WELL-POSEDNESS AND SCATTERING FOR THE ZAKHAROV SYSTEM IN FOUR DIMENSIONS

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Abstract. The Cauchy problem for the Zakharov system in four dimensions is considered. Some new well-posedness results are obtained. For small initial data, global well-posedness and scattering results are proved, including the case of initial data in the energy space. None of these results is restricted to radially symmetric data.

1. Introduction and Main Results

Let $\alpha > 0$. The Zakharov system

$$\begin{cases} i\dot{u} - \Delta u = nu, \\ \ddot{n}/\alpha^2 - \Delta n = -\Delta |u|^2, \end{cases} \tag{1.1}$$

with initial data

$$u(0,x) = u_0,\ n(0,x) = n_0,\ \dot{n}(0,x) = n_1, \tag{1.2}$$

is considered as a simplified mathematical model for Langmuir waves in a plasma, which couples the envelope $u : \mathbb{R}^{1+d} \to \mathbb{C}$ of the electric field and the ion density $n : \mathbb{R}^{1+d} \to \mathbb{R}$, neglecting magnetic effects and the vector field character of the electric field, see [28, Chapter V] and [30].

The parameter $\alpha > 0$ is called the ion sound speed. Formally, as $\alpha \to \infty$, (1.1) reduces to the focusing cubic Schrödinger equation

$$i\dot{u} - \Delta u = |u|^2 u, \tag{1.3}$$

which is energy-critical in dimension $d = 4$, see for example [19, 20, 4] and the references therein concerning recent developments on global-wellposedness, blow-up and scattering for (1.3). For rigorous results on the subsonic limit (as $\alpha \to \infty$) of (1.1) to (1.3) we refer the reader to [26, 24, 22].

Strong solutions (u, n) of the Zakharov system preserve the mass

$$\int_{\mathbb{R}^d} |u|^2 dx = \int_{\mathbb{R}^d} |u_0|^2 dx, \tag{1.4}$$

and the energy, with $D := \sqrt{-\Delta}$,

$$E(u, n, \dot{n}) = \int_{\mathbb{R}^d} |\nabla u|^2 + \frac{|D^{-1}\dot{n}|^2}{2\alpha^2} + \frac{|n|^2}{2} - n|u|^2 dx = E(u_0, n_0, n_1). \tag{1.5}$$

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In view of (1.5), a natural space for the initial data is the energy space

$$(u_0, n_0, n_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d). \tag{1.6}$$

For initial data in the energy space, the Zakharov system is known to be globally well-posed if $d = 1$ (see [5]) and locally well-posed if $d = 2, 3$ (see [3]). A low regularity local well-posedness theory has been developed in [5] in all dimensions, with further extensions in [1] if $d = 2$, and in [2] if $d = 3$, see also the references therein for previous work. In the case of the torus $\mathbb{T}^d$ well-posedness results were proved in [29, 21].

In [23] blow-up results in finite or infinite time for initial data of negative energy were proved if $d = 3$, and if $d = 2$ blow-up in finite time was derived in [8, 7]. Concerning the final data problem in weighted Sobolev spaces, we refer to [27, 6, 25].

Recently, the asymptotic behaviour as $t \to \infty$ for the initial data problem was studied in dimension $d = 3$: In [11], small data energy scattering in the radial case was obtained by using a normal form technique and the improved Strichartz estimates for radial functions from [15]. In [12], a dichotomy between scattering and grow-up was obtained for radial solutions with energy below the ground state energy. In the non-radial case in dimension $d = 3$, scattering was obtained in [16] under the assumption that the initial data are small enough and have sufficient regularity and decay. This result was improved recently in [10, 9], where scattering was shown for small initial data belonging to the energy space with some additional angular regularity.

In the present paper, we continue the analysis of the initial value problem (1.1) and focus on the energy-critical dimension $d = 4$. In particular, we will address the small data global well-posedness and scattering problem in the energy space, i.e.

$$(u_0, n_0, n_1) \in H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \times \dot{H}^{-1}(\mathbb{R}^4), \tag{1.7}$$

with no additional symmetry or decay assumption.

We reduce the wave equation to first order equation as usual: Let

$$N := n - iD^{-1}\dot{n}/\alpha, \tag{1.8}$$

then $n = \operatorname{Re} N = (N + \bar{N})/2$ and the Zakharov system for (u, N) reads as follows:

$$\begin{cases} (i\partial_t - \Delta)u = Nu/2 + \bar{N}u/2, \\ (i\partial_t + \alpha D)N = \alpha D|u|^2. \end{cases} \tag{1.9}$$

The Hamiltonian then becomes

$$E(u, n, \dot{n}) = E_Z(u, N) := \int_{\mathbb{R}^4} |\nabla u|^2 + \frac{|N|^2}{2} - \operatorname{Re} N|u|^2 dx. \tag{1.10}$$

We will restrict ourselves to the system (1.9). Our first main result is a small data global well-posedness and scattering result.

Theorem 1.1. *There exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that for any $(s, l) \in \mathbb{R}^2$ satisfying*

$$\begin{aligned} &l \geq 0, \quad s < 4l + 1, \quad (s, l) \neq (2, 3), \\ &\max(\frac{l+1}{2}, l - 1) \leq s \leq \min(l + 2, 2l + \frac{11}{8}), \end{aligned} \tag{1.11}$$

and for any initial data $(u_0, N_0) \in H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4)$ satisfying

$$\|(u_0, N_0)\|_{H^{1/2}(\mathbb{R}^4) \times L^2(\mathbb{R}^4)} < \varepsilon_0, \tag{1.12}$$

there exists a unique global solution $(u, N) \in C(\mathbb{R}; H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4))$ *of* (1.9) *with some space-time integrability. The solution map is continuous in the norms*

$$H^s \times H^l \ni (u_0, N_0) \mapsto (u, N) \in L^\infty(\mathbb{R}; H^s \times H^l). \tag{1.13}$$

Moreover, there exist $(u^\pm, N^\pm) \in H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4)$ *such that*

$$\lim_{t \to \pm\infty} (\|u(t) - S(t)u^\pm\|_{H^s} + \|N(t) - W_\alpha(t)N^\pm\|_{H^l}) = 0, \tag{1.14}$$

where $S(t) = e^{-it\Delta}$ *and* $W_\alpha(t) = e^{it\alpha D}$ *are the free propagators.*

In the above statement, we need the space-time integrability to ensure the uniqueness. For example, for any $T > 0$,

$$u \in L^2((0, T); B^{1/2}_{4,2}(\mathbb{R}^4)) \tag{1.15}$$

is sufficient for uniqueness on $[0, T]$, where $B^{1/2}_{4,2}$ is the inhomogeneous Besov space. See Propositions 3.1, 5.1 and 5.2 for more detail of the space-time integrability.

Very recently, we learned about an independent work of Kato and Tsugawa [17]. By a different method, they prove the small data scattering for $l = s - 1/2 \geq 0$, using bilinear estimates in U^p-V^p spaces for the standard iteration. While their iteration scheme is more direct, our estimates are more elementary and we cover a wider range of (s, l).

Our second result is a large data local well-posedness result for the same range of regularity (s, l) as above, except for the energy space $H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$.

Theorem 1.2. *Let* $(s, l) \in \mathbb{R}^2$ *satisfy* (1.11) *and* $(s, l) \neq (1, 0)$. *Then, for any* $(u_0, N_0) \in H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4)$, *there exists* $T = T(u_0, N_0) > 0$ *and a unique local solution* $(u, N) \in C([-T, T]; H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4))$ *to* (1.9) *satisfying some space-time integrability* ((1.15) *is enough for the uniqueness*). *Both* $T > 0$ *and* (u, N) *depend continuously on* (u_0, N_0).

In dimension $d = 4$, Ginibre-Tsutsumi-Velo [5] proved local well-posedness in the range $l \leq s \leq l + 1$, $l > 0, 2s > l + 1$, see Figure 2. Their method is the standard Picard iteration argument in the $X^{s,b}$ spaces. Theorem 1.2 gives further local well-posedness results in some new region indicated in Figure 1, while Theorem 1.1 covers the same range of exponents as well as the energy space $(s, l) = (1, 0)$, which is missing from the large data result Theorem 1.2.

The proofs for Theorems 1.1 and 1.2 use the normal form technique and Strichartz estimates as in [11] and the follow-up papers [12, 10, 9] and related work on the Klein-Gordon-Zakharov system [14, 13]. Our argument is somewhat simpler than [5] and it also implies some scattering results.

There is a qualitative difference in our proof between $s < l + 1$ and $s > l + 1$. Since the Strichartz norm of $W_\alpha(t)$ is worse than that of $S(t)$, we use only the H^l_x norm for N, while keeping the full Strichartz norm for u, for $s < l+1$. For $s > l+1$, however, this strategy is prevented by the normal form of u, so we need to modify the Strichartz norm for u, and to use that of N. Consequently, we can not recover all the Strichartz norms of $S(t)$ for u, in spite of the scattering. See Proposition 5.2 for the precise statement. This is consistent with that [5] is restricted to $s \leq l + 1$ and $X^{s,b}$ implies the full range of Strichartz norm.

The energy space $(s, l) = (1, 0)$ is at the intersection of $s = l + 1$ and $l = 0$, where our multilinear estimates actually break down. More precisely, we can not close any Strichartz bound for the normal form of u when $(s, l) = (1, 0)$. This is

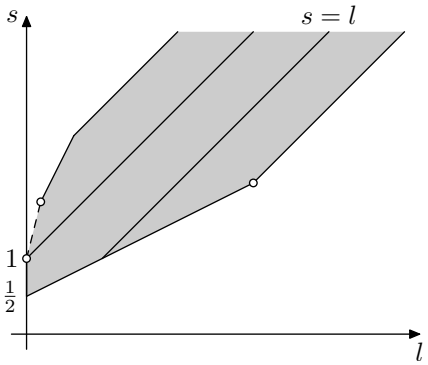

FIGURE 1. Range of (s, l) obtained in Theorems 1.1 and 1.2.

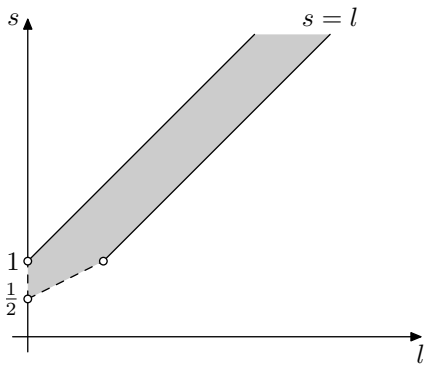

FIGURE 2. Range of (s, l) obtained in [5].

why $(1, 0)$ is excluded from Theorem 1.2. Fortunately enough, with the help of the conservation law (1.10) and using the well-posedness in nearby (s, l), we are still able to show global well-posedness and scattering in the energy space $(s, l) = (1, 0)$ for small data as in Theorem 1.1. Since the limit NLS (1.3) is critical in the energy space $H^1(\mathbb{R}^4)$, it may have blow-up with bounded $H^1 \times L^2$ norm for large data, which suggests that there may be essential difference between large and small data.

At the other excluded endpoint $(s, l) = (2, 3)$, we can prove a strong ill-posedness result, both by instant exit and by non-existence.

Theorem 1.3. *There exists a radial function $u_0 \in H^2(\mathbb{R}^4)$ such that for any $\varepsilon > 0$, any $N_0 \in H^3(\mathbb{R}^4)$, and any $T_0 > 0$, the system (1.9) has no solution $(u, N) \in C([0, T_0]; \mathcal{S}'(\mathbb{R}^4)^2)$ satisfying $(u(0), N(0)) = (\varepsilon u_0, N_0)$, the equation (1.9) in the distribution sense, and*

$$(u, N) \in L^2((0, T_0); H^1(\mathbb{R}^4) \times H^3(\mathbb{R}^4)). \tag{1.16}$$

Moreover, the unique local solution $(u, N) \in C([-T, T]; H^2 \times H^2)$ given by Theorem 1.2 satisfies $N(t) \notin H^3(\mathbb{R}^4)$ for all $t \in [-T, T] \setminus \{0\}$.

Note that (1.16) is weaker than the usual weak solutions, as it does not require $(u(t), N(t)) \in H^s \times H^l$ for all t near 0. The above ill-posedness is due to the mismatch of regularity between u and N in the normal form for N.

