

Covering a Finite Group by the Conjugates of a Coset

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Abstract

We study pairs (G, A) where G is a finite group and $A < G$ is maximal, satisfying $\bigcup_{g \in G} (Ax)^g = G - \{1_G\}$ for all $x \in G - A$. We prove that this condition defines a class of permutation groups, denoted CCI, which is a subclass of the class of primitive permutation groups. We prove that CCI contains the class of 2-transitive groups. We also prove that groups in CCI are either affine or almost simple. In the affine case each CCI group must be 2-transitive, while an almost simple CCI group needs not be 2-transitive. We give various results on the almost simple case and compare between the CCI class and other recently studied classes of groups which lie between the 2-transitive and the primitive permutation groups.

Keywords: Finite Groups, Conjugacy Classes, Cosets, Primitive permutation groups, 2-transitive groups.

1 Introduction

Let G be a finite group¹ and $A < G$. It is well-known (e.g., [10] exercise 1A.7) that $G \neq \bigcup_{g \in G} A^g$ or, equivalently, that A does not intersect each conjugacy class

¹Unless otherwise stated, all our groups are assumed to be finite.

of G ². Consider now a coset Ax of A in G such that $Ax \neq A$ (a *non-trivial coset*). Clearly Ax cannot intersect each conjugacy class of G since $1 \notin Ax$. However, for certain choices of triples (G, A, x) we get:

$$\bigcup_{g \in G} (Ax)^g = G^\#, \quad (1)$$

($G^\# := G - \{1_G\}$), or equivalently, the coset Ax intersects each non-trivial conjugacy class of G . As an immediate example which will be generalized later take $G = S_3$ (the Symmetric group on three letters), any $A < G$ of order 2 and any $x \in G - A$.

From now on we assume $|G| \geq 3$, to avoid the degenerate case $|G| = 2, A = 1$. Furthermore, if $A < M < G$ and A satisfies Condition 1, then so does M and therefore we add the simplifying assumption that A is a maximal subgroup of G to our discussion. Denoting by $Cl(G)$ the set of all conjugacy classes of G , and by $Cl^\#(G)$ the set of all non-trivial conjugacy classes of G , we are ready to define the main concept studied in this paper.

Definition 1 *Let G be a group and A a maximal subgroup of G .*

a. *Let $x \in G - A$. The right coset Ax is Conjugacy Class Intersecting (henceforth CCI) if $Ax \cap C \neq \phi$ for every $C \in Cl^\#(G)$ (equivalently, Ax is CCI if Equation 1 holds).*

b. *The pair (G, A) is CCI if all non-trivial right cosets of A are CCI. We shall also say that A is a CCI-subgroup of G .*

c. *G is a CCI permutation group (for short a CCI group) if G is a primitive permutation group with a point stabilizer A such that (G, A) is CCI. We shall also say that the action of G is CCI.*

In order to motivate the definition of a CCI permutation group, recall that G acts transitively, by right multiplication, on $\Omega_A := \{Ag | g \in G\}$. If (G, A) is CCI then this action is primitive as A is maximal in G . Moreover, the action is faithful since A cannot contain a non-trivial conjugacy class of G , and therefore A_G (the normal core of A in G) is trivial. Thus G is a primitive permutation group. We also have the following basic permutation group theory characterization of a CCI permutation group on Ω : Each non-trivial conjugacy class of G acts transitively on Ω . More precisely:

Proposition 2 *Let G be a primitive permutation group on a set Ω . Then the following are equivalent:*

- (i) *G is CCI,*
- (ii) *For each distinct $\alpha, \beta \in \Omega$ and each $C \in Cl^\#(G)$ there exists $x \in C$ such that $\alpha^x = \beta$.*

Proof. The claim follows from the equivalence of the action of G on Ω to its right multiplication action on Ω_A , where A is any point stabilizer of the action of G on Ω , and from Definition 1. ■

² A intersects B will always mean $A \cap B \neq \phi$.

Thus we study the class of CCI permutation groups, which is a subclass of the class of all primitive permutation. The following proposition is an elementary but important starting point for this study:

Proposition 3 *A 2-transitive group is CCI.*

Using the classification of the primitive groups by the O’Nan-Scott Theorem ([6, Theorem 4.1A]), we can locate the class of CCI groups more precisely. For the regular socle case of this classification we get full equivalence between CCI groups and 2-transitive groups. In fact, it suffices to assume the existence of one CCI coset of the point stabilizer.

Theorem 4 *Let G be a primitive group with a regular socle N , and a point stabilizer A . Let $x \in G - A$ be arbitrary, and suppose that Ax is CCI. Then N is elementary abelian, G is of the affine type and 2-transitive, and, in particular, G is CCI.*

Thus, combining Proposition 3 and Theorem 4 we can conclude that for affine primitive groups the notions of a 2-transitive group and a CCI group are equivalent.

The next theorem completes the picture of the possible O’Nan-Scott types for CCI groups.

Theorem 5 *Let G be a CCI permutation group. Then G is either:*

- a. Of the affine type, that is, $\text{soc}(G)$ is a regular elementary abelian p -group for some prime p , and G is isomorphic to a subgroup of the affine group $\text{AGL}_m(p)$ where the positive integer m is defined by $|\text{soc}(G)| = p^m$; or*
- b. G is an almost simple group, that is $T \leq G \leq \text{Aut}(T)$ for some simple non-abelian group T .*

In view of the last two theorems we can limit our search of CCI groups to almost simple groups. However, the class of almost simple CCI groups is much harder to study. In Section 4 we develop some general tools for studying the almost simple case, and then prove the next two theorems which give a complete answer for two concrete infinite families of almost simple groups.

Theorem 6 *Let G be an almost simple primitive permutation group with $\text{soc}(G) = A_n$, $n \geq 5$. Then G is CCI if and only if G is 2-transitive.*

Theorem 7 *Let $G = \text{PSL}(2, 2^f)$ with $f \geq 2$, viewed as a primitive permutation group. Then G is CCI if and only if G is 2-transitive.*

In contrast, the following example shows that an almost simple CCI group need not be 2-transitive. Henceforth we call a primitive permutation group which is CCI but not 2-transitive a *special CCI group*. Let G be any group acting on a set Ω . Set $P_2(\Omega) := \{\{a, b\} \mid a \neq b \in \Omega\}$ - the set of all 2-subsets of Ω . Then we have an induced action of G on $P_2(\Omega)$ via $\{a, b\}g = \{ag, bg\}$.

Example 8 Let $G = M_{11}$ the Mathieu group which acts sharply 4-transitively on a set Ω of 11 points. Let A be a maximal subgroup of G which is isomorphic to $M_9 : 2$ (in particular $|G : A| = 55$). A GAP calculation shows that (G, A) is special CCI. We observe that A is a point stabilizer of the transitive action of G on $P_2(\Omega)$ (Note that $|P_2(\Omega)| = 55$).

Thus, the class of CCI groups properly contains the class of 2-transitive groups. Recently, the study of Černey's conjecture about the minimal length of a reset word for a finite automaton motivated the introduction and investigation of various classes of primitive permutation groups which properly contain the class of 2-transitive groups. Here we refer the reader to Peter Cameron's set of lecture notes on synchronization ([5]) which reviews this very interesting research area, and where all of the definitions and results required for our next theorem can be found.

Theorem 9 a. A 2-set transitive group needs not be CCI and a CCI group needs not be 2-set transitive.

b. A spreading group needs not be CCI and a CCI group needs not be spreading.

c. A CCI group needs not be 3/2-transitive and a 3/2-transitive group needs not be CCI.

d. $(M_{11}, M_9 : 2)$ is synchronizing.

Finally, Hung Tong-Viet communicated to us the existence of two more special CCI groups among the Mathieu groups. We currently collaborate with him on classifying special CCI groups in the sporadic as well as other families of almost simple groups.

