Spatial birth-and-death Markov dynamics of finite particle systems *†

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August 6, 2015

Abstract

Spatial birth-and-death processes with time dependent rates are obtained as solutions to certain stochastic equations. Existence, uniqueness, uniqueness in law and the Markov property of unique solutions are proven when the integral of the birth rate over $\mathbb{R}^{\mathrm{d}}$ grows not faster than linearly with the number of points of the system. Martingale properties of the constructed process provide a formal connection to the heuristic generator.

We also study pathwise behavior of aggregation model. We estimate the probability of extinction and the speed of growth of the number of particles of the process conditioned on non-extinction.

1 Introduction

We consider spatial birth-and-death processes with time dependent rates. At each moment of time the system is represented as a finite collection of motionless points in $\mathbb{R}^{\mathrm{d}}$. We interpret the points as particles, or individuals. Existing particles may die and new particles may appear. Each particle is characterized by its location.

The state space of a spatial birth-and-death Markov process on $\mathbb{R}^{\mathrm{d}}$ with finite number of points is the space of finite subsets of $\mathbb{R}^{\mathrm{d}}$,

$$\Gamma_0(\mathbb{R}^{\mathrm{d}}) = \{\eta \subset \mathbb{R}^{\mathrm{d}} : |\eta| < \infty\},$$

where $|\eta|$ is the number of points of η.

*Keywords: *birth-and-death processes, population dynamics, aggregation model, stochastic integral equations*

†MSC subject classification: 60K35, 60J25

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Denote by $\mathscr{B}(\mathbb{R}^{\mathrm{d}})$ the Borel σ-algebra on $\mathbb{R}^{\mathrm{d}}$. The evolution of the spatial birth-and-death process on $\mathbb{R}^{\mathrm{d}}$ admits the following description. Two measurable functions characterize the development in time, the birth rate $b : \mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+ \times \Gamma_0(\mathbb{R}^{\mathrm{d}}) \to [0, \infty)$ and the death rate $d : \mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+ \times \Gamma_0(\mathbb{R}^{\mathrm{d}}) \to [0, \infty)$. If the system is in state $\eta \in \Gamma_0$ at time t, then the probability that a new particle appears (a "birth") in a bounded set $B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$ over time interval $[t; t + \Delta t]$ is

$$\Delta t \int_B b(x, t, \eta) dx + o(\Delta t),$$

the probability that a particle $x \in \eta$ is deleted from the configuration (a "death") over time interval $[t; t + \Delta t]$ is

$$d(x, t, \eta)\Delta t + o(\Delta t),$$

and no two events happen simultaneously. By an event we mean a birth or a death. Using a slightly different terminology, we can say that the rate at which a birth occurs in B is $\int_B b(x, t, \eta) dx$, the rate at which a particle $x \in \eta$ dies is $d(x, t, \eta)$, and no two events happen at the same time.

Such processes, in which the birth and death rates depend on the spatial structure of the system as opposed to classical $\mathbb{Z}_+$-valued birth-and-death processes (see e.g. [Har63, Page 116], [AN72, Page 109], and references therein), were first studied by Preston [Pre75]. A heuristic description similar to that above appeared already there. Our description resembles the one in [GK06].

We say that the rates b and d, or the corresponding birth-and-death process, are time-homogeneous if b and d do not depend on time. By abuse of notation we write in this case $b(x, s, \eta) = b(x, \eta)$, $d(x, s, \eta) = d(x, \eta)$. The (heuristic) generator of a time-homogeneous spatial birth-and-death process should be of the form

$$LF(\eta) = \int_{x \in \mathbb{R}^{\mathrm{d}}} b(x, \eta)[F(\eta \cup x) - F(\eta)]dx + \sum_{x \in \eta} d(x, \eta)(F(\eta \setminus x) - F(\eta)), \tag{1}$$

for F in an appropriate domain, where $\eta \cup x$ and $\eta \setminus x$ are shorthands for $\eta \cup \{x\}$ and $\eta \setminus \{x\}$, respectively.

Garcia and Kurtz [GK06] obtained birth-and-death processes as solutions to certain stochastic integral equations. The systems treated there involves an infinite number of particles. In the earlier work [LG95] of Garcia another approach was used: birth-and-death processes were obtained as projections of Poisson point processes. A further development of the projection method appears in [GK08]. Fournier and Méléard [FM04] used a similar equation for the construction of the Bolker–Pacala–Dieckmann–Law process with finitely many particles. Following ideas of

[GK06] and [FM04], we construct the birth-and-death process described above as solution to a stochastic equation.

Holley and Stroock [HS78] constructed the spatial birth-and-death process as a Markov family of unique solutions to the corresponding martingale problem. For the most part, they consider a process contained in a bounded volume, with bounded birth and death rates. They also proved the corresponding result for the nearest neighbor model in $\mathbb{R}^1$ with an infinite number of particles.

Belavkin and Kolokoltsov [BK03] discuss, among other things, a general structure of a Feller semigroup on disjoint unions of Euclidean spaces (see also references therein for the construction of the Markov processes with a given generator). We note that time-homogeneous birth-and-death processes need not have the Feller property. Eibeck and Wagner [EW03] discuss convergence of particle systems to limiting kinetic equations. In particular, they construct the stochastic process corresponding to the particle system as a minimal jump process, or pure jump type Markov process in the terminology of Kallenberg [Kal02]. The jump kernel is assumed to be locally bounded.

Kondratiev and Skorokhod [KS06] constructed a contact process in continuum with an infinite number of particles. The contact process in continuum can be described as the spatial birth-and-death process with

$$b(x,\eta) = \lambda \sum_{y\in\eta} a(x-y), \quad d(x,\eta) \equiv 1,$$

where $\lambda > 0$ and $0 \leq a \in L^1(\mathbb{R}^{\mathrm{d}})$. Under some additional assumptions, they showed existence of the process for a broad class of initial conditions.

The scheme proposed by Etheridge and Kurtz [EK14] covers a wide range of interactions and applies to discrete and continuous models. Their approach is based on, among other things, assigning a certain mark ('level') to each particle and letting this mark evolve according to some law. A critical event, such as birth or death, occurs when the level hit some threshold. Shcherbakov and Volkov [SV15] consider the long term behavior of birth-and-death processes on a finite graph with constant death rate and the birth rate of a special form. Bezborodov [Bez15] obtains the spatial time-homogeneous birth-and-death process as a unique solution to the equation slightly different from the one we use here. Various questions not treated in this paper are considered there, for example the possibility of an explosion, continuous dependence on initial conditions and related semigroup of operators.

In the aforementioned references as well as in the present work the system is represented by a Markov process. An alternative approach consists in using the concept of statistical dynamics that substitutes the notion of a Markov stochastic process. This approach is based on considering evolutions of measures and their correlation functions. For details see e.g. [FKK12], [FKKZ14], and references therein.

Finkelshtein et al. [FKKZ14] consider different aspects of statistical dynamics for the aggregation model. In this model the death rate is given by

$$d(x,\eta) = \exp\left(- \sum_{y \in \eta \setminus x} \phi(x-y) \right),$$

where ϕ is a positive measurable function. For more details see [FKKZ14] and references therein. In this paper we present a detailed long time microscopic description of the process. Also, we estimate the probability of extinction and the speed of growth of the average number of points.

The paper is organized as follows. Notation, definitions and results are given in Section 2. Proofs are given in Sections 3 and 4, with two auxiliary results located to Section 5.

2 The set-up and main results

2.1 Construction and basic properties

Let X be a locally compact separable metric space (typically X will be a subset of $\mathbb{R}^{\mathrm{d}}$). Denote by $\Gamma(X)$ the space of locally finite subsets of X

$$\Gamma(X) = \{\gamma \subset X \mid |\gamma \cap K| < \infty\},$$

also called *the space of configurations over* X. The space $\Gamma(X)$ can be endowed with a σ-field generated by the projection maps

$$\Gamma(X) \ni \gamma \mapsto |\gamma \cap B| \in \mathbb{Z}_+$$

where B is an arbitrary Borel subset of X. For more details about the notions introduced here see e.g. [DVJ08], [Kal02, Chapter 12] or [KK02], and references therein. Throughout this paper Γ_2 stands for $\Gamma(\mathbb{R}_+ \times \mathbb{R}_+)$. Let π be the distribution of a Poisson random measure on $(\Gamma_2, \mathscr{B}(\Gamma_2))$, with the intensity measure being the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$. We denote by $\mathscr{B}_t(\Gamma_2)$ the smallest sub-σ-algebra of $\mathscr{B}(\Gamma_2)$ such that for every $A_1 \in \mathscr{B}([0,t])$, $A_2 \in \mathscr{B}(\mathbb{R}_+)$ the map

$$\Gamma_2 \ni \gamma \mapsto \gamma(A_1 \times A_2) \in \mathbb{R}_+$$

is $\mathscr{B}_t(\Gamma_2)$-measurable. Similarly, define $\mathscr{B}_{>t}(\Gamma_2)$ as the smallest sub-σ-algebra of $\mathscr{B}(\Gamma_2)$ such that for every $A_1 \in \mathscr{B}((t,\infty))$, $A_2 \in \mathscr{B}(\mathbb{R}_+)$ the map

$$\Gamma_2 \ni \gamma \mapsto \gamma(A_1 \times A_2) \in \mathbb{R}_+$$

is $\mathscr{B}_{>t}(\Gamma_2)$-measurable.

Let η_0 be a (random) initial configuration, and let $\hat{\eta}_0$ be the point process on $\mathbb{R}^{\mathrm{d}} \times \Gamma_2$ obtained by associating to each point in η_0 an independent Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$, with the distribution π. That is, for $\eta_0 = \sum_{i=1}^{\infty} \delta_{x_i}$, set

$$\hat{\eta}_0 = \sum_{i=1}^{\infty} \delta_{(x_i,\gamma_i)}.$$

Consider the stochastic equation with Poisson noise

$$\begin{aligned} \eta_t(B) = &\int\limits_{(0,t]\times B\times[0,\infty)\times\Gamma_2} I_{[0,b(x,s,\eta_{s-})]}(u) I\{\int_s^t I_{[0,d(x,r,\eta_{r-})]}(v)\gamma(dr,dv) = 0\} N(ds,dx,du,d\gamma) \\ &+ \int\limits_{B\times\Gamma_2} I\{\int_0^t I_{[0,d(x,r,\eta_{r-})]}(v)\gamma(dr,dv) = 0\}\hat{\eta}_0(dx,d\gamma), \end{aligned} \tag{2}$$

where $(\eta_t)_{t\geq 0}$ is a cadlag Γ_0-valued solution process, N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+ \times \Gamma_2$, the mean measure of N is $ds \times dx \times du \times \pi$. We require the processes N and η_0 to be independent of each other. Equation (2) is understood in the sense that the equality holds a.s. for every bounded $B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$ and $t \geq 0$.

Remark. In the first integral on the right-hand side of (2) x is the place and s is the time of birth of a new particle. This particle is alive as long as $\int_s^t I_{[0,d(x,r,\eta_{r-})]}(v)\gamma(dr,dv) = 0$, where $(x,s,u,\gamma) \in N$. Thus, γ is the process 'responsible' for death. In the death term lies the main difference to the equation considered by Garcia and Kurtz [GK06]. Adapted to our notation, the equation there is of the form

$$\begin{aligned} \eta_t(B) = &\int\limits_{(0,t]\times B\times[0,\infty)\times[0,\infty)} I_{[0,b(x,\eta_{s-})]}(u) I\{\int_s^t d(x,\eta_{r-})dv < r\}\tilde{N}(ds,dx,du,dr) \\ &+ \int\limits_{B\times[0,\infty)} I\{\int_s^t d(x,\eta_{r-})dv < r\}\tilde{\eta}_0(dx,dr), \end{aligned} \tag{3}$$

where $\tilde{N}$ is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+ \times \mathbb{R}_+$ with mean measure $ds \times dx \times du \times e^{-r}dr$, and $\tilde{\eta}_0$ is obtained from η_0 by attaching an independent unit exponential to each point. At first glance, (2) is more complicated than (3), since the death mechanism requires a whole Poisson random measure on $[0;\infty)^2$ instead of just one exponential random variable. However, it is not clear how to a priori define a filtration $\{\tilde{\mathscr{F}}_t\}_{t\geq 0}$ compatible with $\tilde{N}$ and such that a solution to (3), if unique, should possess the Markov property with respect to $\{\tilde{\mathscr{F}}_t\}_{t\geq 0}$.