The rest of paper is organized as follows. In Section 2, we recall the normal form reduction from [11], and then gather multilinear estimates used in the later sections. They easily follow from the Littlewood-Paley decomposition, Coifman-Meyer bilinear estimate, Strichartz and Sobolev inequalities. Using these estimates and the standard contraction argument, we first prove the small data scattering in $H^s \times H^l$ for $s \le l + 1$ in Section 3, and then the local well-posedness for large data in $H^{1/2} \times L^2$ in Section 4. In Section 5, we extend these results to higher regularity by persistence of regularity, except for the energy space $(s, l) = (1, 0)$. Theorem 1.1 for $(s, l) \neq (1, 0)$ follows from Propositions 3.1, 5.1 and 5.2. Similarly, Theorem 1.2 follows from Propositions 4.2, 5.1 and 5.2. In Section 6, we prove Theorem 1.1 in the energy space $(s, l) = (1, 0)$, using the results in $(s, 0)$ for $s < 1$ and in $(1, l)$ for $l > 0$. In Section 7, we prove the ill-posedness Theorem 1.3 at $(s, l) = (2, 3)$.

2. Normal form and multilinear estimates

In this section, we set up integral equations and basic estimates for solving the equation. Our analysis is based on the normal form reduction devised in [11].

2.1. **Review of the normal form reduction and notation from** [11]**.** Let $\hat{\phi} = \mathcal{F}\phi$ denote the Fourier transform of ϕ. We use $S(t)$ and $W_\alpha(t)$ to denote the Schrödinger and wave semigroup, respectively:

$$S(t)\phi = \mathcal{F}^{-1}(e^{it|\xi|^2}\hat{\phi}), \quad W_\alpha(t)\phi = \mathcal{F}^{-1}(e^{i\alpha t|\xi|}\hat{\phi}).$$

Fix a radial, smooth, bump function $\eta_0 : \mathbb{R}^4 \to [0,1]$ with support in the ball $B_{\frac{8}{5}}(0)$, which is equal to 1 in the smaller ball $B_{\frac{4}{5}}(0)$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$ and $\chi_{\leq k}(\xi) = \eta_0(\xi/2^k)$, and let $P_k, P_{\leq k}$ denote the corresponding Fourier multipliers.

For two functions u, v and a fixed $K \in \mathbb{N}$, $K \geq 5$, we define the paraproduct type operators

$$\begin{aligned} (uv)_{LH} :=& \sum_{k\in\mathbb{Z}} (P_{\leq k-K}u)(P_k v), \quad (uv)_{HL} := (vu)_{LH}, \\ (uv)_{HH} :=& \sum_{\substack{|k_1-k_2|\leq K-1 \\ k_1,k_2\in\mathbb{Z}}} (P_{k_1}u)(P_{k_2}v), \end{aligned} \tag{2.1}$$

so that $uv = (uv)_{LH} + (uv)_{HL} + (uv)_{HH}$. We also define

$$\begin{aligned} (uv)_{\alpha L} :=& \sum_{\substack{|k-\log_2\alpha|\leq 1, \\ k\in\mathbb{Z}}} (P_k u)(P_{\leq k-K}v), \quad (uv)_{L\alpha} := (vu)_{\alpha L}, \\ (uv)_{XL} :=& \sum_{\substack{|k-\log_2\alpha|> 1, \\ k\in\mathbb{Z}}} (P_k u)(P_{\leq k-K}v), \quad (uv)_{LX} := (vu)_{XL}, \end{aligned} \tag{2.2}$$

so that $(uv)_{HL} = (uv)_{\alpha L} + (uv)_{XL}$.

Moreover, for any $* = HH, LH, HL, \alpha L$, etc., we denote the symbol (multiplier) of the bilinear operator $(uv)_*$ by $\mathcal{P}_*$. We denote finite sums of these bilinear operators in the obvious way, e.g. $(uv)_{LH+HH} = (uv)_{LH} + (uv)_{HH}$. With these notations, it was shown in [11] that (1.9) is equivalent –at least for smooth solutions– to the following integral equation. Henceforth, for simplicity, we replace the nonlinear term $\operatorname{Re} Nu/2$ with Nu as in [11], because the complex conjugation here makes no essential difference for our arguments.

$$\begin{aligned} u(t) =& S(t)u_0 - S(t)\Omega(N,u)(0) + \Omega(N,u)(t) \\ & - i\int_0^t S(t-s)\Omega(\alpha D|u|^2, u)(s)ds - i\int_0^t S(t-s)\Omega(N,Nu)(s)ds \\ & - i\int_0^t S(t-s)(Nu)_{LH+HH+\alpha L}(s)ds, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} N(t) =& W_\alpha(t)N_0 - W_\alpha(t)D\tilde{\Omega}(u,u)(0) + D\tilde{\Omega}(u,u)(t) \\ &- i\int_0^t W_\alpha(t-s)\alpha D(u\bar{u})_{HH+\alpha L+L\alpha}ds \\ &- i\int_0^t W_\alpha(t-s)(D\tilde{\Omega}(Nu,u) + D\tilde{\Omega}(u,Nu))(s)ds, \end{aligned} \tag{2.4}$$

where Ω, $\tilde{\Omega}$ are the bilinear Fourier multiplication operators

$$\begin{aligned} \Omega(f,g) =& \mathcal{F}^{-1}\int \mathcal{P}_{XL}\frac{\hat{f}(\xi-\eta)\hat{g}(\eta)}{-|\xi|^2+\alpha|\xi-\eta|+|\eta|^2}d\eta, \\ \tilde{\Omega}(f,g) =& \mathcal{F}^{-1}\int \mathcal{P}_{XL+LX}\frac{\alpha\hat{f}(\xi-\eta)\hat{\tilde{g}}(\eta)}{|\xi-\eta|^2-|\eta|^2-\alpha|\xi|}d\eta. \end{aligned}$$

The equations after normal form reduction can be written as

$$\begin{aligned} &(i\partial_t + D^2)(u-\Omega(N,u)) = (Nu)_{LH+HH+\alpha L} + \Omega(\alpha D|u|^2,u) + \Omega(N,Nu), \\ &(i\partial_t + \alpha D)(N - D\tilde{\Omega}(u,u)) = \alpha D|u|^2_{HH+\alpha L+L\alpha} + D\tilde{\Omega}(Nu,u) + D\tilde{\Omega}(u,Nu). \end{aligned} \tag{2.5}$$

2.2. **Function spaces and Strichartz estimates.** Let $s,l\in\mathbb{R}$ and $1\le p,q\le\infty$. We use $B^s_{p,q}, \dot{B}^s_{p,q}$ to denote the standard Besov space, with norms

$$\|f\|_{B^s_{p,q}} = \|P_{\le 0}f\|_p + \Big(\sum_{k=1}^\infty 2^{ksq}\|P_kf\|_p^q\Big)^{1/q}, \quad \|f\|_{\dot{B}^s_{p,q}} = \Big(\sum_{k=-\infty}^\infty 2^{ksq}\|P_kf\|_p^q\Big)^{1/q},$$

with obvious modifications if $q=\infty$, and we simply write $B^s_p = B^s_{p,2}, \dot{B}^s_p = \dot{B}^s_{p,2}$.

For the exponents $s\le l+1$, we use the following resolution spaces

$$\begin{aligned} &u\in X^s := C(\mathbb{R};H^s(\mathbb{R}^4))\cap L^\infty(\mathbb{R};H^s(\mathbb{R}^4))\cap L^2(\mathbb{R};B^s_4(\mathbb{R}^4)), \\ &N\in Y^l := C(\mathbb{R};H^l(\mathbb{R}^4))\cap L^\infty(\mathbb{R};H^l(\mathbb{R}^4)). \end{aligned} \tag{2.6}$$

For any Banach function space Z on $\mathbb{R}^{1+4}$ and any interval $I\subset\mathbb{R}$, the restriction of Z onto I is denoted by $Z(I)$. For example,

$$X^s([0,T]) = C([0,T];H^s(\mathbb{R}^4))\cap L^2((0,T);B^s_4(\mathbb{R}^4)). \tag{2.7}$$

We will use the following well-known Strichartz estimates for the wave and the Schrödinger equation in dimension $d=4$.

Lemma 2.1 (Strichartz estimates, see [18])**.** *For any $s\in\mathbb{R}$ and any functions $\phi(x), f(t,x)$, we have*

$$\begin{aligned} \|S(t)\phi\|_{L^\infty_t H^s_x\cap L^2_t B^s_4} \lesssim& \|\phi\|_{H^s} \\ \Big\|\int_0^t S(t-s)f(s)ds\Big\|_{L^\infty_t L^2_x\cap L^2_t B^0_4} \lesssim& \|f\|_{L^1_tL^2_x+L^2_tB^0_{4/3}}. \end{aligned}$$

$$\begin{aligned} \|W_\alpha(t)\phi\|_{L^\infty_t L^2_x\cap L^2_t \dot{B}^{-5/6}_6} \lesssim& \|\phi\|_{L^2} \\ \Big\|\int_0^t W_\alpha(t-s)f(s)ds\Big\|_{L^\infty_t L^2_x\cap L^2_t \dot{B}^{-5/6}_6} \lesssim& \|f\|_{L^1_tL^2_x}. \end{aligned}$$

2.3. **Multi-linear estimates for quadratic and cubic terms.** Next, we prove multi-linear estimates for the nonlinear terms in (2.5) in the Besov spaces of $x \in \mathbb{R}^4$. For t, only Hölder's inequalities will be used, which needs no explanation. In the following, we ignore the dependence of constants on (s,l), but distinguish by $C(K)$ when it is not uniform for K. The main tools are Littlewood-Paley theory and certain Coifman-Meyer type bilinear Fourier multiplier estimates. Roughly speaking, the multipliers Ω and $\widetilde{\Omega}$ act like

$$\Omega(f,g) \sim D^{-1}\langle D\rangle^{-1}(fg)_{XL}, \quad \widetilde{\Omega}(f,g) \sim D^{-1}\langle D\rangle^{-1}(f\bar{g})_{XL+LX}, \tag{2.8}$$

in product estimates in the Besov spaces. Hence the proof is reduced to usual computation of exponents as in the paraproduct. We only sketch the proof.

Lemma 2.2 (Quadratic terms)**.** *Let $K \geq 5$.*
(1) *Assume that $s, l \geq 0$. Then for any $N(x)$ and $u(x)$,*

$$\begin{aligned} \|(Nu)_{LH+\alpha L}\|_{B^s_{4/3}} &\lesssim \|N\|_{H^l}\|u\|_{B^s_4}, \\ \|(Nu)_{HH}\|_{B^s_{4/3}} &\lesssim C(K)\|N\|_{H^l}\|u\|_{B^s_4}. \end{aligned} \tag{2.9}$$

(2) *Assume $0 \leq l+1 \leq 2s$. Then for any $u(x)$ and $v(x)$,*

$$\begin{aligned} \|D(uv)_{HH}\|_{H^l} &\lesssim C(K)\|u\|_{B^s_4}\|v\|_{B^s_4}, \\ \|D(uv)_{\alpha L+L\alpha}\|_{H^l} &\lesssim \|u\|_{B^s_4}\|v\|_{B^s_4}. \end{aligned} \tag{2.10}$$

Proof. The estimates above follow directly from Bony's paraproduct and Hölder's inequality. For example,

$$\|P_k(Nu)_{LH}\|_{L^{4/3}} \lesssim \sum_{j=k-2}^{k+2} \|(P_{\leq j-K}N)(P_j u)\|_{L^{4/3}} \lesssim \sum_{j=k-2}^{k+2} \|N\|_{L^2}\|P_j u\|_{L^4}. \tag{2.11}$$

Then, we sum up the squares with respect to k. The other estimates follow in a similar manner. This argument loses the summability for HH at the 0 regularity ($s = l = 0$ for (1) and $s = l+1 = 0$ for (2)), but then we can simply use Hölder in x together with the embedding $B^0_p \subset L^p$ and $L^{p'} \subset B^0_{p'}$ for $2 \leq p \leq \infty$. □

Similarly to [12, Lemma 4.4] and [13, Lemma 4.4], we will exploit in the proof of local well-posedness and persistence of regularity that the boundary contributions, as well as cubic terms, can be made small by choosing $K \geq 5$ large.

Lemma 2.3 (Boundary terms)**.** *There exist $\theta_j(s,l) \geq 0$ such that for all $K \geq 5$, and for any $N(x), u(x), v(x)$, we have the following:*
(1) *If $l \geq \max(0, s-2)$ and $(s,l) \neq (2,0)$,*

$$\|\Omega(N,u)\|_{H^s} \lesssim 2^{-\theta_1 K}\|N\|_{H^l}\|u\|_{H^s}, \quad \theta_1 > 0 \text{ for } s < l+2. \tag{2.12}$$

(2) *If $l \leq \min(2s-1, s+1)$ and $(s,l) \neq (2,3)$,*

$$\|D\tilde{\Omega}(u,v)\|_{H^l} \lesssim 2^{-\theta_2 K}\|u\|_{H^s}\|v\|_{H^s}, \quad \theta_2 > 0 \text{ for } l < s+1. \tag{2.13}$$

(3) *If $l \geq \min(0, s-1)$ and $(s,l) \neq (1,0)$,*

$$\|\Omega(N,u)\|_{B^s_4} \lesssim 2^{-\theta_3 K}\|N\|_{H^l}\|u\|_{B^s_4}, \quad \theta_3 > 0 \text{ for } s < l+1. \tag{2.14}$$

(4) *If $l \leq \min(2s-1/2, s+3/2)$ and $(s,l) \neq (2,7/2)$,*

$$\|\langle D\rangle^l \tilde{\Omega}(u,v)\|_{\dot{B}^{1/6}_6} \lesssim 2^{-\theta_4 K}[\|u\|_{B^s_4}\|v\|_{H^s} + \|v\|_{B^s_4}\|u\|_{H^s}], \tag{2.15}$$

where $\theta_4 > 0$ for $l < s+3/2$.

