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The asymptotic number of integral cubic polynomials with bounded heights and discriminants

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Abstract

Let P denote a cubic integral polynomial, and let $D(P)$ and $H(P)$ denote the discriminant and height of P respectively. Let $N(Q,X)$ be the number of cubic integer polynomials P such that $H(P)\le Q$ and $|D(P)|\le X$. We obtain the asymptotic formula of $N(Q,X)$ for $Q^{14/5}\ll X\ll Q^4$ and as $Q\to\infty$. Using this result, for $0\le\eta\le 0.9$ we prove that

$$\sum_{\substack{H(P)\le Q\\ 1\le|D(P)|\ll Q^{4-\eta}}} |D(P)|^{-1/2} \asymp Q^{2-\frac{\eta}{3}}$$

for all sufficiently large Q, where the sum is taken over irreducible polynomials. This improves upon a result of Davenport who dealt with the case $\eta=0$. We also consider an application of the main theorem to some outstanding problems of transcendental number theory.

1 Introduction and results

Let $P(x)=a_nx^n+\cdots+a_1x+a_0\in\mathbb{Z}[x]$ denote a polynomial of degree n, let $H(P)=\max_{0\le i\le n}|a_i|$ denote the height of P, and let $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{C}$ denote the roots of P. The number

$$D(P)=a_n^{2n-2}\prod_{1\le i<j\le n}(\alpha_i-\alpha_j)^2 \tag{1}$$

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is called the discriminant of P. Properties of $D(P)$ where P is an integral polynomial have numerous applications in transcendental number theory. In particular, B. Volkmann's proof [19] of the cubic case of Mahler's conjecture [17] was based purely on a summation formula for discriminants proved by H. Davenport in [13]. The behavior of $D(P)$ is also closely related to the problem of the separation of conjugate algebraic numbers, which has been recently studied in some depth in [3, 8, 9, 14]. Recently lower bounds for the number of integral polynomials with given heights and discriminants (or a discriminant divisible by a large prime power) have been obtained in [6, 7].

Staying strictly within polynomials of degree 3, we obtain the asymptotic formula for the number of integral polynomials with bounded discriminants (Theorem 1 below). Using this asymptotics, we extend Davenport's summation formula (Theorem 2 below). In the last section we consider an application of our main result to finding the Hausdorff dimension of real numbers with a certain approximation property by cubic polynomials.

Throughout, $\#M$ denotes the number of elements in a set M, and $\mathrm{mes}_k\, M$ denotes the k–dimensional Lebesgue measure of a set $M \subset \mathbb{R}^n$ $(k \le n)$. We will also use the Vinogradov symbol $\ll$. The expression $f \ll g$ is equivalent to that the inequality $f \le cg$ holds for some absolute constant c. The expression $f \asymp g$ indicates that $g \ll f \ll g$. The expressions like $f \ll_{x_1,\dots,x_k} g$ or $f \asymp_{x_1,\dots,x_k} g$ mean that corresponding implicit constants depend only on parameters $x_1, \dots, x_k$.

Given $n \in \mathbb{N}$, $Q > 1$ and $v \ge 0$, define

$$\mathcal{P}_n(Q) = \{P \in \mathbb{Z}[x] : \deg P = n,\ H(P) \le Q\}, \tag{2}$$

$$\mathcal{P}_n(Q, v) = \{P \in \mathcal{P}_n(Q) : |D(P)| \le \gamma_n Q^{2n-2-2v}\}, \tag{3}$$

where the constant γ_n depends only on the degree n and is defined by

$$\gamma_n := \sup_{\substack{P \in \mathbb{Z}[x] \\ \deg P = n}} \frac{|D(P)|}{(H(P))^{2n-2}}.$$

Note, it is known (see [20]) that the discriminant $D(P)$ is a homogeneous polynomial of degree $2n-2$ in coefficients of P. Therefore, $\gamma_n < +\infty$, and $H(P) \le Q$ obviously implies $|D(P)| \le \gamma_n Q^{2n-2}$.

The following lower bound for $\#\mathcal{P}_n(Q, v)$ has been shown in [6] and [4]:

$$\#\mathcal{P}_n(Q, v) \gg_n Q^{n+1-2v}, \tag{4}$$

where $0 < v < \frac{1}{2}$. Using the recent results of Beresnevich [2] for the number of rational points near non-degenerate analytic manifolds in $\mathbb{R}^n$, the validity of (4) can also be extended to the range $0 < v < 1$. In [15] it was proved that $\#\mathcal{P}_3(Q, v) \ll Q^{4-5v/3}$. Heuristic arguments suggested that the estimate for $\#\mathcal{P}_3(Q, v)$ in (4) is the best possible up to a constant, and the result from [15] doesn't contradict it. However, the following main

result of this paper, which gives upper and lower bounds for $\#\mathcal{P}_3(Q,v)$, shows that this expectation is clearly wrong.

Let us define the following quantity

$$N(X) = N(Q,X) := \#\{P \in \mathcal{P}_3(Q) : |D(P)| \le X\}.$$

Theorem 1. *For X satisfying $0 \le X \le Q^4/27$, the following equality holds:*

$$N(Q,X) = \kappa\, Q^{2/3}\, X^{5/6} + O\left(X\left|\ln\left(Q^4/X\right)\right| + X + Q^3\right), \tag{5}$$

where κ is an absolute constant defined by

$$\kappa = (3^{4/3}-2)\cdot\frac{2^{7/3}}{\sqrt{3}}\left(2\int_1^\infty \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} + \int_{-1}^1 \sqrt{t^3+1}\,dt\right); \tag{6}$$

an implicit constant in the symbol $O(\cdot)$ is also absolute.

