ON PERMUTATION POLYTOPES - NOTIONS OF EQUIVALENCE

BARBARA BAUMEISTER AND MATTHIAS GRÜNINGER

Abstract. By assigning a permutation polytope to a group we produce new interesting polytopes. For the effective use of this construction method it is desirable to understand which groups are leading to affine equivalent polytopes. Therefore, the notion of effective equivalence has been introduced [BHNP09]. In this note we clarify the notion of effective equivalence and characterize geometrically the effective equivalence permutation groups. Moreover, we present examples showing that the latter do not correspond to affinely equivalent polytopes. We also apply the characterization to the examples. Last but not least notice that we are also providing a framework for the use of representation theoretic methods in the study of permutation polytopes.

1. Introduction

The permutation polytopes are an interesting class of polytopes, see for instance [BHNP09, BiSa96, BHNP11, BHNP12, BaSt03, BLO11, DLY09, EFRS06]. In particular they provide a tool to produce new interesting polytopes, which moreover can be studied with group theoretic and representation theoretic methods. Therefore it is also important to understand which groups are leading to affine equivalent polytopes. This study has been started in [BHNP09]. Abstractly isomorphic permutation groups do not need to have affinely equivalent permutation polytopes: For instance $\langle (12), (34) \rangle$ and $\langle (12)(34),\ (13)(24) \rangle$ are isomorphic groups, but the associated permutation polytopes are a quadrangle and a tetrahedron, respectively, and therefore not affinely equivalent.

On the other hand, the notion of isomorphism of permutation groups is too restrictive to describe the affine equivalent permutation polytopes. In [BHNP09], Section 2.1, it has been observed that there are two permutation groups which are not isomorphic as permutation groups but whose permutation polytopes are affinely equivalent: The permutation polytopes of the permutation groups

$$\langle (1234) \rangle \leq \mathrm{Sym}(4), \langle (1234)(5) \rangle \leq \mathrm{Sym}(5), \langle (1234)(56) \rangle \leq \mathrm{Sym}(6)$$

are all tetrahedrons and therefore all affinely equivalent, but the underlying groups are not isomorphic as permutation groups.

The notion of isomorphism of permutation groups has been generalized to the notion of effectively equivalent permutation groups in [BHNP09]; for the definition see the next section. In fact, all the permutation groups listed in the last paragraph are effectively equivalent permutation groups.

The hope was that two permutation groups are effectively equivalent if and only if the groups are isomorphic and the corresponding permutation polytopes are affinely equivalent. Here we present an example of two permutation groups which are isomorphic as abstract groups and whose permutation polytopes are affinely equivalent, but which are not effectively equivalent thereby answering Question 2.12 of [BHNP09].

Up to now the tools in studying permutation polytopes were mainly connected to convex geometry and group theory. In this note we provide a representation theoretical basis for the study of these polytopes. Moreover, we use representation theory to prove a criterion which determines when two permutation groups G_1 and G_2 are effective equivalent in terms of their permutation polytopes $P(G_1)$ and $P(G_2)$ (see Theorem 4.3, Corollary 4.5 and Theorem 4.6).

The organization of the note is as follows: We start by introducing the notation and recall the relevant previous results. In the third section we present the examples. The representation theoretical approach is presented in the fourth section. There we also characterize the effectively equivalent groups. These results are applied to our examples in the last, the fifth, section.

Acknowledgments. The authors like to thank for support by the DFG through the SFB 701 "Spectral Structures and Topological Methods in Mathematics". Moreover, they thank Benjamin Nill for his very useful comments which helped to improve the paper.

2. Notation and previous results

The convex and the affine hull of a set S in a real vector space will be denoted by $\operatorname{conv}(S)$ and by $\operatorname{aff}(S)$, respectively.

2.1. **Permutation polytopes.** An injective homomorphism $\pi : G \to \operatorname{Sym}(n)$ is called *permutation representation*. The pair (G, π) is called *permutation group*. In this case, we obtain a representation polytope as follows.

The symmetric group $\operatorname{Sym}(n)$ acts on the set $\{1, \dots, n\}$. Let V be an n-dimensional $\mathbb{R}$-vector space with basis $\{e_1, \dots, e_n\}$ and let $\operatorname{Sym}(n)$ act on this vector space by permuting the indices of the vectors in the basis. Then V is the *permutation module for* $\operatorname{Sym}(n)$. This module induces a representation $R : \operatorname{Sym}(n) \to \operatorname{GL}(V) = \operatorname{GL}(\mathbb{R}^n)$, $g \mapsto M_g$, and thereby identifies the symmetric group $\operatorname{Sym}(n)$ with the set of $n \times n$ *permutation matrices*, i.e. the set of matrices whose entries are 0 or 1 such that in every

column and every row there is a unique 1. The polytope

$$P(G,\pi) := \mathrm{conv}(\pi(G)) \subseteq \mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n\times n}$$

is called the *permutation polytope* associated to (G,π). If π or G is clear from the context then we abbreviate $P(G,\pi)$ by $P(G)$ or by $P(\pi)$.