Proof. Since they are all straightforward, we prove only (2.14)-(2.15), leaving (2.12)-(2.13) to the reader. By [11, Lemma 3.5] and using (2.8) with Bernstein, we have

$$\begin{aligned}&\|P_k\langle D\rangle D\Omega(P_{k_0}N, P_{k_1}u)\|_{L^p_x} \lesssim \|P_{k_0}N\|_{L^{p_0}_x}\|P_{k_1}u\|_{L^{p_1}_x}\\ &\lesssim 2^{4k_0(1/p_0-1/q_0)+4k_1(1/p_1-1/q_1)}\|P_{k_0}N\|_{L^{q_0}_x}\|P_{k_1}u\|_{L^{q_1}_x},\end{aligned} \tag{2.16}$$

for any $k, k_0, k_1 \in \mathbb{Z}$ and any $p, p_0, p_1, q_0, q_1 \in [1,\infty]$ satisfying $1/p = 1/p_0 + 1/p_1$ and $q_j \le p_j$. The same estimate holds for the bilinear operator $\widetilde{\Omega}$. For the low frequency part, say if $k_1 \le k_0 - K$, we can replace P_{k_1} with $P_{\le k_1}$. The above with $(p, p_0, p_1, q_0, q_1) = (4, 4, \infty, 2, 4)$ and the HL restriction $|k - k_0| \le 1$ in Ω yields

$$\|\Omega(N,u)\|_{B^s_4} \lesssim \Big\| 2^{k^+(s-1-l)}\|P_kN\|_{H^l} \sum_{k_1\le k-K} 2^{k_1-k_1^+s}\|P_{k_1}u\|_{B^s_4}\Big\|_{l^2_k}, \tag{2.17}$$

where $k^+ := \max(k, 0)$, using $P_{\le 0}B^s_p \subset \dot{B}^0_{p,\infty}$ for the lower frequency component. The summation over $k_1 \le k - K$ is bounded by

$$k \le K \implies 2^{k-K}, \quad k > K \implies \begin{cases} 2^{(1-s)^+(k-K)} & (s \ne 1)\\ k - K & (s = 1).\end{cases} \tag{2.18}$$

This and $\|P_kN\|_{H^l} \in \ell^2_k$ lead to (2.14), with the small factor $2^{-\theta_3 K}$ for $s < 1$ and for $1 \le s < l + 1$. The conditions $l \ge 0$ and $l \ge s - 1$ ensure uniform boundedness of the coefficient after the summation, respectively for $s < 1$ and for $s > 1$, while the endpoint $(s, l) = (1, 0)$ is excluded due to the logarithmic growth at $s = 1$. Similarly with $(p, p_0, p_1, q_0, q_1) = (6, 6, \infty, 4, 2)$, we have

$$\begin{aligned}&\|P_k\langle D\rangle^l\tilde{\Omega}(u,v)_{HL}\|_{\dot{B}^{1/6}_6}\\ &\lesssim 2^{k^+(l-1-s)-k/2}\sum_{k_1\le k-K} 2^{2k_1-k_1^+s}\|P_ku\|_{B^s_4}\|P_{k_1}v\|_{H^s}.\end{aligned} \tag{2.19}$$

Using this and $\|P_ku\|_{B^s_4} \in \ell^2_{k\ge 0}$ lead to (2.15), with the small factor for $s < 2$ and for $2 \le s < l - 3/2$. □

Lemma 2.4 (Cubic terms)**.** *There exist $\theta_j(s, l) \ge 0$ such that for all $K \ge 5$, and for any $M(x), N(x), u(x), v(x), w(x)$, we have the following:*
(1) *If $s \ge 1/2$, then $\theta_1 > 0$ and*

$$\|\Omega(D(uv), w)\|_{H^s} \lesssim 2^{-\theta_1 K}[\|u\|_{H^s}\|v\|_{B^{1/2}_4} + \|v\|_{H^s}\|u\|_{B^{1/2}_4}]\|w\|_{B^{1/2}_4}. \tag{2.20}$$

(2) *If $l \ge 0$, $-l < s \le l + 2$, $s \le 2l + 1$ and $(s, l) \ne (1, 0)$,*

$$\|\Omega(M, Nu)\|_{B^s_{4/3}} \lesssim 2^{-\theta_2 K}\|M\|_{H^l}\|N\|_{H^l}\|u\|_{B^s_4}, \quad \theta_2 > 0 \text{ for } s < l + 2. \tag{2.21}$$

(3) *If $s \ge 1/2$, $-s < l \le s + 1$, $l \le 2s$, and $(s, l) \ne (1, 2)$,*

$$\|D\tilde{\Omega}(Nu, v)\|_{H^l} + \|D\tilde{\Omega}(v, Nu)\|_{H^l} \lesssim 2^{-\theta_3 K}\|N\|_{H^l}\|u\|_{B^s_4}\|v\|_{B^s_4}, \tag{2.22}$$

where $\theta_3 > 0$ for $l < s + 1$.

Proof. For (2.20), we can use a standard product inequality for $s \ge 1/2$:

$$\|uv\|_{B^s_{8/5}} \lesssim \|u\|_{H^s}\|v\|_{B^{1/2}_4} + \|v\|_{H^s}\|u\|_{B^{1/2}_4}, \tag{2.23}$$

which easily follows using $B_4^{1/2} \subset L^8$, e.g. by the paraproduct calculus. Putting $f := uv$, we obtain from (2.16) with $(p, p_0, p_1, q_0, q_1) = (2, 2, \infty, 8/5, 4)$

$$\|P_k\Omega(Df, w)\|_{H^s} \lesssim 2^{k/2-k^+} \sum_{k_1 \le k-K} 2^{k_1-k_1^+/2} \|f\|_{B^s_{8/5}} \|w\|_{B_4^{1/2}}, \tag{2.24}$$

which leads to (2.20) with a small factor, in the same way as in the previous lemma.

For (2.21) and (2.22), we can use a standard product inequality:

$$\sigma \le \min(s, l, s+l-1) \implies \|Nu\|_{H^\sigma} \lesssim \|N\|_{H^l} \|u\|_{B^s_4}, \tag{2.25}$$

which holds for $s + l > 0$ unless $s = 1$ and $\sigma = l$. Putting $g := Nu$, we obtain from (2.16) with $(p, p_0, p_1, q_0, q_1) = (4/3, 2, 4, 2, 2)$

$$\|P_k\Omega(M, g)\|_{B^s_{4/3}} \lesssim \sum_{k_1 \le k-K} 2^{k^+(s-1-l)-k+k_1-k_1^+\sigma} \|P_k M\|_{H^l} \|P_{k_1} g\|_{H^\sigma}. \tag{2.26}$$

First, the low frequency part $k \le 0$ is bounded using Young on $\mathbb{Z}$

$$\begin{aligned} &\|P_{\le 0}\Omega(M, g)\|_{B^s_{4/3}} \lesssim \|P_k\Omega(M, g)\|_{\ell^1_{k\le 0} L^{4/3}_x} \\ &\lesssim \|P_k M\|_{\ell^2_{k\le 0} H^l_x} \Big\| \sum_{k_1 \le k-K} 2^{-k+k_1} \|P_{k_1} g\|_{H^\sigma_x} \Big\|_{\ell^2_{k\le 0}} \lesssim 2^{-K} \|M\|_{H^l} \|g\|_{H^\sigma}. \end{aligned} \tag{2.27}$$

For $0 < k \le K$, the summation over k_1 is bounded by $2^{k(s-l-1)-K} \|P_k M\|_{H^l} \in \ell^2_k$ with the small factor for $s < l + 2$. For $K < k$, it is bounded by

$$\begin{cases} \sigma < 1 \implies 2^{k(s-1-l-\sigma)} 2^{-K(1-\sigma)} \\ \sigma > 1 \implies 2^{k(s-2-l)}. \end{cases} \tag{2.28}$$

The case $\sigma < 1$ is fine if $\sigma = l$ by $s \le 2l+1$, if $\sigma \le s+1$ by $l \ge 0$, and if $\sigma = s+l-1$ by $l \ge 0$. In the critical case $s = 1$ for the product inequality, we have $s < 2l + 1$ and $l > 0$ by the exclusion $(s, l) \ne (1, 0)$, so that we can choose $\sigma = l - \varepsilon$. The case $\sigma > 1$ is fine by $s \le l + 2$. Then the only remaining case is $(s, l) = (3, 1)$, where we are forced to choose $\sigma = 1$ then we should replace (2.26) for $k > K$ with

$$\|P_k\Omega(M, g)\|_{B^s_{4/3}} \lesssim 2^{k(s-2-l)} \|P_k M\|_{H^l} \|P_{\le k-K} g\|_{H^1}, \tag{2.29}$$

which is bounded using $\|P_k M\|_{H^l} \in \ell^2_k$. Thus we obtain (2.21).

Similarly, from (2.16) with $(p, p_0, p_1, q_0, q_1) = (2, 2, \infty, 2, 4)$, we have

$$\begin{aligned} &\|P_k D\widetilde{\Omega}(g, v)_{HL}\|_{H^l} + \|P_k D\widetilde{\Omega}(v, g)_{LH}\|_{H^l} \\ &\lesssim \sum_{k_1 \le k-K} 2^{k^+(l-1-\sigma)+k_1-k_1^+ s} \|P_k g\|_{H^\sigma} \|P_{k_1} v\|_{B^s_4}, \end{aligned} \tag{2.30}$$

for which the low frequencies $k \le K$ are easily bounded using the factor 2^{k_1}, while for $k > K$ the summation is bounded by

$$\begin{cases} s < 1 \implies 2^{k(l-s-\sigma)} 2^{-K(1-s)}, \\ s = 1 \implies 2^{k(l-1-\sigma)}(k-K), \\ s > 1 \implies 2^{k(l-1-\sigma)}. \end{cases} \tag{2.31}$$

The case $s < 1$ is fine if $\sigma = s$ by $l \le 2s$, and if $\sigma = s + l - 1$ by $s \ge 1/2$. The case $s > 1$ is fine if $\sigma = s$ by $l \le s + 1$, and obviously if $\sigma = l$. The critical case $s = 1$ is also fine, as none of the conditions is on the boundary thanks to $(s, l) \ne (1, 2)$.

For the other HL interaction, choosing $(p, p_0, p_1, q_0, q_1) = (2, 4, 4, 2, 4)$ we have

$$\begin{aligned} &\|P_k D\widetilde{\Omega}(g, v)_{LH}\|_{H^l} + \|P_k D\widetilde{\Omega}(v, g)_{HL}\|_{H^l} \\ &\lesssim \sum_{k_1 \le k-K} 2^{k^+(l-1-s)+k_1-k_1^+\sigma} \|P_k v\|_{B_4^s} \|P_{k_1} g\|_{H^\sigma}, \end{aligned} \tag{2.32}$$

which is also easy for $k \le K$. For $k > K$, the summation is bounded by

$$\begin{cases} \sigma < 1 \implies 2^{k(l-\sigma-s)} 2^{-K(1-\sigma)} \\ \sigma > 1 \implies 2^{k(l-1-s)}. \end{cases} \tag{2.33}$$

The case $\sigma < 1$ is the same as the case $s < 1$ in (2.31). The case $\sigma > 1$ is OK by $l \le s+1$. When $l = s+1 \ge 3/2$, we can choose $\sigma = \min(s, l, s+l-1) = s \ne 1$ thanks to $(s, l) \ne (1, 2)$. In the critical case $s = 1$, we can choose $\sigma < \min(s, l, s+l-1) \le 1$ such that $l - s - \sigma < 0$, since $l < 2s = 2$. This concludes the proof of (2.22). □

3. Small data scattering for $s \le l + 1$

Using the multilinear estimates in the previous section, it is now easy to obtain global well-posedness and scattering for small initial data in $H^s \times H^l$ in the range (1.11) under $s \le l + 1$. In Section 5 we will show that we only need smallness in $H^{1/2} \times L^2$ for all regularities by persistence of regularity argument. Fix $K = 5$. As in [11, Section 4], for fixed initial data $(u_0, N_0) \in H^s \times H^l$, we define a mapping $(u, N) \mapsto (u', N') = \Phi_{u_0,N_0}(u, N)$ by the right-hand sides of the equations (2.3)-(2.4). Then for small initial data (u_0, N_0), we see that Φ_{u_0,N_0} is a contraction in a small ball around 0 of $X^s \times Y^l$. Indeed, from the estimates in the previous section, we obtain

$$\begin{aligned} &\|u'\|_{X^s} \lesssim \|u_0\|_{H^s} + \|N\|_{Y^l}\|u\|_{X^s} + \|u\|_{X^s}^3 + \|N\|_{Y^l}^2\|u\|_{X^s}, \\ &\|N'\|_{Y^l} \lesssim \|N_0\|_{H^l} + \|u\|_{X^s}^2 + \|N\|_{Y^l}\|u\|_{X^s}^2, \end{aligned} \tag{3.1}$$

where we need $s \le l + 1$ in using (2.14) for $\Omega(N, u)$. By the contraction mapping principle, we have a unique solution in a small ball in $X^s \times Y^l$, and the Lipschitz continuity of the solution map $H^s \times H^l \to X^s \times Y^l$ follows from the standard argument.