Corollary 1. *For any $v \in \left[0, \frac{3}{5}\right)$ and all sufficiently large Q, we have*

$$\#\mathcal{P}_3(Q,v) \asymp Q^{4-\frac{5}{3}v}, \tag{7}$$

where an implicit constant in the symbol $\asymp$ is absolute.

We shall use the result of Theorem 1 to prove the following generalization of Davenport's summation formula [13] for $|D(P)|^{-1/2}$.

Theorem 2. *For X satisfying $0 \le X \le Q^4/27$ and all sufficiently large Q, we have*

$$\sum_{\substack{H(P)\le Q\\ 1\le|D(P)|\le X}} |D(P)|^{-1/2} = \frac{7}{6}\kappa\, Q^{2/3}X^{1/3} + O\left(X^{1/2}\left|\ln\left(\frac{Q^4}{X}\right)\right| + X^{1/2} + Q^{1.7}\right), \tag{8}$$

where the summation is taken over irreducible polynomials P of degree 3; the constant κ is defined by (6); an implicit constant in the symbol $O(\cdot)$ is absolute.

Assuming $X = \gamma_3 Q^{4-\eta}$ in (8) we have Davenport's formula [13] corresponds to the case $\eta = 0$. For $0 < \eta < 0.9$ we obtain the following result.

Corollary 2. *Suppose that $0 \le \eta < 9/10$. Then for all sufficiently large Q*

$$\sum_{\substack{H(P)\le Q\\ 1\le|D(P)|\le \gamma_3 Q^{4-\eta}}} |D(P)|^{-1/2} \asymp Q^{2-\frac{\eta}{3}}, \tag{9}$$

where an implicit constant in the symbol $\asymp$ is absolute.

2 Auxiliary statements

For the proof of Theorem 1 we shall need the following lemmas. The expression $\|\mathbf{x}\|_\infty$ denotes the maximum norm of a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Lemma 1. *Let $f(\mathbf{x}) \in C(\mathbb{R}^n)$ and let $f(t\mathbf{x}) = t^d f(\mathbf{x})$ for all $t \in \mathbb{R}$, $t > 0$. Let*

$$G(\delta, R) = \{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \le \delta,\ \|\mathbf{x}\|_\infty \le R\}$$

and

$$\sigma(\delta) = \mathrm{mes}_{n-1}\{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \le \delta,\ \|\mathbf{x}\|_\infty = 1\}.$$

Then

$$\mathrm{mes}_n\, G(\delta, R) = \int_0^R r^{n-1}\sigma\left(\frac{\delta}{r^d}\right) dr. \tag{10}$$

Proof. Let us consider the subsets of $G(\delta, R)$:

$$G_i(\delta, R) = \{\mathbf{x} = (x_1, \dots, x_n) \in G(\delta, R) : |x_i| \ge |x_k|, i \ne k\}.$$

Obviously, we have

$$\mathrm{mes}_n\, G(\delta, R) = \sum_{i=1}^n \mathrm{mes}_n\, G_i(\delta, R).$$

Without loss of generality we consider the case of $G_1(\delta, R)$

$$\mathrm{mes}_n\, G_1(\delta, R) = \int_{G_1(\delta,R)} dx_1\, dx_2 \dots dx_n.$$

We change variables by formulas $|x_1| = r$, $x_i = r\theta_i$, $2 \le i \le n$. Jacobian of this transformation is equal to $\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(r, \theta_2, \dots, \theta_n)} = r^{n-1}$. Thus, we have

$$\mathrm{mes}_n\, G_1(\delta, R) = \int_0^R \left(\int_{\Gamma_1^+(\delta,r)} d\theta_2 \dots d\theta_n + \int_{\Gamma_1^-(\delta,r)} d\theta_2 \dots d\theta_n \right) r^{n-1} dr,$$

where

$$\Gamma_1^+(\delta, r) := \{(\theta_2, \dots, \theta_n) \in \mathbb{R}^{n-1} : |\theta_i| \le 1,\ |f(r, r\theta_2, \dots, r\theta_n)| \le \delta\},$$
$$\Gamma_1^-(\delta, r) := \{(\theta_2, \dots, \theta_n) \in \mathbb{R}^{n-1} : |\theta_i| \le 1,\ |f(-r, r\theta_2, \dots, r\theta_n)| \le \delta\}.$$

Let us denote $\sigma_i(\delta) = \mathrm{mes}_{n-1}\{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \le \delta,\ \|\mathbf{x}\|_\infty = 1,\ |x_i| = 1\}$. It is clear that $\sigma(\delta) = \sum_{i=1}^n \sigma_i(\delta)$. Since the function f is homogeneous, we obtain

$$\int_{\Gamma_1^+(\delta,r)} d\theta_2 \dots d\theta_n + \int_{\Gamma_1^-(\delta,r)} d\theta_2 \dots d\theta_n = \sigma_1\left(\frac{\delta}{r^d}\right).$$

The lemma is proved. □

Let us denote $\mathbf{p} = (p_n, \dots, p_1, p_0) \in \mathbb{R}^{n+1}$. Let us define a mapping $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $R\mathbf{p} = (p_0, \dots, p_{n-1}, p_n)$. There is an equivalent definition of the mapping R:

$$P(x) = \sum_{i=0}^{n} p_i x^i \longmapsto R(P)(x) = \sum_{i=0}^{n} p_{n-i} x^i = x^n P\left(\frac{1}{x}\right).$$

Lemma 2. *Let $D(\mathbf{p})$ denote a discriminant of a polynomial $P(x) = \sum_{i=0}^{n} p_i x^i$ as function of the coefficients of the polynomial. Then*

$$D(-\mathbf{p}) = D(\mathbf{p}), \tag{11}$$

$$D(R\mathbf{p}) = D(\mathbf{p}). \tag{12}$$

Proof. The functional equation (11) directly follows from (1).