The special case $G = \mathrm{Sym}(n)$ yields the well-known nth *Birkhoff polytope* $B_n := P(\pi(\mathrm{Sym}(n)))$ (see e.g. [BiSa96]). The concept of a permutation polytope can be generalized to a representation polytope by considering any real representation of G instead of a permutation representation, see [BHNP09].

2.2. **Notions of equivalence of polytopes.** For a standard reference on polytopes we refer to [Zie95]. If the vertices of a polytope $P \subseteq \mathbb{R}^m$ are a subset of a full dimensional lattice Λ in $\mathbb{R}^m$, then we call P a lattice polytope. In this sense every permutation polytope is a lattice polytope, as the vertices all lie in $\mathrm{Mat}_n(\mathbb{Z})$.

As moreover, every vertex of a permutation polytope is a matrix whose entries are only 0 and 1, it is also a 0/1-*polytope*, i.e. a polytope whose vertices are in the set $\{0,1\}^d$ for some $d \in \mathbb{N}$.

There are several notions of equivalence of (lattice) polytopes (see [Zie95]):

Definition 2.1. Two polytopes $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$ are *affinely equivalent* if there is an affine isomorphism of the affine hulls $\phi\colon \mathrm{aff}(P) \to \mathrm{aff}(Q)$ that maps P onto Q, write $P \approx_{aff} Q$. For *lattice equivalence* we additionally require that ϕ is an isomorphism of the affine lattices $(\mathrm{aff}\, P)\cap\Lambda \to (\mathrm{aff}\, Q)\cap\Lambda'$. *Combinatorial equivalence* is an equivalence of the face lattices as posets.

2.3. **Notions of equivalence of groups.** To identify permutation groups that define affinely equivalent permutation polytopes the notion of effective equivalence has been introduced (see [BHNP09]).

We use the following non-standard notation, which does not conflict with the usual notation. For $K = \mathbb{R}$ or $\mathbb{C}$ we denote by $\mathrm{Irr}_K(G)$ the set of pairwise non-equivalent irreducible K-representations, i.e. homomorphisms $G \to \mathrm{GL}(W)$ where W is a K-vector space which does not contain a proper G-invariant subspace. For instance, there is the *trivial representation*, 1_G: $G \to \mathrm{GL}(K)$, $g \mapsto 1$. Every representation $\rho : G \to \mathrm{GL}(V)$ over K splits into irreducible representations. We denote the pairwise non-equivalent *irreducible factors* of ρ by $\mathrm{Irr}_K(\rho) \subseteq \mathrm{Irr}_K(G)$.

Definition 2.2. Two real representations ρ_1 and ρ_2 of G are *stably equivalent* if they contain equivalent non-trivial irreducible factors. Two faithful real representations $\rho_i : G_i \to \mathrm{GL}(V_i)$ (for $i = 1,2$) of finite groups are *effectively equivalent* if there exists an isomorphism $\varphi : G_1 \to G_2$ such that ρ_1 and $\rho_2 \circ \varphi$ are stably equivalent G_1-representations, write $\rho_1 \approx_{eff} \rho_2$. Moreover, we say $G_1 \leq \mathrm{Sym}(n_1)$ and $G_2 \leq \mathrm{Sym}(n_2)$ are *effectively equivalent permutation groups* if $G_1 \hookrightarrow \mathrm{Sym}(n_1)$ and $G_2 \hookrightarrow \mathrm{Sym}(n_2)$ are effectively equivalent permutation representations.

The permutation groups given in the introduction are effective equivalent. One may think that two transitive permutation groups G_1 and G_2 which are effective equivalent already have to be equal. This is not the case as demonstrated in the next example (see also Example 3.1).

Example. Let $G = \mathrm{PSL}_2(13)$ and consider the actions π_1 and π_2 on the coset spaces G/H_1 and G/H_2 where H_1 is subgroup of G isomorphic to D_{14} and H_2 a subgroup isomorphic to D_{12}. According to Atlas-notation the permutation characters are $1a + 12abc + 13a + 14aa$ and $1a + 12abc + 13aa + 14aa$, respectively, see [CNPW85]. Thus π_1 and π_2 are two transitive effective equivalent presentations which are different.

An immediate consequence of the definition of effective equivalence is the following.

Lemma 2.3. *If $G_1 = (G, \pi_1)$ and $G_2 = (G, \pi_2)$ are two permutation groups such that $P(\pi_1) = P(\pi_2)$, then G_1 and G_2 are effective equivalent.*

Proof. In this case $\pi_1^{-1}\pi_2$ is automorphism of G and therefore, G_1 and G_2 are effective equivalent. □

In [BHNP09, 2.3] it has been shown that if ρ and $\bar{\rho}$ are two stably equivalent real representations of a finite group G, then $P(\rho)$ and $P(\bar{\rho})$ are affinely equivalent. If π_1 and π_2 are effectively equivalent permutation representations, then π_1 and $\pi_2 \circ \varphi$ are stably equivalent for some isomorphism $\varphi : G_1 \to G_2$. As $P(\pi_2) = P(\pi_2 \circ \varphi)$ the following holds as well:

Theorem 2.4. *The permutation polytopes related to two effectively equivalent permutation representations are affinely equivalent.*

Notice that Example 2.7 in [BHNP09] shows that effectively equivalent permutation representations do not necessarily have lattice equivalent permutation polytopes. This example shows as well that the volumes of two permutation polytopes associated to effectively equivalent permutation representations may be different.