Now we derive scattering for (u, N) in $H^s \times H^l$, assuming (s, l) satisfying (1.11), $(u, N) \in X^{1/2} \times Y^0$ with small norm and the scattering of the transformed variables, namely for

$$\Psi(u, N) := (u - \Omega(N, u), N - D\tilde{\Omega}(u, u)) \tag{3.2}$$

there exist $(u_\pm, N_\pm) \in H^s \times H^l$ with small norm in $H^{1/2} \times L^2$ such that

$$\Psi(u, N) - (S(t)u_\pm, W_\alpha(t)N_\pm) \to 0 \text{ in } H^s \times H^l \quad (t \to \pm\infty). \tag{3.3}$$

In the current case $s \le l + 1$, the latter assumption (3.3) obviously holds in view of that $(u, N) \in X^s \times Y^l$ and the Strichartz estimate with the global bounds on the nonlinear terms.

The bilinear estimate for the normal form in Lemma 2.3 implies that the above transform Ψ is invertible for small data in $H^{1/2} \times L^2$ and bi-Lipschtiz. More precisely, for any $(u', N') \in H^{1/2} \times L^2$, the inverse image $\Psi^{-1}(u', N')$ is the fixed points of the map

$$(u, N) \mapsto \Psi_{u',N'}(u, N) := (u' + \Omega(N, u), N' + D\widetilde{\Omega}(u, u)). \tag{3.4}$$

Lemma 2.3 implies that $\Psi_{u',N'}$ is a contraction in a small ball of $H^{1/2}\times L^2$ if (u',N') is small, hence the unique small $(u,N)\in\Psi^{-1}(u',N')$ is given by the iteration

$$(u,N)=\lim_{k\to\infty}(\Psi_{u',N'})^k(0,0). \tag{3.5}$$

By (3.3), we get

$$(u,N)-\Psi^{-1}(S(t)u_\pm,W_\alpha(t)N_\pm)\to 0 \text{ in } H^{1/2}\times L^2 \quad (t\to\pm\infty).$$

To show the scattering for (u,N), it suffices to show

$$\Psi^{-1}(S(t)u_\pm,W_\alpha(t)N_\pm)\to(S(t)u_\pm,W_\alpha(t)N_\pm) \text{ in } H^{1/2}\times L^2 \quad (t\to\pm\infty). \tag{3.6}$$

By the construction of inverse, we get

$$(u_\pm^n(t),N_\pm^n(t))\to\Psi^{-1}(S(t)u_\pm,W_\alpha(t)N_\pm) \text{ in } L_t^\infty(H^{1/2}\times L^2) \quad (n\to\infty). \tag{3.7}$$

where $(u_\pm^0,N_\pm^0)=(0,0)$, and for $n=1,2,\cdots$,

$$\begin{aligned} u_\pm^{n+1}=&S(t)u_\pm+\Omega(N_\pm^n,u_\pm^n),\\ N_\pm^{n+1}=&W_\alpha(t)N_\pm+D\tilde\Omega(u_\pm^n,u_\pm^n).\end{aligned}$$

Thus, to show (3.6), it suffices to show for any n

$$(u_\pm^n(t),N_\pm^n(t))\to(S(t)u_\pm,W_\alpha(t)N_\pm) \text{ in } H^{1/2}\times L^2 \quad (t\to\pm\infty), \tag{3.8}$$

for which by induction on n and bilinear estimates it suffices to show

$$(\Omega(N_F,u_F),D\tilde\Omega(u_F,u_F))\to 0 \text{ in } H^s\times H^l \quad (t\to\pm\infty) \tag{3.9}$$

for all free solutions (u_F,N_F) in $H^s\times H^l$. The density argument with the bilinear estimate allows us to restrict to the case $u_F(0),N_F(0)\in C_0^\infty(\mathbb{R}^4)$, then the above is almost obvious by the dispersive decay of $S(t)$ and $W_\alpha(t)$ (we omit the details).

For higher regularity $(s,l)\neq(1/2,0)$, we do not have smallness in $H^s\times H^l$, so we should replace Lemma 2.3 with the following set of estimates

$$\begin{aligned} &\|\Omega(N,u)\|_{H^s}\lesssim\|N\|_{H^l}\|u\|_{B_u}, \quad \|\Omega(N,u)\|_{B_u}\lesssim\|N\|_{B_N}\|u\|_{B_u},\\ &\|D\widetilde\Omega(u,u)_{HL}\|_{H^l}\lesssim\|u\|_{H^s}\|u\|_{B_u},\end{aligned} \tag{3.10}$$

where the Besov spaces B_u and B_N are defined by

$$B_u:=B_p^{s-\varepsilon}, \quad B_N:=B_p^{l-\varepsilon}, \quad 1/p=1/2-\varepsilon/4 \tag{3.11}$$

for some small $\varepsilon>0$ such that $H^s\times H^l\subset B_u\times B_N$ by the sharp Sobolev embedding. (3.10) implies that $\Psi_{u',N'}$ is a contraction with respect to the equivalent norm

$$\|(u,N)\|_Z:=\|u\|_{H^s}+\|N\|_{H^l}+\delta^{-2}\|u\|_{B_u} \tag{3.12}$$

for $0<\delta\ll 1$, on the closed set

$$F:=\{(u,N)\in H^s\times H^l;\|(u,N)\|_Z\le 1/\delta, \quad \|N\|_{B_N}\le\delta, \quad \|u\|_{B_u}\le\delta^3\}, \tag{3.13}$$

provided that $2(u',N')\in F$. Indeed, (3.10) yields for any $(u,N)\in F$,

$$\|(\Omega(N,u),D\widetilde\Omega(u,u))\|_{H^s\times H^l}\lesssim\delta^2, \quad \|\Omega(N,u)\|_{B_u}\lesssim\delta^4, \tag{3.14}$$

hence $\|(\Omega(N,u), D\widetilde{\Omega}(u,u))\|_Z \lesssim \delta^2$ and $\Psi_{u',N'}(u,N) \in F$. For the difference, we have from (3.10), for any $(v,M) \in H^s \times H^l$,

$$\begin{aligned}
&\|(\Omega(N,v), D\widetilde{\Omega}(u,v)_{HL})\|_{H^s\times H^l} \lesssim \|(u,N)\|_{H^s\times H^l}\|v\|_{B_u} \lesssim \delta\|(v,M)\|_Z,\\
&\|(\Omega(M,u), D\widetilde{\Omega}(v,u)_{HL})\|_{H^s\times H^l} \lesssim \|(v,M)\|_{H^s\times H^l}\|u\|_{B_u} \lesssim \delta^3\|(v,M)\|_Z, \qquad (3.15)\\
&\|\Omega(N,v)+\Omega(M,u)\|_{B_u} \lesssim \|N\|_{B_N}\|v\|_{B_u} + \|M\|_{B_N}\|u\|_{B_u} \lesssim \delta^3\|(v,M)\|_Z.
\end{aligned}$$

Since the scattering of $\Psi(u,N)$ implies $\|\Psi(u,N)\|_{B_u\times B_N} \to 0$ as $t\to\infty$, choosing $\delta>0$ small enough ensures that $2\Psi(u,N)\in F$ for large t. Then (u,N) given by (3.5) is the same as the fixed point in F. Since we can take $\delta>0$ arbitrarily small, (3.14) implies that $\|(u,N)-\Psi(u,N)\|_{H^s\times H^l}\to 0$ as $t\to\infty$, hence the scattering of (u,N) in $H^s\times H^l$.

Since all the estimates are uniform and global in time, the same argument works for the final state problem, namely to find the solution for a prescribed (small) scattering data at $t=\infty$. Thus we obtain

Proposition 3.1. *Let $(s,l)\in\mathbb{R}^2$ satisfy (1.11), $s\le l+1$ and $(s,l)\neq(1,0)$. Then there exists $\varepsilon_1=\varepsilon_1(s,l)>0$ such that for any $(u_0,N_0)\in H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4)$ satisfying $\|(u_0,N_0)\|_{H^s\times H^l}\le\varepsilon_1$, there exists a unique global solution $(u,N)\in X^s\times Y^l$ of (1.9). Moreover, there exists $(u^+,N^+)\in H^s\times H^l$ such that*

$$\lim_{t\to\infty}\|u(t)-S(t)u^+\|_{H^s_x} + \|N(t)-W_\alpha(t)N^+\|_{H^l_x} = 0. \qquad (3.16)$$

Conversely, for any $(u^+,N^+)\in H^s\times H^l$ with $\|(u^+,N^+)\|_{H^s\times H^l}\le\varepsilon_1$, there exists a unique solution $(u,N)\in X^s\times Y^l$ satisfying (3.16). Both the maps $(u_0,N_0)\mapsto(u,N)$ and $(u^+,N^+)\mapsto(u,N)$ are Lipschitz continuous from the ε_1-ball into $X^s\times Y^l$.

The uniqueness without the smallness is proved in the next section. For the question if (u,N) obtained above really solves the equation (1.9) before the normal form, see Remark 5.3.

4. Large data local well-posedness for $s<l+1$

For large data, the proof in the previous section does not immediately work, in particular at the endpoint $(s,l)=(1/2,0)$. The main difficulty is the lack of flexibility in the choice of the Strichartz norm for the boundary term and the bilinear term $(Nu)_{LH}$. More precisely, $L^\infty_t H^l_x$ is the only choice among the Strichartz norms of $W_\alpha(t)$ for N, to estimate $\Omega(N,u)$ in $L^\infty_t H^s_x$, and to avoid losing regularity in $(Nu)_{LH}$. For the former term, we can play with the frequency gap parameter K in the normal form to extract a small factor. For the latter term, we use the following

Lemma 4.1. *Let $0<T\le\infty$ and $N\in C([0,T);L^2(\mathbb{R}^4))$. Suppose that $W_\alpha(-t)N(t)$ is strongly convergent in L^2_x as $t\to T-0$. Then for any $\varepsilon>0$, there exists a finite increasing sequence $0=T_0<T_1<\cdots<T_{n+1}=T$ such that*

$$\|N\|_{(L^\infty_t L^2_x + L^2_t L^4_x)(T_j,T_{j+1})} < \varepsilon \qquad (4.1)$$

for each $j=0,\dots,n$.

Note that the $L^2_tL^4_x$ norm is not controlled by the Strichartz estimate for $W_\alpha(t)$, but it is bounded for nice initial data. The case $T=\infty$ will be used for large data scattering. For $T<\infty$, the assumption on N is equivalent to $N\in C([0,T];L^2)$.