We will prove the equation (12). It is easy to see that the polynomial $Q(x) = R(P)(x)$ has roots $\beta_i = 1/\alpha_i$, $1 \le i \le n$, and leading coefficient $q_n = p_0 = p_n \prod_{i=1}^{n} \alpha_i$. We put q_n and β_i, $1 \le i \le n$, into (1) and obtain the equation (12). Note, in the proof we assume that $\alpha_i \ne 0$, $1 \le i \le n$, i.e. $p_0 \ne 0$. But discriminant is a continuous function (polynomial) of the coefficients $\mathbf{p}$. Hence we have that the equation (12) is true in the case $p_0 = 0$. □

Given a polynomial

$$P(t) = zt^3 + yt^2 + xt + u \tag{13}$$

of degree 3, or binary cubic form

$$P(\tau, \chi) = z\tau^3 + y\tau^2\chi + x\tau\chi^2 + u\chi^3, \tag{14}$$

its discriminant is well known to be

$$D(P) = x^2y^2 - 4uy^3 - 27u^2z^2 - 4x^3z + 18uxyz. \tag{15}$$

Let us consider the discriminant surface given by $D(P) = \delta$ in the space $\mathbb{R}^4$; that is

$$\mathcal{S}(\delta) = \left\{(u, x, y, z) \in \mathbb{R}^4 : x^2y^2 - 4uy^3 - 27u^2z^2 - 4x^3z + 18uxyz = \delta\right\}. \tag{16}$$

Since the polynomial (15) is quadratic with respect to u, we may solve (16) with respect to u.

Lemma 3. *The surface $\mathcal{S}(\delta)$ given by* (16) *in the $\mathbb{R}^4$ has the explicit form*

$$\begin{aligned} u_1(x, y, z) &= u_1(x, y, z, \delta) = \tfrac{9xyz - 2y^3 - \sqrt{S - 27z^2\delta}}{27z^2}, \\ u_2(x, y, z) &= u_2(x, y, z, \delta) = \tfrac{9xyz - 2y^3 + \sqrt{S - 27z^2\delta}}{27z^2}, \end{aligned} \tag{17}$$

where u_1, u_2 are the two branches of the function u, and

$$S = S(x, y, z) := 4(y^2 - 3xz)^3. \tag{18}$$

The domain of definition of $u_1(x, y, z)$ and $u_2(x, y, z)$ is given by

$$S - 27z^2\delta \ge 0. \tag{19}$$

The following two Lemmas 4 and 5 will be used to obtain the lower bound.

Lemma 4. *Let*

$$M_z(\delta) = \left\{ (x,y) \in \mathbb{R}^2 : |x| \le 1 - \sqrt{\frac{|\delta|}{3}},\ |y| \le 1,\ S(x,y,1) \ge 27\delta \right\}. \tag{20}$$

Then $|u_{1,2}(x,y,1,\delta)| \le 1$ *for all* $(x,y) \in M_z(\delta)$ *for* $|\delta| \le 1/27$.

Proof. It is easy to see that

$$|u_{1,2}(x,y,1,\delta)| \le \frac{9|xy| + 2|y|^3 + \sqrt{S} + \sqrt{27|\delta|}}{27}.$$

The conditions $|x| \le 1$, $|y| \le 1$ imply the inequality $0 \le S \le 4(y^2+3|x|)^3 \le 16^2$. Hence, we have $|u_{1,2}(x,y,1,\delta)| \le \frac{9|x|+18+\sqrt{27|\delta|}}{27}$, and therefore, the condition $9|x|+\sqrt{27|\delta|} \le 9$ implies the desired bound $|u_{1,2}(x,y,1,\delta)| \le 1$. This condition is equivalent to $|x| \le 1 - \sqrt{\frac{|\delta|}{3}}$. The lemma is proved. □

Lemma 5. *Let*

$$M_y(\delta) = \left\{ (x,z) \in \mathbb{R}^2 : |x| \le 1,\ \frac{1}{3} \le |z| \le 1,\ \frac{2}{9} \le xz,\ S(x,1,z) \ge 27z^2\delta \right\}. \tag{21}$$

Then $|u_{1,2}(x,1,z,\delta)| \le 1$ *for all* $(x,z) \in M_y(\delta)$ *for* $|\delta| \le 1/27$.

Proof. It is easy to observe that

$$|u_{1,2}(x,1,z,\delta)| \le \frac{|9xz-2| + \sqrt{S} + \sqrt{27z^2|\delta|}}{27z^2},$$

where $S = 4(1-3xz)^3$.

From $S - 27z^2\delta \ge 0$, we get the inequality $3xz \le 1 - 3\cdot\sqrt[3]{\frac{z^2}{4}\delta}$. The inequality $xz \ge 2/9$ yields $S \le 4/27$.

$$|u_{1,2}(x,1,z,\delta)| \le \frac{1}{27z^2}\left(1 + \frac{2}{3\sqrt{3}} - 9\cdot\sqrt[3]{\frac{z^2\delta}{4}} + \sqrt{27z^2|\delta|}\right) \le$$

$$\le \frac{1}{3}\left(1 + \frac{2}{3\sqrt{3}}\right) + \sqrt[3]{\frac{3|\delta|}{4}} + \sqrt{\frac{|\delta|}{3}}.$$

For $|\delta| \le 1/27$ for all $(x,z) \in M_y(\delta)$ it holds $|u_{1,2}(x,1,z,\delta)| \le 1$. □

For any given $\mathbf{p} = (u,x,y,z) \in \mathbb{R}^4$ we write $D(\mathbf{p}) = x^2y^2 - 4uy^3 - 27u^2z^2 - 4x^3z + 18uxyz$ (cf. (15)).

