MODULI ALGEBRAS OF SOME SIMPLEST NON-SEMIQUASIHOMOGENEOUS SINGULARITIES

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Abstract. We consider a class of non-semiquasihomogeneous singularities, for which we find the dimension of the moduli algebra and, under some restrictions, an explicit basis of this algebra.

There are only few explicit descriptions known of the moduli algebras of isolated singularities, which are not quasihomogeneous or semiquasihomogeneous. Recall that the singularity (at the origin) given by an equation $f(x_1, ..., x_n) = 0$ is called *quasihomogeneous* if there exist positive weights w_1, ..., w_n of variables with respect to which all the monomials occurring in the Taylor series for f have the same degree (which implies that f is a polynomial). It is called *semiquasihomogeneous* if $f = f_0 + f_>$ where $f_0 = 0$ is a quasihomogeneous singularity with respect to some weights w_1, ..., w_n such that all monomials in $f_>$ have the degrees larger than those in f_0. In the latter case, if the singularity $f_0 = 0$ is isolated, the singularity $f = 0$ is isolated as well.

By abuse of language, we will sometimes say about a singularity f having in mind the singularity given by the equation $f = 0$. We will always assume that f does not contain monomials of the form x_i and $x_i x_j$ with $i \neq j$.

The *moduli algebra* A_f of a singularity f is defined by

$$\mathrm{A}_f := \mathbb{C}\left[\!\left[x_1, ..., x_n\right]\!\right]/J(f) \tag{1}$$

where

$$J(f) := (\partial f/\partial x_1, ..., \partial f/\partial x_n) \tag{2}$$

is the *Jacobian ideal* of f. A quasihomogeneous singularity is isolated if and only if its moduli algebra is finite-dimensional.

Quasihomogeneity of a singularity f implies existence of a grading on the algebra A_f, which makes life simpler. For semiquasihomogeneous singularities the following theorem is known.

Theorem 0.1 ([AGV, 12.2, Corollary]). *A monomial basis of the moduli algebra A_{f_0} of the quasihomogeneous part f_0 of a semiquasihomogeneous singularity $f = f_0 + f_>$ is also a basis of A_f.*

One can prove (see Theorem 1.3) that, under our assumption about the absence of monomials $x_i x_j$ with $i \neq j$, a singularity $f(x_1, ..., x_n)$ can be isolated only if f contains at least n monomials. It is thus natural to consider as the simplest isolated non-semiquasihomogeneous singularities those of the form

$$f(x_1, ..., x_n) = f_0(x_1, ..., x_n) + u, \tag{3}$$

where

$$f_0(x_1, ..., x_n) = u_1 + ... + u_n \tag{4}$$

is an isolated quasihomogeneous singularity containing n monomials u_1, ..., u_n, whereas the monomial u is of (weighted) degree strictly less than that of u_1, ..., u_n.

In this paper, we are first going to list all possible f_0 of this type. Their description can be extracted, more or less, from [AGV] and some other subsequent papers, see e. g. [KS]. For convenience of the reader, we give a proof of this result in the paper.

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We will then investigate what happens when one adds to such f_0 a monomial u as above. In fact we will assume that u satisfies the following stronger condition:

Condition 0.2. The degree vector of u lies strictly inside the convex hull of the degree vectors of $u_1, ..., u_n$ together with the origin.

Under this condition, we will derive a formula for the number

$$\operatorname{def}(f) = \dim \mathrm{A}_{f_0} - \dim \mathrm{A}_f, \tag{5}$$

which we will call the *defect* of f.

We will first derive this formula using the formula of Kouchnirenko [K]. Then we will employ an alternative approach which will provide an additional conceptual insight into our formula. Finally, in the particular case, when f_0 is a so called *Brieskorn-Pham singularity*, i. e.,

$$f_0(x_1, ..., x_n) = x_1^{p_1} + ... + x_n^{p_n}, \tag{6}$$

we will develop yet another approach which will actually yield an explicit monomial basis of A_f.

We would like to note that our monomial basis differs from the one obtained using the Gröbner basis with respect to any monomial ordering. The point is that the most natural monomial basis of the moduli algebra of the Brieskorn-Pham singularity (6) is the so called *parallelepipedal* basis

$$P(p_1, ..., p_n) := \{x_1^{k_1} \cdots x_n^{k_n} \mid 0 \leqslant k_i < p_i, i = 1, ..., n\}, \tag{7}$$

and it is not clear how to obtain it using Gröbner bases techniques. Our basis of A_f is obtained from $P(p_1, ..., p_n)$ by deleting the monomials which become zero in A_f; the number of such is precisely $\operatorname{def}(f)$.

Remark 0.3. Under a monomial we mean a product of some (non-negative integer) powers of the variables taken with coefficient 1. However, if the degree vectors of the monomials $u_1, ..., u_n$ from (4) are linearly independent (which is definitely the case, if f_0 defines an isolated singularity), then, multiplying the variables by some non-zero numbers, one can make f_0 a linear combination of $u_1, ..., u_n$ with arbitrary non-zero coefficients. Similarly, if the degree vectors of the monomials $u_1, ..., u_n, u$ are affinely independent (which is definitely the case, if Condition 0.2 holds), then, multiplying the variables and the very polynomial f by some non-zero numbers, one can make it a linear combination of $u_1, ..., u_n, u$ with arbitrary non-zero coefficients. We will sometimes use this observation.