3. The examples.

In this section we present an example of a group with two non effectively equivalent permutation representations such that the related permutation polytopes are affinely equivalent. But first we show the following "almost example". It consists of two permutation groups which are not stably equivalent, but whose permutation polytopes are even equal. It is not really an example to our question as the permutation groups are effectively equivalent.

3.1. **An "almost example".** Let $G = \mathrm{Alt}(6)$. Then G contains two different subgroups H_1 and H_2 which are both isomorphic to $\mathrm{Alt}(5)$, but not conjugate in G. We may choose H_1 as the stabilizer of 1 in the action of G on the set $[6] := \{1, \ldots, 6\}$. Then H_2 is transitive on $[6]$. The group G acts on both coset spaces G/H_1 and G/H_2, which yields two permutation representations π_1 and π_2. These representations are not stably equivalent, as they contain different irreducible constituents, see for instance [CNPW85], p. 4. On the other hand, as $|G : H_i| = 6$ for $i = 1, 2$, both representations π_1 and π_2 induce embeddings of G into $\mathrm{Sym}(6)$. Since in $\mathrm{Sym}(6)$ there is only one subgroup isomorphic to $\mathrm{Alt}(6)$, it follows that $\pi_1(G) = \pi_2(G)$. Thus by Lemma 2.3 the two representations π_1 and π_2 are effectively equivalent.

Example 3.1 provides two transitive effective equivalent permutation representations π_1 and π_2 of the same degree which are different. But $\pi_1 = \pi_2 \circ \varphi$ for some $\varphi \in \mathrm{Aut}(G)$. We would like to know whether this is always the case:

Question 3.1. *Do two transitive effective equivalent permutation representations (G, π_1) and (G, π_2) of the same degree always only differ by an automorphism of G?*

3.2. **The example.** Let $A = (\mathbb{Z}_2)^2$ be the direct product of two cyclic groups of order two, $B = \mathbb{Z}_4$ and $C = \mathbb{Z}_3$ cyclic groups of order 4 and 3, and let $G := A \times B \times C$. In the following we define two different permutation representations of G:

The permutation representation π_1. Let O_1 be the disjoint union of the right coset spaces $O_{11} := G/A$ and $O_{12} := G/(B \times C)$ and let G act by left multiplication on O_1. Then O_{11} and O_{12} are the G-orbits. The kernels of the action of G on O_{11} and O_{12} are A and $B \times C$, respectively. By Lemma 2.7 and Theorem 3.5 of [BHNP09] $P(\pi_1)$ is the combinatorial product of an 11-simplex with a 3-simplex.

Notice, if $G = H \times K$ for some subgroups H and K of G, then we can extend every irreducible complex representation φ_H of H to an irreducible complex representation φ of G by sending every element of K to the identity. Therefore, we can embed $\mathrm{Irr}_{\mathbb{C}}(H)$ into $\mathrm{Irr}_{\mathbb{C}}(G)$. In this sense $\mathrm{Irr}_{\mathbb{C}}(\pi_1)$ is the union of $\mathrm{Irr}_{\mathbb{C}}(B \times C)$ and $\mathrm{Irr}_{\mathbb{C}}(A)$. As for an abelian group $\mathrm{Irr}_{\mathbb{C}}(G) \cong G$, see Paragraph 6, 6.4 in [Hup67], it follows that $\mathrm{Irr}_{\mathbb{C}}(\pi_1)$ is the union of two subgroups isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_3$ and $(\mathbb{Z}_2)^2$, respectively.

The permutation representation π_2. Let O_2 be the disjoint union of the right coset spaces $O_{21} := G/B$ and $O_{22} := G/(A \times C)$ and let G act by left multiplication on O_2. Then O_{21} and O_{22} are the G-orbits. The kernels of the action of G on O_{21} and O_{22} are B and $A \times C$, respectively. By Lemma 2.7 and Theorem 3.5 of [BHNP09] $P(\pi_2)$ is again the combinatorial product of an 11-simplex with a 3-simplex.

Here $\mathrm{Irr}_{\mathbb{C}}(\pi_2)$ is the union of $\mathrm{Irr}_{\mathbb{C}}(A \times C)$ and $\mathrm{Irr}_{\mathbb{C}}(B)$ and therefore, the union of two subgroups isomorphic to $(\mathbb{Z}_2)^2 \times \mathbb{Z}_3$ and $\mathbb{Z}_4$.

It follows that $P(\pi_1)$ and $P(\pi_2)$ are affinely equivalent. In $\mathrm{Irr}_{\mathbb{C}}(\pi_1)$ there is an irreducible representation of order 12, while every element in $\mathrm{Irr}_{\mathbb{C}}(\pi_2)$ has order at most 6. This shows that the induced real representations π_1 and $\pi_2 \circ \varphi$ do not contain the same irreducible factors for every automorphism $\varphi \in \mathrm{Aut}(G)$. Thus π_1 and π_2 are not effectively equivalent.