Conditions on b, d and η_0. Unless stated otherwise, we assume that the birth rate b satisfies the following conditions: sublinear growth on the second variable in the sense that

$$\int_{\mathbb{R}^{\mathrm{d}}} \sup_{s>0} b(x,s,\eta)dx \leq c_1|\eta| + c_2, \tag{4}$$

for some constants $c_1, c_2 > 0$, and that $b(x,\cdot,\eta)$ and $d(x,\cdot,\eta)$ are right-continuous for any $x \in \mathbb{R}^{\mathrm{d}}$ and $\eta \in \Gamma_0$.

We also assume that

$$E|\eta_0| < \infty. \tag{5}$$

We say that N is *compatible* with a right-continuous complete filtration $\{\mathscr{F}_t\}$ if

$$N([0,q] \times B \times C \times \Xi)$$

is $\mathscr{F}_t$-measurable for any $q \in [0,t], B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}}), C \in \mathscr{B}(\mathbb{R}_+)$, and $\Xi \in \mathscr{B}_t(\Gamma_2)$, and if

$$N((t,t+q'] \times B' \times C' \times \Xi')$$

is independent of $\mathscr{F}_t$ for any $q' \geq 0, B' \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$, $C' \in \mathscr{B}(\mathbb{R}_+)$, and $\Xi' \in \mathscr{B}_{>t}(\Gamma_2)$.

Sometimes we will use the representations

$$N = \sum_{q\in\mathcal{I}} \delta_{(s_q,x_q,u_q,\gamma_q)}, \quad \hat{\eta}_0 = \sum_{q\in\mathcal{J}} \delta_{(x_q,\gamma_q)},$$

where $\mathcal{I}$ and $\mathcal{J}$ are some countable disjoint sets.

Definition 2.1. A *(weak) solution* of equation (2) is a triple $((\eta_t)_{t\geq 0}, N)$, $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\geq 0})$, where

(i) $(\Omega, \mathscr{F}, P)$ is a probability space, and $\{\mathscr{F}_t\}_{t\geq 0}$ is an increasing, right-continuous and complete filtration of sub-σ-algebras of $\mathscr{F}$,

(ii) N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+ \times \Gamma_2$ with intensity $ds \times dx \times du \times \pi$,

(iii) η_0 is a random $\mathscr{F}_0$-measurable element in Γ_0 satisfying (5),

(iv) the processes N and η_0 are independent, N is compatible with $\{\mathscr{F}_t\}_{t\geq 0}$,

(v) $(\eta_t)_{t\geq 0}$ is a cadlag Γ_0-valued process adapted to $\{\mathscr{F}_t\}_{t\geq 0}$, $\eta_t\big|_{t=0} = \eta_0$,

(vi) all integrals in (2) are well-defined,

$$E\int_0^t ds\Big[\int_{\mathbb{R}^d} b(x,s,\eta_{s-}) + \sum_{x\in\eta_{s-}} d(x,s,\eta_{s-})\Big] < \infty, \quad t>0$$

and

(vii) equality (2) holds a.s. for all $t\in[0,\infty]$ and all Borel sets B.

Let

$$\mathscr{S}_t^0 = \sigma\big\{\eta_0, N([0,q]\times B\times C\times \Xi),$$
$$q\in[0,t], B\in\mathscr{B}(\mathbb{R}^d), C\in\mathscr{B}(\mathbb{R}_+), \Xi\in\mathscr{B}_t(\Gamma_2)\big\},$$

and let $\mathscr{S}_t$ be the completion of $\mathscr{S}_t^0$ under P. Note that $\{\mathscr{S}_t\}_{t\geq 0}$ is a right-continuous filtration.

Definition 2.2. A solution of (2) is called *strong* if $(\eta_t)_{t\geq 0}$ is adapted to $(\mathscr{S}_t, t\geq 0)$.

Remark 2.3. In the definition above we considered solutions as processes indexed by $t\in[0,\infty)$. The reformulations for the case $t\in[0,T]$, $0<T<\infty$, are straightforward. This remark also applies to many of the results below.

Definition 2.4. We say that pathwise uniqueness holds for equation (2) and an initial distribution ν if, whenever the triples $((\eta_t)_{t\geq 0}, N)$, $(\Omega,\mathscr{F},P)$, $(\{\mathscr{F}_t\}_{t\geq 0})$ and $((\bar\eta_t)_{t\geq 0}, N)$, $(\Omega,\mathscr{F},P)$, $(\{\bar{\mathscr{F}}_t\}_{t\geq 0})$ are weak solutions of (2) with $P\{\eta_0=\bar\eta_0\}=1$ and $Law(\eta_0)=\nu$, we have $P\{\eta_t = \bar\eta_t, t\in[0,\infty)\}=1$ (that is, the processes $\eta,\bar\eta$ are indistinguishable).

Definition 2.5. We say that *joint uniqueness in law* holds for equation (2) with an initial distribution ν if any two (weak) solutions $((\eta_t),N)$ and $((\eta'_t),N')$ of (2), $Law(\eta_0)=Law(\eta'_0)=\nu$, have the same joint distribution:

$$Law((\eta_t),N) = Law((\eta'_t),N').$$

Theorem 2.6. *Pathwise uniqueness, strong existence and joint uniqueness in law hold for equation (2). The unique solution is a Markov process. If b and d are time-homogeneous, then the family of push-forward measures $\{P_\alpha, \alpha\in\Gamma_0\}$ defined in Remark 3.3 constitutes a Markov process, or a Markov family of probability measures, on $D_{\Gamma_0}[0,\infty)$.*

Proof. The statement is a consequence of Proposition 3.2, Remark 3.3 and Proposition 3.8. In particular, the Markov property of $\{P_\alpha,\alpha\in\Gamma_0\}$ follows from Corollary 3.7.

We call the unique solution of (2) (or, sometimes, the corresponding family of measures on $D_{\Gamma_0}[0,\infty)$) a *(spatial) birth-and-death Markov process.*

Remark. For time-homogeneous b and d, the transition probabilities of the embedded Markov chain (see e.g. [Kal02, Chapter 12]) of the birth-and-death process are completely described by

$$\begin{aligned} Q(\eta, \{\eta \setminus \{x\}\}) &= \frac{d(x,\eta)}{(B+D)(\eta)}, \qquad x \in \eta, \quad \eta \in \Gamma_0, \\ Q(\eta, \{\eta \cup \{x\}, x \in U\}) &= \frac{\int_{x \in U} b(x,\eta)dx}{(B+D)(\eta)}, \quad U \in \mathscr{B}(\mathbb{R}^{\mathrm{d}}), \eta \in \Gamma_0, \end{aligned} \tag{6}$$

where $(B+D)(\eta) = \int_{\mathbb{R}^{\mathrm{d}}} b(x,\eta)dx + \sum_{x\in\eta} d(x,\eta)$.

The following two propositions establish a rigorous relation between the unique solution to (2) and L defined by (1). To formulate the first of them, let us consider the class $\mathscr{C}_b$ of cylindrical functions $F : \Gamma_0 \to \mathbb{R}_+$ with bounded increments. We say that F has bounded increments if

$$\sup_{\eta\in\Gamma_0, x\in\mathbb{R}^{\mathrm{d}}} \big(F(\eta \cup \{x\}) - F(\eta)\big) < \infty.$$

We say that F is cylindrical if for some $R = R_F > 0$

$$F(\eta) = F(\zeta) \text{ whenever } \eta \cap \mathbf{B}(\mathbf{o}_{\mathrm{d}}, R) = \zeta \cap \mathbf{B}(\mathbf{o}_{\mathrm{d}}, R),$$

where $\mathbf{B}(x,R)$ is the closed ball of radius R around x, and $\mathbf{o}_{\mathrm{d}}$ is the origin in $\mathbb{R}^{\mathrm{d}}$. We recall that the filtration $\{\mathscr{S}_t, t \geq 0\}$ appeared before Definition 2.2.

Proposition 2.7. *Let $(\eta_t)_{t\geq 0}$ be a weak solution to (2). Then for any $F \in \mathscr{C}_b$ the process*

$$\begin{aligned} F(\eta_t) - \int_0^t \Big\{ \int_{\mathbb{R}^{\mathrm{d}}} & b(x,s,\eta_{s-})[F(\eta_{s-} \cup \{x\}) - F(\eta_{s-})]dx \\ & - \sum_{x\in\eta_{s-}} d(x,s,\eta_{s-})[F(\eta_{s-} \setminus \{x\}) - F(\eta_{s-})]\Big\} ds \end{aligned} \tag{7}$$

is an $\{\mathscr{S}_t, t \geq 0\}$-martingale. In particular, the integral in (7) is well-defined a.s.

2.2 Aggregation model

Here we consider a specific time-homogeneous model, which we call an aggregation model. This model has a property that the death rate decreases as the number of neighbors grows. We treat here the death rate given below in (8), and, in addition to previous assumptions, we require the birth rate to grow linearly on the number of points in configuration in the sense (9). We prove in Proposition 2.8 that the probability of extinction is small if the initial configuration has many points in some fixed Borel set $\Lambda \subset \mathbb{R}^{\mathrm{d}}$. Propositions 2.9, 2.10 and Theorem 2.11 describe the pathwise behavior of the process.

Let

$$d(x,\eta) = \exp\{-\sum_{y\in\eta}\varphi(x-y)\}, \tag{8}$$

where φ is a nonnegative measurable function. Proposition 3.2 ensures existence and uniqueness of solutions, and that the unique solution is a pure jump type Markov process.

We want to show that, if the initial configuration has m points in some bounded region, then, under some assumption on b and φ, the probability of extinction declines at least exponentially fast by m. Also, we will give a few statements describing the pace of growth of the number of points of the system.

More specifically, let Λ be a measurable subset of $\mathbb{R}^d$, the birth rate and the initial condition η_0 satisfy the same condition as in Proposition 3.2, and, besides that, the inequalities

$$\int\limits_{\Lambda} b(x,\eta)dx \geq c|\eta\cap\Lambda|, \quad \eta\in\Gamma_0, \tag{9}$$

and

$$b(x,\eta^1) \leq b(x,\eta^2), \quad \eta^1,\eta^2\in\Gamma_0, \eta^1\subset\eta^2, \tag{10}$$

hold for some positive c. We assume also that

$$\inf_{x,y\in\Lambda}\varphi(x-y) \geq \log a, \tag{11}$$

where $a > 1$.

We say that the process $(\eta_t)_{t\geq 0}$ *extincts* if $\inf\{t\geq 0 : \eta_t = \varnothing\} < \infty$. This infimum is called the *time of extinction*.