1. "Minimal" isolated quasihomogeneous singularities

Let us thus begin with quasihomogeneous polynomials f_0 of the form (4). We are interested in those of them having an isolated singularity at the origin, i. e. such that the algebra A_{f_0} is finite-dimensional. Such polynomials have been classified in [KS]. They are disjoint sums (i. e. sums whose summands have no variables in common) of polynomials of two types described in the following examples.

Example 1.1. We will call a polynomial of the form

$$C = C_{k_1,...,k_n}(x_1, ..., x_n) := x_1^{k_1} x_2 + x_2^{k_2} x_3 + ... + x_n^{k_n} x_1 \; (k_1, ..., k_n > 1)$$

a *cycle*. It is quasihomogeneous, with weights of variables

$$\begin{aligned}
w_1 &= k_2 \cdots k_{n-1} k_n - k_3 \cdots k_{n-1} k_n + k_4 \cdots k_{n-1} k_n - ... \pm k_n \mp 1, \\
w_2 &= k_3 \cdots k_n k_1 - k_4 \cdots k_n k_1 + k_5 \cdots k_n k_1 - ... \pm k_1 \mp 1, \\
&..., \\
w_n &= k_1 \cdots k_{n-1} - k_2 \cdots k_{n-1} + k_3 \cdots k_{n-1} - ... \pm k_{n-1} \mp 1.
\end{aligned}$$

With these weights, the degree of (each monomial of) C is $d := k_1 \cdots k_n - (-1)^n$.

Computing the partial derivatives, we see that the moduli algebra of C is obtained from the algebra of formal power series by imposing the relations

$$\begin{aligned} x_1^{k_1} &= -k_2 x_2^{k_2-1} x_3, \\ &..., \\ x_{n-1}^{k_{n-1}} &= -k_n x_n^{k_n-1} x_1, \\ x_n^{k_n} &= -k_1 x_1^{k_1-1} x_2. \end{aligned}$$

These relations imply that the moduli algebra is finite-dimensional. Indeed, consider any monomial $x_1^{l_1}...x_n^{l_n}$. If it is not in $P(k_1,...,k_n)$ (see notation (7)), i.e. $l_i \geqslant k_i$ for some i, then one can replace $x_i^{k_i}$ with the right hand side of the corresponding relation obtaining a monomial of the same (weighted) degree with a coefficient, whose module is strictly greater than 1. Repeating this procedure, we either obtain a scalar multiple of a monomial from $P(k_1,...,k_n)$ or twice obtain different scalar multiples of the same monomial. In the latter case we conclude that the original monomial is zero in the moduli algebra. Thus, in the moduli algebra every monomial becomes either zero or a scalar multiple of a monomial from $P(k_1,...,k_n)$. Hence, the dimension of the moduli algebra does not exceed $k_1 \cdots k_n$. In particular, the singularity is isolated.

On the other hand, the dimension of the moduli algebra can be calculated using the following well known formula for the dimension of the moduli algebra A of an isolated quasihomogeneous singularity:

$$\dim A = \prod_{i=1}^{n} \left(\frac{d}{w_i} - 1 \right)$$

(see, e. g., [MO] or [AGV]). In our case this gives $k_1 \cdots k_n$.

This means that the monomials from $P(k_1,...,k_n)$ constitute a basis of the moduli algebra.

Example 1.2. We will call a polynomial of the form

$$T = T_{k_1,...,k_n}(x_1,...,x_n) := x_1^{k_1} x_2 + x_2^{k_2} x_3 + ... + x_{n-1}^{k_{n-1}} x_n + x_n^{k_n},\ (k_1,...,k_n > 1)$$

a *tadpole*. It is quasihomogeneous, with weights of variables

$$w_i = k_1 \cdots k_{i-1}(k_{i+1} \cdots k_n - k_{i+2} \cdots k_n + k_{i+3} \cdots k_n - ... \pm k_n \mp 1).$$

The degree of (each monomial of) T is $d := k_1 \cdots k_n$.

Again computing the partial derivatives, we see that the moduli algebra of T is obtained from the algebra of formal power series by imposing the relations

$$\begin{aligned} x_1^{k_1} &= -k_2 x_2^{k_2-1} x_3, \\ &..., \\ x_{n-2}^{k_{n-2}} &= -k_{n-1} x_{n-1}^{k_{n-1}-1} x_n, \\ x_{n-1}^{k_{n-1}} &= -k_n x_n^{k_n-1}, \\ x_1^{k_1-1} x_2 &= 0. \end{aligned}$$

It is easy to see that these relations imply the following ones:

$$\begin{aligned} x_n^{k_n} &= 0, \\ x_1^{k_1-1} x_2 &= 0, \\ x_1^{k_1-1} x_3^{k_3-1} x_4 &= 0, \\ x_1^{k_1-1} x_3^{k_3-1} x_5^{k_5-1} x_6 &= 0, \\ &... \end{aligned}$$