Remark 3.2. *If $G = A \times B \times C$ is an abelian group such that A and B are non-isomorphic groups of the same size, then we can always construct such an example with π_1 and π_2 the actions of G on the unions of coset spaces $O_1 := G/A \cup G/(B \times C)$ and $O_2 := G/B \cup G/(A \times C)$, respectively.*

4. Characterization of effectively equivalence.

Let $G = (G, \pi)$ be a permutation group of degree n with permutation module V. Then the affine hull of the polytope $P = P(\pi)$ is

$$\mathrm{aff}(\pi(G)) = \{\sum_{g\in G} \lambda_g M_g \mid \lambda_g \in \mathbb{R}, \sum_{g\in G} \lambda_g = 1\} = E_n + U_\pi, \text{ where}$$

$$U_\pi := \{\sum_{g\in G} \lambda_g M_g \mid \lambda_g \in \mathbb{R}, \sum_{g\in G} \lambda_g = 0\} = \{\sum_{g\in G} \lambda_g (M_g - M_e) \mid \lambda_g \in \mathbb{R}\}$$
$$= \langle M_g - M_h \mid g, h \in G\rangle_{\mathbb{R}} \leq \mathrm{End}(V).$$

The $\mathbb{R}$-vector space U_π is a G-module through the definition

$$hM := M_h M = \pi(h)M, \text{for } h \in G \text{ and } M \in U_\pi,$$

as π is a group homomorphism from G into $\mathrm{Sym}(n) \leq \mathrm{GL}_n(\mathbb{R})$. Notice, that if in particular $M = \pi(g)$, then $hM = \pi(hg)$.

In order to describe the structure of U_π nicely we introduce more notation. For $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ let $V(\chi)$ be an irreducible $\mathbb{R}G$-module with character χ. Then $\mathrm{End}_{\mathbb{R}G}(V(\chi))$ is either isomorphic to $\mathbb{R}$, $\mathbb{C}$ or to the quaternions $\mathbb{H}$ and thus $d_\chi := 1/\dim_{\mathbb{R}} \mathrm{End}_{\mathbb{R}G}(V(\chi))$ is either 1, $\frac{1}{2}$ or $\frac{1}{4}$. Set

$$V_\pi := \{\sum_{g\in G} \lambda_g M_g \mid \lambda_g \in \mathbb{R}\} \text{ and } \epsilon_\pi := \sum_{g\in G} M_g.$$

Then $V_\pi = U_\pi \oplus \mathbb{R}\epsilon_G$.

Theorem 4.1. *Let $G = (G, \pi)$ be a permutation group of degree n with permutation module V. Then U_π is isomorphic to*

$$\sum_{\chi\in \mathrm{Irr}_{\mathbb{R}}(\pi)\setminus\{1_G\}} (d_\chi \cdot \chi(1))V(\chi)$$

as an $\mathbb{R}G$-module, where for d a natural number $dV(\chi)$ is the direct sum of d to $V(\chi)$ isomorphic $\mathbb{R}G$-modules.

Proof. Extend π linearly to an $\mathbb{R}$-algebra epimorphism $\pi : \mathbb{R}G \to V_\pi$. This then is $\mathbb{R}G$-linear. By Maschke's Theorem, the group algebra $\mathbb{R}G$ is semi-simple. Thus V_π is semi-simple as well, and by a theorem by Wedderburn (see for instance [Hup67] Chapter V, Hauptsatz 4.4) we get $\mathbb{R}G = \oplus_{i=1}^m A_i$

with A_i simple. Since A_i is simple, either $\pi|A_i$ is injective or $\pi(A_i) = 0$. Thus V_π is semi-simple as well and we can assume that there is an integer $k \leq m$ such that $V_\pi \cong \oplus_{i=1}^k A_i$. Moreover, we can assume that $A_1 = \mathbb{R}\epsilon$ with $\epsilon = \sum_{g\in G} g$. Since $\pi(\epsilon) = \epsilon_\pi$, we have $U_\pi = \oplus_{i=2}^k A_i$.

Let $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ be an irreducible representation of G. Then $V(\chi)$ is a composition factor of the G-module V if and only if there is an index $1 \leq i \leq k$ such that the $\mathbb{R}$-linear extension of χ does not vanish on A_i or equivalently, $A_i = e_\chi \mathbb{R}G$ with

$$e_\chi = \sum_{\psi\in \mathrm{Irr}_{\mathbb{C}}(G|\chi)} e_\psi \;= \sum_{\psi\in \mathrm{Irr}_{\mathbb{C}}(G|\chi)} \frac{1}{|G|}\psi(1) \sum_{g\in G} \overline{\psi(g)}g$$

the central idempotent corresponding to χ. In this case,

$$\dim_{\mathbb{R}} A_i = \dim_{\mathbb{R}} End_{\mathbb{R}G}(V(\chi)) \cdot (\dim_{End_{\mathbb{R}G}(V(\chi))} V(\chi)))^2$$

$$= (1/d_\chi) \cdot (d_\chi \cdot \chi(1))^2 = d_\chi \cdot \chi(1)^2$$

and $A_i \cong \mathrm{End}_{\mathbb{R}}(V(\chi)) \cong (d_\chi \cdot \chi(1))V(\chi)$ as an $\mathbb{R}G$-module, see [Hup67], Chapter V, Satz 4.5. Since ϵ is the central idempotent corresponding to the trivial representation, the claim follows. □

The following lemma shows that affine maps between permutation polytopes are always induced by linear maps of their linear hulls.