2 2-transitivity and CCI affine groups

We have defined (Definition 1) the notion of a CCI coset for a right coset Ax . Similarly, one can define CCI left cosets and CCI double cosets by requiring, respectively, $xA \cap C \neq \phi$ and $AxA \cap C \neq \phi$ for every $C \in Cl^\#(G)$. Note that $Ax \cap C \neq \phi$ iff $xA \cap C \neq \phi$ since $xA \cap C = x(Ax \cap C)x^{-1}$. Also, for any $a \in A$, $Ax \cap C \neq \phi$ iff $Axa \cap C \neq \phi$ since $Axa \cap C = a^{-1}(Ax \cap C)a$, and similarly, $xA \cap C \neq \phi$ iff $axA \cap C \neq \phi$. Therefore, for any $a_1, a_2 \in A$ and for any $x \in G - A$, Axa_1 is CCI iff a_2xA is CCI iff AxA is CCI.

Proof of Proposition 3. Let G be a 2-transitive group and let A be a point stabilizer. Let $x \in G - A$ be arbitrary. Recall that $G = A \cup AxA$ (disjoint union). Let $C \in Cl^\#(G)$. Then $A_G = \{1_G\}$ implies $C \not\subseteq A$ and therefore $AxA \cap C \neq \phi$. Hence AxA is CCI for any $x \in G - A$. ■

Proof of Theorem 4. Let $N = soc(G)$. Then, by regularity of N we have $G = AN$, and $A \cap N = \{1_G\}$. Since Ax is CCI and N contains a non-trivial conjugacy class of G we get $Ax \cap N \neq \phi$. Therefore $Ax = Ay$ for some $y \in N^\#$, and $Ax \cap N = Ay \cap N = (A \cap N)y = \{y\}$. For any $g \in G$ we have $(Ax \cap N)^g = (Ax)^g \cap N = \{y^g\}$. Since $N^\# \subseteq G^\#$, Condition 1 implies $N^\# \subseteq$

$\bigcup_{g \in G} (Ax)^g$. Hence, using the fact that intersection is distributive over union and $1_G \notin \bigcup_{g \in G} (Ax)^g$, we get:

$$N^\# = N \cap N^\# = N \cap \left(\bigcup_{g \in G} (Ax)^g \right) = \bigcup_{g \in G} ((Ax)^g \cap N) = \bigcup_{g \in G} \{y^g\} = y^G,$$

where y^G is the conjugacy class of y in G . In particular, all elements of $N^\#$ have the same order, which implies, together with $N = \text{soc}(G)$, that N is an elementary abelian, and hence G is of the affine type. Moreover, since N is abelian, $y^N = y$, and we get $AyA = Ay^{NA} = Ay^G = AN^\# = G - A$, implying that G is 2-transitive. ■

3 Product and diagonal action types

3.1 Proof of Theorem 5

The following discussion refers to the O’Nan-Scott Theorem as it is formulated [6], (Theorem 4.1A) and uses the notation there. In view of Theorem 4, in order to prove Theorem 5 it remains to rule out the possibility that G is of the "diagonal type" (case b(ii) of Theorem 4.1A of [6]) and the possibility that G is of the "product action" type (case b(iii) of Theorem 4.1A of [6]). In both these cases $\text{soc}(G) \cong T^m$, where T is a simple non-abelian group and $m \geq 2$ an integer.

First we deal with case (b)(iii). Let G be a primitive permutation group of the type described in (b)(iii). We will prove that there exists $x \in G - A$ for which Condition 1 does not hold, and hence G is not CCI. Let $e < m$ be a positive integer divisor of m , and set $d := m/e$ ($d \geq 2$) and $W = U \text{ wr } S_d = B \rtimes S_d$ where $B \cong U^d$ is the base group. Furthermore, U acts on a set Σ , and W acts on $\Omega := \Sigma^d$ via the product action. We have $\text{soc}(U) = T^e$, and $\text{soc}(G) = (\text{soc}(U))^d \trianglelefteq G \leq W$. In particular the product action of W on Ω induces an action of G on Ω . We take $A = G_{(\sigma, \dots, \sigma)}$ - the point stabilizer of the d -tuple (σ, \dots, σ) , where $\sigma \in \Sigma$. We have $A = G \cap W_{(\sigma, \dots, \sigma)}$, and $W_{(\sigma, \dots, \sigma)} = U_\sigma \text{ wr } S_d$. Since U is a primitive permutation group on Σ , and $\text{soc}(U) \trianglelefteq U$, we have that $\text{soc}(U)$ acts transitively on Σ ([6] Theorem 1.6A(v)). Therefore $\text{soc}(U) - U_\sigma$ is non-empty. Choose $y \in \text{soc}(U) - U_\sigma$ and set $x := (y, \dots, y)$ (a d -tuple). Since $\text{soc}(G) = (\text{soc}(U))^d$, we have $x \in G - A$. An element of Ax takes the form $(u_1, \dots, u_d) s (y, \dots, y) = (u_1 y, \dots, u_d y) s$, where $u_i \in U_\sigma$ for all $1 \leq i \leq d$ and $s \in S_d$. Since $y \in \text{soc}(U) - U_\sigma$, we get $u_i y \neq 1_U$ for all $1 \leq i \leq d$. Now consider $z = (1, y_2, \dots, y_d) \in B$ where $y_i \in \text{soc}(U) - U_\sigma$ for all $2 \leq i \leq d$, are arbitrary (since $d > 1$ such an element exists). Note that $1_G \neq z \in G$. We’ll show that $z \notin \bigcup_{g \in G} (Ax)^g$. Suppose by contradiction that $z \in \bigcup_{g \in G} (Ax)^g$. Then there exist

$u_1, \dots, u_d \in U_\sigma$, $s \in S_d$ and $g \in G$ such that

$$z = ((u_1y, \dots, u_dy) s)^g = (u_1y, \dots, u_dy)^g s^g.$$

Since $B \trianglelefteq W$, and $S_d \cap B = 1$, we get $s^g \notin B$. But both z and $(u_1y, \dots, u_dy)^g \in B$, so the last relation forces $s^g = 1$, whence $z = (u_1y, \dots, u_dy)^g$. This implies that $u_iy = 1$ for some $1 \leq i \leq d$ - a contradiction.

For case (b)(ii) of Theorem 4.1A of [6] we need the following:

Lemma 10 *Let G be a group, $A < G$, $1 < B \trianglelefteq G$, and $x \in G - A$ such that Ax is CCI. Then there exists $b \in B - A \cap B$ such that $Ax = Ab$ and*

$$B^\# = \bigcup_{g \in G} ((A \cap B) b)^g. \quad (2)$$

Moreover, if G is CCI then Equation 2 holds for all $b \in B - A \cap B$.

Proof. Since A is core-free we have $B \not\leq A$ and hence, by maximality of A , $G = AB$. Therefore there exists $b \in B - A \cap B$ such that $Ax = Ab$, and:

$$\begin{aligned} B^\# = G^\# \cap B &= \left(\bigcup_{g \in G} (Ab)^g \right) \cap B = \bigcup_{g \in G} ((Ab)^g \cap B) = \bigcup_{g \in G} (Ab \cap B)^g \\ &= \bigcup_{g \in G} ((A \cap B) b)^g. \end{aligned}$$

The last claim follows from $B - A \cap B \subseteq G - A$. ■

Remark 11 *Lemma 10 can be restated as follows: If (G, A) is CCI and $B \trianglelefteq G$ then $(B, A \cap B)$ is "weakly CCI" in the sense that one has to conjugate the non-trivial $A \cap B$ coset by some automorphism group of B containing $\text{Inn}(B)$. Indeed, $(B, A \cap B)$ needs not be CCI. Take $G = {}^2G_2(3)$ and $B = \text{soc}(G) \cong \text{PSL}(2, 8)$. Then G is 2-transitive with a point stabilizer A of index 28 (see [2]). On the other hand $A \cap B \cong D_{18}$ is maximal in B but $(B, A \cap B)$ is not CCI (see Theorem 7 and its proof). It should be mentioned, however, that this is the only example where a 2-transitive almost simple group induces a non-2-transitive action of its socle ([4] Table 7.4).*