It follows that a monomial basis of this algebra can be chosen, as in the previous example, to be contained in $P(k_1,...,k_n)$, but does not coincide with it: the above relations imply that monomials divisible by one of

$$
\begin{aligned}
&x_1^{k_1-1}x_2,\\
&x_1^{k_1-1}x_3^{k_3-1}x_4,\\
&x_1^{k_1-1}x_3^{k_3-1}x_5^{k_5-1}x_6,\\
&...
\end{aligned}
\tag{1}
$$

are zero in the moduli algebra. To find the cardinality of the set of such monomials, let us partition it into the following disjoint subsets:

$$
\begin{aligned}
&\{x_1^{k_1-1}x_2^{i_2}x_3^{i_3}\cdots x_n^{i_n} \mid 0 < i_2 < k_2, 0 \leqslant i_3 < k_3, ..., 0 \leqslant i_n < k_n\},\\
&\{x_1^{k_1-1}x_3^{k_3-1}x_4^{i_4}x_5^{i_5}\cdots x_n^{i_n} \mid 0 < i_4 < k_4, 0 \leqslant i_5 < k_5, ..., 0 \leqslant i_n < k_n\},\\
&\{x_1^{k_1-1}x_3^{k_3-1}x_5^{k_5-1}x_6^{i_6}x_7^{i_7}\cdots x_n^{i_n} \mid 0 < i_6 < k_6, 0 \leqslant i_7 < k_7, ..., 0 \leqslant i_n < k_n\},\\
&...
\end{aligned}
$$

The cardinality of the union of these sets is

$$(k_2-1)k_3\cdots k_n + (k_4-1)k_5\cdots k_n + (k_6-1)k_7\cdots k_n + ...,$$

where the sum ends with $k_n - 1$ if n is even and with 1 if n is odd.

On the other hand, employing the above formula from [AGV], we see that

$$\prod_{i=1}^{n}\left(\frac{d}{w_i}-1\right) = k_1\cdots k_n - (k_2-1)k_3\cdots k_n - (k_4-1)k_5\cdots k_n - (k_6-1)k_7\cdots k_n - ...,$$

so that all the remaining monomials from the parallelepiped (i. e. not divisible by those in (1)) form a basis of the moduli algebra.

Theorem 1.3 ([KS]). *Let $f_0(x_1, ..., x_n)$ be a polynomial with at most n monomials having an isolated singularity at the origin. Then it has exactly n monomials and decomposes into a disjoint sum of cycles and tadpoles.*

Proof. For each i, the polynomial f_0 must contain a monomial of the form x_i^k or $x_i^k x_j$ with $i \neq j$, because otherwise all the points of the i-th coordinate axis would be singular for f_0. It follows that f_0 contains exactly n monomials, and, moreover, for each i, there is a unique monomial of the above form.

Further, for any given j, there cannot be two monomials of the form $x_i^k x_j$. Indeed, suppose that f_0 contains monomials $x_i^k x_j$ and $x_{i'}^{k'} x_j$ (with different i, i', j). Then all the partial derivatives of f_0 but the j-th one vanish on the coordinate plane of the variables x_i and $x_{i'}$. The remaining partial derivative vanishes on some curve on this plane passing through the origin, so all the points of this curve are singular for f_0, which contradicts the isolatedness of the singularity of f_0.

The assertion of the theorem immediately follows. □

2. Calculations using the formula of Kouchnirenko

Consider now a singularity f of the form (3), satisfying Condition 0.2.

For any formal power series g and any face Δ of its Newton polyhedron denote by g_Δ the sum of those terms of g whose degree vectors belong to Δ.

In order to apply the formula of Kouchnirenko [K] to both f_0 and f, we have to check that, for any compact face Δ of the Newton polyhedron of f_0, the system

$$\left(x_i\frac{\partial f_0}{\partial x_i}\right)_\Delta = 0, \quad i = 1, ..., n$$

has no solutions in $(\mathbb{C}^*)^n$, and similarly for f. However, it is known, that this condition automatically holds for any simplicial face Δ, so it suffices to show that all compact faces of the Newton polyhedra of f_0 and f are simplicial.

The Newton polyhedron of f_0 has a unique compact face Δ_0, namely, the simplex spanned by the degree vectors of the monomials $u_1, ..., u_n$.

As for f, it follows from Condition 0.2 that the compact faces of its Newton polyhedron are the simplices $\Delta_1, ..., \Delta_n$, where Δ_i is spanned by the degree vectors of $u_1, ..., u_{i-1}, u_{i+1}, ..., u_n$ together with the degree vector of u.

Thus, we can apply the formula of Kouchnirenko. It has the form

$$\dim \mathrm{A}_f = n!V_n - (n-1)!V_{n-1} + ... + (-1)^{n-1}1!V_1 + (-1)^n,$$

resp.,

$$\dim \mathrm{A}_{f_0} = n!V_n^{(0)} - (n-1)!V_{n-1}^{(0)} + ... + (-1)^{n-1}1!V_1^{(0)} + (-1)^n,$$

where V_n, resp. $V_n^{(0)}$, is the volume of the union Γ_-, resp. $\Gamma_-^{(0)}$, of all straight segments connecting the origin with all points of the simplices $\Delta_1, ..., \Delta_n$, resp. of the simplex Δ_0, whereas V_k, resp. $V_k^{(0)}$, is the sum of the k-dimensional volumes of the intersections of Γ_-, resp. $\Gamma_-^{(0)}$, with all k-dimensional coordinate planes.

Now comes the crucial observation.

Lemma 2.1. *One has*

$$V_k = V_k^{(0)}$$

for all $k < n$, while

$$V_n = V_n^{(0)} - \mathrm{Vol}\ \Delta_u,$$

where Δ_u is the simplex spanned by u, u_1, ..., u_n.