Lemma 6. *Let* $\sigma(\delta) = \mathrm{mes}_3\{\mathbf{p} \in \mathbb{R}^4 : |D(\mathbf{p})| \le \delta,\ \|\mathbf{p}\|_\infty = 1\}$. *Then for* $0 < \delta \le 1/27$ *we have*

$$\sigma(\delta) = c_1\delta^{5/6} + O(\delta), \tag{22}$$

where c_1 *is an absolute constant, which doesn't depend on* δ*; an implicit constant in the symbol* $O(\cdot)$ *is also absolute.*

Proof. Let $\delta > 0$, and let $\mathbf{p} = (u, x, y, z)$.

The properties of discriminant (see Lemma 2) imply that we need to consider only two faces $z = 1$ and $y = 1$ of the box $\|\mathbf{p}\|_\infty = 1$. Let us define

$$\sigma_z(\delta) = \mathrm{mes}_3\{\mathbf{p} \in \mathbb{R}^4 : |D(\mathbf{p})| \le \delta,\ \|\mathbf{p}\|_\infty = 1,\ z = 1\},$$
$$\sigma_y(\delta) = \mathrm{mes}_3\{\mathbf{p} \in \mathbb{R}^4 : |D(\mathbf{p})| \le \delta,\ \|\mathbf{p}\|_\infty = 1,\ y = 1\}.$$

Then

$$\sigma(\delta) = 4(\sigma_z(\delta) + \sigma_y(\delta)).$$

Obviously, we have

$$\sigma_z(\delta) = \iiint_{\mathcal{D}_z} du\,dx\,dy, \qquad \sigma_y(\delta) = \iiint_{\mathcal{D}_y} du\,dx\,dz,$$

where

$$\mathcal{D}_z := \{(u, x, y) \in \mathbb{R}^3 : |D(u, x, y, 1)| \le \delta,\ \max\{|u|, |x|, |y|\} \le 1\},$$
$$\mathcal{D}_y := \{(u, x, z) \in \mathbb{R}^3 : |D(u, x, 1, z)| \le \delta,\ \max\{|u|, |x|, |z|\} \le 1\}.$$

For $\delta > 0$ we consider the auxiliary function

$$h(\delta) = h(x, y, z, \delta) = \begin{cases} (u_1(\delta) - u_1(-\delta)) + (u_2(-\delta) - u_2(\delta)), & S > 27z^2\delta, \\ u_2(-\delta) - u_1(-\delta), & |S| \le 27z^2\delta, \end{cases}$$

where $u_j(\delta) = u_j(x, y, z, \delta)$, $j = 1, 2$.

In the notation introduced above, using Lemmas 4 and 5, we have

$$\iint_{M_z} h(x, y, 1, \delta)\,dx\,dy \le \sigma_z(\delta) \le \iint_{G_z} h(x, y, 1, \delta)\,dx\,dy,$$
$$\iint_{M_y} h(x, 1, z, \delta)\,dx\,dz \le \sigma_y(\delta) \le \iint_{G_y} h(x, 1, z, \delta)\,dx\,dz,$$

where

$$G_z = G_z(\delta) := \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} \le 1,\ S(x, y, 1) \ge -27\delta\},$$
$$G_y = G_y(\delta) := \{(x, z) \in \mathbb{R}^2 : \max\{|x|, |z|\} \le 1,\ S(x, 1, z) \ge -27z^2\delta\}.$$

and

$$M_z := M_z(-\delta) = \left\{(x,y) \in G_z(\delta) : |x| \le 1 - \sqrt{\frac{\delta}{3}}\right\},$$
$$M_y := M_y(-\delta) = \left\{(x,z) \in G_y(\delta) : \frac{1}{3} \le |z|,\ \frac{2}{9} \le xz\right\}.$$

Now, we are ready to prove asymptotic formulas for $\sigma_z(\delta)$ and $\sigma_y(\delta)$. Note that the function $h(x,y,z,\delta)$ has the form

$$h(x,y,z,\delta) = \begin{cases} \dfrac{4\delta}{\sqrt{S+27z^2\delta}+\sqrt{S-27z^2\delta}}, & \text{if } S > -27z^2\delta, \\ \dfrac{2\sqrt{S+27z^2\delta}}{27z^2}, & \text{if } |S| \le 27z^2\delta, \end{cases} \tag{23}$$

where the function $S = S(x,y,z)$ is defined by (18).

In calculations of integrals we will use the substitution $\frac{S}{27z^2\delta} = t^3$, which leads to the following equalities:

$$\frac{1}{3}\sqrt[3]{\frac{4}{z^2\delta}}\,(y^2 - 3xz) = t; \qquad x = \frac{y^2}{3z} - \sqrt[3]{\frac{\delta}{4z}}\,t; \qquad dx = -\sqrt[3]{\frac{\delta}{4z}}\,dt. \tag{24}$$

Lemma 7. *For $0 < \delta \le 1/27$, we have*

$$\sigma_z(\delta) = c_z \cdot \delta^{5/6} + O(\delta), \tag{25}$$

where

$$c_z = \frac{4^{2/3}}{\sqrt{27}}\left(2\int_1^\infty \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} + \int_{-1}^1 \sqrt{t^3+1}\,dt\right). \tag{26}$$

Proof. Firstly, we calculate the integral $\iint_{G_z} h(x,y,1,\delta)\,dx\,dy$. Accordingly to (23), we divide the domain G_z into two subdomains $G_z = G_z^{(1)} \sqcup G_z^{(2)}$, where

$$G_z^{(1)} = \{(x,y) \in G_z : S(x,y,1) > 27\delta\},$$
$$G_z^{(2)} = \{(x,y) \in G_z : |S(x,y,1)| \le 27\delta\}.$$

For the domain $G_z^{(1)}$ we have $I_z^{(1)} := \iint_{G_z^{(1)}} h(x,y,1,\delta)dxdy$. We apply the substitution (24). After this transformation we obtain

$$I_z^{(1)} = \delta^{\frac{5}{6}} \cdot \frac{2\cdot 4^{\frac{2}{3}}}{\sqrt{27}} \int_0^1 dy \int_1^{\tau(y)} \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}},$$

where $\tau(y) := \frac{1}{3}\sqrt[3]{\frac{4}{\delta}}(y^2+3)$.