Proof. Put $N^+ := \lim_{t\to T-0} W_\alpha(-t)N(t) \in L^2_x$. By the strong convergence, there exists $T' \in (0,T)$ such that $\sup_{T'\le t<T} \|N(t) - W_\alpha(t)N^+\|_{L^2_x} < \varepsilon/4$. Since $C_0^\infty \subset L^2_x$ is dense, there exists $N_0 \in C_0^\infty$ such that $\|N_0 - N^+\|_{L^2_x} < \varepsilon/4$. The dispersive decay of $W_\alpha(t)$ implies that $W_\alpha(t)N_0 \in L^2_t L^4_x(\mathbb{R})$. Define N' by

$$N'(t) := P_{\le k}N(t)\ (0 \le t \le T'), \quad N'(t) := W_\alpha(t)N_0\ (T' < t < T). \tag{4.2}$$

By the above choice of T' and N_0, we have $\|N - N'\|_{L^\infty((T',T);L^2_x)} < \varepsilon/2$. Since $N \in C([0,T'];L^2_x)$ and $[0,T']$ is compact, $N'(t) \to N(t)$ in L^2_x uniformly on $t \in [0,T']$ as $k \to \infty$. Hence for large k we have $\|N - N'\|_{L^\infty([0,T'];L^2_x)} < \varepsilon/2$. Hence

$$\|N - N'\|_{L^\infty([0,T);L^2_x)} < \varepsilon/2, \quad N' \in L^2_t([0,T);L^4_x). \tag{4.3}$$

Choosing $T_1 < T_2 < \cdots < T_n$ appropriately ensures that $\|N'\|_{L^2_t L^4_x(T_j,T_{j+1})} < \varepsilon/2$ for each j, then we get the desired estimate. □

Now we are ready to prove the local well-posedness for large data in $H^{1/2} \times L^2$. For any initial data $(u_0, N_0) \in H^{1/2} \times L^2$, let

$$u_F := S(t)(u_0 - \Omega(N_0, u_0)), \quad N_F := W_\alpha(t)(N_0 - D\widetilde{\Omega}(u_0, u_0)), \tag{4.4}$$

and apply Lemma 4.1 to N_F. Then for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\|u_F\|_{L^2_t B^{1/2}_4(0,T)} + \|N_F\|_{L^\infty_t L^2_x + L^2_t L^4_x(0,T)} < \varepsilon. \tag{4.5}$$

Putting $\mathcal{H} := H^{1/2} \times L^2$ and $m := \|(u_F(0), N_F(0))\|_{\mathcal{H}}$, we look for a unique local solution on $(0,T)$ as a fixed point of the map Φ_{u_0,N_0} in the closed set

$$K^\varepsilon_m := \left\{ (u,N) \in C([0,T];\mathcal{H}) \;\middle|\; \begin{array}{l} \|(u,N)\|_{L^\infty_t(0,T;\mathcal{H})} \le 2m, \\ \|u\|_{L^2_t B^{1/2}_4(0,T)} + \|N\|_{L^\infty_t L^2_x + L^2_t L^4_x(0,T)} \le 2\varepsilon \end{array} \right\}.$$

From the multilinear estimates in Section 2, we have

$$\begin{aligned} &\|\Omega(N,u)\|_{X^{1/2}} \lesssim 2^{-\theta K}\|N\|_{L^\infty_t L^2_x}\|u\|_{X^{1/2}}, \\ &\|\Omega(D|u|^2,u)\|_{L^1_t H^{1/2}_x} \lesssim 2^{-\theta K}\|u\|_{L^\infty_t H^{1/2}}\|u\|^2_{L^2_t B^{1/2}_4}, \\ &\|\Omega(N,Nu)\|_{L^2_t B^{1/2}_{4/3}} \lesssim 2^{-\theta K}\|N\|^2_{L^\infty_t L^2_x}\|u\|_{L^2_t B^{1/2}_4}, \\ &\|D\widetilde{\Omega}(u,u)\|_{L^\infty_t L^2_x} \lesssim 2^{-\theta K}\|u\|^2_{L^\infty_t H^{1/2}_x}, \\ &\|D\widetilde{\Omega}(Nu,u)\|_{L^1_t L^2_x} \lesssim 2^{-\theta K}\|N\|_{L^\infty_t L^2_x}\|u\|^2_{L^2_t B^{1/2}_4}, \end{aligned} \tag{4.6}$$

and the same estimate on $D\widetilde{\Omega}(u,Nu)$, as well as for the difference. Taking K large makes these estimates contractive. For the remaining two terms,

$$\begin{aligned} &\|D|u|^2_{HH+\alpha L+L\alpha}\|_{L^1_t L^2_x} \lesssim C(K)\|u\|^2_{L^2_t B^{1/2}_4}, \\ &\|(Nu)_{LH+HH+\alpha L}\|_{L^2_t B^{1/2}_{4/3}+L^1_t H^{1/2}_x} \lesssim C(K)\|N\|_{L^\infty_t L^2_x + L^2_t L^4_x}\|u\|_{L^2_t B^{1/2}_4}, \end{aligned} \tag{4.7}$$

which is also made contractive on the interval $[0,T]$ by choosing $\varepsilon > 0$ small enough such that $C(K)\varepsilon \ll 1$ after fixing K. Then Φ_{u_0,N_0} becomes a contraction on K^ε_m.

The uniqueness of solution in the class $X^{1/2} \times Y^0$ is obtained in the same fashion: Let $(u_j, N_j) \in X^{1/2} \times Y^0$ for $j = 0, 1$ be two solutions. For any $\varepsilon > 0$, applying Lemma 4.1, we can find $T' \in (0,T)$ such that for $j = 0, 1$

$$\|u_j\|_{L^2_t B^{1/2}_4(0,T')} + \|N_j\|_{L^\infty_t L^2_x + L^2_t L^4_x(0,T')} < \varepsilon, \tag{4.8}$$

so that both the solutions belong to K_m^ε on $[0,T']$, hence $(u_0,N_0)=(u_1,N_1)$ as long as they are solutions in the above class.

The continuous dependence is also obtained in the same way, because

$$H^{1/2}\times L^2 \ni (u_0,N_0)\mapsto (u_F,N_F)\in X^{1/2}\times Y^0 \tag{4.9}$$

is continuous. Take a strongly convergent sequence of initial data. If the smallness condition (4.5) is satisfied by the limit, then so is it by those sufficiently close to the limit. Then we can estimate the difference from the limit in the same way as above, leading to the strong continuity.

We have worked out at the lowest regularity $(s,l)=(1/2,0)$, but the same argument works as long as we have the small factor $2^{-\theta K}$, namely for $|s-l|<1$. Thus we obtain

Proposition 4.2. *Let $(s,l)\in\mathbb{R}^2$ satisfy (1.11) and $|s-l|<1$. For any $(u_0,N_0)\in H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4)$, there exists a unique local solution $(u,N)\in (X^s\times Y^l)([0,T])$ of (1.9) for some $T>0$, where both T and (u,N) depend continuously on (u_0,N_0). More precisely, if $(u_{0,n},N_{0,n})\to(u_0,N_0)$ in $H^s\times H^l$, then $T_n\to T$ and for any $0<T'<T$, we have $\|u_n-u\|_{X^s([0,T'])}+\|N_n-N\|_{Y^l([0,T'])}\to 0$.*

5. Persistence of regularity except for $(s,l)=(1,0)$

Once we have the unique solution at the lowest regularity $(s,l)=(1/2,0)$, it gains as much regularity as the initial data. To prove this, we will focus on the derivation of a priori estimates, assuming that all relevant norms are finite, which is justified by the local well-posedness in higher regularity by Proposition 4.2.

For solutions $(u,N)\in (X^{1/2}\times Y^0)([0,T))$ with $(u(0),N(0))\in H^s\times H^l$ and $0<T\le\infty$, we will improve the regularity up to $H^s\times H^l$ by the following steps.

(1) Improve u to $s<l+1$.
(2) Improve N to $l\le 2s-1$, $l\le s+1$, and $(s,l)\neq(2,3)$, for $s<l+1$.
(3) Improve u to $1<s<4l+1$, $s\le 2l+11/8$ and $s\le l+2$.

The persistence of regularity is a general phenomenon in nonlinear wave equations, but we encounter some difficulties. One is the same as in the previous section, which is solved by Lemma 4.1. Another difficulty for $s\ge l+1$ is that the normal form can not keep the full Strichartz norm of u, which is why we separate (3).

5.1. **Regularity upgrade for u in $s<l+1$.** Let $(s,l)\in\mathbb{R}^2$ satisfy (1.11) and $s<l+1$. Let $(u_0,N_0)\in H^s\times H^l$ and let $(u,N)\in(X^{1/2}\times Y^l)([0,T))$ be a solution for some $0<T\le 0$. If $T=\infty$, we also assume that N scatters in H^l_x. From the estimates in Section 2, we have for $s<l+1$,

$$\begin{aligned}
&\|(Nu)_{LH+HH+\alpha L}\|_{L^2_tB^s_{4/3}+L^1_tH^s_x}\le C_1(K)\|N\|_{L^\infty_tL^2_x+L^2_tL^4_x}\|u\|_{L^2_tB^s_4},\\
&\|\Omega(N,u)\|_{L^\infty_tH^s_x}\le C_02^{-\theta K}\|N\|_{L^\infty_tL^2_x}\|u\|_{L^\infty_tH^s_x},\\
&\|\Omega(N,u)\|_{L^2_tB^s_4}\le C_02^{-\theta K}\|N\|_{L^\infty_tH^l_x}\|u\|_{L^2_tB^s_4},\\
&\|\Omega(D|u|^2,u)\|_{L^1_tH^s_x}\le C_02^{-\theta K}\|u\|_{L^\infty_tH^s_x}\|u\|^2_{L^2_tB^{1/2}_4},\\
&\|\Omega(N,Nu)\|_{L^2_tB^s_{4/3}}\le C_02^{-\theta K}\|N\|^2_{L^\infty_tH^l_x}\|u\|_{L^2_tB^s_4},
\end{aligned}\tag{5.1}$$

for some constants $\theta(s,l)>0$, $C_0(s,l)>0$ and $C_1(K,s,l)>0$. Note that $C_0\to\infty$ as $(s,l)\to(1,0)$ in the third and the last estimates, and the small factor $2^{-\theta K}$ is

lost for $s = l + 1$ in the third estimate. Anyway, taking $K = K(s, l)$ large ensures smallness of the right side in the latter 4 estimates:

$$C_0 2^{-\theta K} \left\{ \|N\|_{L_t^\infty H_x^l} + \|u\|^2_{L_t^2 B_4^{1/2}} + \|N\|^2_{L_t^\infty H_x^l} \right\} \ll 1. \tag{5.2}$$

After fixing such K, choose $\varepsilon > 0$ such that $C_1(K)\varepsilon \ll 1$, and apply Lemma 4.1 to N, which yields a finite sequence $0 = T_0 < T_1 < \cdots < T_{n+1} = T$ such that

$$\|N\|_{(L_t^\infty L_x^2 + L_t^2 L_x^4)(T_j, T_{j+1})} < \varepsilon. \tag{5.3}$$

Then on each subinterval we obtain from the above estimates

$$\|u\|_{X^s(T_j, T_{j+1})} \le C_2 \|u(T_j)\|_{H^s} + \|u\|_{X^s(T_j, T_{j+1})}/2, \tag{5.4}$$

for some constant $C_2(s) > 0$. Hence if $u(0) \in H^s$, then by induction on j, we deduce that $u \in X^s([0, T))$. If $T = \infty$, this implies the scattering of u in H^s, via the argument in Section 3.

For continuous dependence on the initial data, consider a sequence of solutions (u_n, N_n) such that $(u_n(0), N_n(0)) \to (u(0), N(0))$ in $H^s \times H^l$, $u_n \to u$ in $X^{1/2}(I)$ and $N_n \to N$ in $Y^l(I)$ for some interval $I \subset [0, T)$. For large n, (u_n, N_n) satisfies similar bounds to (5.2) and (5.3) within I, with slightly bigger bounds. Then the same estimates as above for $(u_n - u, N_n - N)$ yield the convergence in $(X^s \times Y^l)(I)$.