Now consider case (b)(ii) of Theorem 4.1A of [6], where G is a primitive group of the diagonal type (see [6] Section 4.5. p.121-124 for the details of the "diagonal action" construction). As before, T is a simple non-abelian group ($\text{Inn}(T) \cong T$), and $m \geq 2$ an integer. Let $\Gamma := \{1, \dots, m\}$ and $S_m := \text{Sym}(\Gamma)$. Set:

$$\begin{aligned} V &:= \text{Aut}(T) \text{ wr}_\Gamma S_m = U \rtimes S_m \text{ where } U := (\text{Aut}(T))^m, \\ B &:= \{(\tau_1, \dots, \tau_m) \mid \forall 1 \leq i \leq m, \tau_i \in \text{Inn}(T)\} (B \leq U), \\ C &:= \{(\tau, \dots, \tau) \mid \tau \in \text{Aut}(T), s \in S_m\}, \text{ and} \\ N &:= \{(\tau_1, \dots, \tau_m) \mid s \in S_m, \forall 1 \leq i \leq m, \tau_i \in \text{Aut}(T), \\ &\quad \forall 1 \leq i, j \leq m, \tau_i \text{Inn}(T) = \tau_j \text{Inn}(T)\}. \end{aligned}$$

Note that B, C and N are all subgroups of V . By [6, Lemma 4.5B, Theorem 4.5A] and their proofs, we can identify G as a subgroup of N which contains B ($B = \text{soc}(G)$ under this identification). Moreover $B \trianglelefteq N$, and $A := G \cap C$ is a point stabilizer of G . Suppose by contradiction that G is CCI. By Lemma 10

$$B^\# = \bigcup_{g \in G} ((A \cap B)b)^g,$$

for all $b \in B - A \cap B$. Since $A = G \cap C$ and $B \leq G$ we have $A \cap B = C \cap B = \{(\tau, \dots, \tau) \mid \tau \in \text{Inn}(T)\}$. Fix some non-trivial $t \in \text{Inn}(T)$ and set $b := (1, t, 1, \dots, 1) \in B - A \cap B$. We get $(A \cap B)b = \{(\tau, \tau t, \tau \dots, \tau) \mid \tau \in \text{Inn}(T)\}$. Let $g \in G$ be arbitrary. Then $g = (\sigma_1, \dots, \sigma_m)s \in N$, where the $\sigma_i \in \text{Aut}(T)$, $1 \leq i \leq m$, all lie in the same $\text{Inn}(T)$ coset and $s \in S_m$. We have, for any $\tau \in \text{Inn}(T)$:

$$(\tau, \tau t, \tau \dots, \tau)^g = (\tau^{\sigma_1}, \tau^{\sigma_2} t^{\sigma_2}, \tau^{\sigma_3} \dots, \tau^{\sigma_m})^s.$$

In order to derive a contradiction it suffices to show that there exists $1 \neq u \in B$ which is not of the form $(\tau^{\sigma_1}, \tau^{\sigma_2} t^{\sigma_2}, \tau^{\sigma_3} \dots, \tau^{\sigma_m})^s$, as above. First suppose that $m \geq 3$. Since T is simple non-abelian, it splits into at least three distinct $\text{Aut}(T)$ orbits (choose, for instance, the orbits of T elements having three distinct non-trivial orders). Hence $u \in B$ with three components which belong to three distinct $\text{Aut}(T)$ orbits is not of the form $(\tau^{\sigma_1}, \tau^{\sigma_2} t^{\sigma_2}, \tau^{\sigma_3} \dots, \tau^{\sigma_m})^s$. Now suppose that $m = 2$. An element of the form $(\tau^{\sigma_1}, \tau^{\sigma_2} t^{\sigma_2})^s$ has a single trivial component if and only if either $\tau = 1$, in which case the element takes the form $(1, t^{\sigma_2})^s$, or $\tau t = 1$ in which case the element takes the form $(\tau^{\sigma_1}, 1)^s = (t^{-\sigma_1}, 1)^s$. Since not all of the elements of $(\text{Inn}(T))^\#$ are in the $\text{Aut}(T)$ orbits of either t or t^{-1} , there exists $u \in B$, with precisely one non-trivial component, which is not of the form $(\tau^{\sigma_1}, \tau^{\sigma_2} t^{\sigma_2})^s$.

4 Almost simple CCI groups

4.1 General Properties of CCI Groups

In this section we derive some general results, which can be utilized for classifying CCI groups in the almost simple case. These include bounds on the sizes of conjugacy classes or on their numbers. The first set of results is used for proving Theorem 6 and Theorem 7.

Lemma 12 *Let G be a group, $A < G$, $x \in G$, and $C \in \text{Cl}(G)$. Then $|AxA \cap C| = |A \cap Cx^{-1}|$ for all $a \in A$ and*

$$|AxA \cap C| = |A \cap Cx^{-1}| \frac{|A|}{|A \cap A^x|}.$$

Proof. AxA is a disjoint union of $\frac{|A|}{|A \cap A^x|}$ right cosets of the form Axa with $a \in A$. Hence $AxA \cap C = \bigcup_{a \in A} (Axa \cap C)$, where distinct sets $Axa \cap C$ are

disjoint. Furthermore, for all $a \in A$ we have $Axa \cap C = (A \cap Ca^{-1}x^{-1})xa$, and hence $|Axa \cap C| = |A \cap Ca^{-1}x^{-1}|$. Moreover:

$$|A \cap Ca^{-1}x^{-1}| = |a(A \cap Ca^{-1}x^{-1})| = |aA \cap aCa^{-1}x^{-1}| = |A \cap Cx^{-1}|.$$

We proved $|Axa \cap C| = |A \cap Cx^{-1}|$ for all $a \in A$. The claim follows. ■

Corollary 13 *Let G be a group, $A < G$, $C \in Cl(G)$, and $\{1 = x_1, x_2, \dots, x_r\}$ a set of r representatives of the distinct double cosets of A in G . Then:*

$$|C| = \sum_{i=1}^r |A \cap Cx_i^{-1}| \frac{|A|}{|A \cap Ax_i|} = |A \cap C| + \sum_{i=2}^r |A \cap Cx_i^{-1}| \frac{|A|}{|A \cap Ax_i|}.$$

The following is a kind of dual concept to the concept of a CCI coset.

Definition 14 *Let G be a group, $A < G$, and $C \in Cl^\#(G)$. Then C is A -Coset Intersecting (henceforth A -CoI) if $C \cap Ax \neq \phi$ for all $x \in G - A$.*

Lemma 15 *Let G be a group, and $A < G$.*

- a. *(G, A) is CCI if and only if every $C \in Cl^\#(G)$ is A -CoI.*
- b. *If $C \in Cl^\#(G)$ is A -CoI then $|C| \geq |A \cap C| + |G : A| - 1$.*
- c. *Suppose $C \in Cl^\#(G)$ is A -CoI and let $\{1 = x_1, x_2, \dots, x_r\}$ be a set of distinct double coset representatives of A in G . Then $|C| \geq |A \cap C| + \sum_{i=2}^r \frac{|A|}{|A \cap Ax_i|}$.*

Proof. a. Obvious from the definitions.

b. Let $T = \{1 = x_1, x_2, \dots, x_m\}$ be a right transversal of A in G where $m := |G : A|$. Then $|C| = \sum_{i=1}^m |C \cap Ax_i|$. Since C is A -CoI, $|C \cap Ax_i| \geq 1$ for all $2 \leq i \leq m$ and (b) follows.

c. Since C is A -CoI, $|C \cap Ax_i| \geq 1$ for all $2 \leq i \leq r$, and the claim follows from Corollary 13. ■

Lemma 16 *Let G be a group, $A < G$. Then $C \in Cl^\#(G)$ is A -CoI if and only if $AC = G - A$ or $AC = G$, according to whether $A \cap C = \phi$ or $A \cap C \neq \phi$.*

Proof. We have $AC = ACA = \bigcup_{x \in C} AxA$. Since C is A -CoI, the set $\{AxA | x \in C\}$ contains all non-trivial double cosets of A . The claim follows. ■

In the following we'll denote the solvable radical of a group G by $R(G)$. Recall that if G is almost simple then $R(G) = 1$.

