DISTRIBUTION OF COMPLEX ALGEBRAIC NUMBERS

FRIEDRICH GÖTZE, DZIANIS KALIADA, AND DMITRY ZAPOROZHETS

Abstract. For a region $\Omega \subset \mathbb{C}$ denote by $\Psi(Q;\Omega)$ the number of complex algebraic numbers in Ω of degree $\le n$ and naive height $\le Q$. We show that

$$\Psi(Q;\Omega) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_\Omega \psi(z)\,\nu(dz) + O\left(Q^n\right), \quad Q \to \infty,$$

where ν is the Lebesgue measure on the complex plane and the function ψ will be given explicitly.

1. Introduction

Baker and Schmidt [2] showed that the set of *real* algebraic numbers of degree at most n forms a *regular system*: there exists a constant c_n depending on n only such that for any interval $I \subset \mathbb{R}$ for all sufficiently large $Q \in \mathbb{N}$ there exist at least

$$c_n |I|\, Q^{n+1}/(\log Q)^{3n(n+1)}$$

algebraic numbers $\alpha_1, \dots, \alpha_k$ of degree at most n and height at most Q satisfying

$$|\alpha_i - \alpha_j| \ge (\log Q)^{3n(n+1)}/Q^{n+1}, \quad 1 \le i < j \le k.$$

Later Beresnevich [3] proved that the logarithmic factors can be omitted. Bernik and Vasil'ev [5] showed that the analogous result holds for the set of *complex* algebraic numbers. For a more detailed discussion of the literature we refer to an excellent survey monograph by Bugeaud [8].

Note that the number of algebraic numbers of degree at most n and height at most Q is of order Q^{n+1} as $Q \to \infty$. Thus the results of such type show that for any fixed n the algebraic numbers of sufficiently large height are distributed quite regularly. However they describe the behaviour of a part of algebraic numbers only.

In 1971 Brown and Mahler [6] introduced a natural generalization of the Farey sequences: the Farey sequence of degree n and order Q is the set of all real roots of integral polynomials of degree n and height at most Q. An important question in this respect had been asked by Mahler in his letter to Sprindžuk in 1985: what is the distribution of algebraic numbers of a fixed degree $n \ge 2$?

The following answer to this question was suggested in [22] (see also [21], [23] for the case $n = 2$). Fix $n \ge 2$ and consider an arbitrary interval $I \subset \mathbb{R}$. Denote by $\Phi(Q;I)$ the number of real algebraic numbers in I of degree at most n and height at most Q. Then

$$\Phi(Q;I) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_I \varphi(x)\,dx + O\left(Q^n \log^{l(n)} Q\right), \quad Q \to \infty, \tag{1}$$

2010 *Mathematics Subject Classification.* 11N45 (primary), 11C08 (secondary).

Key words and phrases. algebraic numbers, distribution of algebraic numbers, generalized Farey sequences, integral polynomials.

Supported by CRC 701, Bielefeld University (Germany).

where $\zeta(\cdot)$ denotes the Riemann zeta function and $l(n)$ is defined by

$$l(n) = \begin{cases} 1, & n = 2, \\ 0, & n \geq 3. \end{cases}$$

The limit density φ is given by the formula

$$\varphi(x) = \int_{B_x} \left| \sum_{k=1}^{n} k t_k x^{k-1} \right| dt_1 \dots dt_n,$$

where the domain B_x is defined by

$$B_x = \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n \; : \; \max_{1 \leq k \leq n} |t_k| \leq 1, \; |t_n x^n + \dots + t x| \leq 1 \right\}.$$

If $x \in [-1 + 1/\sqrt{2}, 1 - 1/\sqrt{2}]$, then $\varphi(x)$ can be simplified as follows:

$$\varphi(x) = \frac{2^{n-1}}{3} \left(3 + \sum_{k=1}^{n-1} (k+1)^2 x^{2k} \right).$$

In [18] this result was generalized to the limit correlations between real algebraic conjugates. Note in passing that $2^{-n-1}\varphi$ coincides with the density of the real roots of a random polynomial with independent coefficients uniformly distributed on $[-1, 1]$ (see, e.g., [17, Section 3]).

