P}_m \setminus (S \cup \{*\})} \mathcal{Q}_m - s \right) \right| \\ &= \sum_{S \subset \mathcal{P}_m \setminus \{*\}} \left| (\mathcal{Q}_m - \{*\}) \cap \left( \left( \bigcap_{s \in S} \mathcal{Q}_m - s \right) \setminus \left( \bigcup_{s \in (\mathcal{P}_m \setminus \{*\}) \setminus S} \mathcal{Q}_m - s \right) \right) \right| \\ &= |\mathcal{Q}_m - \{*\}| \\ &= |\mathcal{Q}_m|. \end{aligned}$$

Furthermore, for  $S \subset \mathcal{P}_m \setminus \{*\}$ , one has

$$\begin{aligned} &\mathcal{Q}_{S \cup \{*\}, \mathcal{P}}^m \cup \mathcal{Q}_{S, \mathcal{P}}^m \\ &= (\mathcal{Q}_m - \{*\}) \cap \left( \left( \bigcap_{s \in S} \mathcal{Q}_m - s \right) \setminus \left( \bigcup_{s \in (\mathcal{P}_m \setminus \{*\}) \setminus S} \mathcal{Q}_m - s \right) \right) \dot{\cup} \\ &\quad \left( \bigcap_{s \in S} \mathcal{Q}_m - s \right) \setminus (\mathcal{Q}_m - \{*\}) \cap \left( \left( \bigcap_{s \in S} \mathcal{Q}_m - s \right) \setminus \left( \bigcup_{s \in (\mathcal{P}_m \setminus \{*\}) \setminus S} \mathcal{Q}_m - s \right) \right) \\ &= \left( \bigcap_{s \in S} \mathcal{Q}_m - s \right) \setminus \left( \bigcup_{s \in (\mathcal{P}_m \setminus \{*\}) \setminus S} \mathcal{Q}_m - s \right) \\ &= \mathcal{Q}_{S, \mathcal{P} \setminus \{*\}}^m \end{aligned}$$

wherefore, by induction hypothesis,

$$\sum_{S \subset \mathcal{P}_m \setminus \{*\}} |S| (q_{S, \mathcal{P}}^m + q_{S \cup \{*\}, \mathcal{P}}^m) = \sum_{S \subset \mathcal{P}_m \setminus \{*\}} |S| q_{S, \mathcal{P} \setminus \{*\}}^m = (|\mathcal{P}_m| - 1) |\mathcal{Q}_m|.$$

This completes the proof.  $\square$

**Lemma 3.** For any finite subsets  $\mathcal{P}$  and  $\mathcal{Q}$  of  $\mathbb{Z}^2$  and square-free  $d$ , one has

$$\sum_{\substack{(\nu_p)_{p|d} \\ 0 \leq \nu_p \leq |\mathcal{P}_p|}} \prod_{p|d} \left( (\nu_p + |\mathcal{Q}_p|) \sum_{\substack{S \subset \mathcal{P}_p \\ |S| = |\mathcal{P}_p| - \nu_p}} q_{S, \mathcal{P}}^p \right) = \prod_{p|d} (p^2 |\mathcal{P}_p| + p^2 |\mathcal{Q}_p| - |\mathcal{P}_p| |\mathcal{Q}_p|).$$

*Proof.* This is proved by induction on the number  $\omega(d)$  of prime factors of  $d$ . For the induction basis  $\omega(d) = 1$ , say  $d = p$ , note that by (A2) and Lemma 2 one indeed has

$$\begin{aligned} \sum_{0 \leq \nu_p \leq |\mathcal{P}_p|} \left( (\nu_p + |\mathcal{Q}_p|) \sum_{\substack{S \subset \mathcal{P}_p \\ |S| = |\mathcal{P}_p| - \nu_p}} q_{S, \mathcal{P}}^p \right) &= \left( |\mathcal{Q}_p| \sum_{S \subset \mathcal{P}_p} q_{S, \mathcal{P}}^p \right) + \sum_{S \subset \mathcal{P}_p} (|\mathcal{P}_p| - |S|) q_{S, \mathcal{P}}^p \\ &= |\mathcal{Q}_p| p^2 + |\mathcal{P}_p| p^2 - |\mathcal{P}_p| |\mathcal{Q}_p|. \end{aligned}$$