Theorem 17 *Let G be a group, $A < G$ a maximal subgroup, and suppose that (G, A) is CCI. Suppose in addition that $R(G) = 1$. Then $Z(A) = 1$.*

Proof. Suppose by contradiction that $Z(A) > 1$. Let $b \in Z(A)$, $b \neq 1$. Clearly $A \leq C_G(b)$ and since $R(G) = 1$ and A is maximal, $C_G(b) = A$. Let $C = b^G$. Now $b \in A \cap C$ so $A \cap C \neq \phi$. Since C is A -CoI, $AC = G$ by Lemma 16. Since $|C| = |G : C_G(b)| = |G : A|$, $AC = G$ implies that C is a transversal of A in G . By a result of Stein [11] $\langle C \rangle$ is solvable, contradicting $R(G) = 1$. ■

Theorem 18 *Let G be a group, $A < G$ and $x \in G - A$. Then Ax is CCI if and only if for every $C \in Cl^\#(G)$, we have $A \cap Cx^{-1} \neq \phi$. If Ax is CCI then $C_A(x)$ acts by conjugation on $A \cap Cx^{-1}$. Let $t(A, C, x)$ be the number of orbits of this action and let $t(A, x)$ be the number of orbits of the conjugation action of $C_A(x)$ on A , then:*

$$\sum_{C \in Cl^\#(G)} t(A, C, x) = t(A, x). \quad (3)$$

In particular $k - 1 \leq t(A, x)$.

Proof. By assumption, for every $C \in Cl^\#(G)$ we have $Ax \cap C \neq \phi$ which is equivalent to $A \cap Cx^{-1} \neq \phi$. Hence:

$$Ax = Ax \cap G^\# = Ax \cap \left(\bigcup_{C \in Cl^\#(G)} C \right) = \bigcup_{C \in Cl^\#(G)} (Ax \cap C),$$

is a disjoint union of $|Cl^\#(G)| = k - 1$ non-empty sets. It follows that

$$A = \left(\bigcup_{C \in Cl^\#(G)} (Ax \cap C) \right) x^{-1} = \bigcup_{C \in Cl^\#(G)} (A \cap Cx^{-1}),$$

is a disjoint union of $k - 1$ non-empty sets. Observe that $C_A(x)$ acts by conjugation on $Cx^{-1} \cap A$, for every $C \in Cl^\#(G)$, hence, by definition of $t(A, C, x)$, Equation 3 follows immediately. Since $t(A, C, x) \geq 1$ for any $C \in Cl^\#(G)$, we get $k - 1 \leq t(A, x)$. ■

Theorem 18 can be applied in order to rule out certain potential candidates for CCI pairs (G, A) . One looks for $x \in G - A$ which minimizes $t(A, x)$, and on the other hand for conjugacy classes C for which $t(A, C, x) > 1$ (Since (G, A) is CCI we already have $t(A, C, x) \geq 1$). This can be achieved by examination of the distinct elements orders that can occur in $A \cap Cx^{-1}$.

The next lemma provides a general upper bound on $t(A, x)$.

Lemma 19 *Let G be a group, $A < G$, $x \in G - A$. Then*

$$t(A, x) = \frac{|A|}{|C_A(x)|} \sum_{a \in C_A(x)} \frac{1}{|a^A|}, \quad (4)$$

where a^A is the A -conjugacy class of a . If in addition $Z(A) = 1$ then $t(A, x) \leq \frac{1}{2} |A| \left(1 + \frac{1}{|C_A(x)|} \right)$, and $t(A, x) = |A|$ if and only if $C_A(x) = 1$. Otherwise $t(A, x) \leq \frac{3}{4} |A|$.

Remark 20 *By Theorem 26 below, if (G, A) is special CCI the existence of $x \in G - A$ such that $C_A(x) \neq 1$ is guaranteed.*

Proof of Lemma 19. The orbit counting lemma ([4] Theorem 2.2) gives:

$$t(A, x) = \frac{1}{|C_A(x)|} \sum_{a \in C_A(x)} |C_A(a)| = \frac{|A|}{|C_A(x)|} \sum_{a \in C_A(x)} \frac{1}{|a^A|}.$$

If $Z(A) = 1$ then we have $|a^A| > 1$ and the rest of the claim follows. ■

The next set of results provides further tools for classifying almost simple CCI permutation groups, although we do not demonstrate their use in this paper. The starting point is the condition $t(A, x) = |A|$ (see Lemma 19). The discussion involves the following well studied concept (see [1]):

Definition 21 Let G be a group, and $A < G$. Then A is called a *CC-subgroup* of G if for every $a \in A^\#$ it holds that $C_G(a) \leq A$.

Remark 22 Note that the condition $C_G(a) \leq A$ for every $a \in A^\#$ is equivalent to the condition $C_A(x) = 1$ for every $x \in G - A$.

Lemma 23 Let G be a group, and $A < G$ with $Z(A) = 1$.

1. Let p be a prime divisor of $|A|$. Suppose that $t(A, x) = |A|$ for every p -element $x \in G - A$. Then $|G : A|$ is a p' -number (equivalently, A contains a Sylow p -subgroup of G).

2. If $t(A, x) = |A|$ for every p -element $x \in G - A$, for every prime divisor p of $|G|$, then A is a CC-subgroup of G . In particular A is a Hall subgroup of G , and $|A|$ and $|C_G(x)|$ are coprime for any $x \in G - A$.

Proof. 1. Assume by contradiction that $Q < P$, where Q is a (non-trivial) Sylow p -subgroup of A , and P is a Sylow p -subgroup of G . Since $N_P(Q) > Q$ there exists $x \in N_P(Q) - Q \subseteq G - A$, and since $Z(N_P(Q)) \cap Q \neq 1$ we get $C_A(x) \neq 1$ in contradiction to Lemma 19.

2. Suppose that $t(A, x) = |A|$ for every p -element $x \in G - A$, for every prime divisor p of $|G|$. Let $x \in G - A$. Then there exists a prime divisor p of $o(x)$ such that $x_p \in G - A$, where $x_p = x^{\alpha_p}$ (α_p some positive integer) is the non-trivial p -part of x . We get that $C_A(x) \leq C_A(x^{\alpha_p}) = C_A(x_p) = 1$ implies $C_A(x) = 1$. Thus A is a CC-subgroup of G . Such a subgroup must be a Hall subgroup of G ([1] Theorem A or part 1 of the lemma). Since $A < G$ there is a conjugacy class C of G such that $C \cap A = \phi$ or, equivalently, $C \subseteq G - A$. Since $C_A(x) = 1$ for all $x \in G - A$, it follows that the conjugation action of A on C is semi-regular, and so $|C|$ is divisible by $|A|$. Therefore, for any $x \in C$ we have that $|C| = |G : C_G(x)|$ is divisible by $|A|$, or, equivalently, $|C_G(x)|$ is coprime to $|A|$. ■

Proposition 24 Let G be a group, $A < G$, and $C \neq D$ conjugacy classes of involutions of G . Suppose that Ax intersects both C and D . Then $A \cap A^x$ has even order.

Proof. Let $c \in C$ and $d \in D$ be such that $c, d \in Ax$. Then $cd^{-1} = cd \in A$. Using the fact that any two involutions generate a dihedral subgroup, we have

that $\langle c, d \rangle = \langle c, cd \rangle$ is a dihedral subgroup of order $2o(cd)$. If $o(cd)$ is odd $\langle c \rangle$ and $\langle d \rangle$ are Sylow 2-subgroups of $\langle c, d \rangle$ and hence they are conjugate in $\langle c, d \rangle$. Therefore c and d are conjugate in G - a contradiction. It follows that $o(cd)$ is even. Now $(cd)^{-1} = d^{-1}c^{-1} = dc \in A$ and $c(cd)c^{-1} = dc$. Therefore $dc \in A \cap A^c$. But $c \in Ax$ so $A \cap A^c = A \cap A^x$. In particular, $A \cap A^x$ is even order. ■

Proposition 25 *Let G be a group, A a CCI-subgroup of G , and assume $R(G) = 1$. If $C_A(y) = 1$ for every involution $y \in G - A$, then $A \cap A^x$ is odd order for every $x \in G - A$, and G has a single conjugacy class of involutions I . If, in addition, A has even order, then $A \cap I$ is a single conjugacy class of involutions of A , and $|I| = |G : A| |I \cap A|$.*

Proof. By the assumptions and Theorem 17, we get $Z(A) = 1$. Let $x \in G - A$ be arbitrary. If A is odd order, then clearly $A \cap A^x$ is also odd order. Suppose that $|A|$ is even. Then $|G|$ is even and $G - A$ contains involutions (since $G - A$ intersects every non-trivial conjugacy class of G). Therefore there exists an involution $s \in G - A$ such that $x = as$ for some $a \in A$ and $A^x = A^{as} = A^s$. Thus we may assume that x is an involution, and therefore $\langle x \rangle$ acts by conjugation on $A \cap A^x$. Since $C_A(x) = 1$, this action has no non-trivial fixed points, and hence $|A \cap A^x|$ is odd. By Proposition 24, G has a single conjugacy class of involutions I .