The aim of this note is to obtain a *complex* counterpart of (1): we are interested in the distribution of the "complex" Farey sequence of degree n and large order Q. The real and complex cases are quite different from each other. In particular, the result for the non-real numbers can not be deduced from the real case. Therefore we use a different approach to solve this problem.

Let us give a very brief overview of some related works. There are numerous papers studying the distribution of distances between algebraic conjugates. In this active research area, notable results were obtained in [11], [16], [4], [12], [9], [10]. This problem is closely related with the problem of the distribution of polynomial discriminants [7], [19].

In [4] Beresnevich, Bernik and Götze obtained the following result. Let $n \geq 2$ and $0 < \rho \leq \frac{n+1}{3}$. Then for all sufficiently large Q and any interval $I \subset [-\frac{1}{2}, \frac{1}{2}]$ there exist at least $\frac{1}{2} Q^{n+1-2\rho} |I|$ real algebraic numbers α of degree n and height $H(\alpha) \asymp_n Q$ having a real conjugate α^* such that $|\alpha - \alpha^*| \asymp_n Q^{-\rho}$.

Using potential theory Pritsker [27] considered the case when $n \to \infty$ and found the asymptotic distribution of the roots of an integral polynomial whose generalized Mahler measure satisfies some conditions. As a corollary he obtained the solution of Schur's problem on traces of algebraic numbers. The paper [27] also contains a number of references on this subject. Pritsker's results are closely related to the problem of the distribution of the complex roots of random polynomials with i.i.d. coefficients when $n \to \infty$. The landmark result of Erdős and Turán [15] implies that the arguments of the complex roots are asymptotically uniformly distributed (see [20] for the proof without any additional assumption). Moreover, under some quite general assumptions the roots are clustered near the unit circle (see [30], [20]).

Some papers are devoted to the asymptotic behavior of the number of algebraic elements α of a fixed degree n and a bounded multiplicative Weil height $\mathcal{H}(\alpha) \leq X$

over some base number field (as X tends to infinity). Let $\overline{\mathbb{Q}}_n(X)$ denote the number of such elements over the field of rational numbers $\mathbb{Q}$. Masser and Vaaler [26] established the following asymptotic formula using a result by Chern and Vaaler [13]:

$$\overline{\mathbb{Q}}_n(X) = \sigma_n X^{n(n+1)} + O\left(X^{n^2} \log^{l(n)} X\right), \quad X \to \infty,$$

where the explicit factor σ_n and the implicit big-O-notation constant depend on n only. Here the Weil height $\mathcal{H}(\alpha)$ can be expressed in terms of the Mahler measure by $\mathcal{H}(\alpha) = M(\alpha)^{1/n}$. Note that X is of order $Q^{1/n}$, where Q is the upper bound for the corresponding naive heights. In [25] Masser and Vaaler generalized this result to arbitrary base number fields. References and some historical results related to the topic can be found in [24, Chapter 3, §5]. Note that these results are based on the use of the Weil height and do not overlap with ours.

2. Main result

For an integral polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_n \neq 0,$$

its height is defined as $H(p) := \max_{0 \le i \le n} |a_i|$.

A *minimal polynomial* p of an algebraic number α is an integral nonzero polynomial of the minimal degree with coprime coefficients such that $p(\alpha) = 0$. Given an algebraic number α, its degree $\deg(\alpha)$ and height $H(\alpha)$ are defined as degree and height of the corresponding minimal polynomial.

We always assume that degree n is arbitrary but *fixed*. Hence the constants in different asymptotic relations (as $Q \to \infty$) in this paper might depend on n.

For a complex region $\Omega \subset \mathbb{C}$ denote by $\Psi(Q; \Omega)$ the number of algebraic numbers in Ω of degree at most n and height at most Q. We always assume that Ω does not intersect the real axis and that its boundary consists of a finite number of algebraic curves.