For the induction step, consider  $d$  with  $\omega(d) = r + 1$ , where  $r \geq 1$ , say  $d = p_1 \dots p_r p_{r+1}$ . Then

$$\begin{aligned} &\sum_{0 \leq \nu_p \leq |\mathcal{P}_p|} \prod_{\substack{(\nu_p)_p | d \\ p | d}} \left( (\nu_p + |\mathcal{Q}_p|) \sum_{\substack{S \subset \mathcal{P}_p \\ |S| = |\mathcal{P}_p| - \nu_p}} q_{S, \mathcal{P}}^p \right) \\ &= \sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} \sum_{\substack{(\nu_p)_p | d \\ 0 \leq \nu_p \leq |\mathcal{P}_p| \\ \nu_{p_{r+1}} = i}} (i + |\mathcal{Q}_{p_{r+1}}|) \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S| = |\mathcal{P}_{p_{r+1}}| - i}} q_{S, \mathcal{P}}^{p_{r+1}} \prod_{p | \frac{d}{p_{r+1}}} \left( (\nu_p + |\mathcal{Q}_p|) \sum_{\substack{S \subset \mathcal{P}_p \\ |S| = |\mathcal{P}_p| - \nu_p}} q_{S, \mathcal{P}}^p \right) \\ &= \sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} (i + |\mathcal{Q}_{p_{r+1}}|) \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S| = |\mathcal{P}_{p_{r+1}}| - i}} q_{S, \mathcal{P}}^{p_{r+1}} \sum_{\substack{(\nu_p)_p | \frac{d}{p_{r+1}} \\ 0 \leq \nu_p \leq |\mathcal{P}_p|}} \prod_{p | \frac{d}{p_{r+1}}} \left( (\nu_p + |\mathcal{Q}_p|) \sum_{\substack{S \subset \mathcal{P}_p \\ |S| = |\mathcal{P}_p| - \nu_p}} q_{S, \mathcal{P}}^p \right), \end{aligned}$$

which is, by induction hypothesis,

$$\begin{aligned} &\sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} (i + |\mathcal{Q}_{p_{r+1}}|) \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S| = |\mathcal{P}_{p_{r+1}}| - i}} q_{S, \mathcal{P}}^{p_{r+1}} \prod_{p | \frac{d}{p_{r+1}}} (p^2 |\mathcal{P}_p| + p^2 |\mathcal{Q}_p| - |\mathcal{P}_p| |\mathcal{Q}_p|) \\ &= \prod_{p | \frac{d}{p_{r+1}}} (p^2 |\mathcal{P}_p| + p^2 |\mathcal{Q}_p| - |\mathcal{P}_p| |\mathcal{Q}_p|) \sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S| = |\mathcal{P}_{p_{r+1}}| - i}} (i + |\mathcal{Q}_{p_{r+1}}|) q_{S, \mathcal{P}}^{p_{r+1}}. \end{aligned}$$

It thus remains to show that

$$\sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S| = |\mathcal{P}_{p_{r+1}}| - i}} (i + |\mathcal{Q}_{p_{r+1}}|) q_{S, \mathcal{P}}^{p_{r+1}} = p_{r+1}^2 |\mathcal{P}_{p_{r+1}}| + p_{r+1}^2 |\mathcal{Q}_{p_{r+1}}| - |\mathcal{P}_{p_{r+1}}| |\mathcal{Q}_{p_{r+1}}|,$$

which is clear since, by (A2) and Lemma 2 again, one has

$$\begin{aligned}
 & \sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S|=|\mathcal{P}_{p_{r+1}}|-i}} (i + |\mathcal{Q}_{p_{r+1}}|) q_{S, \mathcal{P}}^{p_{r+1}} \\
 &= \sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S|=|\mathcal{P}_{p_{r+1}}|-i}} i q_{S, \mathcal{P}}^{p_{r+1}} + \sum_{i=0}^{|\mathcal{P}_{p_{r+1}}|} \sum_{\substack{S \subset \mathcal{P}_{p_{r+1}} \\ |S|=|\mathcal{P}_{p_{r+1}}|-i}} |\mathcal{Q}_{p_{r+1}}| q_{S, \mathcal{P}}^{p_{r+1}} \\
 &= \sum_{S \subset \mathcal{P}_{p_{r+1}}} (|\mathcal{P}_{p_{r+1}}| - |S|) q_{S, \mathcal{P}}^{p_{r+1}} + |\mathcal{Q}_{p_{r+1}}| \sum_{S \subset \mathcal{P}_{p_{r+1}}} q_{S, \mathcal{P}}^{p_{r+1}} \\
 &= |\mathcal{P}_{p_{r+1}}| p_{r+1}^2 - |\mathcal{P}_{p_{r+1}}| |\mathcal{Q}_{p_{r+1}}| + |\mathcal{Q}_{p_{r+1}}| p_{r+1}^2.
 \end{aligned}$$