Proposition 2.8. *Let $\tilde{C} > 0$. Then there exists $m_0 = m_0(\tilde{C}) \in \mathbb{N}$ such that, whenever $m \geq m_0$,*

$$P_\alpha\{(\eta_t)_{t\geq 0} \text{ extincts } \} \leq \tilde{C}^{-m}$$

for all α satisfying $|\alpha\cap\Lambda| = m$

Proposition 2.9. *For all $\alpha\in\Gamma_0$,*

$$P_\alpha\Big(\{|\eta_t\cap\Lambda|\to\infty\}\cup\{\exists t' : \forall t\geq t', |\eta_t\cap\Lambda| = \varnothing\}\Big) = 1. \tag{12}$$

Remark. Note that we do not require $b(\cdot,\varnothing)\equiv 0$; if $\int_\Lambda b(x,\varnothing)dx > 0$, then (12) implies

$$P_\alpha\{|\eta_t\cap\Lambda|\to\infty\} = 1.$$

The next proposition is a consequence of the exponentially fast decay of the death rate.

Proposition 2.10. *With probability* 1 *only a finite number of deaths inside* Λ *occur:*

$$P_\alpha\Big\{|\eta_t\cap\Lambda| - |\eta_{t-}\cap\Lambda| = -1 \text{ for infinitely many different } t\geq 0\Big\} = 0, \ \alpha\in\Gamma_0.$$

Theorem 2.11. *Let* $\alpha\in\Gamma_0$. *For* P_α*-almost all* $\omega\in F := \{\lim_{t\to\infty}|\eta_t\cap\Lambda| = \infty\}$ *we have*

$$\liminf_{t\to\infty}\frac{|\eta_t\cap\Lambda|}{e^{ct}} > 0. \tag{13}$$

Corollary 2.12. *For all configurations* α *with* $\alpha\cap\Lambda\neq\varnothing$,

$$\inf_{t>0}\frac{E_\alpha|\eta_t\cap\Lambda|}{e^{ct}} > 0. \tag{14}$$

Remark. If Λ has a finite volume and the birth rate is given constant within Λ, that is

$$b(x,\eta) = c_3 > 0, \quad x\in\Lambda,$$

then from the proofs we can conclude that Theorem 2.11 and Corollary 2.12 still hold provided that we replace (13) by

$$\liminf_{t\to\infty}\frac{|\eta_t\cap\Lambda|}{c_3\mathrm{Vol_d}(\Lambda)} > 0. \tag{15}$$

and (14) by

$$\inf_{t>0}\frac{E_\alpha|\eta_t\cap\Lambda|}{c_3\mathrm{Vol_d}(\Lambda)} > 0. \tag{16}$$

Here $c_3\mathrm{Vol_d}(\Lambda)$ is the d-dimensional volume of Λ. These two growth estimates stand in contrast to the mesoscopic behavior of the system [FKKZ14]. Theorem 5.3 in [FKKZ14] says that for some values of parameters the solution to the mesoscopic equation started from sufficiently small initial condition stays bounded. On the contrary, the microscopic system grows whenever it survives, and the density always grows.

3 Proof of Theorem 2.6 and Proposition 2.7

Let us start with the equation

$$\overline{\eta}_t(B) = \int\limits_{(0,t]\times B\times[0,\infty)\times\Gamma_2} I_{[0,\bar{b}(x,s,\eta_{s-})]}(u)N(ds,dx,du,d\gamma) + \eta_0(B), \tag{17}$$

where $\bar{b}(x,\eta) := \sup_{s>0,\xi\subset\eta} b(x,s,\xi)$. Note that $\bar{b}$ satisfies sublinear growth condition (4) if b does.

This equation is of the type (2), with $\bar{b}$ being the birth rate, and the zero function being the death rate, and all definitions of existence and uniqueness of solutions are applicable here. Later a unique solution of (17) will be used as a majorant of a solution to (2).

Proposition 3.1. *Under assumptions* (4) *and* (5)*, strong existence and pathwise uniqueness hold for equation* (17)*. In particular, the unique solution* $(\bar{\eta}_t)_{t\geq 0}$ *satisfies*

$$E|\bar{\eta}_t| < \infty, \quad t \geq 0. \tag{18}$$

Proof. For $\omega \in \{\int\limits_{\mathbb{R}^d} \bar{b}(x,\eta_0)dx = 0\}$, set $\zeta_t \equiv \eta_0$, $\sigma_n = \infty$, $n \in \mathbb{N}$.

For $\omega \in F := \{\int\limits_{\mathbb{R}^d} \bar{b}(x,\eta_0)dx > 0\}$, we define the sequence of random pairs $\{(\sigma_n, \zeta_{\sigma_n})\}$, where

$$\sigma_{n+1} = \inf\{t > 0 : \int\limits_{(\sigma_n, \sigma_n + t] \times B \times [0,\infty) \times \Gamma_2} I_{[0,\bar{b}(x,\zeta_{\sigma_n})]}(u) N(ds,dx,du,d\gamma) > 0\} + \sigma_n, \quad \sigma_0 = 0,$$

and

$$\zeta_0 = \eta_0, \quad \zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup \{z_{n+1}\}$$

for $z_{n+1} = \{x \in \mathbb{R}^d : N(\{\sigma_{n+1}\} \times \{x\} \times [0, \bar{b}(x,\zeta_{\sigma_n})] \times \Gamma_2) > 0\}$. The points z_n are uniquely determined almost surely on F. Furthermore, $\sigma_{n+1} > \sigma_n$ a.s., and σ_n are finite a.s. on F (in particular because $\bar{b}(x,\zeta_{\sigma_n}) \geq \bar{b}(x,\eta_0)$). For $\omega \in F$, we define $\zeta_t = \zeta_{\sigma_n}$ for $t \in [\sigma_n, \sigma_{n+1})$. Then by induction on n it follows that σ_n is a stopping time for each $n \in \mathbb{N}$, and ζ_{σ_n} is $\mathscr{F}_{\sigma_n} \cap F$-measurable. By direct substitution we see that $(\zeta_t)_{t\geq 0}$ is a strong solution to (17) on the time interval $t \in [0, \lim\limits_{n\to\infty} \sigma_n)$. Although we have not defined what is a solution, or a strong solution, on a random time interval, we do not discuss it here. Instead we are going to show that

$$\lim_{n\to\infty} \sigma_n = \infty \quad \text{a.s.} \tag{19}$$

This relation is evidently true on the complement of F. If $P(F) = 0$, then (19) is proven.

If $P(F) > 0$, define a probability measure on F, $Q(A) = \frac{P(A)}{P(F)}$, $A \in \mathscr{I} := \mathscr{F} \cap F$, and define $\mathscr{I}_t = \mathscr{F}_t \cap F$.

The process N is independent of F, therefore it is a Poisson point process on the probability space $(F, \mathscr{I}, Q)$ with the same intensity, compatible with $\{\mathscr{I}_t\}_{t\geq 0}$. From now on and until other is specified, we work on the filtered probability space $(F, \mathscr{I}, \{\mathscr{I}_t\}_{t\geq 0}, Q)$. We use the same symbols for random processes and random variables, having in mind that we consider their restrictions to F.

The process $(\zeta_t)_{t\in[0, \lim\limits_{n\to\infty}\sigma_n)}$ has the Markov property, because the process N has the strong Markov property and independent increments, see Appendix. Indeed, conditioning on $\mathscr{I}_{\sigma_n}$,

$$E\big[I_{\{\zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup x \text{ for some } x \in B\}} \mid \mathscr{I}_{\sigma_n}\big] = \frac{\int\limits_B \bar{b}(x,\zeta_{\sigma_n})dx}{\int\limits_{\mathbb{R}^d} \bar{b}(x,\zeta_{\sigma_n})dx},$$

thus the chain $\{\zeta_{\sigma_n}\}_{n\in Z_+}$ is a Markov chain, and, given $\{\zeta_{\sigma_n}\}_{n\in Z_+}$, $\sigma_{n+1}-\sigma_n$ are distributed exponentially:

$$E\{I_{\{\sigma_{n+1}-\sigma_n>a\}} \mid \{\zeta_{\sigma_n}\}_{n\in Z_+}\} = \exp\{-a\int\limits_{\mathbb{R}^d} \bar{b}(x,\zeta_{\sigma_n})dx\}.$$

Therefore, the random variables $\gamma_n = (\sigma_n - \sigma_{n-1})(\int\limits_{\mathbb{R}^d} \bar{b}(x,\zeta_{\sigma_n})dx)$ constitute a sequence of independent random variables exponentially distributed with parameter 1, independent of $\{\zeta_{\sigma_n}\}_{n\in Z_+}$. Theorem 12.18 in [Kal02] implies that $(\zeta_t)_{t\in[0,\lim\limits_{n\to\infty}\sigma_n)}$ is a pure jump type Markov process.

The jump rate of $(\zeta_t)_{t\in[0,\lim\limits_{n\to\infty}\sigma_n)}$ is given by

$$c(\alpha) = \int\limits_{\mathbb{R}^d} \bar{b}(x,\alpha)dx.$$

Condition (4) implies that $c(\alpha) \le c_1|\alpha| + c_2$. Consequently,

$$c(\zeta_{\sigma_n}) \le c_1|\zeta_{\sigma_n}| + c_2 = c_1|\zeta_0| + c_1 n + c_2.$$

We see that $\sum_n \frac{1}{c(\zeta_{\sigma_n})} = \infty$ a.s., hence Proposition 12.19 in [Kal02] implies that $\sigma_n \to \infty$.

Now we return again to our initial probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge0}, P)$. We have proved the existence of a strong solution. The uniqueness follows by induction on jumps of the process. Namely, let $(\tilde{\zeta}_t)_{t\ge0}$ be another solution of (17). From (vii) of Definition 2.1 and the equality

$$\int\limits_{(0,\sigma_1)\times\mathbb{R}^d\times[0,\infty]\times\Gamma_2} I_{[0,\bar{b}(x,\eta_0)]}(u)N(ds,dx,du,d\gamma) = 0,$$

it follows that $P\{\tilde{\zeta}$ has a birth before $\sigma_1\} = 0$. At the same time, the equality

$$\int\limits_{\{\sigma_1\}\times\mathbb{R}^d\times[0,\infty]\times\Gamma_2} I_{[0,\bar{b}(x,\eta_0)]}(u)N(ds,dx,du,d\gamma) = 1,$$

which holds a.s., yields that $\tilde{\zeta}$ has a birth at the moment σ_1, and in the same point of space at that. Therefore, $\tilde{\zeta}$ coincides with ζ up to σ_1 a.s. Similar reasoning shows that they coincide up to σ_n a.s., and, since $\sigma_n \to \infty$ a.s.,

$$P\{\tilde{\zeta}_t = \zeta_t \text{ for all } t \ge 0\} = 1.$$

Thus, pathwise uniqueness holds.