Theorem 2.1. *We have that*

$$\Psi(Q; \Omega) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_\Omega \psi(z)\nu(dz) + O\left(Q^n\right), \quad Q \to \infty, \tag{2}$$

where ν is the Lebesgue measure on the complex plane. The limit density ψ is given by the formula

$$\psi(z) = \frac{1}{|\operatorname{Im} z|} \int_{D_z} \left| \sum_{k=1}^{n-1} t_k \left((k+1) z^k - \frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z} \right) \right|^2 dt_1 \ldots dt_{n-1}. \tag{3}$$

The integration is performed over the region

$$D_z = \Bigg\{ (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : \max_{1 \le k \le n-1} |t_k| \le 1,$$
$$\left| z \sum_{k=1}^{n-1} t_k \left(z^k - \frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z} \right) \right| \le 1, \; \left| \frac{1}{\operatorname{Im} z} \sum_{k=1}^{n-1} t_k \operatorname{Im} z^{k+1} \right| \le 1 \Bigg\}.$$

The implicit constant in the big-O-notation in (2) depends on n, the number of the algebraic curves that form the boundary $\partial\Omega$, and their maximal degree only.

The proof of Theorem 2.1 is given in Section 3. Now let us derive several properties of the limit density ψ.

Proposition 2.2. *The function ψ is positive on $\mathbb{C}$ and satisfies the following functional equations:*

$$\psi(-z) = \psi(\bar{z}) = \psi(z),$$
$$\psi\left(\frac{1}{z}\right) = |z|^4 \psi(z). \tag{4}$$

Proof. The positiveness as well as the first relation are trivial. To prove (4), note that for any integral irreducible polynomial $g(z)$ of degree n, the polynomial $z^n g(z^{-1})$ is also irreducible and has the same degree and the same height. Hence for any region $\Omega \subset \mathbb{C}$ it holds

$$\Psi(Q; \Omega) = \Psi(Q; \Omega^{-1}),$$

where Ω^{-1} is defined as $\Omega^{-1} = \left\{z^{-1} \in \mathbb{C} : z \in \Omega\right\}$. Letting Q tend to infinity, we get by applying Theorem 2.1

$$\int_\Omega \psi(z)\, \nu(dz) = \int_{\Omega^{-1}} \psi(z)\, \nu(dz).$$

On the other hand, after the substitution $z \to 1/z$, we obtain

$$\int_\Omega \psi(z)\, \nu(dz) = \int_{\Omega^{-1}} \psi(z^{-1}) |z|^{-4}\, \nu(dz).$$

Since the class of regions Ω is sufficiently large, (4) follows. □

Proposition 2.3. *Near the real line the density ψ admits the following asymptotic approximation:*

$$\psi(x_0 + iy) = A|y| \cdot (1 + o(1)), \quad y \to 0, \tag{5}$$

where the constant A does not depend on y and can be written explicitly as follows:

$$A = \int\limits_{\widetilde{D}_{x_0}} \left| \sum_{k=1}^{n-1} k(k+1) t_k x_0^{k-1} \right|^2 dt_1 \ldots dt_{n-1}.$$

Here the integration is performed over the region

$$\widetilde{D}_{x_0} = \Bigg\{ (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : \max_{1 \le k \le n-1} |t_k| \le 1, \\ \left| \sum_{k=1}^{n-1} k t_k x_0^{k+1} \right| \le 1, \ \left| \sum_{k=1}^{n-1} (k+1) t_k x_0^k \right| \le 1 \Bigg\}.$$

Relation (5) may be regarded as a "repulsion" of exponent 1 of complex roots from the real axis.

Proof. Since

$$\frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z} = \frac{z^{k+1} - \bar{z}^{k+1}}{z - \bar{z}} = \sum_{j=0}^{k} z^{k-j}\, \bar{z}^j,$$

it follows that

$$(k+1)z^k - \frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z} = \sum_{j=0}^{k} z^{k-j}\left(z^j - \bar{z}^j\right)$$
$$= (z-\bar{z})\sum_{j=1}^{k} z^{k-j}\sum_{m=0}^{j-1} z^{j-1-m}\bar{z}^m = (z-\bar{z})\sum_{s=1}^{k} s z^{s-1}\bar{z}^{k-s}.$$