Lemma 4.2. *If V is an $\mathbb{R}$-vector space and $\psi : \mathrm{aff}(P(\pi)) \to V$ an affine map, then ψ can be uniquely lifted to a linear map $\Psi : V_\pi \to V$.*

Proof. Since $P(\pi)$ contains a basis of V_π, the space V_π is the affine hull of $\{0\}$ and $P(\pi)$. Moreover, every element in $\mathrm{aff}(P(\pi))$ is a matrix whose rows and columns all have sum 1. This shows that $0 \notin \mathrm{aff}(P(\pi))$. Thus there is a unique affine map $\Psi : V_\pi \to V$ such that $\Psi|\,\mathrm{aff}(P(\pi)) = \psi$ and $\Psi(0) = 0$. Since $\Psi(0) = 0$, the map Ψ is linear. □

For $\varphi \in \mathrm{Aut}(G)$ we say that two G-modules U_1 and U_2 are *φ-isomorphic* if there exists an isomorphism $\phi : U_1 \to U_2$ with $\phi(gu) = \varphi(g)\phi(u)$ for all $u \in U_1$ and all $g \in G$. Theorem 4.1 implies the following characterization of the effectively equivalent permutation groups:

Theorem 4.3. *Let $G_1 = (G, \pi_1)$ and $G_2 = (G, \pi_2)$ be two permutation groups. Then the following are equivalent:*

(a) *G_1 and G_2 are effectively equivalent.*
(b) *There is a $\varphi \in \mathrm{Aut}(G)$ such that U_{π_1} and U_{π_2} are φ-isomorphic.*
(c) *There is a $\varphi \in \mathrm{Aut}(G)$ such that V_{π_1} and V_{π_2} are φ-isomorphic.*
(d) *There is an affine isomorphism $\phi : \mathrm{aff}(P(\pi_1)) \to \mathrm{aff}(P(\pi_2))$ which maps $P(\pi_1)$ onto $P(\pi_2)$ and which restricted to $\pi_1(G)$ is a group homomorphism.*

Proof. Suppose that (b) holds. Then U_{π_1} and $U_{\pi_2\circ\varphi}$ are isomorphic G-modules. By Theorem 4.1 $U_{\pi_2\circ\varphi}$ and U_{π_1} have the same irreducible constituents. Thus $\pi_2\circ\varphi$ and π_1 are stably equivalent and G_1 and G_2 are effectively equivalent; so (a) holds. Statements (b) and (c) are equivalent since V_{π_i} and U_{π_i} only differ by the trivial $\mathbb{R}G$-module.

Suppose that (a) holds. Then there is an automorphism φ of G such that π_1 and $\pi_2\circ\varphi$ are stably equivalent. Thus if $(\lambda_g)_{g\in G}$ is a family of real numbers with $\sum_{g\in G}\lambda_g = 0$, then by [BHNP09] Theorem 2.4

$$\sum_{g\in G}\lambda_g\pi_1(g) = 0 \text{ if and only if } \sum_{g\in G}\lambda_g\pi_2(\varphi(g)) = 0.$$

Thus the map

$$\phi : \operatorname{aff}(P(\pi_1)) \to \operatorname{aff}(P(\pi_2)), \sum_{g\in G}\lambda_g\pi_1(g) \mapsto \sum_{g\in G}\lambda_g\pi_2(\varphi(g)),$$

where $\sum_{g\in G}\lambda_g = 1$, is a well-defined affine map.

By the same argument, ϕ is injective, and as the image of ϕ affinely spans $\operatorname{aff}(P(\pi_2))$, the map ϕ is surjective as well. As π_1, π_2 and φ are group homomorphisms, the restriction of ϕ to $\pi_1(G)$ is a group homomorphism onto $\pi_2(G)$. This shows that (a) implies (d).