Now we turn our attention to (18). We can write

$$
\begin{aligned}
|\zeta_t| &= |\eta_0| + \sum_{n=1}^{\infty} I\{|\zeta_t| - |\eta_0| \geq n\} \\
&= |\eta_0| + \sum_{n=1}^{\infty} I\{\sigma_n \leq t\}.
\end{aligned} \tag{20}
$$

Since $\sigma_n = \sum_{i=1}^{n} \frac{\gamma_i}{\int_{\mathbb{R}^{\mathrm{d}}} \bar{b}(x, \zeta_{\sigma_i}) dx}$, we have

$$
\{\sigma_n \leq t\} = \{\sum_{i=1}^{n} \frac{\gamma_i}{\int_{\mathbb{R}^{\mathrm{d}}} \bar{b}(x, \zeta_{\sigma_i}) dx} \leq t\} \subset \{\sum_{i=1}^{n} \frac{\gamma_i}{c_1 |\zeta_{\sigma_i}| + c_2} \leq t\}
$$

$$
\subset \{\sum_{i=1}^{n} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \leq t\} = \{Z_t - Z_0 \geq n\},
$$

where (Z_t) is the Yule process, i.e. the birth process on $\mathbb{Z}_+$ with transition rates $p_{k,k+1} = (c_1 + c_2)k$, $p_{k,l} = 0$, $l \neq k+1$, see, e.g., [AN72, Chapter 3, Section 5]. Here (Z_t) is defined as follows: $Z_t - Z_0 = n$ when

$$
\sum_{i=1}^{n} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \leq t < \sum_{i=1}^{n+1} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)},
$$

and $Z_0 = |\eta_0|$. Thus, we have $|\zeta_t| \leq Z_t$ a.s., hence $E|\zeta_t| \leq EZ_t < \infty$. The constructed solution is strong. □

Proposition 3.2. *Under assumptions (4)-(5), pathwise uniqueness and strong existence hold for equation (2). The unique solution (η_t) satisfies*

$$
E|\eta_t| < \infty, \quad t \geq 0. \tag{21}
$$

Proof. Let us define stopping times with respect to $\{\mathscr{F}_t, t \geq 0\}$, $0 = \theta_0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq ...$, and the sequence of (random) configurations $\{\eta_{\theta_j}\}_{j \in \mathbb{N}}$ as follows: as long as

$$
\theta_{n+1} = \theta_{n+1}^{b} \wedge \theta_{n+1}^{\mathrm{d}} + \theta_n < \infty,
$$

where

$$
\theta_{n+1}^{b} = \inf\{t > 0 : \int_{(\theta_n, \theta_n + t] \times \mathbb{R}^{\mathrm{d}} \times [0,\infty) \times \Gamma_2} I_{[0, b(x, s, \eta_{\theta_n})]}(u) N(ds, dx, du, d\gamma) > 0\},
$$

$$
\theta_{n+1}^{\mathrm{d}} = \inf\{t > 0 : \sum_{\substack{q \in \mathcal{I} \cup \mathcal{J}, \\ x_q \in \eta_{\theta_n}}} \int_{(\theta_n, \theta_n + t] \times [0,\infty)} I_{[0, d(x_i, r, \eta_{\theta_n})]}(v) \gamma_q(dr, dv) > 0\},
$$

we set $\eta_{\theta_{n+1}} = \eta_{\theta_n} \cup \{z_{n+1}\}$ if $\theta^b_{n+1} \le \theta^{\rm d}_{n+1}$, where $\{z_{n+1}\} = \{z \in \mathbb{R}^{\rm d} : N(\{\theta_n + \theta^b_{n+1}\} \times \{z\} \times \mathbb{R}_+ \times \Gamma_2) > 0\}$; $\eta_{\theta_{n+1}} = \eta_{\theta_n} \setminus \{z_{n+1}\}$ if $\theta^b_{n+1} > \theta^{\rm d}_{n+1}$, where $\{z_{n+1}\} = \{x_q \in \eta_{\theta_n} : \gamma_q(\{\theta_n + \theta^{\rm d}_{n+1}\} \times \mathbb{R}_+) > 0\}$; the configuration $\eta_{\theta_0} = \eta_0$ is the initial condition of (2), $\eta_t = \eta_{\theta_n}$ for $t \in [\theta_n, \theta_{n+1})$. Note that

$$P\{\theta^b_{n+1} = \theta^{\rm d}_{n+1} \mid \min\{\theta^b_{n+1}, \theta^{\rm d}_{n+1}\} < \infty\} = 0,$$

the points z_n are a.s. uniquely determined, and

$$P\{z_{n+1} \in \eta_{\theta_n} \mid \theta^b_{n+1} \le \theta^{\rm d}_{n+1}\} = 0.$$

If for some n

$$\theta_{n+1} = \infty,$$

we set $\theta_{n+k} = \infty$, $k \in \mathbb{N}$, and $\eta_t = \eta_{\theta_n}$, $t \ge \theta_n$.

Random variables $\theta_n, n \in \mathbb{N}$, are stopping times with respect to the filtration $\{\mathscr{F}_t, t \ge 0\}$. Using the strong Markov property of a Poisson point process, we see that, on $\{\theta_n < \infty\}$, the conditional distribution of θ^b_{n+1} given $\mathscr{F}_{\theta_n}$ is

$$P\{\theta^b_{n+1} > p\} = \exp\{-\int_{\theta_n}^{\theta_n + p} b(x, s, \eta_{\theta_n}) ds\},$$

and the conditional distribution of $\theta^{\rm d}_{n+1}$ given $\mathscr{F}_{\theta_n}$ is

$$P\{\theta^{\rm d}_{n+1} > p\} = \exp\{-\int_{\theta_n}^{\theta_n + p} d(x, s, \eta_{\theta_n}) ds\}.$$

In particular, $\theta^b_n, \theta^{\rm d}_n > 0$, $n \in \mathbb{N}$.

We are going to show that a.s.

$$\theta_n \to \infty, \quad n \to \infty. \tag{22}$$

Denote by θ'_k the moment of the k-th birth. It is sufficient to show that $\theta'_k \to \infty$, $k \to \infty$, because only finitely many deaths may occur between any two births, since there are only finitely particles. By induction on k' we can see that $\{\theta'_k\}_{k' \in \mathbb{N}} \subset \{\sigma_i\}_{i \in \mathbb{N}}$, where σ_i are the moments of births of $(\overline{\eta}_t)_{t \ge 0}$, the solution of (17), and $\eta_t \subset \overline{\eta}_t$ for all $t \in [0, \lim_n \theta_n)$. For instance, let us show that $(\overline{\eta}_t)_{t \ge 0}$ has a birth at θ'_1. We have $\overline{\eta}_{\theta'_1 -} \supset \overline{\eta}_0 = \eta_0$, and $\eta_{\theta'_1 -} \subset \eta_t \mid_{t=0} = \eta_0$, hence for all $x \in \mathbb{R}^{\rm d}$

$$\overline{b}(x, \overline{\eta}_{\theta'_1 -}) \ge \overline{b}(x, \eta_{\theta'_1 -}) \ge b(x, \eta_{\theta'_1 -})$$

The latter implies that at time moment θ'_1 a birth occurs for the process $(\overline{\eta}_t)_{t\geq 0}$ in the same point. Hence, $\eta_{\theta'_1} \subset \overline{\eta}_{\theta'_1}$, and we can go on. Since $\sigma_k \to \infty$ as $k \to \infty$, we also have $\theta'_k \to \infty$, and therefore $\theta_n \to \infty$, $n \to \infty$.

Let us now prove the inequality from item (vi) of Definition 2.1,

$$E \int_0^t ds \Big[\int_{\mathbb{R}^d} b(x, s, \eta_{s-}) + \sum_{x \in \eta_{s-}} d(x, s, \eta_{s-}) \Big] < \infty, \quad t > 0. \tag{23}$$

Denote the number of births and deaths before t by b_t and d_t respectively, i.e.

$$b_t = \#\{s : |\eta_s| - |\eta_{s-}| = 1\} = \int_{(0,t]\times\mathbb{R}^d\times[0,\infty)\times\Gamma_2} I_{[0,b(x,s,\eta_{s-})]}(u) N(ds, dx, du, d\gamma)$$

and

$$d_t = \#\{s : |\eta_s| - |\eta_{s-}| = -1\} = \int_{(0,t]\times[0,\infty)} \sum_{\substack{q\in\mathcal{I}\cup\mathcal{J}, \\ x_q\in\eta_{r-}}} I_{[0,d(x_i,r,\eta_{r-})]}(v) \gamma_q(dr, dv).$$

Note that $|\eta_t| = b_t - d_t + |\eta_0|$ and θ_k are the moments of jumps for $c_t := b_t + d_t$, so that

$$c_t = \sum_{k\in\mathbb{N}} I\{\theta_k \leq t\}, \quad t \geq 0.$$

For $n \in \mathbb{N}$ define

$$c_t^{(n)} = \int_{(0,t]\times\mathbb{R}^d\times[0,\infty)\times\Gamma_2} I_{[0,b(x,s,\eta_{s-})\wedge n]}(u) I\{|x| \leq n\} N(ds, dx, du, d\gamma)$$

$$+ \int_{0,t]\times[0,\infty)} \sum_{\substack{q\in\mathcal{I}\cup\mathcal{J} \\ x_q\in\eta_{r-}}} I_{[0,d(x_i,r,\eta_{r-})\wedge n]}(v) I\{|x| \leq n\} \gamma_q(dr, dv).$$

Then

$$M_t^{(n)}(x) = c_t^{(n)}(x) - \int_0^t \int_{x:|x|\leq n} \big(b(x, s, \eta_{s-}) \wedge n\big) ds - \int_0^t \int_{x:|x|\leq n} \big(d(x, s, \eta_{s-}) \wedge n\big) ds$$

is a martingale with respect to $\{\mathscr{S}_t\}$. By the optional stopping theorem $EM^{(n)}_{\theta_1\wedge t} = 0$, hence

$$\int_0^{\theta_1\wedge t} \int_{x:|x|\leq n} \big(b(x, s, \eta_{s-}) \wedge n + d(x, s, \eta_{s-}) \wedge n\big) ds \leq P\{\theta_1 < t\} \leq 1.$$

Similarly,

$$\int_{\theta_m\wedge t}^{\theta_{m+1}\wedge t}\int_{x:|x|\le n}\big(b(x,s,\eta_{s-})\wedge n + d(x,s,\eta_{s-})\wedge n\big)ds \le P\{\theta_m < t\}.$$

Consequently,

$$\int_0^t\int_{x:|x|\le n}\big(b(x,s,\eta_{s-})\wedge n + d(x,s,\eta_{s-})\wedge n\big)ds$$

$$\le \sum_{m=1}^{\infty} E\int_{\theta_m}^{\theta_{m+1}}\int_{x:|x|\le n}\big(b(x,s,\eta_{s-})\wedge n + d(x,s,\eta_{s-})\wedge n\big)ds$$

$$\le \sum_{m=1}^{\infty} P\{\theta_m \le t\} = \sum_{m=1}^{\infty} P\{c_t \ge m\} = Ec_t.$$

Letting $n\to\infty$, we get by the monotone convergence theorem

$$\int_0^t\int_{\mathbb{R}^d}\big(b(x,s,\eta_{s-}) + d(x,s,\eta_{s-})\big)ds \le Ec_t$$

Only existing particles may disappear, hence the number of deaths d_t satisfies

$$d_t \le b_t + |\eta_0|.$$

Thus,

$$Ec_t \le 2Eb_t + E|\eta_0| \le 2E|\overline{\eta}_t| + E|\eta_0| < \infty. \tag{24}$$

Since $\eta_t \subset \overline{\eta}_t$ a.s., Proposition 3.1 implies (21).

If follows from the above construction, (22) and (23) that (η_t) is a strong solution to (2). Similarly to the proof of Proposition 3.1, we can show by induction on n that equation (2) has a unique solution on $[0,\theta_n]$. Namely, each two solutions coincide on $[0,\theta_n]$ a.s. Thus, any solution coincides with (η_t) a.s. for all $t\in[0,\theta_n]$.