Hence $\psi(z)$ and D_z can be rewritten as follows:

$$\psi(z) = 4\,|\operatorname{Im} z| \int_{D_z} \left| \sum_{k=1}^{n-1} t_k \sum_{s=1}^{k} s z^{s-1}\bar{z}^{k-s} \right|^2 dt_1 \ldots dt_{n-1},$$

and

$$D_z = \Bigg\{ (t_1,\ldots,t_{n-1}) \in \mathbb{R}^{n-1} : \max_{1\le k\le n-1} |t_k| \le 1,$$
$$\left| \sum_{k=1}^{n-1} t_k \sum_{j=1}^{k} z^{k-j+1}\,\bar{z}^j \right| \le 1,\ \left| \sum_{k=1}^{n-1} t_k \sum_{j=0}^{k} z^{k-j}\,\bar{z}^j \right| \le 1 \Bigg\}.$$

Note that $\widetilde{D}_{x_0} = D_{x_0+0\cdot i}$. Letting $\operatorname{Im} z \to 0$ concludes the proof. □

Proposition 2.4. *For $|z| \ge 1$, the function $\psi(z)$ can be estimated by*

$$\psi(z) \asymp_n \frac{|\operatorname{Im} z|}{|z|^6},$$

where the implicit constant depends on n only.

Proof. It follows from Proposition 2.3 that $\psi(z) \asymp_n |\operatorname{Im} z|$ for $|z| \le 1$. Hence (4) yields the proof. □

If $|z|$ is relatively small or relatively large, then it is possible to write the limit density in a simpler form.

Proposition 2.5. *If $|z| \le 1 - 1/\sqrt{2}$, then*

$$\psi(z) = \frac{2^{n-1}}{3\,|\operatorname{Im} z|} \sum_{k=1}^{n-1} \left| (k+1)z^k - \frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z} \right|^2.$$

If $|z| \ge 2 + \sqrt{2}$, then

$$\psi(z) = \frac{2^{n-1}}{3\,|\operatorname{Im} z|} \sum_{k=1}^{n-1} \frac{1}{|z|^{4k+4}} \left| (k+1)\bar{z}^k - \frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z} \right|^2.$$

Proof. For $|z| \le 1 - 1/\sqrt{2}$ it holds

$$\sum_{k=2}^{n} (k-1)|z|^k \le 1, \text{ and } \sum_{k=2}^{n} k|z|^{k-1} \le 1,$$

which leads to

$$D_z = [-1,1]^{n-1},$$

and a straightforward integration yields the first relation. The second statement follows from the first one and (4). □

Let us conclude the section by considering the case $n = 2$.

Example. In the case of quadratic algebraic numbers the density function takes the form

$$\psi(z) = \frac{2}{|\operatorname{Im} z|} \int_{D_z} |t \operatorname{Im} z|^2 \, dt,$$

where

$$D_z = \left\{ t \in \mathbb{R} : |t| \leq \min\left(1, \frac{1}{|z|^2}, \frac{1}{2|\operatorname{Re} z|}\right) \right\}.$$

By some elementary transformations, we obtain

$$\psi(x+iy) = \begin{cases} \frac{8}{3}y, & \text{if } x^2+y^2 \leq 1, \text{ and } |x| \leq \frac{1}{2}, \\ \frac{y}{3x^3}, & \text{if } (|x|-1)^2 + y^2 \leq 1, \text{ and } |x| > \frac{1}{2}, \\ \frac{8y}{3(x^2+y^2)^3}, & \text{if } (|x|-1)^2 + y^2 > 1, \text{ and } x^2+y^2 > 1. \end{cases}$$

3. Proof of Theorem 2.1

We start with some notation.

For any Borel set $A \subset \mathbb{R}^d$ denote by $\operatorname{Vol}(A)$ the Lebesgue measure of A, denote by $\lambda(A)$ the number of points in A with integer coordinates, and denote by $\lambda^*(A)$ the number of points in A with coprime integer coordinates. The Riemann zeta function is denoted by $\zeta(\cdot)$ and the Möbius function is denoted by $\mu(\cdot)$.