Suppose that (d) holds. We want to show (c). First note that ϕ maps $\pi_1(G)$ bijectively onto $\pi_2(G)$, since these are vertices of the corresponding polytopes. Thus ϕ induces a group isomorphism between $\pi_1(G)$ and $\pi_2(G)$ which we will also call ϕ. Then $\varphi := \pi_2^{-1}\circ\phi\circ\pi_1$ is a group automorphism of G. If $u = \sum_{g\in G}\lambda_g\pi_1(g)$ with $\sum_{g\in G}\lambda_g = 1$ and $x \in G$, then

$$\phi(xu) = \phi(x\sum_{g\in G}\lambda_g\pi_1(g)) = \phi(\sum_{g\in G}\lambda_g x\pi_1(g)) = \sum_{g\in G}\lambda_g\phi(\pi_1(xg)) =$$

$$\sum_{g\in G}\lambda_g\phi(\pi_1(x))\phi(\pi_1(g)) = \sum_{g\in G}\lambda_g\pi_2(\varphi(x))\phi(\pi_1(g))$$

$$= \sum_{g\in G}\lambda_g\varphi(x)\phi(\pi_1(g)) = \varphi(x)\sum_{g\in G}\lambda_g\phi(\pi_1(g)) = \varphi(x)\phi(u).$$

by the definition of the action of G on U_{π_1} and U_{π_2}. By Lemma 4.2 we can extend ϕ to a linear isomorphism $\Psi : V_{\pi_1} \to V_{\pi_2}$, for which one easily sees that $\Psi(gu) = \varphi(g)\Psi(u)$ holds for all $g \in G$ and all $u \in V_{\pi_1}$. □

The equivalence between (a) and (d) yields another possibility to describe effective equivalence.

Definition 4.4. If P and Q are two polytopes, G a group which acts as automorphism group on both P and Q, then an affine isomorphism $\phi : \operatorname{aff}(P) \to \operatorname{aff}(Q)$ with $\phi(P) = Q$ is called an *affine G-isomorphism* if there is an automorphism φ of G such that $\phi(gx) = \varphi(g)\phi(x)$ holds for all $g \in G$ and all vertices x of P.

If π is a permutation representation of a finite group G, then left multiplication defines a natural action of G on $P(\pi)$ as we saw above. Then the equivalence between (a) and (d) in Theorem 4.3 gives us:

Corollary 4.5. *If $G_1 = (G, \pi_1)$ and $G_2 = (G, \pi_2)$ are two permutation groups, then they are effectively equivalent if and only if there is an affine G-isomorphism between the corresponding permutation polytopes.*

Let $G_i = (G, \pi_i)$ be permutation groups for $i = 1, 2$. Suppose that there is an affine isomorphism $\phi : P(\pi_1) \to P(\pi_2)$. Then we get an isomorphism $\hat{\phi} : \mathrm{Aut}(P(\pi_1)) \to \mathrm{Aut}(P(\pi_2))$ by setting $\hat{\phi}(g)(x) := \phi(g(\phi^{-1}(x))$ for $g \in \mathrm{Aut}(P(\pi_1))$ and $x \in P(\pi_2)$. For $g \in G$ let $\psi_i(g) \in \mathrm{Aut}(P(\pi_i))$ be the automorphism defined by $\psi_i(g)(u) = \pi_i(g)u$ for $u \in \mathrm{aff}(P(\pi_i))$. Hence $\psi_i : G \to \mathrm{Aut}(P(\pi))$ is a monomorphism. Note that $\psi_i(G)$ acts regularly on the set of vertices of $P(\pi_i)$.

Theorem 4.6. *G_1 and G_2 are effectively equivalent if and only if there is an affine isomorphism $\phi : \mathrm{aff}(P(\pi_1)) \to \mathrm{aff}(P(\pi_2))$ mapping $P(\pi_1)$ onto $P(\pi_2)$ such that $\hat{\phi}(\psi_1(G))$ and $\psi_2(G)$ are conjugate in $\mathrm{Aut}(P(\pi_1))$.*

Proof. If G_1 and G_2 are effectively equivalent, then by 4.3 there is an affine isomorphism $\phi : \mathrm{aff}(P(\pi_1)) \to \mathrm{aff}(P(\pi_2))$ with $\Phi(P(\pi_1)) = P(\pi_2)$ and an isomophism φ of G with $\phi(xu) = \varphi(x)\phi(u)$ for all $x \in G$ and $u \in P(\pi_1)$. Therefore it follows immediately that $\psi_2(G) = \hat{\phi}(\psi_1(G))$. Now suppose there is $\phi : \mathrm{aff}(P(\pi_1)) \to \mathrm{aff}(P(\pi_2))$ such that $\phi(P(\pi_1)) = P(\pi_2)$ and $\psi_2(G)$ and $\hat{\phi}(\psi_1(G))$ are conjugate. We may assume that $\phi(\pi_1(1)) = \pi_2(1)$. Since $\psi_2(G)$ acts transitively on the vertices of $P(\pi_2)$, we may assume that there is an $a \in \mathrm{Aut}(P(\pi_2))_{\pi_2(1)}$ such that $\psi_2(G)^a = \hat{\phi}(\psi_1(G))$. After replacing ϕ by $a^{-1} \circ \phi$ we may assume $\hat{\phi}(\psi_1(G)) = \psi_2(G)$. Thus there is a bijective map $\varphi : G \to G$ such that $\psi_2(\varphi(g)) = \hat{\phi}(\psi_1(g))$ for all $g \in G$. Since $\hat{\phi}, \psi_1$ and ψ_2 are isomorphisms, it follows that φ is an automorphism of G. Thus we have for all $g \in G$ and all $u \in \mathrm{aff}(P(\pi_1))$:

$$\phi(\psi_1(g)(u)) = \phi(\psi_1(g)(\phi^{-1}(\phi(u)))) = \hat{\phi}(\psi_1(g)(\phi(u)) = \psi_2(\varphi(g))(\phi(u)).$$

Since $\phi(\pi_1(g)) = \pi_2(g)$, we get

$$\phi(\pi_1(g)) = \phi(\psi_1(g)(\pi_1(1))) = \psi_2(\varphi(g))\pi_2(1) = \pi_2(\varphi(g)).$$

Thus $\phi|\pi_1(G)$ is a group homomorphism and the claim follows by Theorem 4.3. □

As a consequence this lemma establishes Conjecture 5.4 of [BHNP09] for our examples.