Since $\tau(y) \geq \tau_0 := \sqrt[3]{\frac{4}{\delta}}$ and $\int_{\tau(y)}^{\infty} \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} \leq \int_{\tau_0}^{\infty} t^{-3/2}dt = 2^{2/3}\delta^{1/6}$, we have the following asymptotics for $I_z^{(1)}$:

$$I_z^{(1)} = \delta^{\frac{5}{6}} \cdot \frac{2 \cdot 4^{\frac{2}{3}}}{\sqrt{27}} \int_1^{\infty} \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} + O(\delta). \tag{27}$$

Applying (24) yields

$$I_z^{(2)} := \iint_{G_z^{(2)}} h(x,y,1,\delta)dxdy = \delta^{\frac{5}{6}} \cdot \frac{4^{\frac{2}{3}}}{\sqrt{27}} \int_{-1}^{1} \sqrt{t^3+1}\, dt \tag{28}$$

for the domain $G_z^{(2)}$.

Therefore, we have

$$\iint_{G_z} h(x,y,1,\delta)\, dx\, dy = c_z \cdot \delta^{5/6} + O(\delta),$$

where c_z is defined by (26).

Since

$$\mathrm{mes}_2(G_z \setminus M_z) = O(\delta^{1/2}), \qquad \sup_{(x,y)\in G_z\setminus M_z} h(x,y,1,\delta) = O(\delta^{1/2}),$$

we obtain

$$\iint_{G_z\setminus M_z} h(x,y,1,\delta)\, dx\, dy = O(\delta).$$

Thus, Lemma 7 is proved. □

Lemma 8. *For $0 < \delta \leq 1/27$, we have*

$$\sigma_y(\delta) = c_y \cdot \delta^{5/6} + O(\delta), \tag{29}$$

where

$$c_y = 3(\sqrt[3]{3}-1) \cdot \frac{4^{2/3}}{\sqrt{27}} \left(2\int_1^{\infty} \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} + \int_{-1}^{1} \sqrt{t^3+1}\, dt \right). \tag{30}$$

Proof. Let us calculate the integral $\iint_{G_y} h(x,1,z,\delta)\, dx\, dz$. According to (23), for the domain G_y we have $G_y = G_y^{(1)} \sqcup G_y^{(2)}$, where

$$G_y^{(1)} = \{(x,z) \in G_y : S(x,1,z) > 27z^2\delta\},$$
$$G_y^{(2)} = \{(x,z) \in G_y : |S(x,1,z)| \leq 27z^2\delta\}.$$

We consider the domain $G_y^{(1)}$. Let us apply the substitutions (24) to the integral

$$I_y^{(1)} := \iint_{G_y^{(1)}} h(x,1,z,\delta)dxdz = 8\delta \int_0^1 dz \int_{x_1(z)}^{x_2(z)} \frac{dx}{\sqrt{S+27z^2\delta}+\sqrt{S-27z^2\delta}},$$

where

$$x_1(z) = \begin{cases} \frac{1}{3z} - \sqrt[3]{\frac{\delta}{4z}}, & -1 \le z \le -z_\delta, \\ -1, & -z_\delta < z \le 1, \end{cases}$$

$$x_2(z) = \begin{cases} 1, & -1 \le z < z_\delta, \\ \frac{1}{3z} - \sqrt[3]{\frac{\delta}{4z}}, & z_\delta \le z \le 1, \end{cases}$$

and $z_\delta \in [0,1]$ is the real solution of the equation $\frac{1}{3z} - \sqrt[3]{\frac{\delta}{4z}} = 1$, which is equivalent to

$$\delta = \frac{4}{z^2}\left(\frac{1}{3} - z\right)^3. \tag{31}$$

Note that $z_\delta \in \left[\frac{1}{7}, \frac{1}{3}\right]$ for $\delta \in [0,1]$, and $\lim_{\delta\to 0} z_\delta = \frac{1}{3}$.

After the substitutions (24) the limits of integration are given by

$$t_1(z) = \begin{cases} 1, & -1 \le z \le -z_\delta, \\ \frac{1}{3}\sqrt[3]{\frac{4}{z^2\delta}}(1+3z), & -z_\delta < z \le 1; \end{cases}$$

$$t_2(z) = \begin{cases} \frac{1}{3}\sqrt[3]{\frac{4}{z^2\delta}}(1-3z), & -1 \le z < z_\delta, \\ 1, & z_\delta \le z \le 1. \end{cases}$$

Hence the integral $I_y^{(1)}$ may be rewritten as

$$I_y^{(1)} = \delta^{\frac{5}{6}} \cdot \frac{2\cdot 4^{\frac{2}{3}}}{\sqrt{27}} \int_0^1 \frac{dz}{z\sqrt[3]{z}} \int_{t_2(z)}^{t_1(z)} \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}}.$$

The right hand side of the integral can be written as a sum of two integrals, which can be estimated as follows:

$$J_1 := \int_0^{z_\delta} \frac{dz}{z\sqrt[3]{z}} \int_{t_2(z)}^{t_1(z)} \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} \le \int_0^{z_\delta} \frac{dz}{z\sqrt[3]{z}} \int_{t_2(z)}^{t_1(z)} \frac{dt}{\sqrt{t^3}} \le$$

$$\le \delta^{\frac{1}{6}} \cdot 6\sqrt[3]{4}\sqrt{3} \int_0^{\frac{1}{3}} \frac{dz}{\sqrt{1-9z^2}\cdot(\sqrt{1+3z}+\sqrt{1-3z})},$$

$$J_2 := \int_{z_\delta}^{1} \frac{dz}{z\sqrt[3]{z}} \int_{t_2(z)}^{t_1(z)} \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} = \\ = 3(z_\delta^{-1/3}-1)\int_1^\infty \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} + O(\delta^{1/6}).$$

It is easy to obtain from (31) that $\left| z_\delta^{-1/3} - \sqrt[3]{3} \right| = O(\delta^{1/3})$.