5.2. **Regularity upgrade for N in $s < l + 1$.** Let $(s, l) \in \mathbb{R}^2$ satisfy (1.11) and $s < l + 1$. Let $(u_0, N_0) \in H^s \times H^l$ and let $(u, N) \in (X^s \times Y^{l'})([0, T))$ for some $0 < T \le \infty$ and some $l' \in (s - 1, l)$. From the estimates in Section 2, we have

$$\begin{aligned}
&\|D(|u|^2)_{HH+L\alpha+\alpha L}\|_{L_t^1 H_x^l} \le C_1(K)\|u\|^2_{L_t^2 B_4^s}, \\
&\|D\widetilde{\Omega}(u, u)\|_{L_t^\infty H_x^l} \le C_0 \|u\|^2_{L_t^\infty H_x^s}, \\
&\|D\widetilde{\Omega}(Nu, u)\|_{L_t^1 H_x^l} + \|D\widetilde{\Omega}(u, Nu)\|_{L_t^1 H_x^l} \le C_0 \|N\|_{L_t^\infty H_x^l} \|u\|^2_{L_t^2 B_4^s},
\end{aligned} \tag{5.5}$$

for some constants $C_0(s, l) > 0$ and $C_1(K, s, l) > 0$, and the same for $D\widetilde{\Omega}(u, Nu)$. Choose $\varepsilon > 0$ so small that $C_0\varepsilon^2 \ll 1$. Since $u \in L_t^2 B_4^s(0, T)$, there exists a finite sequence $0 = T_0 < T_1 < \cdots < T_{n+1} = T$ such that

$$\|u\|_{L_t^2 B_4^s(T_j, T_{j+1})} < \varepsilon \tag{5.6}$$

for each j. Then on each subinterval we have from the above estimates

$$\begin{aligned}
\|N\|_{L_t^\infty H_x^l(T_j, T_{j+1})} \le{}& C_2 \|N(T_j)\|_{H^l} + \|N\|_{L_t^\infty H_x^l(T_j, T_{j+1})}/2 \\
&+ C_1(K)\varepsilon^2 + C_0 \|u\|^2_{L_t^\infty H_x^s(T_j, T_{j+1})},
\end{aligned} \tag{5.7}$$

for some constant $C_2(l) > 0$. Hence if $N(0) \in H^l$, then by induction on j, we deduce that $N \in L_t^\infty H^l(0, T)$. If $T = \infty$, then we have the scattering of N from the argument in Section 3. We also obtain the Strichartz norm of N using (2.15) for the normal form. We can also upgrade continuous dependence, using the same estimates for the difference from the limit, see the previous subsection for more detail. Combining the results in this and the previous subsections yields

Proposition 5.1. *Let $(s, l) \in \mathbb{R}^2$ satisfy (1.11) and $s < l + 1$. Let $(u, N) \in (X^{1/2} \times Y^0)(I)$ be a solution of (1.9) on an interval $I \subset \mathbb{R}$, and suppose that*

$(u(t_0), N(t_0)) \in H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4)$ at some $t_0 \in I$. Then $(u, N) \in (X^s \times Y^l)(I)$ and moreover,

$$N \in L_t^2(I; \dot{B}_6^{l-5/6} \cap \dot{B}_6^{-5/6}). \tag{5.8}$$

If $I \supset (t_0, \infty)$, then (u, N) scatters in $H^s \times H^l$ as $t \to \infty$. If $(u_n(t_0), N_n(t_0)) \to (u(t_0), N(t_0))$ in $H^s \times H^l$ and the corresponding sequence of solutions $(u_n, N_n) \to (u, N)$ in $(X^{1/2} \times Y^0)(J)$ on some interval $t_0 \in J \subset I$, then the convergence holds in $(X^s \times Y^l)(J)$. The same convergence result holds for the scattering data, if $I \cap J \supset (t_1, \infty)$ for some $t_1 < \infty$.

5.3. **Regularity upgrade for u in $s \geq l + 1$.** Let $(s, l) \in \mathbb{R}^2$ satisfy $s \geq l + 1$. Then $l > 0$ and $s > 1$. Let $(u_0, N_0) \in H^s \times H^l$ and let $(u, N) \in (X^{s'} \times Y^l)([0, T))$ for some $0 < T \leq \infty$ and some $s' \in (1, s)$. In this case, the normal form estimate is not good enough to keep the full Strichartz bound of u. Hence we decompose

$$\begin{cases} u = u' + \Omega(N, u), \\ (i\partial_t - \Delta)u' = (Nu)_{LH\cdots} + \Omega(\alpha D|u|^2, u) + \Omega(N, Nu), \end{cases} \tag{5.9}$$

where $LH\cdots := LH + HH + \alpha L$ for brevity, and look for closed estimates in

$$\begin{aligned} &u' \in X^s, \quad u \in X' := L_t^\infty H_x^s \cap L_t^{2/(1-\gamma)} L_x^\infty, \\ &N \in L_t^\infty H_x^l \cap L_t^{2/\gamma} \mathcal{B}, \quad \mathcal{B} := \dot{B}_{q_1}^{l-5\gamma/6} \cap \dot{B}_{q_1}^{-5\gamma/6}, \end{aligned} \tag{5.10}$$

where $1/q_1 := 1/2 - \gamma/3$, for some $\gamma \in [0, 3/4]$ satisfying

$$\gamma + 1 < s, \quad 2l + \gamma/2 + 1 \geq s. \tag{5.11}$$

Such γ exists if and only if $1 < s < 4l + 1$ and $s \leq 2l + 11/8$. Also note that

$$X^s \subset L_t^{2/(1-\gamma)} B_{4/(1+\gamma)}^s \subset L_t^{2/(1-\gamma)} L_x^\infty \tag{5.12}$$

since $\gamma + 1 < s$. Similarly, $L_t^{2/\gamma} \mathcal{B}$ is a wave-Strichartz norm in H^l, cf. (5.8).

We write $(Nu)_{LH\cdots} = (Nu')_{LH\cdots} + (N\Omega(N, u))_{LH\cdots}$. From the estimates in Section 2, we have for $k > 2$,

$$\begin{aligned} &\|(Nu')_{LH\cdots}\|_{L_t^2 B_{4/3}^s + L_t^1 H_x^s} \leq C_1(K) \|N\|_{L_t^\infty L_x^2 + L_t^2 L_x^4} \|u'\|_{L_t^2 B_4^s}, \\ &\|\Omega(D|u|^2, u)\|_{L_t^1 H_x^s} \leq C_0 2^{-\theta K} \|u\|_{L_t^2 B_4^{1/2}}^2 \|u\|_{L_t^\infty H_x^s}, \\ &\|P_{>k} \Omega(N, u)\|_{L_t^\infty H_x^s} \leq C_0 \|N_{>k-1}\|_{L_t^\infty H_x^l} \|u\|_{L_t^\infty H_x^s}, \end{aligned} \tag{5.13}$$

for some constants $C_0(s, l) > 0$, $\theta(s) > 0$ and $C_1(K, s, l) > 0$. We need some more estimates. Since $H^{l+2} \subset L_x^\infty$, we have for $k > 2$,

$$\|P_{>k} \Omega(N, u)\|_{L_t^{2/(1-\gamma)} L_x^\infty} \leq C_0 \|N_{>k-1}\|_{L_t^\infty H_x^l} \|u\|_{L_t^{2/(1-\gamma)} L_x^\infty}. \tag{5.14}$$

It remains to estimate $\Omega(N, Nu), (N\Omega(N, u))_{LH\cdots}$. If $\mathcal{B} \subset L_x^4$, then for $k \gg \langle \log \alpha \rangle$,

$$\begin{aligned} &\|P_{>k} \Omega(N, Nu)\|_{B_{4/3}^{l+2}} \lesssim \|N_{>k-1}\|_{H_x^l} \|Nu\|_{L_x^4} \lesssim \|N_{>k-1}\|_{H_x^l} \|N\|_{\mathcal{B}} \|u\|_{L_x^\infty}, \\ &\|P_{>k}(N\Omega(N, u))_{LH\cdots}\|_{B_{4/3}^{l+2}} \lesssim \|N\|_{\mathcal{B}} \|N_{>k-K-3}\|_{H_x^l} \|u\|_{L_x^\infty}. \end{aligned} \tag{5.15}$$

If $\mathcal{B} \not\subset L_x^4$ but $l \geq 5\gamma/6$, then putting

$$1/q_2 := 1/q_1 - (l - 5\gamma/6)/4 = 1/2 - \gamma/8 - l/4, \quad 1/q_3 := 1/2 + 1/q_2, \tag{5.16}$$

we have $\mathcal{B} \subset L^{q_2}$, $B^s_{4/3} \supset B^{2l+\gamma/2+1}_{4/3} \supset B^{l+2}_{q_3}$, and so

$$\begin{aligned}
&\|P_{>k}\Omega(N,Nu)\|_{B^s_{4/3}} \lesssim \|N_{>k-1}\|_{H^l_x}\|Nu\|_{L^{q_2}} \lesssim \|N_{>k-1}\|_{H^l_x}\|N\|_{\mathcal{B}}\|u\|_{L^\infty_x},\\
&\|P_{>k}(N\Omega(N,u))_{LH\cdots}\|_{B^s_{4/3}} \lesssim \|N\|_{\mathcal{B}}\|N_{>k-K-3}\|_{H^l}\|u\|_{L^\infty_x}.
\end{aligned} \tag{5.17}$$

If $l < 5\gamma/6$, then using

$$B^s_{4/3} \supset B^{2l+\gamma/2+1}_{4/3} \supset B^{2l-5\gamma/6+2}_{q_4}, \quad 1/q_4 := 1/2 + 1/q_1 = 1 - \gamma/3, \tag{5.18}$$

we have

$$\begin{aligned}
\|P_{>k}(N\Omega(N,u))_{LH\cdots}\|_{B^s_{4/3}} &\lesssim \|N\|_{\mathcal{B}}\|\Omega(N,u)_{>k-K-2}\|_{H^{l+2}}\\
&\lesssim \|N\|_{\mathcal{B}}\|N_{>k-K-3}\|_{H^l}\|u\|_{L^\infty_x}.
\end{aligned} \tag{5.19}$$

For the other term, putting $\sigma := 5\gamma/6 - l > 0$ and $\beta := l/(l+\sigma) \in (0,1)$, we have the complex interpolation

$$[H^l, B^{-\sigma}_{q_1}]_\beta = B^0_{q_5} \subset L^{q_5}, \quad [H^l, B^{-\sigma}_{q_1}]_{1-\beta} = B^{l-\sigma}_{q_6}, \tag{5.20}$$

where $1/q_5 := (1-\beta)/2 + \beta/q_1$ and $1/q_6 := 1/q_4 - 1/q_5$, whereas

$$\begin{aligned}
&\|P_{>k}\Omega(N,Nu)\|_{B^s_{4/3}} \lesssim \|P_{>k}\Omega(N,Nu)\|_{B^{l-\sigma+2}_{q_4}}\\
&\lesssim \|N_{>k-1}\|_{B^{l-\sigma}_{q_6}}\|Nu\|_{L^{q_5}} \lesssim \|N_{>k-1}\|_{B^{l-\sigma}_{q_6}}\|N\|_{B^0_{q_5}}\|u\|_{L^\infty_x}.
\end{aligned} \tag{5.21}$$

Hence by the interpolation inequality,

$$\|\Omega(N,Nu)\|_{B^s_{4/3}} \lesssim \|N_{>k-1}\|^\beta_{H^l}\|N\|^{1-\beta}_{H^l}\|N\|_{\mathcal{B}}\|u\|_{L^\infty_x}. \tag{5.22}$$

Therefore, in any case we have some $\beta(l,\gamma) \in (0,1]$ such that

$$\begin{aligned}
&\|P_{>k}\Omega(N,Nu)\|_{L^2_t B^s_{4/3}} \le C_2\|N_{>k-1}\|^\beta_{L^\infty_t H^l_x}\|N\|^{1-\beta}_{L^\infty_t H^l_x}\|N\|_{L^{2/\gamma}_t\mathcal{B}}\|u\|_{X'},\\
&\|P_{>k}(N\Omega(N,u))_{LH\cdots}\|_{L^2_t B^s_{4/3}} \le C_2\|N_{>k-K-3}\|_{L^\infty_t H^l_x}\|N\|_{L^{2/\gamma}_t\mathcal{B}}\|u\|_{X'},
\end{aligned} \tag{5.23}$$

for some constant $C_2(s,l) > 0$. Choose $K \gg 1$ so large that $C_0 2^{-\theta K}\|u\|^2_{L^2_t B^{1/2}_4} \ll 1$, and then choose $\varepsilon > 0$ so small and $k \gg K$ so large that

$$\begin{aligned}
&C_1(K)\varepsilon + C_0\|N_{>k-1}\|_{L^\infty_t H^l_x} \ll 1,\\
&C_2\|N_{>k-K-3}\|^\beta_{L^\infty_t H^l_x}\|N\|^{1-\beta}_{L^\infty_t H^l_x}\|N\|_{L^{2/\gamma}_t\mathcal{B}} \ll 1.
\end{aligned} \tag{5.24}$$

Applying Lemma 4.1, we obtain a finite sequence $0 = T_0 < T_1 < \cdots < T_{n+1} = T$ such that (5.3) holds. Then from the above estimates on each subinterval,

$$\begin{aligned}
&\|u'_{>k}\|_{X^s(T_j,T_{j+1})} \le C_3\|u'(T_j)\|_{H^s} + \delta\|u'\|_{X^s(T_j,T_{j+1})} + \delta\|u\|_{X'(T_j,T_{j+1})},\\
&\|\Omega(N,u)_{>k}\|_{X'(T_j,T_{j+1})} \le \delta\|u\|_{X'(T_j,T_{j+1})},
\end{aligned} \tag{5.25}$$

for some small constant $\delta > 0$, while the frequencies $\le k$ are bounded by $X^{1/2}$. Using $u = u' + \Omega(N,u)$ and $X^s \subset X'$, and adding the low frequencies, we obtain

$$\begin{aligned}
&\|u'\|_{X^s(T_j,T_{j+1})} + \|\Omega(N,u)\|_{X'(T_j,T_{j+1})}\\
&\quad\le 2C_3\|u'(T_j)\|_{H^s} + 2^{k(s-1/2)}\|u\|_{X^{1/2}(T_j,T_{j+1})}.
\end{aligned} \tag{5.26}$$

By induction on j starting from $\|u'(0)\|_{H^s} < \infty$, we thus obtain

$$\|u\|_{X'(0,T)} \lesssim \|u'\|_{X^s(0,T)} + \|\Omega(N,u)\|_{X'(0,T)} < \infty. \tag{5.27}$$

If $T = \infty$, then u is scattering by the argument in Section 3. Thus we have obtained

Proposition 5.2. *Let $(s,l) \in \mathbb{R}^2$ satisfy (1.11) and $s \geq l+1$. Let $(u,N) \in (X^{1/2} \times Y^0)(I)$ be a solution of (1.9) on an interval $I \subset \mathbb{R}$, and suppose that $(u(t_0), N(t_0)) \in H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4)$ at some $t_0 \in I$. Then we have $u - \Omega(N,u) \in X^s(I)$, as well as (5.8), and for all $\gamma \in [0, 3/4]$ satisfying (5.11),*

$$u \in C(I; H^s_x) \cap L^\infty_t(I; H^s_x) \cap L^{2/(1-\gamma)}(I; L^\infty_x). \tag{5.28}$$

We also have scattering and continuous dependence similar to Proposition 5.1, but in the above space (5.28).