This completes the proof.  $\square$

The proof of the main result below needs the well-known estimate

$$(A3) \quad |B_\rho(x) \cap \mathbb{Z}^2| = \pi \rho^2 + O(\rho) + O(1),$$

approximating the number of points of  $\mathbb{Z}^2$  in a large ball  $B_\rho(x)$  (the last error term being required only when  $\rho^2 < 1$ ). This is obtained by dividing  $B_\rho(x)$  into fundamental regions for  $\mathbb{Z}^2$ , each of volume 1 and containing one point of  $\mathbb{Z}^2$ , with the error terms arising from fundamental regions that overlap the boundary of  $B_\rho(x)$ . A more precise version is given as [10, Prop. 1].

**Theorem 9.** *The measure  $\nu$  is ergodic.*

*Proof.* We have to show that

$$\nu(B_{\mathcal{P}}) \nu(B_{\mathcal{Q}}) = \lim_{R \rightarrow \infty} \frac{1}{\pi R^2} \sum_{x \in B_R(0) \cap \mathbb{Z}^2} \nu(x + B_{\mathcal{P}} \cap B_{\mathcal{Q}})$$

for any finite admissible subsets  $\mathcal{P}$  and  $\mathcal{Q}$  of  $\mathbb{Z}^2$ ; cf. [41, Thm. 1.17(i)]. By Corollary 1, the latter is equivalent to

$$\prod_p \left(1 - \frac{|\mathcal{P}_p|}{p^2}\right) \prod_p \left(1 - \frac{|\mathcal{Q}_p|}{p^2}\right) = \lim_{R \rightarrow \infty} \frac{1}{\pi R^2} \sum_{x \in B_R(0) \cap \mathbb{Z}^2} \nu(x + B_{\mathcal{P}} \cap B_{\mathcal{Q}}).$$

Let  $\mu$  denote the Möbius function for the rest of this proof. The left hand side is equal to

$$\begin{aligned}
 \prod_p \left(1 - \frac{|\mathcal{P}_p|}{p^2} - \frac{|\mathcal{Q}_p|}{p^2} + \frac{|\mathcal{P}_p| |\mathcal{Q}_p|}{p^4}\right) &= \prod_p \left(1 - \left(\frac{|\mathcal{P}_p|}{p^2} + \frac{|\mathcal{Q}_p|}{p^2} - \frac{|\mathcal{P}_p| |\mathcal{Q}_p|}{p^4}\right)\right) \\
 &= \sum_{d \text{ square-free}} \mu(d) \prod_{p|d} \left(\frac{|\mathcal{P}_p|}{p^2} + \frac{|\mathcal{Q}_p|}{p^2} - \frac{|\mathcal{P}_p| |\mathcal{Q}_p|}{p^4}\right) \\
 &= \sum_{d \text{ square-free}} \frac{\mu(d)}{d^2} \prod_{p|d} \left(|\mathcal{P}_p| + |\mathcal{Q}_p| - \frac{|\mathcal{P}_p| |\mathcal{Q}_p|}{p^2}\right).
 \end{aligned}$$