Now suppose in addition that $|A|$ is even. Since A is maximal and (G, A) is CCI, A has $m = |G : A|$ distinct conjugates. Let $T = \{x_1, \dots, x_m\}$ be any right transversal of A . Then $\bigcup_{i=1}^m A^{x_i}$ is a normal subset of G . Since $|A|$ is even, $A \cap I \neq \phi$, which in turn implies $I \subseteq \bigcup_{i=1}^m A^{x_i}$. Hence $I = \bigcup_{i=1}^m (I \cap A^{x_i})$. Since $|A^{x_i} \cap A^{x_j}|$ is odd order for $i \neq j$, we get $(I \cap A^{x_i}) \cap (I \cap A^{x_j}) = \phi$. Furthermore, $I \cap A^{x_i} = (I \cap A)^{x_i}$ implies $|I \cap A^{x_i}| = |I \cap A|$, whence $|I| = m |I \cap A|$. Finally, let $a \in A$ be an involution. Then $C_G(a) = C_A(a)$ for otherwise there exists $x \in G - A$ such that $x \in C_G(a)$, contradicting the fact that $|A \cap A^x|$ is odd. Hence $|I| = |G : C_G(a)| = |G : C_A(a)| = m |A : C_A(a)|$. Consequently $|I \cap A| = |A : C_A(a)|$ and $I \cap A$ is a single conjugacy class of involutions of A . ■

Theorem 26 *Let G be an almost simple group, and suppose that (G, A) is special CCI. Then there exists $x \in G - A$ such that $C_A(x) \neq 1$.*

Proof. We will use Arad and Herfort classification of groups having CC-subgroups ([1] Theorem A) in order to show that A is not a CC-subgroup of G . Since G is almost simple we have $Z(A) = 1$ by Theorem 17. Set $\pi := \pi(A)$ - the set of prime divisors of $|A|$. By Theorem A of [1] and the maximality of A , we get that A is a Hall π -subgroup of G . Moreover, (G, A) falls into one of the four cases (i)-(iv) of that theorem. Cases (ii) and (iv) are ruled out since $Z(A) = 1$ and hence A is not nilpotent. Case (iii) splits into two subcases: (iii)(a) for which G is solvable, contradicting $R(G) = 1$, and (iii)(b) for which

$G \cong PSL(2, q)$ where $q \equiv 3 \pmod{4}$ and $|A| = \frac{q(q-1)}{2}$. In this case A is a point stabilizer of the action of G on the projective line of $q+1$ points and this action is 2-transitive ([8] Theorem 2.8.2). Thus case (iii) is ruled out.

In Case (i) A is non-nilpotent and of even order. Furthermore, Case (i) splits into three subcases: (i)(a) for which G is a Frobenius group and A is a Frobenius complement. The Frobenius kernel is a normal nilpotent subgroup of G contradicting $R(G) = 1$. In subcase (i)(b) $G \cong PSL(2, 2^n)$ with $n \geq 2$. By Theorem 7, whose proof does not rely on the current theorem, all CCI-subgroups of G are point stabilizers of 2-transitive actions. In subcase (i)(c) $G \cong Sz(q)$ with $q = 2^{2n+1}$, $n \geq 1$ and A is solvable. Here we use the classification of maximal subgroups of the Suzuki groups (see [3] Theorem 7.3.3). Set $s := \sqrt{2q}$. A maximal subgroup A of $Sz(q)$ is either:

1. A normalizer of a Sylow 2-subgroup of G which is of order $q^2(q-1)$. In this case A is a point stabilizer of a 2-transitive action of G on a set of q^2+1 points. Thus (G, A) is 2-transitive.

2. A normalizer of a cyclic subgroup of order $q-1$ or of a cyclic subgroup of order $q+s+1$ or of a cyclic subgroup of order $q-s+1$. The order of A is, respectively, $2(q-1)$ or $4(q+s+1)$ or $4(q-s+1)$. In this case A is not a Hall π -subgroup since the 2-part of its order is not equal to the 2-part of $|G|$ (which is at least 8).

3. A Suzuki group $Sz(q_0)$ where $q = q_0^r$, $q_0 \neq 2$ and r is a prime. But then $q_0 \geq 8$ and hence $Sz(q_0)$ is non-solvable, contradicting the assumption of (i)(c). ■

Remark 27 *There are examples of 2-transitive almost simple groups with a point stabilizer which is a CC-subgroup (see [1]). As a concrete example take $G = A_5$ and $A = A_4$.*

4.2 CCI-subgroups of $PSL(2, 2^f)$

In this section we prove Theorem 7. Set $q := 2^f$, $f \geq 2$, $G := PSL(2, q)$, and let $k = q+1$ be the number of conjugacy classes of G . We use the detailed information available on maximal subgroups (Dickson's classification) and conjugacy classes of G - see [8] Chapter 2, and [9] Chapter II.

Lemma 28 *Let F be a maximal Frobenius subgroup of G ($|F| = q(q-1)$), and P its Frobenius kernel. Then P is a Sylow 2-subgroup of G , and:*

1. F intersects each G conjugacy class of elements of order dividing $q-1$.
2. Let $1 \neq y \in F$ be of order dividing $q-1$. Let C be the conjugacy class of y in G . Then $C \cap F = Py \cup Py^{-1}$, and the union on the right hand side of the last equation is a disjoint union of two conjugacy classes of F .

Proof. We prove (2). Let $x \in G$ be of order $q-1$ such that $y \in \langle x \rangle$. Then $\langle x \rangle = C_G(y) \leq F$ and since $P \trianglelefteq F$ we get that $\langle x \rangle$ acts by conjugation on P . Since $C_G(x) = \langle x \rangle$, we get $C_G(x) \cap P = 1$. Thus, $\langle y \rangle$ acts semiregularly by conjugation on $P^\# = P - \{1_P\}$ and each orbit is of size $o(y) > 1$.

Let $1 \neq \alpha \in P$. Then α is an involution and $\alpha y \alpha = \alpha \alpha^{y^{-1}} y$, and by the preceding argument $\alpha^{y^{-1}} \in P^\#$ and $\alpha \neq \alpha^{y^{-1}}$. It follows that $\alpha \alpha^{y^{-1}} \in P^\#$. Furthermore, if $1 \neq \beta \in P$ and $\beta \neq \alpha$, then, assuming $\alpha \alpha^{y^{-1}} = \beta \beta^{y^{-1}}$ yields, using the fact that P is abelian and α and β are involutions, $\alpha \beta = (\alpha \beta)^{y^{-1}}$ which is a contradiction since $\alpha \beta \in P^\#$ and $\langle y \rangle$ acts regularly by conjugation on $P^\#$. Therefore, the set $\{\alpha y \alpha = \alpha \alpha^{y^{-1}} y \mid \alpha \in P\} = Py$ is a set of q distinct elements of F , all conjugate to y . Now we have, for any $\alpha \in P$:

$$\alpha (Py) \alpha = P(\alpha y \alpha) = P(\alpha \alpha^{y^{-1}}) y = Py.$$

Moreover, using $P \leq F$ and $y \in \langle x \rangle$, we also have $(Py)^x = Py$. Therefore $F = \langle P, x \rangle$ normalizes Py . Since all elements of Py are conjugate to y in F we get that Py is a conjugacy class of F . Now $y \in Py \cap C$ implies $Py \subseteq C$, and similarly $Py^{-1} \subseteq C$.