Corollary 4.7. *If $P(\pi_1) \approx_{aff} P(\pi_2)$, but $\pi_1 \not\approx_{eff} \pi_2$, then $\mathrm{Aut}(P(\pi_1)) \cong \mathrm{Aut}(P(\pi_2))$ contains two non-conjugate regular subgroups which are isomorphic to G. In particular $\mathrm{Aut}(P(\pi_i)) > \psi_i(G)$.*

Proof. The polytopes $P(\pi_1)$ and $P(\pi_2)$ are affinely equivalent while π_1 and π_2 are not effectively equivalent. Thus by Theorem 4.6 the subgroups $\hat{\phi}(\psi_1(G))$ and $\psi_2(G)$ of $\mathrm{Aut}(P(\pi_1))$ are not conjugate and therefore not equal, which yields the assertion. □

We wonder whether these results can be used to prove the following.

Question 4.8. *Let $G_1 = (G, \pi_1)$ and $G_2 = (G, \pi_2)$ be two transitive permutation groups (of the same degree). Are then $P(\pi_1)$ and $P(\pi_2)$ affinely equivalent if and only if π_1 and π_2 are effectively equivalent?*

This question has certainly a positive answer if G is abelian, as in this case every transitive presentation is already regular. If G_1 and G_2 are not transitive then the answer to the question is negative as our examples demonstrate. Permutation polytopes for cyclic groups are of importance in statistics, see for instance [EFRS06]. Therefore, it would be interesting to know whether they behave more nicely.

Question 4.9. *Let $G_1 = (G, \pi_1)$ and $G_2 = (G, \pi_2)$ be two cyclic permutation groups. Are then $P(\pi_1)$ and $P(\pi_2)$ affinely equivalent if and only if π_1 and π_2 are effectively equivalent?*

5. Applications

5.1. **Application of Lemma 2.3 to** $\mathrm{Alt}(6)$**.** We obtain as an immediate consequence the following well-known fact.

Lemma 5.1. *The group $G = \mathrm{Alt}(6)$ has an outer automorphism which interchanges the two conjugacy classes of $\mathrm{Alt}(5)$-subgroups of G.*

Proof. Let H_1 and H_2 be two subgroups of $\mathrm{Alt}(6)$ of the two conjugacy classes and π_1 and π_2 be as in Subsetion 3.1. Then, as both representations yield the same permutation polytope, Lemma 2.3 implies that (G, π_1) and (G, π_2) are effective equivalent. Thus $\pi_1^{-1}\pi_2$ is an automorphism of $\mathrm{Alt}(6)$ that maps H_1 onto H_2. □

5.2. **Application of Theorem 4.3 to the example.** In this section we apply our characterization of the effectively equivalent permutation groups given in Theorem 4.3 to give a new, direct and more geometric proof of the fact that the permutation groups (G, π_1) and (G, π_2) presented in Example 3.2 are not effectively equivalent. We use the notation introduced in Example 3.2.

Suppose that (G, π_1) and (G, π_2) are effectively equivalent. Then according to Theorem 4.3 there is an affine isomorphism $\phi : \mathrm{aff}(P(\pi_1)) \to \mathrm{aff}(P(\pi_2))$ which maps $P(\pi_1)$ to $P(\pi_2)$ and which restricted to $\pi_1(G)$ is a group homomorphism.

Let H be a subgroup of G such that $\pi_1(H)$ is the set of vertices of a face of the polytope $P(\pi_1)$. Then $\pi_1(H)$ is a subgroup of $\pi_1(G)$ which implies

that $\phi(\pi_1(H))$ is a subgroup of $\pi_2(G)$. As ϕ is an affine isomorphism from $\mathrm{aff}(P(\pi_1))$ to $\mathrm{aff}(P(\pi_2))$ as well, the set $\phi(\pi_1(H))$ is the set of vertices of a face of the polytope $P(\pi_2)$.

Now we count all the faces of $P(\pi_i)$ which have 24 vertices and whose set of vertices is a subgroup of $\pi_i(G)$ (for $i = 1, 2$). The polytope $P(\pi_i)$ is the product of an 11-simplex with a 3-simplex. Therefore every face of $P(\pi_i)$ has the shape $E \times F$ where E is a face of the 11-simplex and F a face of the 3-simplex (for $i = 1, 2$).