Thus,

$$I_y^{(1)} = \delta^{\frac{5}{6}} \cdot \frac{2\cdot 4^{\frac{2}{3}}}{\sqrt{27}}(J_1+J_2) = \\ = \delta^{\frac{5}{6}} \cdot \frac{2\cdot 4^{\frac{2}{3}}}{\sqrt{27}}\, 3(\sqrt[3]{3}-1)\int_1^\infty \frac{dt}{\sqrt{t^3+1}+\sqrt{t^3-1}} + O(\delta). \quad (32)$$

For the domain $G_y^{(2)}$ we have

$$I_y^{(2)} := \iint_{G_y^{(2)}} h(x,1,z,\delta)dxdz = 4\int_{z_\delta}^1 dz \int_{x_1(z)}^{x_2(z)} \frac{\sqrt{S+27z^2\delta}}{27z^2}dx,$$

where $x_1(z) = \frac{1}{3z} - \sqrt[3]{\frac{\delta}{4z}}$ and $x_2(z) = \min\left\{\frac{1}{3z} + \sqrt[3]{\frac{\delta}{4z}},\ 1\right\}$.

Let us apply the substitution (24) to the integral $I_y^{(2)}$. Since $\left| z_{\pm\delta} - \frac{1}{3} \right| = O(\delta^{1/3})$, $|x_2(z) - x_1(z)| = O(\delta^{1/3})$ for $z \geq z_\delta$, and $\sup_{(x,z)\in G_y^{(2)}} h(x,1,z,\delta) = O(\delta^{1/2})$, we get the following asymptotics for $I_y^{(2)}$

$$I_y^{(2)} = \delta^{\frac{5}{6}} \cdot \frac{4^{\frac{2}{3}}}{\sqrt{27}} \cdot 3(\sqrt[3]{3}-1)\int_{-1}^1 \sqrt{t^3+1}\, dt + O(\delta^{7/6}). \quad (33)$$

From (32) and (33), we get

$$\iint_{G_y} h(x,1,z,\delta)\, dx\, dz = c_y \cdot \delta^{5/6} + O(\delta),$$

where c_y is defined by (30).

Since

$$\mathrm{mes}_2(G_y \setminus M_y) = O(1), \qquad \sup_{(x,z)\in G_y\setminus M_y} h(x,1,z,\delta) = O(\delta),$$

we obtain

$$\iint_{G_y\setminus M_y} h(x,1,z,\delta)\, dx\, dz = O(\delta).$$

Hence, the proof of Lemma 8 is completed. □

Using Lemmas 7 and 8, we obtain the equality

$$\sigma(\delta) = c_1 \cdot \delta^{\frac{5}{6}} + O(\delta), \tag{34}$$

where $c_1 := 4(c_z + c_y)$, and c_z, c_y are defined by (26), (30).

This is the desired equation, and Lemma 6 is proved. □

3 Proof of Theorem 1

Lemma 9. *Let*

$$V_3(\delta) = \{P \in \mathbb{R}[x] : \deg P = 3,\ H(P) \le 1,\ |D(P)| \le \delta\}.$$

Then for $0 < \delta \le 1/27$

$$\operatorname{mes}_4 V_3(\delta) = \kappa \cdot \delta^{5/6} + O(\delta\,|\ln\delta| + \delta), \tag{35}$$

where $\kappa = \frac{3}{2}c_1$*, and* c_1 *is the same as in* (22)*; an implicit constant in the symbol* $O(\cdot)$ *is absolute.*

Proof. Applying Lemma 1, we have

$$\operatorname{mes}_4 V_3(\delta) = \int_0^1 r^3\,\sigma\left(\frac{\delta}{r^4}\right)\,dr.$$

Using Lemma 6, we have that the asymtotic formula $\sigma\left(r^{-4}\delta\right) = c_1 r^{-10/3}\delta^{5/6} + O(r^{-4}\delta)$ holds on the interval $[r_0, 1]$, where $r_0 := (27\delta)^{1/4}$. Thus, we obtain

$$\int_{r_0}^1 r^3\,\sigma\left(\frac{\delta}{r^4}\right)\,dr = \frac{3}{2}c_1\delta^{5/6} + O(\delta\,|\ln\delta| + \delta).$$

On the interval $[0, r_0)$ we shall use the trivial bound $\sigma\left(r^{-4}\delta\right) \ll 1$. Hence, we have $\int_0^{r_0} r^3\,\sigma\left(\frac{\delta}{r^4}\right)\,dr = O(\delta)$. The lemma is proved. □

Theorem 3 ([10])**.** *Let* $\mathcal{D} \subset \mathbb{R}^d$ *be a bounded region consisting of all points* $(x_1, \dots, x_d)$ *that satisfy all of a finite set of algebraic inequalities*

$$F_i(x_1, \dots, x_d) \ge 0, \qquad 1 \le i \le k,$$

where F_i *is a polynomial with real coefficients of degree* $\deg F_i \le m$*. Let*

$$\Lambda(\mathcal{D}) = \mathcal{D} \cap \mathbb{Z}^d.$$

Then

$$|\#\Lambda(\mathcal{D}) - \operatorname{mes}_d \mathcal{D}| \le C \max(\bar{V}, 1),$$

where C *depends only on* d*,* k*,* m*, and* $\bar{V}$ *is the greatest* r*–dimensional measure of any projection of* $\mathcal{D}$ *on a coordinate space,* $1 \le r \le d-1$*.*

Let $Q \cdot \mathcal{S}$ denote a set obtained by uniform scaling of a set $\mathcal{S}$ in Q times.