□

Remark 3.3. Assume that b and d are time-homogeneous. Let η_0 be a non-random initial condition, $\eta_0\equiv\alpha$, $\alpha\in\Gamma_0$. The solution of (2) with $\eta_0\equiv\alpha$ will be denoted as $(\eta(\alpha,t))_{t\ge 0}$. Let P_α be the push-forward of P under the mapping

$$\Omega \ni \omega \mapsto (\eta(\alpha,\cdot)) \in D_{\Gamma_0}[0,\infty). \tag{25}$$

It can be derived from the proof of Proposition 3.2 that, for fixed $\omega\in\Omega$, the unique solution is jointly measurable in (t,α). Thus, the family $\{P_\alpha\}$ of probability measures on

$D_{\Gamma_0}[0,\infty)$ is measurable in α. We will often use formulations related to the probability space $(D_{\Gamma_0}[0,\infty), \mathscr{B}(D_{\Gamma_0}[0,\infty)), P_\alpha)$; in this case, coordinate mappings will be denoted by η_t,

$$\eta_t(x) = x(t), \quad x \in D_{\Gamma_0}[0,\infty).$$

The processes $(\eta_t)_{t\in[0,\infty)}$ and $(\eta(\alpha,\cdot))_{t\in[0,\infty)}$ have the same law (under P_α and P, respectively). As one would expect, the family of measures $\{P_\alpha, \alpha \in \Gamma_0\}$ is a Markov process, or a Markov family of probability measures; see Proposition 3.8 below. For a measure μ on Γ_0, we define

$$P_\mu = \int P_\alpha \mu(d\alpha).$$

We denote by E_μ the expectation under P_μ.

Remark 3.4. Let b_1, d_1 be another pair of birth and death rates, satisfying all conditions imposed on b and d. Consider a unique solution $(\tilde{\eta}_t)$ of (2) with rates b_1, d_1 instead of b, d, but with the same initial condition η_0 and all the other underlying structures. *If for all* $\zeta \in D$, *where* $D \in \mathscr{B}(\Gamma_0(\mathbb{R}^d))$, $b_1(\cdot,\zeta) \equiv b(\cdot,\zeta)$, $d_1(\cdot,\zeta) \equiv d(\cdot,\zeta)$, *then* $\tilde{\eta}_t = \eta_t$ *for all* $t \le \inf\{s \ge 0 : \eta_s \notin D\} = \inf\{s \ge 0 : \tilde{\eta}_s \notin D\}$. This may be proven in the same way as the theorem above.

Remark 3.5. Assume that all the conditions of Proposition 3.2 are fulfilled except (5). Then we could not claim that (21) holds. However, we would still get a unique solution on $[0,\infty)$. We are mostly interested in the case of a non-random initial condition, therefore we do not discuss the case when (21) is not satisfied.

Remark 3.6. We solved equation (2) ω-wisely. We can deduce from the proof of Proposition 3.2 that θ_n and z_n are measurable functions of η_0 and N in the sense that, for example, $\theta_1 = F_1(\eta_0, N)$ a.s. for a measurable function $F_1 : \Gamma_0 \times \Gamma(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_2) \to \mathbb{R}_+$. As a consequence, there is a functional dependence of the solution process and the "input": the process $(\eta_t)_{t\ge 0}$ is some function of η_0 and N.

The following corollary is a consequence of Proposition 3.2 and Remark 3.6.

Corollary 3.7. *Joint uniqueness in law holds for equation* (2) *with initial distribution* ν *satisfying*

$$\int_{\Gamma_0} |\gamma| \nu(d\gamma) < \infty.$$

As usually, the Markov property of a solution follows from uniqueness.

Proposition 3.8. *The unique solution* $(\eta_t)_{t\in[0,\infty)}$ *of* (2) *is a Markov process.*

Proof. Take arbitrary $t > t_0 > 0$. Consider the equation

$$\xi_t(B) = \int\limits_{(t_0,t]\times B\times[0,\infty)\times\Gamma_2} I_{[0,b(x,s,\eta_{s-})]}(u) I\{\int_s^t I_{[0,d(x,r,\eta_{r-})]}(v)\gamma(dr,dv) = 0\} N(ds,dx,du,d\gamma)$$
$$+ \int\limits_{B\times\Gamma_2} I\{\int_{t_0}^t I_{[0,d(x,r,\eta_{r-})]}(v)\gamma(dr,dv) = 0\}\hat{\eta}_{t_0}(dx,d\gamma) + \eta_{t_0}(B), \quad t \geq t_0. \tag{26}$$

The unique solution of (26) is $(\eta_s)_{s\in[t_0,\infty)}$. As in the proof of Proposition 3.2 we can see that $(\eta_s)_{s\in[t_0,\infty)}$ is measurable with respect to the filtration generated by random variables of the form $N([s,q]\times B\times U\times\Xi)$, and $\eta_{t_0}(B)$, where $B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$, $i \in \mathbb{Z}$, $t_0 \leq s \leq q$, $U \in \mathscr{B}(\mathbb{R}_+)$ $\Xi \in \mathscr{B}_{>t}(\Gamma_2)$. Poisson point process have independent increments, hence for any $\mathcal{U} \in \mathscr{B}(D_{\Gamma_0}[t_0,\infty))$

$$P\{(\eta_t)_{t\in[t_0,\infty)} \in \mathcal{U} \mid \mathscr{F}_{t_0}\} = P\{(\eta_t)_{t\in[t_0,\infty)} \in \mathcal{U} \mid \eta_{t_0}\}$$

almost surely. □

Proof of Proposition 2.7. Let N_1 be the image of N under the projection

$$(s,x,u,\gamma) \mapsto (s,x,u).$$

The process N_1 is a Poisson point process on $\mathbb{R}_+\times\mathbb{R}^{\mathrm{d}}\times\mathbb{R}_+$ with intensity measure $dsdxdu$. We have

$$\eta_t(B) = \int\limits_{(0,t]\times B\times[0,\infty)\times\Gamma_2} I_{[0,b(x,s,\eta_{s-})]}(u) I\{\int_s^t I_{[0,d(x,r,\eta_{r-})]}(v)\gamma(dr,dv) = 0\} N(ds,dx,du,d\gamma)$$

$$+ \int\limits_{B\times\Gamma_2} I\{\int_0^t I_{[0,d(x,r,\eta_{r-})]}(v)\gamma(dr,dv) = 0\}\hat{\eta}_0(dx,d\gamma)$$

$$= \int\limits_{(0,t]\times B\times[0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u) N_1(ds,dx,du) + \eta_0(B)$$

$$- \sum_{q\in\mathcal{I}\cup\mathcal{J}} \int\limits_{(0,t]\times[0,\infty)} I\{x_q \in \eta_{r-}\} I_{[0,d(x,r,\eta_{r-})]}(v)\gamma_q(dr,dv).$$

By Ito's formula for $F \in \mathscr{C}_b$

$$F(\eta_t) - F(\eta_0) = \int\limits_{(0,t]\times\mathbf{B}(\mathbf{o}_d,R_F)\times[0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u)\big\{F(\eta_{s-}\cup x) - F(\eta_{s-})\big\} N_1(ds,dx,du)$$

$$+\sum_{q\in\mathcal{I}\cup\mathcal{J}}\int\limits_{(0,t]\times[0,\infty)} I\{x_q\in\eta_{r-}\}I_{[0,d(x,r,\eta_{r-})]}(v)\big\{F(\eta_{s-}\setminus x)-F(\eta_{s-})\big\}\gamma_q(dr,dv).$$

We can write

$$\int\limits_{(0,t]\times\mathbf{B}(\mathbf{o}_d,R_F)\times[0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u)\big\{F(\eta_{s-}\cup x)-F(\eta_{s-})\big\}N_1(ds,dx,du)$$

$$=\int\limits_{(0,t]\times\mathbf{B}(\mathbf{o}_d,R_F)} b(x,s,\eta_{s-})\big\{F(\eta_{s-}\cup x)-F(\eta_{s-})\big\}dxds$$

$$+\int\limits_{(0,t]\times\mathbf{B}(\mathbf{o}_d,R_F)\times[0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u)\big\{F(\eta_{s-}\cup x)-F(\eta_{s-})\big\}\tilde{N}_1(ds,dx,du),$$

where $\tilde{N}=N-dsdxdu$. Since $F\in\mathscr{C}_b$, the process

$$\int\limits_{(0,t]\times\mathbf{B}(\mathbf{o}_d,R_F)\times[0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u)\big\{F(\eta_{s-}\cup x)-F(\eta_{s-})\big\}\tilde{N}_1(ds,dx,du)$$

is a martingale by item (vi) of Definition 2.1, see e.g. [IW81, Section 3 of Chapter 2]. Similarly,

$$\sum_{q\in\mathcal{I}\cup\mathcal{J}}\int\limits_{(0,t]\times[0,\infty)} I\{x_q\in\eta_{r-}\}I_{[0,d(x,r,\eta_{r-})]}(v)\big\{F(\eta_{s-}\setminus x)-F(\eta_{s-})\big\}\gamma_q(dr,dv)$$

can be decomposed into a sum of

$$+\int\limits_{(0,t]}\sum_{x\in\eta_{r-}} d(x,r,\eta_{r-})]\big\{F(\eta_{s-}\setminus x)-F(\eta_{s-})\big\}dr$$

and a martingale. The desired statement follows. □

3.1 Coupling

Here we discuss the coupling of two birth-and-death processes. As a matter of fact, we have already used the coupling technique in the proof of Proposition 3.2.

Consider two equations of the form (2),

$$\begin{aligned}\xi_t^{(k)}(B)=&\int\limits_{(0,t]\times B\times[0,\infty)\times\Gamma_2} I_{[0,b_k(x,s,\xi^{(k)}_{s-})]}(u)I\{\int_s^t I_{[0,d_k(x,r,\xi^{(k)}_{r-})]}(v)\gamma(dr,dv)=0\}\\ &\times N(ds,dx,du,d\gamma)+\int\limits_{B\times\Gamma_2} I\{\int_0^t I_{[0,d_k(x,r,\xi^{(k)}_{r-})]}(v)\gamma(dr,dv)=0\}\hat{\eta}_0(dx,d\gamma),\quad k=1,2.\end{aligned}\tag{27}$$

Assume that the initial conditions $\xi_0^{(k)}$ and the rates b_k, d_k satisfy the conditions of Proposition 3.2. Let $(\xi_t^{(k)})_{t\in[0,\infty)}$ be the unique strong solutions.

Proposition 3.9. *Assume that almost surely $\xi_0^{(1)} \subset \xi_0^{(2)}$, and for any two finite configurations $\eta^1 \subset \eta^2$,*

$$b_1(x,s,\eta^1) \leq b_2(x,s,\eta^2), \quad x \in \mathbb{R}^{\mathrm{d}}, s \geq 0 \tag{28}$$

and

$$d_1(x,s,\eta^1) \geq d_2(x,s,\eta^2), \quad x \in \eta^1, s \geq 0$$

Then

$$\xi_t^{(1)} \subset \xi_t^{(2)}, \quad t \in [0,\infty). \tag{29}$$

We will show by induction that each moment of birth for $(\xi_t^{(1)})_{t\in[0,\infty)}$ is a moment of birth for $(\xi_t^{(2)})_{t\in[0,\infty)}$ too, and each moment of death for $(\xi_t^{(2)})_{t\in[0,\infty)}$ is a moment of death for $(\xi_t^{(1)})_{t\in[0,\infty)}$ if the dying point is in $(\xi_t^{(1)})_{t\in[0,\infty)}$. Moreover, in both cases the birth or the death occurs at exactly the same point. Here a moment of birth is a random time at which a new point appears, a moment of death is a random time at which a point disappears from the configuration. The statement formulated above is in fact equivalent to (29).