The faces of $P(\pi_1)$ given by a subgroup of size 24. In this case $G = H_1 \times H_2$ where $H_1 \cong \mathbb{Z}_{12}$ and $H_2 \cong \mathbb{Z}_2^2$; and $P(\pi_1) = P(H_1) \times P(H_2)$. Further $P(H_1)$ is an 11-simplex and $P(H_2)$ a 3-simplex. If H is a subgroup of G such that $\pi_1(H)$ is the set of vertices of a face with 24 vertices, then $H = K_1 \times K_2$ such that K_i is a subgroup of H_i (for $i = 1, 2$) and such that $|H| = |K_1| \cdot |K_2| = 24$. Then either $K_1 = H_1$ and K_2 of order 2 or K_1 is of order 6 and $K_2 = H_2$. As there is just one subgroup of order 6 in H_1 and three subgroups of order 2 in H_2, it follows that the 24-vertex faces which are coming from a subgroup are precisely three faces of the shape of a prisma over an 11-simplex and one face which is the product of a 5-simplex with a 3-simplex.

The faces of $P(\pi_2)$ given by a subgroup of size 24. Here we have the factorization $G = M_1 \times M_2$ where $M_1 \cong \mathbb{Z}_2^2 \times 3$ and $M_2 \cong \mathbb{Z}_4$. The polytope $P(\pi_2) = P(M_1) \times P(M_2)$ is the product of an 11-simplex and a 3-simplex. If H is a subgroup of G such that $\pi_2(H)$ is the set of vertices of a face with 24 vertices, then as above $H = K_1 \times K_2$ such that K_i is a subgroup of M_i (for $i = 1, 2$) and such that $|H| = |K_1| \cdot |K_2| = 24$. In this case there are three subgroups of M_1 of size 6 and precisely one subgroup of K_2 of size 2. Therefore, the 24-vertex faces which are coming from a subgroup are precisely one prisma over an 11-simplex and three faces which are the product of a 5-simplex and a 3-simplex.

This contradicts the fact that ϕ maps every face of $P(\pi_1)$ which is induced by a subgroup of G isomorphically onto a face of $P(\pi_2)$ which is induced by a subgroup of G. Thus (G, π_1) and (G, π_2) are not effectively equivalent.

References

[BaSt03] Alexander Barvinok and Tamon Stephen. The distribution of values in the quadratic assignment problem. *Math. Oper. Res.*, 28(1):64–91, 2003.

[BHNP09] Barbara Baumeister, Christian Haase, Benjamin Nill, and Andreas Paffenholz. On permutation polytopes. *Adv. Math.*, 222(2):431–452, 2009.

[BHNP11] Barbara Baumeister, Christian Haase, Benjamin Nill, and Andreas Paffenholz. Permutation polytopes of cyclic groups, September 2011. preprint, `arXiv: 1109.0191`.

[BHNP12] Barbara Baumeister, Christian Haase, Benjamin Nill, and Andreas Paffenholz. Polytopes associated to Dihedral groups, 2012. preprint, `arXiv: 1212:4442`.

[BiSa96] Louis J. Billera and A. Sarangarajan. The combinatorics of permutation polytopes. Billera, Louis J. (ed.) et al., Formal power series and algebraic combinatorics.

Séries formelles et combinatoire algébrique 1994. Invited lectures presented at the 6th international DIMACS workshop, May 23-27, 1994. Providence, RI: American Mathematical Society. DIMACS, Ser. Discrete Math. Theor. Comput. Sci. 24, 1-23 (1996).

[BLO11] Katherine Burggraf, Jesús De Loera, and Mohamed Omar. On volumes of permutation polytopes, 2011. Fields Institute Communications 69, (2013).

[CNPW85] John H. Conway, Robert T. Curtis, Simon P. Norton, Richard A. Parker and Robert A. Wilson. *An ATLAS of finite groups* Oxford University Press, 1985.

[DLY09] Jesús A. De Loera, Fang Liu, Ruriko Yoshida. A generating function for all semi-magic squares and the volume of the Birkhoff polytope. *J. Algebr. Comb.*, 30(1):113–139, 2009.

[EFRS06] Nicholas Eriksson, Stephen E. Fienberg, Alessandro Rinaldo, Seth Sullivant. Polyhedral conditions for the nonexistence of the MLE for hierarchical log-linear models. *J. Symb. Comput.*, 41(2):222–233, 2006.

[Hup67] Bertram Huppert. Endliche Gruppen I, Berlin: Springer-Verlag, 1967.

[Zie95] Günter M. Ziegler. *Lectures on polytopes.* Graduate Texts in Mathematics 152, Berlin: Springer-Verlag, 1995

[Zie00] Günter M. Ziegler, Lectures on 0/1-polytopes. In: Kalai, G. (ed.) et al., Polytopes - combinatorics and computation, DMV-seminar Oberwolfach, Germany, November 1997, Basel: Birkhäuser. DMV Semin. **29**, 1–41, 2000

Barbara Baumeister, Universität Bielefeld, Germany

E-mail address: b.baumeister@math.uni-bielefeld.de

Matthias Grüninger, Universität Würzburg, Germany

E-mail address: matthias.grueninger@mathematik.uni-wuerzburg.de