Next observe that y and y^{-1} are conjugate in G , since $\langle x \rangle$ is contained in a Dihedral subgroup of type $D_{2(q-1)}$ and every power of x is conjugate to its inverse by any reflection r . Hence $Py \cup Py^{-1} \subseteq C$. Finally, in order to show that $C \cap F = Py \cup Py^{-1}$, recall that we have $\frac{q-2}{2}$ distinct conjugacy classes in G of non-trivial elements whose order divides $q-1$. The intersection of each one of them with F is of size at least $|Py \cup Py^{-1}| = 2q$, and since distinct conjugacy classes are disjoint, F contains at least $2q \frac{q-2}{2} = q(q-2)$ non-trivial elements whose order divides $q-1$. On the other hand $|F - P| = |F| - |P| = q(q-2)$, and all elements of $F - P$ are non-trivial of order dividing $q-1$. Thus the bound is saturated and $C \cap F = Py \cup Py^{-1}$. ■

Proof of Theorem 7. We have four possibilities for $A < G$ maximal:

1. $A \cong C_2^f \rtimes C_{q-1}$: Here A is a point stabilizer of a 2-transitive action.
2. $A \cong D_{2(q-1)}$: For any involution $x \in G - A$, the centralizer $C_G(x)$ is a Sylow 2-subgroup of G . Choose an involution $x \in G - A$ such that $C_A(x) = C_G(x) \cap A$ is a Sylow 2-subgroup of A . Thus $C_A(x) = \langle r \rangle$ where r is one of the $q-1$ reflections of A . The conjugacy class C_r^A of r in A contains all $q-1$ reflections. Therefore, substituting $|A| = 2(q-1)$, $|C_A(x)| = 2$, $|C_1| = 1$ and $|C_r^A| = q-1$ in Equation 4 gives $t(A, x) = q$.

Consider $t(A, I, x)$, where I is the unique conjugacy class of involutions of G . Since x and r are two distinct commuting involutions in $C_G(x)$, rx is a third distinct involution in $C_G(x)$. Thus $x \neq rx \in I$, and $\{x, rx\}x^{-1} = \{1, r\} \subseteq A \cap Ix^{-1}$. Clearly, 1 and r belong to two distinct orbits for the conjugation action of $C_A(x)$ on $A \cap Ix^{-1}$, and so $t(A, I, x) \geq 2$. Assuming (G, A) is CCI, gives $t(A, C, x) \geq 1$ for the remaining $k-2$ non-trivial conjugacy classes $C \neq I$. Thus:

$$\sum_{C \in \text{Cl}^\#(G)} t(A, C, x) \geq 2 + (k-2) = q+1 > q = t(A, x),$$

contradicting Theorem 18.

3. $A \cong D_{2(q+1)}$: Let $x \in G - A$ be an arbitrary involution. Then $C_G(x)$ is a Sylow 2-subgroup of G . Observe that A intersects each one of the $q+1$ distinct

Sylow 2-subgroups of G , and so $C_A(x) = C_G(x) \cap A = \langle r \rangle$, r is a reflection, is a Sylow 2-subgroup of A . The computation of $t(A, x)$ proceeds as in the $D_{2(q-1)}$ case, and we get $t(A, x) = q + 2$. Assume that Ax is CCI. Then, in particular, Ax intersects every conjugacy class of elements of order $q - 1$. Note that a cyclic subgroup of G of order $q - 1$ has $\phi(q - 1)$ elements of order $q - 1$, where ϕ is the Euler totient function, and since each such element is conjugate in G to its inverse, there are altogether $\frac{\phi(q-1)}{2}$ conjugacy classes of elements of order $q - 1$ in G . Let $y \in Ax$ be an element of order $q - 1$, and let C be its conjugacy class in G . Since $y \in Ax$ there exists $1_G \neq a \in A$ such that $y = ax$. By Lemma 28, y belongs to a Frobenius subgroup F , and we have $P_y \subseteq C$ where P is the unique Sylow 2-subgroup of G contained in F . Now $P \cap A = \langle s \rangle$ for some involution s . Set $y_1 := sy \in P_y \subseteq C$. Note that $\{y, y_1\}x^{-1} = \{a, sa\} \subseteq Cx^{-1} \cap A$. If a is an involution (a reflection) then sa is a product of two reflections and therefore $sa \in \langle R \rangle$ the subgroup of rotations of A , and is not conjugate to a in A . If a is a rotation then sa is a reflection and again a and sa are not conjugate in A . This shows that $t(A, C, x) \geq 2$. By the above there are $\frac{\phi(q-1)}{2}$ conjugacy classes C of this kind. Moreover, as in the $D_{2(q-1)}$ case we have $t(A, I, x) \geq 2$. We get:

$$\begin{aligned} \sum_{C \in \text{Cl}^\#(G)} t(A, C, x) &\geq 2 \left(\frac{\phi(q-1)}{2} + 1 \right) + \left(k - 1 - \left(\frac{\phi(q-1)}{2} + 1 \right) \right) \\ &= k + \frac{\phi(q-1)}{2}. \end{aligned}$$

Therefore, Theorem 18 gives $q + 1 + \frac{\phi(q-1)}{2} \leq q + 2$, or equivalently, $\frac{\phi(q-1)}{2} \leq 1$, implying $\phi(q - 1) = 2$. Now $\phi(n) = 2$ iff $n \in \{3, 4, 6\}$, so $q = 4$. Indeed, in this case G has a 2-transitive action, with a point stabilizer D_{10} .

4. $A \cong PGL(2, q_0) \cong PSL(2, q_0)$: Here $q = q_0^r$ for some prime r and $q_0 \geq 2$, so $q_0 = 2^{f_0}$ with $f_0 \geq 1$ and $f_0 r = f$. By Theorem 18, if (G, A) is CCI then $k - 1 = q \leq |A|$. Hence we have to consider only q_0 values for which $q \leq |PGL(2, q_0)| = (q_0 + 1)q_0(q_0 - 1)$, which is equivalent to $2^{f_0(r-1)} \leq 2^{2f_0} - 1$. Since the last inequality holds if and only if $r = 2$, (G, A) is not CCI for $r > 2$. Henceforth we assume $r = 2$, or equivalently, $q = q_0^2$. Let $x \in G - A$ be an involution such that the unique Sylow 2-subgroup P of G containing x contains a (unique) Sylow 2-subgroup P_0 of A . Then $C_A(x) = P_0$. All non-trivial elements of $C_A(x)$ are involutions which belong to the unique conjugacy class of involutions of A whose size is $q_0^2 - 1$. Equation 4 gives $t(A, x) = q_0^2 + q_0 - 2$. Now consider $t(A, I, x)$. Since $P_0^\# \subseteq I \cap P_0$ and $P_0^\# x \subseteq P^\# \subseteq I$, we get $P_0^\# \subseteq A \cap Ix^{-1}$. But $x \in I$ implies $1_G \in A \cap Ix^{-1}$ as well, so $P_0 \subseteq A \cap Ix^{-1}$. Since $C_A(x)$ acts trivially on P_0 so we get $t(A, I, x) \geq |P_0| = q_0$. Thus, assuming that Ax is CCI yields:

$$\sum_{C \in \text{Cl}^\#(G)} t(A, C, x) \geq q_0 + k - 2 = q_0^2 + q_0 - 1 > t(A, x),$$

contradicting Theorem 18. ■

4.3 CCI-subgroups of S_n and A_n for $n \geq 5$.

The classification of the CCI-subgroups of S_n and A_n for $n \geq 5$ uses the classification of small index subgroups of S_n and A_n , by Liebeck and earlier work of Jordan (see [6] 5.9 Notes). These results are given in theorems 5.2A and 5.2B of [6]. Both S_n and A_n are represented via their natural action on $\Omega = \{1, \dots, n\}$ and denoted accordingly, $Sym(\Omega)$ and $Alt(\Omega)$. For any group G acting on Ω and any $\Delta \subseteq \Omega$, $G_{(\Delta)}$ is the subgroup of G fixing Δ pointwise and $G_{\{\Delta\}}$ is the subgroup of G fixing Δ setwise, and $S_{\{\Delta\}}$ when $G = Sym(\Omega)$. We also need the following inequalities.