Denote by $\mathcal{P}_Q$ the class of all integral polynomials of degree at most n and height at most Q. The cardinality of this class is $(2Q+1)^{n+1}$. Recall that an integral polynomial is called *prime*, if it is irreducible over $\mathbb{Q}$, primitive (the greatest common divisor of its coefficients equals 1), and its leading coefficient is positive.

For $k \in \{0, 1, \dots, n\}$ denote by γ_k the number of prime polynomials from $\mathcal{P}_Q$ that have exactly k roots lying in Ω. For any algebraic number its minimal polynomial is prime, and any prime polynomial is a minimal polynomial for some algebraic number. Therefore,

$$\Psi(Q;\Omega) = \sum_{k=1}^{n} k\gamma_k. \tag{6}$$

Consider a subset $A_k \subset [-1,1]^{n+1}$ consisting of all points $(t_0, \dots, t_n) \in [-1,1]^{n+1}$ such that the polynomial $t_n x^n + \dots + t_1 x + t_0$ has exactly k roots lying in Ω. Then the number of primitive polynomials from $\mathcal{P}_Q$ which have exactly k roots in Ω is equal to $\lambda^*(QA_k)$. By the definition of a prime polynomial, we have that

$$\left| \gamma_k - \frac{1}{2}\lambda^*(QA_k) \right| \leq R_Q, \tag{7}$$

where R_Q denotes a number of reducible polynomials (over $\mathbb{Q}$) from $\mathcal{P}_Q$. Note that the factor $\frac{1}{2}$ arises in the above inequality because prime polynomials have positive leading coefficient. It is known (see [33]) that

$$R_Q = O\left(Q^n \log^{l(n)} Q\right), \quad Q \to \infty. \tag{8}$$

There do not exist reducible over $\mathbb{Q}$ integral quadratic polynomials having non-real roots. Hence it follows from (6), (7), and (8) that

$$\Psi(Q;\Omega) = \frac{1}{2}\sum_{k=1}^{n} k\lambda^*(QA_k) + O\left(Q^n\right), \quad Q \to \infty. \tag{9}$$

To estimate $\lambda^*(QA_k)$, we need the following lemma.

Lemma 3.1. *Consider a region $A \subset \mathbb{R}^d$, $d \geq 2$, with boundary consisting of a finite number of algebraic surfaces only. Then*

$$\lambda^*(tA) = \frac{\mathrm{Vol}(A)}{\zeta(d)} t^d + O\left(t^{d-1} \log^{l(d)} t\right), \quad t \to \infty. \tag{10}$$

Here the implicit constant in the big-O-notation depends on d, the number of the algebraic surfaces, and their maximal degree only.

The results of this type are well-known, see, e.g., the classical monograph by Bachmann [1, pp. 436–444] (in particular, formulas (83a) and (83b) on pages 441–442). For the readers convenience we include a short proof here.

Proof. Note that

$$\lambda(tA) = \sum_{j=1}^{[Nt]+1} \lambda^*\left(\frac{t}{j}A\right),$$

where N is chosen to be so large that $A \subset [-N, N]^d$. Applying the classical Möbius inversion formula (see, e.g., [28]) yields

$$\lambda^*(tA) = \sum_{j=1}^{[Nt]+1} \mu(j)\, \lambda\left(\frac{t}{j}A\right). \tag{11}$$

By the Lipschitz principle (see [14]) it follows that

$$\left|\lambda\left(\frac{t}{j}A\right) - \left(\frac{t}{j}\right)^d \mathrm{Vol}(A)\right| \leq c \cdot \left(\frac{t}{j}\right)^{d-1} \tag{12}$$

for some constant c depending on the number of the algebraic surfaces and their maximal degree only. Applying this to (11) we get

$$\left|\lambda^*(tA) - \mathrm{Vol}(A) t^d \sum_{j=1}^{[Nt]+1} \frac{\mu(j)}{j^d}\right| \leq c\, t^{d-1} \sum_{j=1}^{[Nt]+1} \frac{1}{j^{d-1}}. \tag{13}$$