Proof. The point is that by our Condition 0.2, if a face of Γ_- has a nontrivial intersection with a coordinate plane, then it is a common face of Γ_- and $\Gamma_-^{(0)}$. As for V_n, it is the volume of the simplex $\Gamma_-^{(0)}$ (whose volume is $V_n^{(0)}$) with Δ_u removed. Hence the lemma. ☐

In order to write an explicit formula for def_f, introduce the following notations:

$$u_i = x_1^{\nu_{i1}} x_2^{\nu_{i2}} \cdots x_n^{\nu_{in}},\ i = 1, ..., n, \quad u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Theorem 2.2. *The defect* $\mathrm{def}(f)$ *is equal to the determinant of the matrix*

$$M_f = \begin{pmatrix} \nu_{11} - a_1 & ... & \nu_{1n} - a_n \\ ... & & \\ \nu_{n1} - a_1 & ... & \nu_{nn} - a_n \end{pmatrix}.$$

Proof. Indeed, the volume of Δ_u is equal to $\frac{1}{n!}$ times the determinant of the matrix made from the degree vectors of $u_1, ..., u_n$ with the degree vector of u subtracted from each. ☐

3. Algebraic approach

Retaining the assumptions and notation of Section 2, we are now going to employ the algebra

$$\mathrm{P}_f := \mathbb{C}[x_1, ..., x_n]/J_{\mathrm{pol}}(f)$$

obtained by replacing the algebra of formal power series with the algebra of polynomials in the definition of A_f. Here $J_{\mathrm{pol}}(f)$ is the ideal of the algebra $\mathbb{C}[x_1, ..., x_n]$ generated by the partial derivatives of f, so that $J(f)$ is the closure of $J_{\mathrm{pol}}(f)$ in the formal topology of the algebra $\mathbb{C}[[x_1, ..., x_n]]$.

Note that for a quasihomogeneous polynomial f_0 with isolated singularity, the algebra P_{f_0} coincides with A_{f_0}, since the ideal $J_{\mathrm{pol}}(f)$ contains all quasihomogeneous polynomials of sufficiently large degree.

As for the non-quasihomogeneous case, one has

Proposition 3.1. *Suppose given an ideal $I = (f_1 + g_1, ..., f_n + g_n)$ in $\mathbb{C}[x_1, ..., x_n]$, where the polynomials f_1, ..., f_n are quasihomogeneous with respect to some positive weights, while*

$$\deg(g_i) < \deg(f_i),\ i = 1, ..., n,$$

with respect to these weights. Then, if the algebra $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_n)$ *is finite-dimensional, the algebra* $\mathbb{C}[x_1, ..., x_n]/I$ *is finite-dimensional of the same dimension, and, moreover, any basis of the former consisting of quasihomogeneous polynomials is also a basis of the latter.*

Proof. The finite-dimensionality of the algebra $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_n)$ means that the (quasihomogeneous) polynomials $f_1, ..., f_n$ constitute a regular sequence in $\mathbb{C}[x_1, ..., x_n]$, which implies that $\mathbb{C}[x_1, ..., x_n]$ is a free module over the subalgebra $\mathbb{C}[f_1, ..., f_n]$ generated by $f_1, ..., f_n$. Moreover, let $H \subset \mathbb{C}[x_1, ..., x_n]$ be a quasihomogeneous subspace complementary to the ideal $(f_1, ..., f_n)$. Then any basis (over $\mathbb{C}$) of H is a basis of $\mathbb{C}[x_1, ..., x_n]$ over $\mathbb{C}[f_1, ..., f_n]$. In other words, any element of $\mathbb{C}[x_1, ..., x_n]$ is uniquely represented as a polynomial in $f_1, ..., f_n$ with coefficients in H. Considering consecutively the highest degree terms, one can see that the same remains true when replacing $f_1, ..., f_n$ with $f_1 + g_1, ..., f_n + g_n$. It follows that H is a subspace complementary to the ideal I as well. □

Corollary 3.2. *Let* f *be a polynomial of the form* $f_0 + f_<$, *where* f_0 *is quasihomogeneous with isolated singularity and* $f_<$ *consists of monomials whose degrees are strictly less than that of* f_0*. Then the algebras* P_f *and* $\mathrm{P}_{f_0} = \mathrm{A}_{f_0}$ *have equal dimensions; moreover, any monomial basis of* P_{f_0} *is also a basis of* P_f.

Remark 3.3. If one tries a similar trick for the terms of the lowest degree instead of highest, then one has to deal with a decreasing filtration. To get the desired result in this case, one only needs completeness of the algebra, so this can be done in the algebra of formal power series. It is thus similarly true that if $\deg(g_i) > \deg(f_i)$, $i = 1, ..., n$, then any monomial basis of $\mathbb{C}[\![x_1, ..., x_n]\!]/(f_1, ..., f_n)$ is also a basis of $\mathbb{C}[\![x_1, ..., x_n]\!]/(f_1 + g_1, ..., f_n + g_n)$. As a corollary one obtains, in particular, Theorem 0.1.

Unlike A_f, which is obviously a local algebra, P_f may have several maximal ideals. Like every finite-dimensional algebra, it is a direct sum of local algebras, one of them being A_f. In our particular case P_f has one specific property which will be crucial for this section.

Lemma 3.4. *Under the assumptions of Section 2, if* $a_1, ..., a_n > 1$, *then in any local quotient algebra* A *of* P_f, *either all variables* x_i *become nilpotent (so that* A *is a quotient of* A_f*), or all variables become invertible.*

Proof. It suffices to show that for any homomorphism $\chi : \mathrm{P}_f \to \mathbb{C}$, if $\chi(x_i) = 0$ for some variable x_i, then all variables map to zero, too.