It is easy to see that

$$N(Q, X) = \#\Lambda \left(Q \cdot V_3(X/Q^4) \right).$$

Assuming $\delta = X/Q^4$ and $\mathcal{D} = Q \cdot V_3(\delta)$ in Theorem 3, we obtain

$$N(Q, X) = Q^4 \cdot \mathrm{mes}_4 \, V_3(X/Q^4) + O(Q^3).$$

Now, Theorem 1 follows from Lemma 9.

By assuming $X = \gamma_3 Q^{4-2v}$ we obtain Corollary 1. Here, the bounds for v are a direct consequence of the condition $Q^3 \ll Q^{2/3} X^{5/6}$, which proves Corollary 1.

4 Proof of Theorem 2

Let us define the following sets of polynomials

$$\mathcal{P}_n^*(Q) = \{P \in \mathcal{P}_n(Q) : P \text{ is irreducible over } \mathbb{Q}\}, \tag{36}$$
$$\mathcal{P}_n^{**}(Q) = \{P \in \mathcal{P}_n(Q) : P \text{ is reducible over } \mathbb{Q}\}. \tag{37}$$

Obviously, we have $\mathcal{P}_n^{**}(Q) = \mathcal{P}_n(Q) \backslash \mathcal{P}_n^*(Q)$.

Lemma 10 ([16, Lemma 1])**.** *The number of reducible polynomials $\#\mathcal{P}_n^{**}(Q)$ has the following order:*

$$\#\mathcal{P}_n^{**}(Q) \asymp_n \begin{cases} Q^n, & n \geq 3, \\ Q^2 \ln Q, & n = 2. \end{cases}$$

Let us define the following functions

$$\nu(X) = \nu(Q, X) := \#\{P \in \mathcal{P}_3^*(Q) : |D(P)| = X\},$$
$$N^*(X) = N^*(Q, X) := \#\{P \in \mathcal{P}_3^*(Q) : |D(P)| \leq X\} = \sum_{1 \leq d \leq X} \nu(d),$$
$$s(X) = s(Q, X) := \sum_{\substack{P \in \mathcal{P}_3^*(Q) \\ |D(P)| \leq X}} |D(P)|^{-1/2}.$$

Firstly, we shall obtain the upper bounds for the functions $N^*(X)$ and $s(X)$, which follow directly from Davenport's results [13].

Lemma 11. *The following upper bounds hold for the functions $N^*(Q, X)$ and $s(Q, X)$:*

$$N^*(Q, X) \ll Q \cdot X^{3/4}, \tag{38}$$
$$s(Q, X) \ll Q \cdot X^{1/4}. \tag{39}$$

Proof. The sum $s(X)$ may be written in form

$$s(X) = \sum_{1 \le d \le X} \nu(d) d^{-1/2}.$$

To apply Davenport's result we need to introduce some terminology and notations. Following [11], two binary cubic forms with integral coefficients are said to be **properly equivalent**, if one can be transformed into the other by a linear substitution with integral coefficients and determinant 1.

Let $h(D)$ denote the number of classes of properly equivalent irreducible binary cubic forms that have discriminant D.

Using the formula (5) from [13], we obtain

$$\nu(d) \ll Q \cdot (h(d) + h(-d)) \cdot d^{-1/4}.$$

This result yields the following estimates for $N^*(X)$ and $s(X)$

$$N^*(X) \ll Q \sum_{1 \le d \le X} (h(d) + h(-d)) \cdot d^{-1/4},$$
$$s(X) \ll Q \sum_{1 \le d \le X} (h(d) + h(-d)) \cdot d^{-3/4}.$$

It was proved in [11, 12] (see formulas (3) and (1) respectively) that

$$\sum_{1 \le |d| \le X} h(d) \ll X.$$

Thus, by partial summation we obtain the bounds (38) and (39). Note that this lemma is a direct extension of Davenport's result (see formula (3) from [13]). □

Now using Theorem 1, we shall get the asymtotic formula for the function $N^*(X)$.

Lemma 12. *For X satisfying $c_2 Q^{14/5} \le X \le \gamma_3 Q^4$, where c_2 is an absolute constant, and sufficiently large Q, the function $N^*(Q, X)$ has the asymptotic formula:*

$$N^*(Q, X) = \kappa\, Q^{2/3}\, X^{5/6} + O\left(X \left|\ln\left(Q^4/X\right)\right| + X + Q^3\right), \tag{40}$$

where the absolute constant κ is the same as in Theorem 1.

Proof. Lemma 10 gives $\#\mathcal{P}_3^{**}(Q) \asymp Q^3$. Thus, with the use of Theorem 1, we have (40). □

We will denote

$$S(Q,X)=\sum_{\substack{P\in\mathcal{P}_3^*(Q)\\ |D(P)|\le X}}|D(P)|^{-1/2}.$$

It naturally follows that

$$S(Q,X)=\sum_{1\le d\le X}\nu(d)d^{-1/2}.$$

The sum can be now split into two parts: $S(Q,X)=S_1(Q)+S_2(Q,X)$, where

$$S_1(Q)=\sum_{1\le d\le c_2Q^{14/5}}\nu(d)d^{-1/2},\qquad S_2(Q,X)=\sum_{c_2Q^{14/5}<d\le X}\nu(d)d^{-1/2}.$$

The upper bound for $S_1(Q)$ follows from (39)

$$S_1(Q)=s(Q,c_2Q^{14/5})\ll Q^{17/10}.$$

Let us estimate $S_2(Q,X)$ by partial summation. We have

$$\sum_{d=D_1}^{D_2}\nu(d)d^{-1/2}=\sum_{d=D_1}^{D_2}(N^*(d)-N^*(d-1))d^{-1/2}=$$

$$=\frac{N^*(D_2)}{\sqrt{D_2+1}}-\frac{N^*(D_1-1)}{\sqrt{D_1}}+\sum_{d=D_1}^{D_2}N^*(d)(d^{-1/2}-(d+1)^{-1/2}),$$