For a fixed  $R > 0$ , the right hand side is

$$\begin{aligned}
& \frac{1}{\pi R^2} \sum_{x \in B_R(0) \cap \mathbb{Z}^2} \prod_p \left( 1 - \frac{|(\mathcal{Q} \cup x + \mathcal{P})_p|}{p^2} \right) \\
&= \frac{1}{\pi R^2} \sum_{x \in B_R(0) \cap \mathbb{Z}^2} \sum_{d \text{ square-free}} \mu(d) \prod_{p|d} \frac{|(\mathcal{Q} \cup x + \mathcal{P})_p|}{p^2} \\
&= \frac{1}{\pi R^2} \sum_{d \text{ square-free}} \mu(d) \sum_{x \in B_R(0) \cap \mathbb{Z}^2} \prod_{p|d} \frac{|(\mathcal{Q} \cup x + \mathcal{P})_p|}{p^2} \\
&= \frac{1}{\pi R^2} \sum_{d \text{ square-free}} \mu(d) \sum_{\substack{(\nu_p)_{p|d} \\ |\mathcal{Q}_p| \leq \nu_p \leq |\mathcal{Q}_p| + |\mathcal{P}_p|}} \sum_{\substack{x \in B_R(0) \cap \mathbb{Z}^2 \\ \forall p|d: |(\mathcal{Q} \cup x + \mathcal{P})_p| = \nu_p}} \prod_{p|d} \frac{\nu_p}{p^2} \\
&= \frac{1}{\pi R^2} \sum_{d \text{ square-free}} \mu(d) \sum_{\substack{(\nu_p)_{p|d} \\ |\mathcal{Q}_p| \leq \nu_p \leq |\mathcal{Q}_p| + |\mathcal{P}_p|}} \prod_{p|d} \frac{\nu_p}{p^2} \sum_{\substack{x \in B_R(0) \cap \mathbb{Z}^2 \\ \forall p|d: |(\mathcal{Q} \cup x + \mathcal{P})_p| = \nu_p}} 1 \\
&= \frac{1}{\pi R^2} \sum_{d \text{ square-free}} \frac{\mu(d)}{d^2} \sum_{\substack{(\nu_p)_{p|d} \\ |\mathcal{Q}_p| \leq \nu_p \leq |\mathcal{Q}_p| + |\mathcal{P}_p|}} \prod_{p|d} \nu_p \sum_{\substack{x \in B_R(0) \cap \mathbb{Z}^2 \\ \forall p|d: |(\mathcal{Q} \cup x + \mathcal{P})_p| = \nu_p}} 1.
\end{aligned}$$

By [10, Prop. 1] and Proposition 2 with  $\mathbb{Z}^2$  replaced by the lattices  $p\mathbb{Z}^2$ , where  $p \mid d$ , one obtains the estimate

$$\left( \frac{\pi R^2}{d^2} + O\left( \left( \frac{R^2}{d^2} \right)^{1-1/2} \right) + O(1) \right) \prod_{p|d} |\{x_p \in (\mathbb{Z}^2)_p : |(\mathcal{Q} \cup x + \mathcal{P})_p| = \nu_p\}|$$

for the inner sum. Substituting this in the above expression and letting  $R$  tend to infinity, one obtains

$$\begin{aligned}
& \sum_{d \text{ square-free}} \frac{\mu(d)}{d^4} \sum_{\substack{(\nu_p)_{p|d} \\ |\mathcal{Q}_p| \leq \nu_p \leq |\mathcal{Q}_p| + |\mathcal{P}_p|}} \prod_{p|d} (\nu_p |\{x_p \in (\mathbb{Z}^2)_p : |(\mathcal{Q} \cup x + \mathcal{P})_p| = \nu_p\}|) \\
&= \sum_{d \text{ square-free}} \frac{\mu(d)}{d^4} \sum_{\substack{(\nu_p)_{p|d} \\ 0 \leq \nu_p \leq |\mathcal{P}_p|}} \prod_{p|d} ((\nu_p + |\mathcal{Q}_p|) |\{x_p \in (\mathbb{Z}^2)_p : |(\mathcal{Q} \cup x + \mathcal{P})_p| = \nu_p + |\mathcal{Q}_p|\}|)
\end{aligned}$$

and the inner product can be rewritten as

$$\begin{aligned}
& \prod_{p|d} \left( (\nu_p + |\mathcal{Q}_p|) |\{x_p \in (\mathbb{Z}^2)_p : x_p \in \mathcal{Q}_{S, \mathcal{P}}^p \text{ for } S \subset \mathcal{P}_p \text{ with } |S| = |\mathcal{P}_p| - \nu_p\}| \right) \\
&= \prod_{p|d} \left( (\nu_p + |\mathcal{Q}_p|) \sum_{\substack{S \subset \mathcal{P}_p \\ |S| = |\mathcal{P}_p| - \nu_p}} q_{S, \mathcal{P}}^p \right).
\end{aligned}$$