If $\chi(x_i) = 0$, then by the condition $a_1, ..., a_n > 1$, the monomial u and all its partial derivatives map to zero, too. It follows that the images of the variables under χ coincide with their images under some homomorphism $\mathrm{P}_{f_0} \to \mathbb{C}$. But as already mentioned, $\mathrm{P}_{f_0} = \mathrm{A}_{f_0}$, and all variables map to zero under the unique homomorphism $\mathrm{A}_{f_0} \to \mathbb{C}$. □

It thus follows that

$$\mathrm{P}_f = \mathrm{A}_f \oplus \mathrm{P}_f[x_1^{-1}, ..., x_n^{-1}],$$

where $\mathrm{P}_f[x_1^{-1}, ..., x_n^{-1}]$ is the localization of P_f obtained by inverting all variables. Due to Remark 0.3, the latter algebra is nothing else than the group algebra of the finite abelian group G with generators x_1, ..., x_n and defining relations $u_1 = ... = u_n = u$. Clearly, the order of G is equal to $\det M_f$, where M_f is the matrix described in Theorem 2.2. Thus, we have recovered the latter theorem.

4. An explicit basis for the Brieskorn-Pham case

We will now present yet another proof of the defect formula for the particular case when f_0 is the Brieskorn-Pham singularity (6). Along the way we will produce two monomial bases of A_f, one of them lying in the fundamental parallelepiped (7).

Making use of Remark 0.3, we may (and will) assume that

$$f = \frac{a_1}{p_1} x_1^{p_1} + ... + \frac{a_n}{p_n} x_n^{p_n} - x_1^{a_1} \cdots x_n^{a_n}. \tag{10}$$

Then

$$\frac{x_i}{a_i}\frac{\partial f}{\partial x_i} - \frac{x_j}{a_j}\frac{\partial f}{\partial x_j} = x_i^{p_i} - x_j^{p_j} \in J(f),$$

and we can factorize $\mathbb{C}\llbracket x_1, ..., x_n \rrbracket$ by $J(f)$ in two steps: first factorize by the ideal $I(f)$ generated by the differences $x_i^{p_i} - x_j^{p_j}$, and then factorize the obtained algebra by $J(f)/I(f)$.

We will first study the algebra $\mathrm{P}_f := \mathbb{C}[x_1, ..., x_n]/J_{\mathrm{pol}}(f)$ defined as in the previous section. It also can be obtained in two steps, factorizing first by the ideal $I_{\mathrm{pol}}(f)$ of $\mathbb{C}[x_1, ..., x_n]$ generated by the differences $x_i^{p_i} - x_j^{p_j}$, and then by $J_{\mathrm{pol}}(f)/I_{\mathrm{pol}}(f)$.

Moreover, it is convenient to embed the algebra $\mathbb{C}[x_1, ..., x_n]$ into the algebra $\mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ of Laurent polynomials. Denote by $J_{\mathrm{Lau}}(f)$ and $I_{\mathrm{Lau}}(f)$ the ideals of $\mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ generated by $J_{\mathrm{pol}}(f)$ and $I_{\mathrm{pol}}(f)$, respectively.

The algebra

$$\mathrm{L}_f^\infty = \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]/I_{\mathrm{Lau}}(f)$$

is nothing else than the group algebra of the (infinite) abelian group G^∞ with generators $g_1, ..., g_n$ and defining relations $g_i^{p_i} = g_j^{p_j}$.

Adding one extra generator g_0, write defining relations of G^∞ in the form

$$g_0 = g_1^{p_1} = ... = g_n^{p_n}. \tag{11}$$

Then a canonical form for an element $u = g_0^{k_0} g_1^{k_1} \cdots g_n^{k_n} \in G^\infty$ is

$$g_0^q g_1^{r_1} \cdots g_n^{r_n} \text{ with } 0 \leqslant r_1 < p_1, ..., 0 \leqslant r_n < p_n. \tag{12}$$

Explicitly, r_i $(i - 1, ..., n)$ is defined so that $k_i = q_i p_i + r_i$ for $i = 1, ..., n$, and $q = k_0 + q_1 + ... + q_n$.

Similarly, the algebra

$$\mathrm{P}_f^\infty = \mathbb{C}[x_1, ..., x_n]/I_{\mathrm{pol}}(f).$$

is nothing else than the semigroup algebra of the abelian semigroup (with unit) G_+^∞ with generators $\tilde{g}_1, ..., \tilde{g}_n$ and defining (semigroup) relations $\tilde{g}_i^{p_i} = \tilde{g}_j^{p_j}$.

Adding one extra generator $\tilde{g}_0$, write defining relations of G_+^∞ in the form

$$\tilde{g}_0 = \tilde{g}_1^{p_1} = ... = \tilde{g}_n^{p_n}.$$

Then a canonical form for an element of G_+^∞ is

$$\tilde{g}_0^q \tilde{g}_1^{r_1} \cdots \tilde{g}_n^{r_n} \text{ with } q \geqslant 0, 0 \leqslant r_1 < p_1, ..., 0 \leqslant r_n < p_n. \tag{13}$$

It follows that the natural homomorphism $G_+^\infty \to G^\infty$, taking $\tilde{g}_i$ to g_i, is an embedding. Its image consists of the monomials $g_0^{k_0} g_1^{k_1} \cdots g_n^{k_n}$ with $k_0, k_1, ..., k_n \geqslant 0$, which we will call *positive*. In what follows we will identify the semigroup G_+^∞ with the subsemigroup in G^∞ formed by the positive monomials (and $\tilde{g}_i$ with g_i).