For any $d\ge 2$ we have

$$(d+1)^{-1/2}=d^{-1/2}-\frac{1}{2}d^{-3/2}+O(d^{-5/2}).$$

Assumming $D_1=c_2Q^{14/5}$ and $D_2=X$, we obtain:

$$\frac{N^*(X)}{\sqrt{X+1}}=\kappa\,Q^{2/3}\,X^{1/3}+O\left(X^{1/2}\left|\ln\left(Q^4/X\right)\right|+X^{1/2}+Q^3X^{-1/2}\right),$$

$$\frac{N^*(D_1-1)}{\sqrt{D_1}}\ll Q^{3-7/5}\ll Q^{8/5}.$$

Using Lemma 12, we obtain

$$\sigma=\sum_{d=D_1}^{D_2}N^*(d)(d^{-1/2}-(d+1)^{-1/2})=\frac{\kappa}{2}Q^{2/3}\cdot\sum_{d=D_1}^{D_2}d^{-2/3}+$$

$$+O\left(\sum_{d=D_1}^{D_2}\left(d^{-1/2}\left|\ln\left(Q^4/d\right)\right|+d^{-1/2}+Q^3d^{-3/2}+Q^{2/3}d^{-5/3}\right)\right).$$

Since

$$\sum_{d=D_1}^{X} d^{-2/3} = \frac{1}{3}X^{1/3} + O(D_1^{1/3} + X^{-2/3}),$$

we have

$$\sigma = \frac{\kappa}{6}Q^{2/3}X^{1/3} + O\left(X^{1/2}\left|\ln\left(Q^4/X\right)\right| + X^{1/2} + Q^{8/5}\right).$$

Theorem 2 is proved.

5 Some applications

In this section, we give an illustration how some results on Hausdorff's dimension and Mahler's Problem for cubic polynomials could be obtained from our estimates. These results are well-known and have been solved [1, 5, 19], but in due time, it was hard problems. We show here how these problems could be solved in cubic case by simple using of our result.

Mahler's Conjecture [18]. *Let $\mathcal{L}_n(w)$ be the set of real numbers such that the inequality*

$$|P(x)| < H(P)^{-w},\ w > n, \tag{41}$$

has infinitely many solutions in integral polynomials with $\deg P \le n$. *Then*

$$\mathrm{mes}_1\, \mathcal{L}_n(w) = 0.$$

Let $x \in \mathcal{L}_3(w)$, and $P \in \mathbb{Z}[x]$, $\deg P \le n$, be a solution of (41). If α_1 is the root of $P(x)$ closest to x, then it is known (see [18]) that

$$|x - \alpha_1| < 6H(P)^{-w}|D(P)|^{-\frac{1}{2}}. \tag{42}$$

Let $\mathcal{L}(t, w)$ be the set of real numbers x such that the inequality (41) has solutions in polynomials belonging to the class

$$\mathcal{P}_t = \{P \in \mathcal{P}_3^*(2^t)\ :\ 2^{t-1} < H(P)\},$$

where the set $\mathcal{P}_3^*(Q)$ is defined according to (36).

Let us cover the set $\mathcal{L}(t, w)$ by intervals $I_\alpha = \{x \in \mathbb{R} : |x - \alpha| < 6H(P)^{-w}|D(P)|^{-\frac{1}{2}}\}$, where $P \in \mathcal{P}_t$, and α is a real root of P. Let us consider the series

$$S_1 := \sum_{t=1}^{\infty} \sum_{\substack{P\in\mathcal{P}_t \\ c_2 2^{2.8t} < |D(P)|}} \left(2^{-(t-1)w}|D(P)|^{-\frac{1}{2}}\right)^{\frac{4}{w+1}} \ll$$

$$\ll \sum_{t=1}^{\infty} 2^{-\frac{4w}{w+1}t} \sum_{c_2 2^{2.8t} < D \le \gamma_3 2^{4t}} N^*(D) D^{-1-\frac{2}{w+1}}.$$

By Lemma 12, we have $N^*(2^t, D) \ll 2^{2t/3} D^{5/6}$. This implies

$$S_1 \ll \sum_{t=1}^{\infty} 2^{t\left(-\frac{4w}{w+1}+4-\frac{8}{w+1}\right)} = \sum_{t=1}^{\infty} 2^{-\frac{4}{w+1}t} < \infty.$$

Considering the case when $|D(P)| \le c_2 2^{2.8t}$, we use the trivial upper bound $N^*(2^t, D) \ll 2^{3t}$. Hence, the series

$$S_2 := \sum_{t=1}^{\infty} \sum_{\substack{P\in\mathcal{P}_t \\ |D(P)|\le c_2 2^{2.8t}}} \left(2^{-(t-1)w}|D(P)|^{-\frac{1}{2}}\right)^{\frac{4}{w+1}} \ll$$

$$\ll \sum_{t=1}^{\infty} 2^{-\frac{4w}{w+1}t} \sum_{1\le D\le c_2 2^{2.8t}} 2^{3t} D^{-1-\frac{2}{w+1}} \ll \sum_{t=1}^{\infty} 2^{\frac{3-w}{w+1}t}$$

converges for $w > n$.

Thus, for the Hausdorff dimension we have $\dim_H \mathcal{L}_3(w) \le \frac{4}{w+1}$ (that agrees with the fact that $\dim_H \mathcal{L}_n(w) = \frac{n+1}{w+1}$, see [1, 5]). This leads to $\mathrm{mes}_1 \mathcal{L}_3(w) = 0$.

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