It is easy to replace L^∞_x with $B^s_{4/(1+\gamma)} + H^{l+2}$ using $u' \in X^s$ and (5.14).

5.4. **Lipschitz continuity of the solution map.** Here we consider local Lipschitz continuity of the flow map. In the above arguments, the Lipschitz dependence is lost only when we seek time intervals with smallness, typically by Lemma 4.1. If $(u_0, N_0) \in H^s \times H^l$ with $s > 1/2$ and $l > 0$, however, it is easy to see that (4.5) holds locally uniformly with respect to the initial data, because we can first dispose of the high frequencies using the higher regularity, and then the remaining low frequencies by Sobolev in x and Hölder in t.

Similarly, the regularity upgrading argument in Section 5.1 works uniformly if $l > 0$ and $T < \infty$, because of (5.3), so does the argument in Section 5.2 for $s > 1/2$, $l < \min(2s-1, s+1)$, and $T < \infty$, because of (5.6), as well as that in Section 5.3 for $l > 0$, $s < \min(2l + 11/8, l+2)$, and $T < \infty$, because of (5.24) and (5.3).

Thus we obtain Lipschitz continuity of the flow map, locally both in time and in the initial data, for all the exponents (s,l) in the range and off the boundary. Since we need to decrease l for the uniform control in (5.24), γ in (5.10) can not be on the boundary, namely $2l + \gamma/2 + 1 > s$, for the local Lipschitz estimate.

The Lipschitz continuity global in time and for the scattering is more tricky, because the $L^2_t L^4_x$ norm in Lemma 4.1 is not bounded by the Strichartz estimate for the wave equation. For small data, we can obtain Lipschitz estimates directly from the contraction mapping argument, but then the smallness on $H^{1/2} \times L^2$ depends on (s,l), which tends to 0 as (s,l) approaches $s = 4l+1$, $(2,3)$ or (∞, ∞). The regularity upgrading for N in Section 5.2 works well for $T = \infty$, because in (5.6) the number of subintervals can be uniformly bounded for each $\varepsilon > 0$, provided that $\|u\|_{L^2_t B^s_4}$ is uniformly bounded. This yields a smallness condition in the form

$$\|(u_0, N_0)\|_{H^{1/2} \times L^2} \leq \varepsilon_2(s,l), \tag{5.29}$$

where $\varepsilon_2(s,l) > 0$ is non-decreasing in l, for global Lipschitz continuity in $H^s \times H^l$.

Remark 5.3. Strictly speaking, we need to prove that the solution to (2.5) obtained above is also a solution of the equation (1.9) before the normal form. The easiest way is to use [5] for existence of solutions for smooth approximating initial data, taking the limit by the continuous dependence proved above. To be self-contained, however, we can directly show that smooth solutions of (2.5) solve (1.9). In fact, if $(u,N) \in (X^s \times Y^s)(I)$ with $s \gg 1$ is a solution of (2.5) on some interval I, then by definition of Ω and $\widetilde{\Omega}$, (2.5) reads

$$\begin{aligned} eq_u &:= (i\partial_t + D^2)u - Nu = -\Omega(eq_N, u) - \Omega(N, eq_u), \\ eq_N &:= (i\partial_t + \alpha D)N - \alpha D|u|^2 = -D\widetilde{\Omega}(eq_u, u) - D\widetilde{\Omega}(u, eq_u). \end{aligned}$$

Since $eq_u, eq_N \in C(I; H^{s-2})$ and $\Omega, D\widetilde{\Omega} : (H^{s-2})^2 \to H^{s-2}$ has a small factor due to K, we deduce that $eq_u = 0 = eq_N$ on I if K is large enough.

6. Small data scattering in the energy space

For $(s,l)=(1,0)$, the failure of Strichartz bound on the normal form $\Omega(N,u)$ cannot be compensated by regularity of N, and so there seems no way to close the estimates as above for $(s,l)=(1,0)$. Instead, we invoke the conservation laws with the weak compactness argument. This type of argument usually yields a weak result, typically without uniqueness. We can however obtain the strong well-posedness for small data as in Theorem 1.1, thanks to that both in the larger space $(s,0)$ with $s<1$ and in the smaller space $(1,l)$ with $l>0$.

Assume $(u_0,N_0)\in H^1\times L^2$. By Proposition 3.1 there is $\varepsilon_0:=\varepsilon_1(1/2,0)\ll 1$ such that if $\|(u_0,N_0)\|_{H^{1/2}\times L^2}\le\varepsilon_0$ then there is a unique global solution (u,N) in $X^{1/2}\times Y^0$, satisfying

$$\|(u,N)\|_{X^{1/2}\times Y^0}\le C\varepsilon_0\ll 1. \tag{6.1}$$

Proposition 5.1 implies that $(u,N)\in X^s\times Y^0$ for all $s\in[1/2,1)$.

Fix a sequence $\{(u_{0,n},N_{0,n})\}\subset\mathcal{S}(\mathbb{R}^4)$ such that $(u_{0,n},N_{0,n})\to(u_0,N_0)$ in $H^1\times L^2$ and $\|(u_{0,n},N_{0,n})\|_{H^{1/2}\times L^2}\le\varepsilon_0$. By Proposition 3.1, for each n, there is a unique global solution (u_n,N_n) satisfying (6.1) and for all $1/2\le s<1$,

$$\sup_n\|(u_n,N_n)\|_{X^s\times Y^0}<\infty. \tag{6.2}$$

Now we claim a uniform bound at the energy level:

$$\sup_{n,t}\|(u_n(t),N_n(t))\|_{H^1\times L^2}<\infty. \tag{6.3}$$

By Proposition 5.1, we have $(u_n,N_n)\in X^8\times Y^9$ for all n, by which we can justify the conservation law $E_Z(u_n(t),N_n(t))=E_Z(u_{0,n},N_{0,n})$. Using (6.1) for N_n together with the Sobolev inequality $\|u\|_{L^4_x}\lesssim\|\nabla u\|_{L^2_x}$ yields

$$E_Z(u_n,N_n)=(1-O(\varepsilon_0))\|\nabla u_n\|_2^2+\|N_n\|_2^2/2, \tag{6.4}$$

which, combined with the lower regularity bound (6.2), implies (6.3).

Next we prove convergence $u_n(t)\to u(t)$ in H^1_x as $n\to\infty$, locally uniformly in $\mathbb{R}$. Take any convergent sequence $t_n\to t_\infty$. From Propositions 3.1 and 5.1, we know that $u_n(t_n)\to u(t_\infty)$ in H^s_x for $s<1$, and $N_n(t_n)\to N(t_\infty)$ in L^2_x. From (6.3), we have $\{u_n(t_n)\}_n$ is bounded in $H^1_x\subset L^4_x$, thus we get $u(t_\infty)\in H^1$, $u_n(t_n)\to u(t_\infty)$ weakly in H^1_x, and $|u_n(t_n)|^2\to|u(t_\infty)|^2$ weakly in L^2_x. Since $N_n(t_n)\to N(t_\infty)$ strongly in L^2_x, we have $\int N_n(t_n)|u_n(t_n)|^2dx\to\int N(t_\infty)|u(t_\infty)|^2dx$, and so,

$$\begin{aligned}E_Z(u(t_\infty),N(t_\infty))&\le\liminf_{n\to\infty}E_Z(u_n(t_n),N_n(t_n))\\&=\liminf_{n\to\infty}E_Z(u_{0,n},N_{0,n})=E_Z(u_0,N_0).\end{aligned} \tag{6.5}$$

By the time reversibility we get $E_Z(u(t_\infty),N(t_\infty))=E_Z(u_0,N_0)$. Indeed, if there is a $t_0\in\mathbb{R}$ such that $E_Z(u(t_0),N(t_0))<E_Z(u_0,N_0)$, then we solve the Zakharov system with initial data $(u(t_0),N(t_0))$ at $t=t_0$. By the uniqueness we get a contradiction. Then the equality in (6.5) implies $\|\nabla u_n(t_n)\|_{L^2}\to\|\nabla u(t_\infty)\|_{L^2}$, from which we conclude that $u_n(t_n)\to u(t_\infty)$ strongly in H^1_x, and so the locally uniform convergence $u_n\to u$ in $C(\mathbb{R};H^1_x)$. Thus we obtain the unique global solution $(u,N)\in(C\cap L^\infty)(\mathbb{R};H^1\times L^2)$. Note that the smoothness of the approximate solutions (u_n,N_n) was used only to ensure the unique existence and the conservation law. Now that we have them for the solutions in the energy space, we can apply the

above argument to a sequence of initial data in $H^1 \times L^2$, which implies continuous dependence of the initial data, locally uniformly in time.

By Propositions 3.1 and 5.1, (u, N) scatters to some (u^+, N^+) in $H^s \times L^2$ for all $s < 1$. Since $u(t) \in L^\infty(\mathbb{R}; H^1_x)$, we have $S(-t)u(t) \to u^+$ weakly in H^1 as $t \to +\infty$. Since $|u(t)|^2$ is bounded in $(H^1_x)^2 \subset B^1_{4/3}$, while $N(t)$ is vanishing in B^{-1}_4 as $t \to \infty$ due to the scattering in $L^2_x \subset B^{-1}_4$, we have

$$\int N(t)|u(t)|^2 dx \to 0 \quad (t \to +\infty), \tag{6.6}$$

and so

$$\begin{aligned}\|\nabla u^+\|_2^2 + \|N^+\|_2^2/2 &\le \liminf_{t\to+\infty} \|\nabla S(-t)u(t)\|_2^2 + \|W_\alpha(-t)N(t)\|_2^2/2 \\ &= \liminf_{t\to+\infty} \|\nabla u(t)|_2^2 + \|N(t)\|_2^2/2 \\ &= \liminf_{t\to+\infty} E_Z(u(t), N(t)) = E_Z(u_0, N_0).\end{aligned}$$

To prove the equality above, we consider the final state problem. Following the argument in Step 1, we fix a sequence $\{(u_n^+, N_n^+)\} \subset \mathcal{S}(\mathbb{R}^4)$ such that $(u_n^+, N_n^+) \to (u^+, N^+)$ in $H^1 \times L^2$. Then by Proposition 3.1, we have a sequence of solutions $(\widetilde{u}_n, \widetilde{N}_n) \in X^{1/2} \times Y^0$ scattering to (u_n^+, N_n^+) as $t \to \infty$, which converges to (u, N) in $X^{1/2} \times Y^0$ as $n \to \infty$. The regularity is upgraded to $X^s \times Y^l$ for all (s, l) in Proposition 5.1. As in Step 1, we have $\sup_{n,t} \|(\widetilde{u}_n, \widetilde{N}_n)\|_{H^1\times L^2} < \infty$, hence $u_n(t) \to u(t)$ weakly in H^1_x. Thus by (6.6)

$$\begin{aligned}E_Z(u(t), N(t)) \le \liminf_{n\to\infty} E_Z(\widetilde{u}_n(t), \widetilde{N}_n(t)) &= \liminf_{n\to\infty} \|\nabla u_n^+|_2^2 + \|N_n^+\|_2^2/2 \\ &= \|\nabla u^+|_2^2 + \|N^+\|_2^2/2.\end{aligned}$$

Hence we get

$$\lim_{t\to+\infty} E_Z(u(t), N(t)) = \|\nabla u^+\|_2^2 + \|N^+\|_2^2/2$$

and so, $S(-t)u(t) \to u^+$ strongly in H^1_x, namely the scattering in H^1_x.