Lemma 29 1. $\frac{1}{2} \binom{2m}{m} > \binom{2m}{2} + 1$ for all $m \geq 4$.
 2. $\binom{n}{2} + 1 < \binom{n}{3}$ for all $n \geq 6$.
 3. $2 \binom{n}{3} + 1 < \binom{n}{4}$ for all $n \geq 12$.
 4. $\frac{1}{2} \binom{2m}{m} > 2 \binom{2m}{3} + 1$ for all $m \geq 6$.

Proof. All four claims take the form $f(n) > g(n)$ for all integers n satisfying $n \geq l$, where l is a given positive integer, and f and g are specified functions. They can all be proven by first checking that $f(l) > g(l)$ and then verifying $f(n+1)/f(n) \geq g(n+1)/g(n)$ for all $n \geq l$. ■

Proof of Theorem 6. I. Let $G = Sym(\Omega)$ and A a CCI-subgroup of G ($A \not\leq Alt(\Omega)$). Let C be the conjugacy class of transpositions. Then, $|C| = \binom{n}{2}$ and Lemma 15 gives:

$$|A \cap C| + |G : A| - 1 \leq \binom{n}{2}. \quad (5)$$

We will show that for all $n \geq 7$, A is a point stabilizer of the natural action of G on Ω ($A \cong S_{n-1}$), which is 2-transitive. A GAP ([7]) calculation shows that for $n = 5, 6$ there are further possibilities for A , but all of them involve a 2-transitive action. Thus we assume $n \geq 7$, and split the discussion into two cases:

a. $|A \cap C| \geq 1$. If $|A \cap C| = 1$ then A contains a unique transposition and hence this transposition is central in A in contradiction to Theorem 17. Thus $|A \cap C| \geq 2$ which implies, by Inequality 5, that $|G : A| < \binom{n}{2}$. Hence A satisfies the assumptions of Theorem 5.2B of [6] with $r = 2$. We have the following possibilities:

(i) There exists $\Delta \subseteq \Omega$ such that $|\Delta| < 2$ and $Alt(\Omega)_{(\Delta)} \leq A \leq G_{\{\Delta\}}$. If $|\Delta| = 0$ we get $Alt(\Omega)_{(\Delta)} = Alt(\Omega)$ contradicting $A \not\leq Alt(\Omega)$. If $|\Delta| = 1$, $A = G_{\{\Delta\}}$ by maximality. In this case the action of G on Ω_A is equivalent to its action on Ω which is 2-transitive.

(ii) $n = 2m$, $m \geq 4$. Here $|G : A| = \frac{1}{2} \binom{2m}{m}$, and Inequality 5 contradicts Lemma 29 (1).

b. $|A \cap C| = 0$. It follows, by Inequality 5, that $|G : A| \leq \binom{n}{2} + 1$. By Lemma 29 (2), $\binom{n}{2} + 1 < \binom{n}{3}$ for $n \geq 6$. Hence A satisfies the assumptions of Theorem 5.2B of [6] with $r = 3$. We have the following possibilities:

(i) There exists $\Delta \subseteq \Omega$ such that $|\Delta| < 3$ and $Alt(\Omega)_{(\Delta)} \leq A \leq G_{\{\Delta\}}$. The $|\Delta| = 0$ possibility is ruled out as in (a). If $|\Delta| = 1$ then, as in (a), $A = G_{\{\Delta\}}$ contradicting $|A \cap C| = 0$. Suppose $|\Delta| = 2$. By maximality $A = G_{\{\{a,b\}\}}$ where $a \neq b \in \Omega$. Thus $A \cong S_2 \times S_{n-2}$ contradicting the assumption $|A \cap C| = 0$ because A contains transpositions.

(ii) $n = 2m$, $m \geq 4$. Here $|G : A| = \frac{1}{2} \binom{2m}{m}$, and Inequality 5 contradicts Lemma 29 (1).

(iii) Since $n \geq 7$ we remain with $(n, r, |G : A|, A) = (7, 3, 30, PSL(3, 2))$, $(8, 3, 30, AGL(3, 2))$. For both cases $|G : A| = 30$, but under our assumptions Inequality 5 gives $|G : A| \leq \binom{n}{2} + 1$, so $|G : A| \leq 22$ for $n = 7$ and $|G : A| \leq 29$ for $n = 8$ - a contradiction.

II. Let $G = A_n$ and A a CCI-subgroup of G . Take C to be the G conjugacy class of three cycles. Then, $|C| = 2 \binom{n}{3}$ and Lemma 15 gives:

$$|A \cap C| + |G : A| - 1 \leq 2 \binom{n}{3}. \quad (6)$$

By Lemma 29 (3) and Inequality 6, we get $|G : A| < \binom{n}{4}$ for all $n \geq 12$.

Assume $n \geq 12$. Theorem 5.2A of [6] with $r = 4$ gives the following possibilities:

(i) There exists $\Delta \subseteq \Omega$ such that $|\Delta| < 4$ and $G_{(\Delta)} \leq A \leq G_{\{\Delta\}}$.

$|\Delta| = 0$: We get $A = G$ contradicting $A < G$.

$|\Delta| = 1$: We get $A = G_{\{\Delta\}}$ by maximality, and the action of G on Ω_A is equivalent to its action on Ω which is 2-transitive.

$|\Delta| = 2, 3$: These cases are treated below in Lemma 30.

(ii) $n = 2m$ and $|G : A| = \frac{1}{2} \binom{2m}{m}$. If $m \geq 6$ Inequality 6 contradicts Lemma 29 (4).

It remains to check $5 \leq n \leq 11$. For $n = 11$ we have $2 \binom{11}{3} = \binom{11}{4} < \binom{11}{5}$. Thus we can apply Theorem 5.2A of [6] with $r = 5$. Assuming Case (i) of Theorem 5.2A of [6], there exists $\Delta \subseteq \Omega$ such that $|\Delta| < 5$ and $G_{(\Delta)} \leq A \leq G_{\{\Delta\}}$. The values $|\Delta| = 0, 1, 2, 3$ are ruled out as before. For $|\Delta| = 4$ we get $A = A_{\{\Delta\}}$, and hence, an analysis similar to the previous cases yields:

$$|G : A| = |Sym(\Omega) : S_{\{\{a,b,c,d\}\}}| = \binom{11}{4} = 2 \binom{11}{3}.$$

Comparing with Inequality 6 we get $|A \cap C| = 1$. This is impossible by our choice of C since $t \in A \cap C$ implies $t^2 \in A \cap C$ and $t \neq t^2$. Note that (ii) and (iii) of Theorem 5.2A of [6] are not relevant for $n = 11$. Finally, the range $5 \leq n \leq 10$ is eliminated using GAP.

III. For $n = 6$ we have used GAP to check the three additional possibilities for G beside A_6 and S_6 . ■

Lemma 30 *Let $n \geq 5$ be an integer, $G := Alt(\Omega)$. If $A = G_{\{\{1,2\}\}}$ or $A = G_{\{\{1,2,3\}\}}$ then (G, A) is not CCI.*

Proof. a. $A = G_{\{\{1,2\}\}}$. Then $S_{\{\{1,2\}\}} = \langle (1,2) \rangle \times \text{Sym}(\{3, \dots, n\})$, and $A = G \cap S_{\{\{1,2\}\}}$. Set $S_n := \text{Sym}(\Omega)$. Then $S_n = S_{\{\{1,2\}\}}G$, whence $S_n/G \cong S_{\{\{1,2\}\}}/G \cap S_{\{\{1,2\}\}} = S_{\{\{1,2\}\}}/A$, yielding $|S_{\{\{1,2\}\}} : A| = 2$. Thus $|A| = (n-2)!$. Since A contains all even permutations of $S_{\{\{1,2\}\}}$, we get $\text{Alt}(\{3, \dots, n\}) \leq A$ and $\text{Alt}(\{3, \dots, n\})(1,2)(3,4) \subseteq A$. It follows (using $|A| = (n-2)!$) that A is the disjoint union

$$A = \text{Alt}(\{3, \dots, n\}) \cup \text{Alt}(\{3, \dots, n\})(1,2)(3,4).$$

Consider the right coset $A(1,3)(2,4)$ (permutations act on the right):

$$A(1,3)(2,4) = \text{Alt}(\{3, \dots, n\})(1,3)(2,4) \cup \text{Alt}(\{3, \dots, n\})(1,4)(2,3).$$