Correspondingly, the natural homomorphism $\mathrm{P}_f^\infty \to \mathrm{L}_f^\infty$ is an embedding, with the image spanned by the positive monomials, so we will identify the algebra P_f^∞ with the subalgebra of L_f^∞ spanned by the positive monomials.

Now the algebra

$$\mathrm{L}_f = \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]/J_{\mathrm{Lau}}(f)$$

is the group algebra of the finite abelian group G obtained from G^∞ by imposing additional relations

$$g_i^{p_i - 1} = g_1^{a_1} \cdots g_{i-1}^{a_{i-1}} g_i^{a_i - 1} g_{i+1}^{a_{i+1}} \cdots g_n^{a_n}. \tag{14}$$

Since these relations imply the relations $g_i^{p_i} = g_j^{p_j}$, the order def_f of G is equal to the absolute value of the determinant of the matrix

$$M_f = \begin{pmatrix} p_1 - a_1 & -a_2 & ... & -a_n \\ -a_1 & p_2 - a_2 & ... & -a_n \\ ... & & & \\ -a_1 & -a_2 & ... & p_n - a_n \end{pmatrix}.$$

One can observe that all the relations (14) reduce to one relation $g_0 = g_1^{a_1} \cdots g_n^{a_n}$ in the group G^∞, so adding them means the factorization by the (infinite) cyclic subgroup $\langle T \rangle$ generated by the element

$$T = g_0 g_1^{-a_1} \cdots g_n^{-a_n}.$$

We will refer to the cosets of $\langle T \rangle$ in G^∞ as to T-lines, so the elements of G can be viewed as T-lines.

Similarly, the algebra P_f is the semigroup algebra of the abelian semigroup G_+ obtained from G_+^∞ by imposing additional relations (14). However, these relations do not any more reduce to one relation, if being considered as semigroup relations.

Let us call two elements $u, v \in G_+^\infty$ equivalent and write $u \sim v$, if the relations (14) imply $u = v$ (in the semigroup sense). A necessary condition for this is that u and v should belong to the same T-line. An element $u \in G_+^\infty$ given in the canonical form (13) is equivalent to Tu if and only if one of the relations (14) is applicable to u, which means that

$$u g_1^{-a_1} \cdots g_{i-1}^{-a_{i-1}} g_i^{-a_i+1} g_{i+1}^{-a_{i+1}} \cdots g_n^{-a_n} \in G_+^\infty$$

for some i. This can be reformulated as the following arithmetic condition:

$$\#\{j | r_j < a_j\} \leqslant \begin{cases} q+1, & \exists\ j\ r_j = a_j - 1, \\ q, & \forall\ j\ r_j \neq a_j - 1. \end{cases} \tag{15}$$

Clearly, two elements of a T-line are equivalent if and only if any two consecutive elements of the interval between them are equivalent. It follows that the equivalence classes lying on a given T-line are some intervals (maybe, reducing to one point), whose union is the intersection of this line with G_+^∞. Let us call them T-intervals. The elements of the semigroup G_+ can be viewed as T-intervals.

Since the degree of T is positive, the degree linearly increases along any T-line. With respect to the natural ordering on a T-line, all sufficiently small elements do not belong to G_+^∞, while all sufficiently large elements belong to G_+^∞ and are equivalent. Thus, there are finitely many T-intervals on any given T-line, one of them being infinite and all the others finite.

When passing from P_f to A_f, infinite T-intervals go to 0, while the images of finite intervals constitute a basis of A_f (see Preliminaries). Thus, we come to the equality

$$\dim \mathrm{A}_f = \dim \mathrm{P}_f - |G| = \dim \mathrm{P}_f - \mathrm{def}(f),$$

recovering the defect formula.

Moreover, the finite intervals are exactly those having a (unique) largest element, and these largest elements are exactly those to which neither of the relations (14) is applicable. Thereby we come to the following

Theorem 4.1. *The images of positive monomials (13) violating the property (15) constitute a basis of the algebra* A_f.

5. Another basis for the Brieskorn-Pham case

Another possibility is to use the smallest elements of T-intervals instead of the largest ones. An element $u \in G_+^\infty$ given in the canonical form (13) is the smallest one in a T-interval if and only if neither of the relations (14) is applicable to u in the opposite direction, that is $u g_i^{-(p_i-1)} \notin G_+^\infty$, which means that $0 \leqslant r_i < p_i - 1$ for $i = 1, ..., n$. This gives the parallelepipedal basis of P_f. The problem is to determine which of these elements belong to infinite T-intervals. We cannot do it in the general case. There is however one case for which we can do it.

Namely one has

Proposition 5.1. *Suppose*

$$p_1 \geqslant n a_1, ..., p_n \geqslant n a_n.$$

Then, a monomial $x_1^{k_1} \cdots x_n^{k_n}$ *with* $0 \leqslant k_i < p_i - 1$, $1 \leqslant i \leqslant n$ *is zero in* A_f *iff it is divisible by one of the* $n!$ *monomials*

$$x_1^{\sigma(1)a_1 - 1} \cdots x_n^{\sigma(n)a_n - 1},$$

where σ runs through all permutations of $\{1,...,n\}$.