To show the continuity of the solution map in $L^\infty_t(\mathbb{R}; H^1_x)$, it remains to prove $u_n(t_n) - u(t_n) \to 0$ in H^1_x in the case $t_n \to \infty$, for a sequence of solutions (u_n, N_n) in the energy space such that $(u_n(0), N_n(0)) \to (u(0), N(0))$ in $H^1 \times L^2$. Since $S(-t)u(t) \to u^+$ in H^1_x, it is equivalent to showing $S(-t_n)u_n(t_n) \to u^+$ in H^1_x. We already know the H^s_x convergence for $s < 1$ as well as the weak convergence in H^1_x. Then the strong convergence is equivalent to $\|u_n(t_n)\|_{H^1_x} \to \|u^+\|_{H^1}$. Since

$$\|N_n(t_n)\|_{B^{-1}_4} \le \|N_n - N\|_{L^\infty_t L^2_x} + \|N(t_n)\|_{B^{-1}_4} \to 0, \tag{6.7}$$

we have $\int N_n(t_n)|u_n(t_n)|^2 dx \to 0$, and so, as $n \to \infty$,

$$\begin{aligned}&\|\nabla u_n(t_n)\|_2^2 + \|N_n(t_n)\|_2^2/2 = E_Z(u_n(t_n), N_n(t_n)) + o(1) \\ &= E_Z(u_n(0), N_n(0)) + o(1) = E_Z(u(0), N(0)) + o(1) \\ &= \|\nabla u^+\|_2^2 + \|N^+\|_2^2/2 + o(1).\end{aligned} \tag{6.8}$$

Since $\|u_n(t_n)\|_2 \to \|u^+\|_2$ and $\|N_n(t_n)\|_2 \to \|N^+\|_2$, the above implies the strong convergence of $S(-t_n)u_n(t_n)$ in H^1_x, and thus $u_n \to u$ in $L^\infty_t(\mathbb{R}; H^1_x)$. This completes the proof of Theorem 1.1 in the case $(s, l) = (1, 0)$.

7. Ill-posedness at $(s,l)=(2,3)$

In this section, we prove Theorem 1.3. The main point is that the multilinear estimates fail only for the boundary quadratic term coming from the initial data. Exploiting the dispersive smoothing, we can prove that the other terms are more regular if the initial data is localized in space.

Proof of Theorem 1.3. First of all, for any initial data $(u_0,N_0)\in H^2\times H^3$, we have a unique local solution for (s,l) in (1.11) satisfying $s\le 2$ and $l\le 3$, say $(u,N)\in (X^2\times Y^2)([0,T])$, by Propositions 4.2 and 5.1. In the Duhamel formula (2.4), the first term on the right is obviously in $C(\mathbb{R};H^3)$. The integral terms are regular thanks to the high regularity. Indeed,

$$\begin{aligned}&\|D|u|^2_{HH+\alpha L+L\alpha}\|_{L^1_tH^3_x}\lesssim \|u\|^2_{L^2_tB^2_4},\\ &\|D\widetilde{\Omega}(Nu,u)\|_{L^1_tH^3_x}\lesssim \|N\|_{L^\infty_tH^2_x}\|u\|^2_{L^2_tB^2_4},\end{aligned}\tag{7.1}$$

and the same for $D\widetilde{\Omega}(u,Nu)$. To bound $D\widetilde{\Omega}(u,u)$ in H^3_x, we use local smoothing for u, assuming that

$$u_0\in W^{2,1}(\mathbb{R}^4)=\{f:\partial^\alpha f\in L^1(\mathbb{R}^4)\text{ for }|\alpha|\le 2\}.\tag{7.2}$$

Then $S(t)u_0\in C((0,\infty);B^2_p)$ for all $p>2$ by the dispersive L^p_x decay estimate for $S(t)$. Moreover, in the Duhamel formula (2.3) of u, the terms except for $(Nu)_{LH\dots}$ easily gain better regularity by

$$\begin{aligned}&\|\Omega(N,u)\|_{H^3_x}\lesssim \|N\|_{H^2_x}\|u\|_{H^2_x},\\ &\|\Omega(D|u|^2,u)\|_{L^1_tH^3_x}\lesssim \|u\|_{L^\infty_tH^2_x}\|u\|^2_{L^2_tB^2_4},\\ &\|\Omega(N,Nu)\|_{L^2_tB^4_{4/3}}\lesssim \|N\|^2_{L^\infty_tH^2_x}\|u\|_{L^2_tB^2_4}.\end{aligned}\tag{7.3}$$

The remaining term is bounded in $C([0,T];B^2_3)$ by

$$\begin{aligned}\|\int_0^t S(t-s)(Nu)_{LH\dots}ds\|_{B^2_3}&\lesssim \int_0^t|t-s|^{2/3}\|(Nu)_{LH\dots}\|_{B^2_{3/2}}ds\\ &\lesssim \int_0^t|t-s|^{2/3}\|N(s)\|_{L^6_x}\|u(s)\|_{H^2_x}ds.\end{aligned}\tag{7.4}$$

Gathering the above estimates, we obtain $u\in C((0,T];H^2\cap B^2_3)$. Since $B^2_3\subset L^\infty$,

$$\|D\widetilde{\Omega}(u,u)\|_{H^3}\lesssim \|u\|_{H^2}\|u\|_{B^2_3},\tag{7.5}$$

and plugging this into the above estimates for N, we deduce that

$$N-W_\alpha(t)D\widetilde{\Omega}(u_0,u_0)\in C((0,T];H^3_x),\tag{7.6}$$

if $u_0\in H^2\cap W^{2,1}(\mathbb{R}^4)$. Hence it suffices to find such a u_0 that $D\widetilde{\Omega}(u_0,u_0)\notin H^3_x$. It is constructed in the next Lemma 7.1. Then $N(t)\notin H^3_x$ for all $0<t<T$, namely the instant exit or the latter part of the theorem.

Thanks to the high regularity, it is easy to translate it to non-existence. Indeed, if $(u,N)\in L^2((0,T);H^1\times H^3)$ then from the equation without the normal form,

$$Nu\in L^1_tH^1_x\implies u\in C_tH^1_x\cap L^2_tB^1_4\implies D|u|^2\in L^1_tL^2_x\implies N\in C_tL^2_x.\tag{7.7}$$

In particular, (u,N) belongs to the uniqueness class at $(s,l)=(1/2,0)$. Hence it should be identical with the exiting solution obtained above, satisfying $N(t)\notin H^3$ for all $t\neq 0$, contradicting $N\in L^2_t((0,T);H^3_x)$. □

It remains to prove the failure of the bilinear estimate:

Lemma 7.1. *There is a radial $u \in (H^2 \cap W^{2,1})(\mathbb{R}^4)$ satisfying $D\widetilde{\Omega}(u,u) \notin H^3(\mathbb{R}^4)$.*

This failure of bilinear estimate comes from that $H^2(\mathbb{R}^4)$ is not an algebra, but we should be careful about cancellation in the nonlinearity. In fact, the proof below implies that $D\widetilde{\Omega}(u,u)$ is bounded in H^3 for real-valued or purely imaginary $u \in H^2$.

Proof. Modulo a bounded operator, the symbol of $D\widetilde{\Omega}$ can be approximated

$$\frac{\alpha|\xi|}{|\xi-\eta|^2-|\eta|^2 \mp \alpha|\xi|} = \frac{\alpha}{|\xi|} + \frac{\alpha(2\xi\cdot\eta \pm \alpha|\xi|)}{|\xi|(|\xi-\eta|^2-|\eta|^2 \mp \alpha|\xi|)} \tag{7.8}$$

in the XL frequency, while in the LX frequency,

$$\frac{\alpha|\xi|}{|\xi-\eta|^2-|\eta|^2 \mp \alpha|\xi|} = \frac{-\alpha}{|\xi|} + \frac{\alpha(2\xi\cdot(\xi-\eta) \pm \alpha|\xi|)}{|\xi|(|\xi-\eta|^2-|\eta|^2 \mp \alpha|\xi|)}, \tag{7.9}$$

where the second terms are $O(|\xi|^{-2}\langle Low\rangle)$ for $|\xi| \gg 1$, and so bounded $H^2 \times H^2 \to H^3$ for high frequency. Hence it suffices to construct $u \in H^2 \cap W^{1,2}$ such that $\operatorname{supp}\hat{u}(\xi) = 0$ for $|\xi| \lesssim 1$ and

$$(u\bar{u})_{HL} - (u\bar{u})_{LH} \notin H^2(\mathbb{R}^4). \tag{7.10}$$

Indeed, this is necessary and sufficient for $D\widetilde{\Omega}(u,u) \notin H^3$ under the condition of $\operatorname{supp}\hat{u}$. Note that the left side is simply zero if $u(\mathbb{R}^4) \subset \mathbb{R}$ or $iu(\mathbb{R}^4) \subset \mathbb{R}$. The remaining is the anti-symmetric part, which can be expanded by putting $u = v + iw$

$$(u\bar{u})_{HL} - (u\bar{u})_{LH} = 2i[(wv)_{HL} - (wv)_{LH}]. \tag{7.11}$$

Now it is easy to avoid the cancellation considering the form

$$v = \sum_{j>J} a_j\varphi_j, \quad w = \sum_{j>J} b_j\varphi_j, \quad \varphi_j(x) = \varphi(2^j x), \tag{7.12}$$

where $J \gg \log\alpha$, $\{a\},\{b\} \subset [0,\infty)$, and $\varphi \in \mathcal{S}(\mathbb{R}^4;\mathbb{R})$ is a non-zero real-valued radial function satisfying

$$0 \le \hat{\varphi} \le 1, \quad \operatorname{supp}\hat{\varphi} \subset \{||\xi|-1| \ll 1\}. \tag{7.13}$$

Put $c := \varphi(0) > 0$. Injecting the above ansatz expands the bilinear form

$$(vw)_{HL} - (vw)_{LH} = \sum_{j>J}\sum_{k>J}^{j-K} (a_jb_k - a_kb_j)\varphi_j\varphi_k. \tag{7.14}$$

Since $\mathcal{F}(\varphi_j\varphi_k)$ is supported around $|\xi| = 2^j$,

$$\|(7.14)\|_{H^2}^2 \sim \sum_{j>J} \|2^{2j} \sum_{k\le j-K} (a_jb_k - a_kb_j)\varphi_j\varphi_k\|_2^2. \tag{7.15}$$

Imposing a support condition on $\{a\},\{b\}$

$$\operatorname{supp} a \cap \operatorname{supp} b = \emptyset, \tag{7.16}$$

we can decouple the above as

$$\|(7.14)\|_{H^2}^2 \sim \sum_{j>J} \|2^{2j} \sum_{k\le j-K} a_jb_k\varphi_j\varphi_k\|_2^2 + \sum_{j>J} \|2^{2j} \sum_{k\le j-K} b_ja_k\varphi_j\varphi_k\|_2^2. \tag{7.17}$$

By rescaling $x \mapsto 2^{-j}x$, and using $\varphi(2^{k-j}x) = c + O(|2^{k-j}x|)$, the L^2_x norm is approximated by

$$\begin{aligned} &\|2^{2j}\sum_{k\le j-K} a_j b_k \varphi_j \varphi_k\|_{L^2_x} = \|a_j\varphi(x)\sum_{k\le j-K} b_k \varphi(2^{k-j}x)\|_{L^2_x} \\ &\ge c|a_j|\|\varphi\|_{L^2_x}\sum_{k\le j-K} b_k - C\|a_j x\varphi(x)\|_{L^2_x}\sum_{k\le j-K} b_k 2^{k-j}. \end{aligned} \tag{7.18}$$

Fix $\theta \in (1/2, 3/4)$ and let

$$a_j = \begin{cases} j^{-\theta} & J < j \text{ is even} \\ 0 & \text{otherwise,} \end{cases} \qquad b_j = \begin{cases} j^{-\theta} & J < j \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \tag{7.19}$$

Then for $j > K + J$,

$$\sum_{k\le j-K} b_k \sim (j-K)^{1-\theta}, \qquad \sum_{k\le j-K} b_k 2^{k-j} \lesssim 2^{-K}, \tag{7.20}$$

and so

$$\|(7.14)\|_{H^2} \gtrsim \|j^{-\theta}(j-K)^{1-\theta}\|_{\ell^2(j>J+K)} - C2^{-K}\|j^{-\theta}\|_{\ell^2(j>J)} = \infty, \tag{7.21}$$

since $-\theta < -1/2 < 1 - 2\theta$. Also we have

$$\|u\|_{H^2} \lesssim \|j^{-\theta}\|_{\ell^2(j>J)} < \infty, \quad \|u\|_{W^{2,1}} \lesssim \|2^{-2j}j^{-\theta}\|_{\ell^1(j>J)} < \infty. \tag{7.22}$$

Thus we have obtained a desired example $u \in H^2 \cap W^{2,1}$. □

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