Choose any $f \in \text{Alt}(\{3, \dots, n\})$. Let $x, y \in \{3, \dots, n\}$ be such that $f(x) = 3$ and $f(y) = 4$. Note that $x, y, 1, 2$ are all distinct. We have:

$$(x)(f(1,3)(2,4)) = 1, \quad (y)(f(1,3)(2,4)) = 2.$$

Thus $1, 2, x, y \in \text{Support}(f(1,3)(2,4))$. Therefore $f(1,3)(2,4)$ moves at least 4 distinct points, and by a similar argument so does $f(1,4)(2,3)$. We have proved that every element in the coset $A(1,3)(2,4)$ moves at least 4 distinct points. It follows that $A(1,3)(2,4)$ does not intersect the conjugacy class of three cycles and hence it is not a CCI coset.

b. $A = G_{\{\{1,2,3\}\}}$. We have $S_{\{\{1,2,3\}\}} = \langle (1,2), (1,2,3) \rangle \times \text{Sym}(\{3, \dots, n\})$, and $A = G \cap S_{\{\{1,2,3\}\}}$. Then $S_n = S_{\{\{1,2,3\}\}}G$, and hence $S_n/G \cong S_{\{\{1,2,3\}\}}/A$, and therefore $|S_{\{\{1,2,3\}\}} : A| = 2$. Thus $|G_{\{\{1,2,3\}\}}| = 3(n-3)!$. Note that A contains all even permutations of $S_{\{\{1,2,3\}\}}$ and hence it contains B and $B(1,2)(4,5)$, where $B : \text{Alt}(\{4, \dots, n\}) \times \langle (1,2,3) \rangle$. These are two disjoint sets whose union has $3(n-3)!$ elements, and therefore $A = B \cup B(1,2)(4,5)$. In particular, $(1,3,2)(1,2)(4,5) = (1,3)(4,5) \in A$ and $(1,2,3)(1,2)(4,5) = (2,3)(4,5) \in A$. Also note that for $n = 5$ ($\text{Alt}(\{4,5\})$ is trivial) $A = G_{\{\{4,5\}\}} \cong S_3$ is discussed in part (a), and for $n = 6$ we have that A is not maximal, since $S_{\{\{1,2,3\}\}} \times S_{\{\{4,5,6\}\}}$ is not maximal in S_6 (see case (ii) in Theorem 5.2A of [6]). Therefore we can assume $n \geq 7$. Consider the right coset $A(1,4)(2,5) = B(1,4)(2,5) \cup B(1,5)(2,4)$. An element $g \in A(1,4)(2,5)$ takes the form $g = f_1 f_2 h$ where $f_1 \in \text{Alt}(\{4, \dots, n\})$, $f_2 \in \langle (1,2,3) \rangle$ and $h \in \{(1,4)(2,5), (1,5)(2,4)\}$. Let $x, y \in \{4, \dots, n\}$ be such that $f_1(x) = 4$ and $f_1(y) = 5$. Note that $x, y, 1, 2$ are all distinct. Moreover, it is easily checked that $(z)g \neq z$ for all $z \in \{x, y, 1, 2\}$. Thus, any $g \in A(1,4)(2,5)$ moves at least four points and therefore $A(1,4)(2,5)$ does not intersect the conjugacy class of three cycles. ■

4.4 Proof of Theorem 9

A useful characterization of Synchronizing groups ([5] Part 3) is based on the following definition:

Definition 31 Let π be a partition of the set Ω . A section or a transversal of π is a subset $S \subseteq \Omega$ which contains exactly one element of each part of π . Let G be a group acting on Ω . We say that π is section-regular for G with section S if for every $g \in G$ we have that Sg is also a section of π .

The verification of the following two basic lemmas is left to the reader.

Lemma 32 Let $n \geq 4$ be an integer. In any collection of n distinct 2-subsets of a set of cardinality n , there exist two 2-subsets with empty intersection.

Lemma 33 Let $n \geq 2$ be an integer. In any collection of at least $\lfloor \frac{n}{2} \rfloor + 1$ distinct 2-subsets of a set of cardinality n there are two distinct 2-subsets which intersect non-emptily.

Proof of Theorem 9. a. A cyclic group of order 3 acting on itself by right multiplication is 2-set transitive, but not CCI. Conversely, it is shown in [5] Part 2, that a 2-set transitive group which is not 2-transitive has odd order. Thus M_{11} (Example 8) is a CCI group which is not 2-set transitive.

b. By (a) a spreading group needs not be CCI. Conversely, extending an argument given at the end of [5] Part 3, the transitive action of M_{11} on $P_2(\{1, \dots, 11\})$ (Example 8) is non-spreading although it is CCI.

c. Any 2-set transitive group is 3/2-transitive, and so by (a) we can conclude that a 3/2-transitive group needs not be CCI. In the other direction we'll prove that the action of $G := M_{11}$ on $P_2(\Omega)$, where $\Omega := \{1, \dots, 11\}$ (Example 8) is not 3/2-transitive. The action of G on $P_2(\Omega)$ is 3/2-transitive if and only if all non-diagonal orbitals (orbits of the induced action of G on $P_2(\Omega) \times P_2(\Omega)$) have the same size. Since G acts 4-transitively on Ω , for any two ordered pairs of 2-sets, $(\{a_1, b_1\}, \{c_1, d_1\})$ and $(\{a_2, b_2\}, \{c_2, d_2\})$ such that $|\{a_1, b_1\} \cap \{c_1, d_1\}| = |\{a_2, b_2\} \cap \{c_2, d_2\}|$, there exists $g \in G$ such that

$$\begin{aligned} (\{a_1, b_1\}, \{c_1, d_1\})g &= (\{a_1, b_1\}g, \{c_1, d_1\}g) = \\ &= (\{a_1g, b_1g\}, \{c_1g, d_1g\}) = (\{a_2, b_2\}, \{c_2, d_2\}). \end{aligned}$$

Thus, the action of G on $P_2(\Omega)$ has two non-diagonal orbitals, O_i , $i \in \{0, 1\}$, where i is the cardinality of the intersection of the pair members. A simple computation gives $|O_0| = \binom{11}{2} \binom{9}{2} = 1980$ and $|O_1| = 11 \cdot 10 \cdot 9 = 990$, so $|O_0| \neq |O_1|$.

d. Retain the notations of (c). By [5] Part 3, Theorem 1, G is synchronizing if and only if there is no non-trivial section-regular partition π of $P_2(\Omega)$ for G with section S (see Definition 31), and by [5] Part 3, Theorem 9, all parts of such a non-trivial section-regular partition must have the same size. This implies that a section-regular partition $\pi = \{\pi_1, \dots, \pi_k\}$ of $P_2(\Omega)$ for G with section S satisfies either $|S| = k = 5$ and $|\pi_i| = 11$ for all $1 \leq i \leq 5$ or $|S| = k = 11$ and $|\pi_i| = 5$ for all $1 \leq i \leq 11$. Suppose that $|S| = k = 5$ and $|\pi_i| = 11$ for all $1 \leq i \leq 5$. By Lemma 32 and Lemma 33 (both applied with $n = 11$), for each $1 \leq i \leq 5$ we have that π_i contains two 2-subsets with an empty intersection and two 2-subsets which intersect in precisely one element. Now pick any two

distinct 2-subsets, A and B , belonging to S and any $1 \leq i \leq 5$. By the previous reasoning, there exist $C, D \in \pi_i$ such that $|A \cap B| = |C \cap D| \in \{0, 1\}$. Since G acts 4-transitively on Ω , we can find $g \in G$ such that $(Ag, Bg) = (C, D)$. This implies $|Sg \cap \pi_i| = 2$ and therefore π cannot be a section-regular partition of $P_2(\Omega)$. The argument for the second case ($|S| = k = 11$ and $|\pi_i| = 5$ for all $1 \leq i \leq 11$) is similar. ■

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