Moreover a monomial $x_1^{k_1}\cdots x_n^{k_n}$ with $0\leqslant k_i<p_i-1$, $1\leqslant i\leqslant n$ is nonzero in A_f iff there is an $m\geqslant 1$ such that $k_i<ma_i-1$ for m different indices i. Thus a basis of A_f is given by those monomials $x_1^{k_1}\cdots x_n^{k_n}$ with $0\leqslant k_i\leqslant p_i-2$ such that either for one of the i's one has $k_i\leqslant a_i-2$, or for two of them $k_i\leqslant 2a_i-2$, or for three $k_i\leqslant 3a_i-2$, ..., or for each i, $k_i\leqslant na_i-2$.

Proof. By the last proof, what we have to show in the first half of the proposition is that the above operator T is applicable infinitely many times without leaving A^+_∞ to a symbol of the form $(0;\sigma(1)a_1-1,...,\sigma(n)a_n-1)$ for some permutation σ. Indeed one easily sees

$$\begin{aligned}
&T(0;\sigma(1)a_1-1,...,\sigma(n)a_n-1)\\
&=(0;(\sigma(1)-1)a_1-1,...,p_{\sigma^{-1}(1)}-1,...,(\sigma(n)-1)a_n),\\
&T^2(0;\sigma(1)a_1-1,...,\sigma(n)a_n-1)\\
&=(0;(\sigma(1)-2)a_1-1,...,p_{\sigma^{-1}(1)}-a_{\sigma^{-1}(1)}-1,...,p_{\sigma^{-1}(2)}-1,...,(\sigma(n)-2)a_n),\\
&\quad...,\\
&T^k(0;\sigma(1)a_1-1,...,\sigma(n)a_n-1)=(0;c_{1k},...,c_{nk}),
\end{aligned}$$

where

$$c_{ik}=\begin{cases}(\sigma(i)-k)a_i-1, & \sigma(i)>k,\\ p_i-(k-\sigma(i))a_i-1, & \sigma(i)\leqslant k\end{cases}$$

for $k<n$, so T remains applicable after each of these steps, and

$$T^n(0;\sigma(1)a_1-1,...,\sigma(n)a_n-1)=(0;p_1-(n-\sigma(1))a_1-1,p_n-(n-\sigma(n))a_n-1).$$

But the latter symbol satisfies

$$p_1-(n-\sigma(1))a_1-1\geqslant\sigma(1)a_1-1,...,p_n-(n-\sigma(n))a_n-1\geqslant\sigma(n)a_n-1,$$

so $(0;\sigma(1)a_1-1,...,\sigma(n)a_n-1)$ indeed represents zero in A_f.

To show the converse, observe the following

Lemma 5.2. *One has $k_i\geqslant\sigma(i)a_i-1$, $1\leqslant i\leqslant n$ for some permutation σ iff for any subset $s\subseteq\{1,...,n\}$ of cardinality $1\leqslant m\leqslant n$ there is an $i\in s$ with $k_i\geqslant ma_i-1$.*

Proof. If $k_i\geqslant\sigma(i)a_i-1$ for all i, then for any s as above there is an i in s with $\sigma(i)\geqslant m$, and for this i we will have $k_i\geqslant ma_i-1$.

Conversely if the condition of the lemma is satisfied, then taking $s=\{1,...,n\}$ we will find an i with $k_i\geqslant na_i-1$. Put $\sigma(i)=n$. Further for $s=\{1,...,n\}\setminus\{i\}$ there is a $j\neq i$ with $k_j\geqslant(n-1)a_j-1$. Put $\sigma(j)=n-1$. Next consider $\{1,...,n\}\setminus\{i,j\}$ and so on. Evidently this will produce a permutation with required properties. □

Returning to our proposition, what remains to prove thus is that T is only applicable finitely many times to $(0;k_1,...,k_n)$ provided $(k_1,...,k_n)$ satisfies negation of the condition of the above lemma. That is, there are some indices $1\leqslant i_1<...<i_m\leqslant n$ with $k_{i_1}<ma_{i_1}-1$, ..., $k_{i_m}<ma_{i_m}-1$.

Indeed,

$$T^{m-1}(0;k_1,...,k_n)=[m-1;k_1-(m-1)a_1,...,k_n-(m-1)a_n]$$

and since we know that there are at least m different indices i with $k_i-(m-1)a_i<a_i-1$, we conclude that the condition (‡) is violated, so that the operator T is not applicable anymore. □

Without the requirement of this proposition we do not have an explicit description of the basis from the fundamental parallelepiped. We can however say more about it using the Hessian of f.

Recall that the Hessian, i. e. the determinant of the matrix $(\frac{\partial^2 f}{\partial x_i\partial x_j})$ has the property that its image in A_f generates the highest degree of the maximal ideal, i. e. any nonzero element of A_f divides it.

Now in our case we have

Proposition 5.3. *The image of the Hessian of f in A_f is represented by (a scalar multiple of) the monomial $x_1^{na_1-2}\cdots x_n^{na_n-2}$.*

Proof. We have

$$\frac{\partial^2 f}{\partial x_i^2} = p_i(p_i-1)x_i^{p_i-2} + \lambda a_i(a_i-1)x_1^{a_1}\cdots x_i^{a_i-2}\cdots x_n^{a_n}$$

and

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \lambda a_j a_k x_1^{a_1}\cdots x_j^{a_j-1}\cdots x_k^{a_k-1}\cdots x_n^{a_n}$$

for $j \neq k$.