Thus, in order to prove the claim, it suffices to show, for square-free  $d$ , the identity

$$\sum_{\substack{(\nu_p)_{p|d} \\ 0 \leq \nu_p \leq |\mathcal{P}_p|}} \prod_{p|d} \left( (\nu_p + |\mathcal{Q}_p|) \sum_{\substack{S \subset \mathcal{P}_p \\ |S| = |\mathcal{P}_p| - \nu_p}} q_{S, \mathcal{P}}^p \right) = \prod_{p|d} (p^2 |\mathcal{P}_p| + p^2 |\mathcal{Q}_p| - |\mathcal{P}_p| |\mathcal{Q}_p|),$$

which is just the content of Lemma 3.  $\square$

APPENDIX B. ISOMORPHISM BETWEEN  $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$  AND  $(\prod_p(\mathbb{Z}^2)_p, \mathbb{Z}^2, \mu)$

Let  $\mathbb{A}_1$  be the Borel subset of  $\mathbb{X}_V = \mathbb{A}$  consisting of the elements  $X \in \mathbb{X}_V$  that satisfy

$$|X_p| = p^2 - 1$$

for any prime  $p$ , i.e.  $X$  misses exactly one coset of  $p\mathbb{Z}^2$  in  $\mathbb{Z}^2$ . There is a natural Borel map

$$\theta: \mathbb{A}_1 \rightarrow \mathbb{T},$$

given by  $X \mapsto ([y_p]_p)$ , where  $[y_p]_p$  is uniquely determined by  $[y_p]_p \notin X_p$ . Note that  $\theta$  fails to be continuous or injective. Clearly,  $\mathbb{A}_1$  is  $\mathbb{Z}^2$ -invariant and  $\theta$  intertwines the  $\mathbb{Z}^2$ -actions. Note also that, for  $X \in \mathbb{A}_1$ , one clearly has

$$(A4) \quad X \subset \varphi(\theta(X)).$$

**Lemma 4.** *One has  $V \in \mathbb{A}_1$ .*

*Proof.* Fix a prime number  $p$  and choose a set  $A$  of  $p^2 - 1$  elements of  $\mathbb{Z}^2$  such that  $|A_p| = p^2 - 1$ . By the Chinese Remainder Theorem (Prop. 2), we may assume that  $A_{p'} = \{(0, 0)_{p'}\}$  for the finitely many primes  $p' < p$ . Then,  $A \in \mathbb{A}$  and, by Theorem 2, there is a translation  $t \in \mathbb{Z}^2$  such that  $t + A \subset V$ . Since

$$p^2 - 1 = |(t + A)_p| \leq |V_p| \leq p^2 - 1,$$

the assertion follows.  $\square$

**Lemma 5.** *One has  $\nu(\mathbb{A}_1) = 1$ .*

*Proof.* The ergodicity of the full measure  $\nu$  (Thms. 5 and 6(b)) implies that  $\nu$ -almost every  $X \in \mathbb{X}_V$  has a dense  $\mathbb{Z}^2$ -orbit; compare [41, Thm. 1.7]. It follows from Lemma 4 that

$$\{X \in \mathbb{X}_V \mid X \text{ has a dense } \mathbb{Z}^2\text{-orbit}\} \subset \mathbb{A}_1.$$

This inclusion is due to the fact that, if  $X$  has a dense orbit, then in particular  $V \in \mathbb{A}_1$  is an element of the orbit closure of  $X$ . Since, for any prime  $p$ , representatives of the  $p^2 - 1$  different elements of  $V_p$  in  $V$  can be chosen within a finite distance from the origin, it is clear from the definition of the topology on  $\mathbb{X}_V$  that there is a translation  $t \in \mathbb{Z}^2$  such that  $t + X$  contains these representatives. Thus

$$p^2 - 1 \geq |X_p| = |(t + X)_p| \geq |V_p| = p^2 - 1$$

and the assertion follows.  $\square$

Hence the push-forward measure of  $\nu$  to  $\mathbb{T}$  by the map  $\theta$  is a  $\mathbb{Z}^2$ -invariant probability measure and thus is the normalised Haar measure  $\mu$ . Set

$$\mathbb{T}_1 := \varphi^{-1}(\mathbb{A}_1),$$

which is a  $\mathbb{Z}^2$ -invariant Borel set with  $\varphi(\mathbb{T}_1) \subset \mathbb{A}_1$ . In particular, this shows that  $\theta \circ \varphi|_{\mathbb{T}_1} = \text{id}_{\mathbb{T}_1}$  and thus the restriction of  $\varphi$  to  $\mathbb{T}_1$  and the restriction of  $\theta$  to  $\varphi(\mathbb{T}_1)$  are injective.