Here we deal only with the base case, the induction step is done in the same way. We have nothing to show if τ_1 is a moment of a birth of $(\xi_t^{(2)})_{t\in[0,\infty)}$ or a moment of death of $(\xi_t^{(1)})_{t\in[0,\infty)}$. Assume that a new point is born for $(\xi_t^{(1)})_{t\in[0,\infty)}$ at τ_1,

$$\xi_{\tau_1}^{(1)} \setminus \xi_{\tau_1-}^{(1)} = \{x_1\}.$$

The process $(\xi^{(1)})_{t\in[0,\infty)}$ satisfies (2), therefore $N_1(\{x\} \times \{\tau_1\} \times [0, b_k(x_1, \tau_1, \xi_{\tau_1-}^{(1)})]) = 1$. Since

$$\xi_{\tau_1-}^{(1)} = \xi_0^{(1)} \subset \xi_0^{(2)} = \xi_{\tau_1-}^{(2)},$$

by (28)

$$N_1(\{x\} \times \{\tau_1\} \times [0, b_k(x_1, \tau_1, \xi_{\tau_1-}^{(2)})]) = 1,$$

hence

$$\xi_{\tau_1}^{(2)} \setminus \xi_{\tau_1-}^{(2)} = \{x_1\}.$$

The case when τ_2 is a moment of death for $(\xi_t^{(2)})_{t\in[0,\infty)}$ is analyzed analogously. □

4 Aggregation model: proofs

The main idea behind our analysis in this section is to couple the process $(\eta_t)_{t\geq 0}$ with another birth-and-death process, to which we can apply Lemma 5.1.

To do so, let us introduce another pair of the birth and death rates, b_1, d_1, and an initial condition $\xi_0 = \eta_0 \cap \Lambda$, such that $b_1(x,\eta) = d_1(x,\eta) = 0$ for $x \notin \Lambda$, $d_1(x,\eta) = a^{-|\eta|}$ for $x \in \Lambda$, $b_1(x,\eta) \leq b(x,\eta)$ for all x, η, and for some constant $c > 0$

$$\int_{\Lambda} b_1(x,\eta)dx = c|\eta \cap \Lambda|, \quad \eta \in \Gamma_0.$$

There exists a function b_1 satisfying these assumptions.

Functions b_1, d_1 satisfy conditions of Theorem 2.6. Furthermore, the conditions of Proposition 3.9 are fulfilled here: for $\eta^1, \eta^2 \in \Gamma_0$, $\eta^1 \subset \eta^2$ we have

$$b_1(x,\eta^1) \leq b(x,\eta^1) \leq b(x,\eta^2)$$

as well as

$$d_1(x,\eta^1) \geq d(x,\eta^1) \geq d(x,\eta^2).$$

Denote by $(\xi_t)_{t\geq 0}$ the unique solution of (2) with the birth and death rates b_1, d_1 and initial condition ξ_0. By Proposition 3.9, $\xi_t \subset \eta_t$ hold a.s. for all $t \geq 0$.

In this section we will work on the canonical probability space

$$\big(D_{\Gamma_0}[0,\infty) \times D_{\Gamma_0}[0,\infty), \mathscr{B}(D_{\Gamma_0}[0,\infty) \times D_{\Gamma_0}[0,\infty)), P_\alpha\big),$$

where P_α is the push-forward of the measure P under

$$\Omega \ni \omega \mapsto (\eta(\alpha,\cdot),(\xi(\alpha,\cdot)) \in D_{\Gamma_0}[0,\infty) \times D_{\Gamma_0}[0,\infty).$$

Consider the embedded Markov chain of the process $(\xi_t)_{t\geq 0}$, $Y_k := \xi_{\tau_k}$, where τ_k are the moments of jumps of (ξ_t). It turns out that the process $u = \{u_k\}_{k\in\mathbb{N}}$, where $u_k := |Y_k|$, is a Markov chain too. Indeed, the equality

$$P_{\alpha_1}\{|Y_1| = k\} = P_{\alpha_2}\{|Y_1| = k\}, \quad k \in \mathbb{N}, \alpha \in \Gamma_0.$$

holds when $|\alpha_1 \cap \Lambda| = |\alpha_2 \cap \Lambda|$, since both sides are equal to

$$\begin{cases} \frac{c}{c+a^{-|\alpha_1\cap\Lambda|}} & \text{if} \quad k = |\alpha_1 \cap \Lambda| + 1, \\ \frac{a^{-|\alpha_1\cap\Lambda|}}{c+a^{-|\alpha_1\cap\Lambda|}} & \text{if} \quad k = |\alpha_1 \cap \Lambda| - 1, \\ 0 & \text{in other cases.} \end{cases}$$

Therefore, Lemma 5.1 is applicable here, with $f(\cdot) = |\cdot|$.

Proof of Proposition 2.8. Having in mind the inclusion $\xi_t \subset \eta_t$ (P_α-a.s.), we will prove this lemma for (ξ_t).

The transition probabilities for the Markov chain $\{u_k\}_{k\in\mathbb{Z}_+}$ are given by

$$p_{i,j} = P_\alpha\{u_k = j \mid |u_{k-1}| = i\} = \begin{cases} \frac{c}{c+a^{-i}} & \text{if} \quad j = i+1, \\ \frac{a^{-i}}{c+a^{-i}} & \text{if} \quad j = i-1, \\ 0 & \text{in other cases,} \end{cases} \tag{30}$$

for $i \in \mathbb{N}, j \in \mathbb{Z}_+$, and $p_{0,j} = I_{\{j=0\}}$, see Remark 5.2 and (6).

Since the zero is a trap and it is accessible from all other states, there are no recurrent states except zero, and the process u has only two possible types of behavior on infinity:

$$P_\alpha\{\exists l \in \mathbb{N} \text{ s.t. } u_l = \varnothing \text{ or } \lim_{m\to\infty} u_m = \infty\} = 1.$$

We will now use properties of countable state space Markov chains, see, e.g., [Chu67, § 12, chapter 1]. Chung considers there Markov chain with a reflecting barrier at 0, but we may still apply those results, adapting them correspondingly. Denote $\varrho_m = \prod_{k=1}^{m} \frac{p_{k,k-1}}{p_{k,k+1}}$. Then the probability $P_\alpha\{\exists k \in \mathbb{N} \text{ s.t. } u_k = 0\}$ equals to 1 if and only if $\sum_{j=1}^{\infty} \varrho_j = \infty$, whichever initial condition α, $|\alpha\cap\Lambda| > 0$, we have. Moreover, if $\sum_{j=1}^{\infty} \varrho_j < \infty$ and $P_\alpha\{u_0 = q\} = 1$ (or, equivalently, $|\alpha \cap \Lambda| = q$), then $p_q := P_\alpha\{\exists k \in \mathbb{N} \text{ s.t. } u_k = 0\} = \frac{\sum_{j=q}^{\infty} \varrho_j}{1+\sum_{j=1}^{\infty} \varrho_j}$. From (30) we see that in our case $\varrho_j = c^{-j} a^{-\frac{j(j+1)}{2}}$, and

$$p_q = \frac{\sum_{j=q}^{\infty} c^{-j} a^{-\frac{j(j+1)}{2}}}{1 + \sum_{j=1}^{\infty} c^{-j} a^{-\frac{j(j+1)}{2}}} \le \frac{\sum_{j=q}^{\infty} c^{-j} a^{-\frac{j^2}{2}}}{1 + \sum_{j=1}^{\infty} c^{-j} a^{-\frac{j^2}{2}}}. \tag{31}$$

Now, for arbitrary $C > 1$ choose $q \in \mathbb{N}$ for which $c^{-1} a^{-\frac{q}{2}} < C^{-1}$. For $j > q$ we have

$c^{-j}a^{-\frac{j^2}{2}} < c^{-j}a^{-\frac{jq}{2}} = (c^{-1}a^{\frac{-q}{2}})^j < C^{-j}$, and

$$\sum_{j=q}^{\infty} c^{-j}a^{-\frac{j^2}{2}} < \sum_{j=q}^{\infty} C^{-j} = \frac{C^{-q}}{1-C^{-1}},$$

so that the statement of the lemma for $(\xi_t)_{t\geq 0}$ follows from (31). □

Note that the number of particles of the process will go to infinity with probability 1 even though the probability of extinction is positive, unless $b(\cdot,\varnothing) = 0$ almost everywhere with respect to the Lebesgue measure. However, if $b(\cdot,\varnothing) \equiv 0$, then

$$P\Big(\{|\xi_t| = 0 \text{ for large } t\ \} \cup \{|\xi_t| \to \infty\ , t \to \infty\}\Big) = 1$$

and

$$P\Big(\{|\xi_t| = 0 \text{ for large } t\ \} \cap \{|\xi_t| \to \infty\ , t \to \infty\}\Big) = 0.$$

The following equality is also taken from [Chu67, § 12, chapter 1]; for $q > s$ and all β with $|\beta \cap \Lambda| = q$,

$$P_\beta\{\exists k \in \mathbb{N} : |u_k| = s\} = \frac{\sum\limits_{j=q}^{\infty} \varrho_j(s)}{1 + \sum\limits_{j=s+1}^{\infty} \varrho_j(s)},$$

where $\varrho_m(s) = \prod\limits_{k=s+1}^{m} \frac{p_{k,k-1}}{p_{k,k+1}} = c^{-(m-s)}a^{-\frac{1}{2}(m-s)(m+s+1)}$; in our case

$$P_\beta\{\exists k \in \mathbb{N} : |u_k| = s\} = \frac{\sum\limits_{j=q}^{\infty} c^{-(j-s)}a^{-\frac{1}{2}(j-s)(j+s+1)}}{1 + \sum\limits_{j=s+1}^{\infty} c^{-(j-s)}a^{-\frac{1}{2}(j-s)(j+s+1)}} := c_{q,s} < 1. \tag{32}$$

Note that

$$c_{q+1,1} \to 0, \qquad q \to \infty \tag{33}$$

Proof of Proposition 2.9. Let $(X_k)_{k\in\mathbb{Z}_+}$ be the embedded chain of $(\eta_t)_{t\geq 0}$. First we will show that for all $m \in \mathbb{N}$ and $\alpha \in \Gamma_0$,

$$P_\alpha\{|X_k \cap \Lambda| = m \text{ infinitely often }\} = 0. \tag{34}$$

Let $\beta \in \Gamma_0$, $|\beta \cap \Lambda| = m$, $m \in \mathbb{N}$ (the case of $m = 0$ is similar, and we do not write it down). Denote $\tilde{k} = \min\{k \in \mathbb{N} : X_k \cap \Lambda \neq X_0 \cap \Lambda\}$. Since $\xi_t \subset \eta_t$ holds P_β - a.s.,

$$P_\beta\big\{|X_k\cap\Lambda|>m,\forall k\ge\tilde k\big\}\ge P_\beta\big\{|Y_k\cap\Lambda|>m,\forall k\ge 1\big\}$$
$$=P_\beta\big\{u_k>m,\ \forall k\ge 1\big\}. \tag{35}$$

By (32), the probability $P_\beta\{u_k>m,\forall k\ge 1\}$ is positive and does not depend on β, $|\beta\cap\Lambda|=m$:

$$s_m:=P_\beta\{u_k>m,\forall k\ge 1\}\ge p_{m,m+1}(1-c_{m+1,m})>0. \tag{36}$$

Define k_i^m, $i\in\mathbb{N}$, subsequently by $k_{j+1}^m=\min\{k>k_j^m : |X_k\cap\Lambda|=m$ and $\exists\bar k<k: |X_{\bar k}\cap\Lambda|\ne m\}$, $k_0^m=0$. Note that for all β

$$P_\beta\bigg\{\exists n_0:|X_n\cap\Lambda|=m \text{ for all } n\ge n_0\bigg\}=0.$$

By the strong Markov property,

$$P_\alpha\bigg\{|X_k\cap\Lambda|=m \text{ infinitely often }\bigg\}\le P_\alpha\bigg\{k_j^m<\infty,\forall j\in\mathbb{N}\bigg\}$$

$$=\prod_{j=1}^{\infty}P_\alpha\big\{k_{j+1}^m<\infty\mid k_j^m<\infty\big\}=0,$$

by (35) and (36). Indeed, if $P_\alpha\{k_j^m<\infty\}>0$, then

$$P_\alpha\{k_{j+1}^m<\infty\mid k_j^m<\infty\}=\frac{E_\alpha I_{\{k_j^m<\infty\}}P_{X_{k_j^m}}\{k_1^m<\infty\}}{E_\alpha I_{\{k_j^m<\infty\}}}$$

$$\le\frac{E_\alpha I_{\{k_j^m<\infty\}}\big(1-P_{X_{k_j^m}}\{|X_k\cap\Lambda|>m,\forall k\ge\tilde k\}\big)}{E_\alpha I_{\{k_j^m<\infty\}}}$$

$$\le\frac{E_\alpha I_{\{k_j^m<\infty\}}\big(1-P_{X_{k_j^m}}\{u_k>m,\forall k\ge 1\}\big)}{E_\alpha I_{\{k_j^m<\infty\}}}=1-s_m<1.$$

Having proved (34), we observe that

$$\big\{|\eta_t\cap\Lambda|\to\infty\big\}\cup\big\{\exists t':\forall t\ge t',|\eta_t\cap\Lambda|=\varnothing\big\}$$
$$=\bigg(\bigcup_{m=1}^{\infty}\{|X_k\cap\Lambda|=m \text{ infinitely often}\}\bigg)^c. \tag{37}$$

Note that if for some element of probability space $\omega\in\Omega$ the process $(\eta_t)_{t\ge0}$ is stuck in a trap γ, $\gamma\cap\Lambda=\varnothing$, then ω belongs to the set on the left-hand side of (37) and does not belong to the set $\big\{|X_k\cap\Lambda|=m \text{ infinitely often}\big\}$, $m\in\mathbb{N}$.