It follows that the Hessian is a linear combination of monomials of the form

$$\prod_{i\in s} x_i^{p_i+ka_i-2}\prod_{j\notin s} x_j^{ka_j-2}$$

where $s \subseteq \{1,...,n\}$ is some subset with $n-k$ elements, $0 \leqslant k \leqslant n$. But for $k>0$ the monomial $x_i^{p_i+ka_i-2}$ is equal to a scalar multiple of $x_1^{a_1}\cdots x_i^{(k+1)a_i-2}\cdots x_n^{a_n}$ in P_f.

It follows that the image of the Hessian in P_f is represented by a binomial of the form

$$\lambda_1 x_1^{p_1-2}\cdots x_n^{p_n-2} + \lambda_2 x_1^{na_1-2}\cdots x_n^{na_n-2}$$

for some scalars λ_1, λ_2. Now the first monomial is divisible by all monomials from the fundamental parallelepiped, and since $\operatorname{def}(f)$ is positive, we know that at least one of these monomials must become zero in A_f, hence $x_1^{p_1-2}\cdots x_n^{p_n-2}$ is zero in A_f. On the other hand we know that the Hessian cannot be zero in A_f, thus $x_1^{na_1-2}\cdots x_n^{na_n-2}$ does not become zero in A_f and the Hessian is represented by its nonzero scalar multiple. □

The last proposition gives us the following additional information about a basis of A_f confined to the fundamental parallelepiped. Applying the transformation T largest allowed amount of times to the symbol $[0; na_1-2,...,na_n-2]$ of the Hessian gives the symbol $(n-1; a_1-2,...,a_n-2)$ (note that if we would still apply T once more we would obtain $(0; p_1-2,...,p_n-2)$; as we saw this is the symbol of another summand of the Hessian which is zero in A_f).

Since each nonzero element of A_f divides the image of the Hessian, we will thus obtain a basis of A_f by considering all monomials from the fundamental parallelepiped which divide some monomial whose symbol belongs to the subinterval ending with $(n-1; a_1-2,...,a_n-2)$ (containing $[0; na_1-2,...,na_n-2]$ somewhere along the way, and starting somewhere in the fundamental parallelepiped). Thus let $(k_1,...,k_n)$ be the n-tuples obtained by considering all the symbols $(q; r_1,...,r_n)$ in this subinterval, replacing each r_i which happens to be p_i-1 by p_i-2 and further replacing any q of the remaining r_i's by p_i-2. Then, the monomial basis of A_i confined to the fundamental parallelepiped consists precisely of those monomials which divide one of the $x_1^{k_1}\cdots x_n^{k_n}$ for the $(k_1,...,k_n)$ obtained as said.

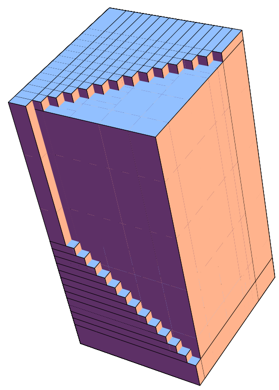
As a corollary we see that for $n=2$ one can always name not more than three monomials from the fundamental parallelepiped such that a basis of A_f is given by those monomials which divide one of these three. Indeed in this case the above procedure gives not more than one monomial $x_1^{k_1}x_2^{k_2}$ with $k_i < p_i-2$, $i=1,2$, and some number of monomials of the form $x_1^{p_1-2}x_2^{k_2'}$ and $x_1^{k_1'}x_2^{p_2-2}$. Then clearly it suffices to leave just one of each kind — namely the one with largest k_1' and the one with largest k_2'. Thus for $n=2$ the number of such "maximal" monomials is not larger than the number required in the case covered by Proposition 5.1

On the other hand, for larger n, although we may always choose maximal basic elements $x_1^{k_1}\cdots x_n^{k_n}$ in the fundamental parallelepiped in such a way that no more than one of them will have $k_i < p_i-2$, $i=1,...,n$, computational experiments show that already for $n=3$ the number of required additional ones lying on the boundary of the fundamental parallelepiped, i. e. with some k_i equal to p_i-2, may be much larger than in the case of Proposition 5.1, which is 6 for $n=3$.

Example 5.4. For the polynomial

$$f(x,y,z) = x^{19} + y^{29} + z^{58} + \lambda x^6 y^{10} z^{19}$$

the basis of A_f represented by monomials from the fundamental parallelepiped consists of all monomials which divide one of those with the following degree vectors: $[4,27,56]$, $[5,26,56]$, $[6,25,56]$, $[7,24,56]$, $[8,23,56]$, $[9,22,56]$, $[10,21,56]$, $[11,20,56]$, $[12,19,56]$, $[13,18,56]$, $[14,17,56]$, $[15,16,56]$, $[16,15,56]$, $[17,14,56]$, $[17,15,39]$, $[17,16,38]$, $[17,17,37]$, $[17,18,36]$, $[17,19,35]$, $[17,20,34]$, $[17,21,33]$, $[17,22,32]$, $[17,23,31]$, $[17,24,30]$, $[17,25,29]$, $[17,26,28]$, $[17,27,27]$, and $[16,27,55]$. All of the points with the above coordinates lie on the boundary of the fundamental parallelepiped, i. e. have $k_i = p_i - 2$ for some i, and none of these may be omitted. Graphically, the corresponding area of the fundamental parallelepiped to which the basis is confined looks like this:



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