**Lemma 6.** *One has  $\theta(\mathbb{A}_1) = \mathbb{T}_1$ .*

*Proof.* It suffices to prove the inclusion  $\theta(\mathbb{A}_1) \subset \mathbb{T}_1$ , since this immediately yields the assertion due to  $\mathbb{T}_1 = \theta(\varphi(\mathbb{T}_1)) \subset \theta(\mathbb{A}_1)$ . So let us assume the existence of an  $X \in \mathbb{A}_1$  with  $\theta(X) \notin \mathbb{T}_1$ , i.e.  $\varphi(\theta(X)) \notin \mathbb{A}_1$ . Then there is a prime number  $p$  such that

$$|(\varphi(\theta(X)))_p| < p^2 - 1.$$

Using (A4), this implies that also  $|X_p| < p^2 - 1$ , a contradiction.  $\square$

In particular, this shows that  $\theta^{-1}(\mathbb{T}_1) = \mathbb{A}_1$  and thus

$$\mu(\mathbb{T}_1) = \nu(\mathbb{A}_1) = 1.$$

It follows that  $\theta$  is a factor map from  $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$  to  $(\prod_p(\mathbb{Z}^2)_p, \mathbb{Z}^2, \mu)$ .

In order to see that  $\theta$  is in fact an isomorphism, let us first note that the Borel map  $\varphi: \mathbb{T} \rightarrow \mathbb{X}_V$  is measure-preserving since, for any  $\rho$ -patch  $\mathcal{P}$ , one has

$$\begin{aligned} \mu(\varphi^{-1}(B_{\mathcal{P}})) &= \mu(\{([y_p]_p) \in \mathbb{T} \mid [y_p]_p \notin \mathcal{P}_p \text{ for all } p\}) \\ &= \prod_p \left( \frac{p^2 - |\mathcal{P}_p|}{p^2} \right) \\ &= \nu(B_{\mathcal{P}}) \end{aligned}$$

by Corollary 1. Next, consider the subset  $\mathbb{A}_1^*$  of elements  $X \in \mathbb{A}_1$  that are maximal elements of  $\mathbb{A}$  with respect to inclusion, i.e.

$$\forall Y \in \mathbb{A} : (X \subset Y \Rightarrow X = Y).$$

Clearly,  $V$  and every translate of  $V$  are elements of  $\mathbb{A}_1^*$ ; see Lemma 4. Using Lemma 6, one can verify that  $\mathbb{A}_1^*$  contains precisely the elements  $X \in \mathbb{A}_1$  with

$$(A5) \quad X = \varphi(\theta(X)).$$

Employing (A5), one further verifies that

$$\mathbb{A}_1^* = \mathbb{A}_1 \cap \varphi(\mathbb{T}).$$

Since  $\varphi(\mathbb{T})$  can be seen to be a Borel set,  $\mathbb{A}_1^*$  is thus a Borel set with measure

$$\nu(\mathbb{A}_1^*) = \nu(\mathbb{A}_1 \cap \varphi(\mathbb{T})) = \nu(\mathbb{A}_1) + \nu(\varphi(\mathbb{T})) - \nu(\mathbb{A}_1 \cup \varphi(\mathbb{T})) = 1.$$

Setting

$$\mathbb{T}_1^* := \varphi^{-1}(\mathbb{A}_1^*),$$

the restrictions  $\varphi: \mathbb{T}_1^* \rightarrow \mathbb{A}_1^*$  and  $\theta: \mathbb{A}_1^* \rightarrow \mathbb{T}_1^*$  are well-defined ( $X \in \mathbb{A}_1^* \Rightarrow \varphi(\theta(X)) = X \in \mathbb{A}_1^*$ ) and can now be shown to be bijective and inverses to each other. Hence  $\theta$  resp.  $\varphi$  are isomorphisms.

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