The statement of the proposition follows from (34) and (37). $\square$

Proof of Proposition 2.10. Define $\widetilde{\eta}_t := \eta_t \cap \Lambda$ and let $\widetilde{X}_k = \widetilde{\eta}_{\varsigma_k}$, where ς_k is the ordered sequence of jumps of $(\widetilde{\eta}_t)_{t\geq 0}$. Of course, the process $\{\widetilde{\eta}_t\}_{t\geq 0}$ is not Markov in general, and neither is $\{\widetilde{X}_k\}_{k\in\mathbb{N}}$. However, for all $\alpha \in \Gamma_0(\mathbb{R}^{\mathrm{d}})$ the inequality

$$P_\alpha\{|\widetilde{X}_1| - |\widetilde{X}_0| = 1\} \geq p_{|\alpha\cap\Lambda|,|\alpha\cap\Lambda|+1}$$

holds, because for every $\zeta \in \Gamma_0$, $\zeta \cap \Lambda = m$, the integral of the birth rate $b(\cdot,\zeta)$ over Λ is larger than cm, and the cumulative death rate in Λ, $\sum\limits_{x\in\zeta\cap\Lambda} d(x,\zeta)$, is less than ma^{-m}.

The probability of the event that absolutely no death occurs is positive, even when the initial configuration contains only one point inside Λ:

$$P_\alpha\Big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\Big\} = P_\alpha\Big\{|\widetilde{X}_{k+1}| - |\widetilde{X}_k| = 1 \text{ for all } \ k \in \mathbb{N}\Big\}$$

$$= \prod_{k\in\mathbb{N}} P_\alpha \Big\{|\widetilde{X}_{k+1}| - |\widetilde{X}_k| = 1 \Big| |\widetilde{X}_k| - |\widetilde{X}_{k-1}| = 1, ..., |\widetilde{X}_1| - |\widetilde{X}_0| = 1\Big\}$$

$$\geq \prod_{k\in\mathbb{N}} \inf_{\substack{\zeta\in\Gamma^0(\mathbb{R}^{\mathrm{d}}),\\ |\zeta\cap\Lambda|=|\alpha\cap\Lambda|+k}} P_\zeta\{|\widetilde{X}_1| - |\widetilde{X}_0| = 1\}$$

$$\geq \prod_{i=|\alpha|}^{\infty} p_{i,i+1} = \prod_{i=|\alpha|}^{\infty} \frac{c}{c+a^{-i}} = \prod_{i=|\alpha|}^{\infty} (1 - \frac{a^{-i}}{c+a^{-i}}) > 0,$$

because the series $\sum\limits_{i=|\alpha|}^{\infty} \frac{a^{-i}}{c+a^{-i}}$ converges. In particular, $\prod\limits_{i=m}^{\infty} p_{i,i+1} \to 1$ as m goes to ∞. Also,

$$P_{\alpha_n}\big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\big\} \to 1, \quad |\alpha_n \cap \Lambda| \to \infty. \tag{38}$$

It is clear only an a.s. finite number of deaths inside Λ occurs on $\{\exists t' : \forall t \geq t', |\eta_t \cap \Lambda| = \varnothing\}$.

By Proposition 2.9, it remains to show that only an a.s. finite number of deaths inside Λ occurs on $\{|\eta_t \cap \Lambda| \to \infty\} = \{|\widetilde{\eta}_t| \to \infty\}$. Let us introduce the stopping times $\sigma_n = \inf\{s \in \mathbb{R} : |\widetilde{\eta}_s| \geq n\}$, which are finite on $\{|\widetilde{\eta}_t| \to \infty\}$. Only a finite number of events (births and deaths) occur until arbitrary finite time P_β-a.s. for all $\beta \in \Gamma_0$, hence for $n \in \mathbb{N}$

$$P_\alpha\Big(\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all but finitely many } \ t \geq 0\} \cap \{|\widetilde{\eta}_t| \to \infty\}\Big)$$

$$\geq P_\alpha\Big(\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq \sigma_n\} \cap \{|\widetilde{\eta}_t| \to \infty\}\Big)$$

$$= P_\alpha I_{\{|\widetilde{\eta}_t|\to\infty\}} P_{\eta_{\sigma_n}}\big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\big\}.$$

From $|\eta_{\sigma_n}| \geq n$ we have by (38)

$$P_{\eta_{\sigma_n}}\big\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all } \ t \geq 0\big\} \to 1, \quad n \to \infty.$$

Therefore,

$$P_\alpha\Big(\{|\widetilde{\eta}_t| - |\widetilde{\eta}_{t-}| \geq 0 \text{ for all but finitely many } \ t \geq 0\} \cap \{|\widetilde{\eta}_t| \to \infty\}\Big) = P_\alpha\{|\widetilde{\eta}_t| \to \infty\}.$$

□

Proposition 2.10 is also applicable to $(\xi)_{t\geq 0}$, since b_1, d_1 satisfy all the conditions imposed on b, d.

Proof of Theorem 2.11. First we prove the Lemma for $(\xi)_{t\geq 0}$: we prove that for P_α-almost all $\omega \in F_1 := \{\lim\limits_{t\to\infty} |\xi_t \cap \Lambda| = \infty\}$,

$$\liminf_{t\to\infty} \frac{|\xi_t \cap \Lambda|}{e^{ct}} > 0. \tag{39}$$

There is no loss in generality in assuming $u_0 = |\alpha \cap \Lambda| > 0$. Let $0 = \tau_0 < \tau_1 < \tau_2 < ...$ be the moments of jumps of $(\xi_t)_{t\geq 0}$, so that $\xi_{\tau_k} = Y_k$. We recall that the random variables $u_n = |Y_n|$ constitute a Markov chain. Denote $\psi(n) = cn + na^{-n}$. Then

$$\int\limits_\Lambda b_1(x, Y_k)dx + \sum_{x\in Y_k} d_1(x, Y_k) = c|Y_k| + |Y_k|a^{-|Y_k|} = \psi(u_k).$$

By Theorem 12.17 in [Kal02] the random variables $\gamma_k := \psi(u_k)(\tau_{k+1} - \tau_k)$, $k \in \mathbb{Z}_+$ are independent and exponentially distributed with parameter 1. Furthermore, the sequence $\{\gamma_k\}$ is independent of Y. In particular, it is independent of $\{u_k\}_{k\in Z_+}$.

From Proposition 2.10 we know that only a finite number of deaths inside Λ occur a.s. on F_1. Particularly, there exists a positive finite random variable $\mathbf{m}$ such that the inequalities

$$u_0 + n \geq u_n \geq u_0 + n - \mathbf{m}(\omega), \quad n \in \mathbb{N} \tag{40}$$

hold with probability 1.

We can write

$$\tau_n = \sum_{k=1}^{n-1}(\tau_{k+1} - \tau_k) = \sum_{k=1}^{n-1} \frac{\gamma_k}{\psi(u_k)} \geq \sum_{k=1}^{n-1} \frac{\gamma_k}{u_0 + ck}.$$

Due to Kolmogorov's two-series theorem, the series $\sum\limits_{k=1}^{\infty} \frac{\gamma_k}{u_0+ck}$ is divergent (we recall that $E\gamma_k = D\gamma_k = 1$). Hence $\tau_n \to \infty$ a.s.

We will show below that

$$c\tau_n \le \ln n + c\tilde{\gamma}, \quad n \in \mathbb{N}, \tag{41}$$

where $\tilde{\gamma}$ is some finite random variable. Using (41), we obtain

$$P_\alpha\big\{|\xi_t| \ge \frac{e^{ct}}{(\mathbf{m}+1)e^{c\tilde{\gamma}}}, t \ge 0\big\} = P_\alpha\big\{|\xi_{\tau_n}| \ge \frac{e^{c\tau_{n+1}}}{(\mathbf{m}+1)e^{c\tilde{\gamma}}}, n \in \mathbb{N}\big\}$$

$$= P_\alpha\big\{u_n \ge \frac{1}{\mathbf{m}+1}e^{c\tau_{n+1}-c\tilde{\gamma}}, n \in \mathbb{N}\big\} = P_\alpha\big\{\ln(u_n) + \ln(\mathbf{m}+1) \ge c\tau_{n+1} - c\tilde{\gamma}, n \in \mathbb{N}\big\} = 1.$$

Therefore, (39) holds.

Inequality (41) follows from the convergence of the series

$$\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\psi(u_k)} - \frac{1}{ck}\Big). \tag{42}$$

To establish the convergence of (42), we note that

$$\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\psi(u_k)} - \frac{\gamma_k}{cu_k}\Big) \tag{43}$$

converges by Kolmogorov's theorem:

$$-\sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\psi(u_k)} - \frac{\gamma_k}{cu_k}\Big) = \sum_{k=1}^{\infty}\gamma_k\frac{u_k a^{-u_k}}{cu_k\psi(u_k)} \le \frac{1}{c^2}\sum_{k=1}^{\infty}\gamma_k\frac{a^{-u_k}}{u_k}$$

$$= \frac{1}{c^2}\sum_{k=1}^{\mathbf{m}} + \frac{1}{c^2}\sum_{k=\mathbf{m}+1}^{\infty} \le \frac{1}{c^2}\sum_{k=1}^{\mathbf{m}}\gamma_k\frac{a^{-u_k}}{u_k} + \frac{1}{c^2}\sum_{j=1}^{\infty}\gamma_k\frac{a^{-j}}{j},$$

and

$$\sum_{k=1}^{\infty}\Big(\frac{1}{ck} - \frac{1}{cu_k}\Big) \tag{44}$$

converges by (40).

The convergence of the series in (42) follows from the fact that (43) and (44) converge.

We have thus proved (39). To establish the statement of the theorem, note that $\tilde{\sigma}_n = \inf\{t > 0 : |\eta_t| \ge n\}$ is finite on F and

$$\big\{\liminf \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \to \infty\big\} \subset \big\{\liminf \frac{|\xi_t|}{e^{ct}} = 0\big\}.$$

It follows from what we have already proved that

$$P_\beta\big\{\liminf \frac{|\xi_t|}{e^{ct}} = 0\big\} = P_\beta\big\{(\xi_t)_{t\ge 0} \text{ extincts}\big\}, \quad \beta \in \Gamma_0.$$

Therefore, by Proposition 2.8 and the strong Markov property (for the latter see e.g. [Kal02, Theorem 12.14])

$$P_\alpha\Big\{\liminf \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \to \infty\Big\} = P_\alpha P_{\eta_{\tilde{\sigma}_n}}\Big\{\liminf \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \to \infty\Big\}$$

$$\leq P_\alpha P_{\eta_{\tilde{\sigma}_n}}\Big\{\liminf \frac{|\xi_t|}{e^{ct}} = 0\Big\} \leq \tilde{C}^{-n},$$

where $\tilde{C}$ is the constant that appeared in Proposition 2.8. Since n is arbitrary,

$$P_\alpha\Big\{\liminf \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \to \infty\Big\} = 0.$$

□

Proof of Corollary 2.12. Let us fix a configuration α, $\alpha \cap \Lambda \neq \varnothing$. We saw in the proof of Theorem 2.11 that for almost all $\omega \in F = \{\omega : \liminf\limits_{t\to\infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} > 0\}$ we have

$$P_\alpha\{|\xi_t| \geq \frac{1}{(\mathbf{m}+1)e^{c\tilde{\gamma}}} e^{ct}, t \geq 0\} = 1.$$

Let F_k be the set $\{\omega : \frac{1}{(\mathbf{m}+1)e^{c\tilde{\gamma}}} \geq \frac{1}{k}\}$. Then $\bigcup\limits_{k\in\mathbb{N}} F_k = F$, and, since $P_\alpha(F) > 0$,

$$P_\alpha(F_k) > 0$$

for some $k \in \mathbb{N}$. Hence

$$E_\alpha|\eta_t \cap \Lambda| \geq E_\alpha|\eta_t \cap \Lambda| I_{F_k} \geq \frac{1}{k} e^{ct} P_\alpha(F_k). \quad \square$$

5 Appendix

5.1 Markovian functions of a Markov chain

Let $(S, \mathscr{B}(S))$ be a Polish (state) space. Consider a (time-homogeneous) Markov chain on $(S, \mathscr{B}(S))$ as a family of probability measures on S^∞. Namely, on the measurable space $(\Omega, \mathscr{F}) = (S^\infty, \mathscr{B}(S^\infty))$ consider a family of probability measures $\{P_s\}_{s\in S}$ such that for the coordinate mappings

$$X_n : \Omega \to S,$$

$$X_n(s_1, s_2, ...) = s_n$$

the process $X = \{X_n\}_{n\in\mathbb{Z}_+}$ is a Markov chain, and for all $s \in S$

$$P_s\{X_0 = s\} = 1,$$

$$P_s\{X_{n+m_j} \in A_j, j = 1, ..., k_1 \mid \mathscr{F}_n\} = P_{X_n}\{X_{m_j} \in A_j, j = 1, ..., k_1\}.$$

Here $A_j \in \mathscr{B}(S)$, $m_j \in \mathbb{N}$, $k_1 \in \mathbb{N}$, $\mathscr{F}_n = \sigma\{X_1, ..., X_n\}$. The space S is separable, hence there exists a transition probability kernel $Q : S \times \mathscr{B}(S) \to [0, 1]$ such that

$$Q(s, A) = P_s\{X_1 \in A\}, \quad s \in S,\ A \in \mathscr{B}(S).$$

Consider a transformation of the chain X, $Y_n = f(X_n)$, where $f : S \to \mathbb{Z}_+$ is a Borel-measurable function, with convention $\mathscr{B}(\mathbb{Z}_+) = 2^{\mathbb{Z}_+}$. In this section we will give sufficient conditions for $Y = \{Y_n\}_{n\in\mathbb{Z}_+}$ to be a Markov chain. A similar question was discussed in [BR58].

Lemma 5.1. *Assume that for any bounded Borel function $h : S \to S$*

$$E_s h(X_1) = E_q h(X_1) \text{ whenever } f(s) = f(q), \tag{45}$$

Then Y is a Markov chain.

Remark. Condition (45) is the equality of distributions of X_1 under two different measures, P_s and P_q.

Proof. For the natural filtrations of the processes X and Y we have an inclusion

$$\mathscr{F}_n^X \supset \mathscr{F}_n^Y, \quad n \in \mathbb{N}, \tag{46}$$

since Y is a function of X. For $k \in \mathbb{N}$ and bounded Borel functions $h_j : \mathbb{Z}_+ \to \mathbb{R}$, $j = 1, 2, ..., k$ (any function on $\mathbb{Z}_+$ is a Borel function),

$$\begin{gathered} E_s\left[\prod_{j=1}^k h_j(Y_{n+j}) \mid \mathscr{F}_n^X\right] = E_{X_n}\prod_{j=1}^k h_j(f(X_j)) = \\ \int_S Q(x_0, dx_1)h_1(f(x_1))\int_S Q(x_1, dx_2)h_2(f(x_2))...\int_S Q(x_{n-1}, dx_n)h_n(f(x_n))\bigg|_{x_0=X_n} \end{gathered} \tag{47}$$

To transform the last integral, we introduce a new kernel: for $y \in f(S)$ chose $x \in S$ with $f(x) = y$, and then for $B \subset \mathbb{Z}_+$ define

$$\overline{Q}(y, B) = Q(x, f^{-1}(B)). \tag{48}$$

The expression on the right-hand side does not depend on the choice of x because of (45). To make the kernel $\overline{Q}$ defined on $\mathbb{Z}_+ \times \mathscr{B}(\mathbb{Z}_+)$, we set

$$\overline{Q}(y, B) = I_{\{0 \in B\}}, \ y \notin f(S).$$

Then, setting $z_n = f(x_n)$, it follows from the change of variables formula for the Lebesgue integral that

$$\int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) = \int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-1}), dz_n) h_n(z_n).$$

Likewise, setting $z_{n-1} = f(x_{n-1})$, we get

$$\int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) =$$

$$\int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-1}), dz_n) h_n(z_n) =$$

$$\int_{\mathbb{Z}_+} \overline{Q}(f(x_{n-2}), dz_{n-1}) h_n(z_{n-1}) \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n).$$

Further proceeding, we obtain

$$\int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) ... \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) =$$

$$\int_{\mathbb{Z}_+} \overline{Q}(z_0, dz_1) h_1(z_1) \int_{\mathbb{Z}_+} \overline{Q}(z_1, dz_2) h_2(z_2) ... \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n),$$

where $z_0 = f(x_0)$.

Thus,

$$E_s \left[\prod_{j=1}^{k} h_j(Y_{n+j}) \mid \mathscr{F}_n^X \right] =$$

$$\int_{\mathbb{Z}_+} \overline{Q}(f(X_0), dz_1) h_1(z_1) \int_{\mathbb{Z}_+} \overline{Q}(z_1, dz_2) h_2(z_2) ... \int_{\mathbb{Z}_+} \overline{Q}(z_{n-1}, dz_n) h_n(z_n).$$

This equality and (46) imply that Y is a Markov chain.

Remark 5.2. The kernel $\overline{Q}$ and the chain $f(X_n)$ are related: for all $s \in S$, $n, m \in \mathbb{N}$ and $M \subset \mathbb{N}$,

$$P_s\{f(X_{n+1}) \in M \mid f(X_n) = m\} = \overline{Q}(m, M)$$

whenever $P_s\{f(X_{n+1}) = m\} > 0$. Informally, we can say that $\overline{Q}$ is the transition probability kernel for the chain $\{f(X_n)\}_{n \in \mathbb{Z}_+}$.

Remark 5.3. Clearly, this result holds for a Markov chain which is not necessarily defined on a canonical state space, because the property of a process to be a Markov chain depends on its distribution only.

5.2 Strong Markov property of the driving process

Let N be compatible with a right-continuous complete filtration $\{\mathscr{F}_t\}_{t\geq 0}$, and τ be a finite a.s. $\{\mathscr{F}_t\}_{t\geq 0}$-stopping time. For $\gamma \in \Gamma_2$, $\gamma = \sum_i \delta_{(s_i,u_i)}$, let $\theta_\tau \gamma = \sum_{i:s_i>\tau} \delta_{(s_i-\tau,u_i)}$. Also, for $\Xi \in \mathscr{B}(\Gamma_2)$ we define the shift

$$\theta_\tau \Xi = \{\gamma \in \Gamma_2 \mid \theta_\tau \gamma \in \Xi\}$$

Introduce another point process $\overline{N}$ on $\mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+ \times \Gamma_2$,

$$\overline{N}([0,s] \times U \times \Xi) = N((\tau, \tau+s] \times U \times \theta_\tau \Xi), \quad s>0,\ U \in \mathscr{B}(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+),\ \Xi \in \mathscr{B}(\Gamma_2).$$

Proposition 5.4. *The process $\overline{N}$ is a Poisson point process with intensity measure $ds \times dx \times du \times \pi$, independent of $\mathscr{F}_\tau$.*

Proof. To prove the proposition, it is enough to show that

(i) for any $b > a > 0$, open bounded $U \subset \mathbb{R}^{\mathrm{d}} \times \mathbb{R}_+$ and open $\Upsilon \subset \Gamma_2$, $\overline{N}((a,b) \times U \times \Xi)$ is a Poisson random variable with mean $(b-a) \times l(U) \times \pi(\Xi)$, where l is the Lebesgue measure on the corresponding space, and

(ii) for any $b_k > a_k > 0$, $k = 1, ..., m$, open bounded $U_k \subset \mathbb{R}^{\mathrm{d}}$ and open $\Xi_k \subset \Gamma_2$ such that $((a_i,b_i) \times U_i \times \Xi_i) \cap ((a_j,b_j) \times U_j \times \Xi_j) = \varnothing$, $i \neq j$, the collection $\{\overline{N}((a_k,b_k) \times U_k \times \Xi_k)\}_{k=1,m}$ is a finite sequence of independent random variables, independent of $\mathscr{F}_\tau$.

Let τ_n be the sequence of $\{\mathscr{F}_t\}_{t\geq 0}$-stopping times, $\tau_n = \frac{k}{2^n}$ on $\{\tau \in (\frac{k-1}{2^n}, \frac{k}{2^n}]\}$, $k \in \mathbb{N}$. Then $\tau_n \downarrow \tau$ and $\tau_n - \tau \leq \frac{1}{2^n}$. The stopping times τ_n take only countably many values. Therefore the process N satisfies the strong Markov property for τ_n: the processes $\overline{N}_n$, defined by

$$\overline{N}_n([0,s] \times U \times \Xi) := N((\tau_n, \tau_n + s] \times U \times \theta_{\tau_n} \Xi),$$

are Poisson point processes, independent of $\mathscr{F}_{\tau_n}$.

To prove (i), note that $\overline{N}_n((a,b) \times U \times \Xi) \to \overline{N}((a,b) \times U \times \Xi)$ a.s. and all random variables $\overline{N}_n((a,b) \times U \times \Xi)$ have the same distribution, therefore $\overline{N}((a,b) \times U \times \Xi)$ is a Poisson random variable with mean $(b-a)l(U)\pi(\Xi)$. The random variables $\overline{N}_n((a,b) \times U \times \Xi)$ are independent of $\mathscr{F}_\tau$, hence $\overline{N}((a,b) \times U \times \Xi)$ is independent of $\mathscr{F}_\tau$, too. Similarly, (ii) follows. $\square$

Acknowledgement

I wish to thank my thesis advisor Yuri Kondratiev for suggesting the problem and superb guidance during the progress of the work. I also thank Vladimir Bogachev for many critical and stimulating comments. The author acknowledges the financial support of the DFG through the SFB 701 (Bielefeld University) and the IRTG (IGK) 1132 "Stochastics and Real World Models".

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