It is well known (see, e.g., [28]) that

$$\sum_{j=1}^{\infty} \frac{\mu(j)}{j^d} = \frac{1}{\zeta(d)}.$$

Therefore,

$$\left|\sum_{j=1}^{[Nt]+1} \frac{\mu(j)}{j^d} - \frac{1}{\zeta(d)}\right| \leq \sum_{j=[Nt]+2}^{\infty} \frac{1}{j^d} \leq \frac{1}{(d-1)(Nt)^{d-1}}. \tag{14}$$

Furthermore, it holds that

$$\sum_{j=1}^{[Nt]+1} \frac{1}{j^{d-1}} \leq \begin{cases} \zeta(d-1), & d \geq 3, \\ \log([Nt]+1) + 1, & d = 2. \end{cases} \tag{15}$$

Combining (13), (14), and (15) completes the proof. □

The right-hand side of (10) is estimated by the right-hand sides of (12) and (14) which are of the same order. The one involving the Möbius function can be made slightly sharper (by a logarithmic factor) using an unconditional estimate for the Mertens function (see, e.g., [29]). Assuming the Riemann hypothesis the latter can be improved more (see [32]). However, the error term in the Lipschitz principle can be made smaller for special type of regions only (see [31]), which is not our case.

Since the boundary of Ω consists of a finite number of algebraic curves, the boundary of A_k consists of a finite number of algebraic surfaces. Thus it follows from Lemma (3.1) that

$$\lambda^*(QA_k) = \frac{\operatorname{Vol}(A_k)}{\zeta(n+1)} Q^{n+1} + O\left(Q^n\right), \quad t \to \infty,$$

which together with (9) implies

$$\Psi(Q;\Omega) = \frac{Q^{n+1}}{2\zeta(n+1)} \sum_{k=1}^{n} k \operatorname{Vol}(A_k) + O\left(Q^n\right), \quad Q \to \infty. \tag{16}$$

To calculate $\sum_{k=1}^{n} k \operatorname{Vol}(A_k)$, we need the following result from the theory of random polynomials. Let $\xi_0, \xi_1, \dots, \xi_n$ be independent random variables uniformly distributed on $[-1,1]$. Consider the random polynomial

$$G(x) = \xi_n x^n + \xi_{n-1} x^{n-1} + \cdots + \xi_1 x + \xi_0.$$

Denote by $N(\Omega)$ the number of the roots of $G(z)$ lying in Ω. By definition,

$$\operatorname{Vol}(A_k) = 2^{n+1}\, \mathbb{P}(N(\Omega) = k),$$

which implies

$$\sum_{k=1}^{n} k \operatorname{Vol}(A_k) = 2^{n+1}\, \mathbb{E}N(\Omega). \tag{17}$$

The right-hand side of the latter relation was calculated in [34] in more general setup: it was shown that if the coefficients $\xi_0, \xi_1, \dots, \xi_n$ have a joint probability density function $p(x_0, x_1, \dots, x_n)$, then $\mathbb{E}N(\Omega)$ is given by the formula

(18)

$$\begin{aligned}
\mathbb{E}N(\Omega) = \int_{\Omega} dr d\alpha & \int_{\mathbb{R}^{n-1}} dt_1 \dots dt_{n-1} \frac{r^2}{\sin\alpha} \\
& \times \left(\left[\sum_{k=1}^{n-1} t_k r^{k-1} \left((k+1)\cos(k+1)\alpha - \cos\alpha \frac{\sin(k+1)\alpha}{\sin\alpha} \right) \right]^2 \right. \\
& \left. + \left[\sum_{k=1}^{n-1} k t_k r^{k-1} \sin(k+1)\alpha \right]^2 \right) \\
& \times p\left(\frac{1}{\sin\alpha} \sum_{k=1}^{n-1} t_k r^{k+1} \sin k\alpha,\ -\frac{1}{\sin\alpha} \sum_{k=1}^{n-1} t_k r^k \sin{(k+1)\alpha},\ t_1,\ \dots,\ t_{n-1} \right),
\end{aligned}$$

where $r = |z|$ and $\alpha = \arg z$ are polar coordinates in the complex plane. The corresponding formula in [34] contains a typo. Here we use the correct version.

In the case when the coefficients are independent and uniformly distributed on $[-1,1]$, their joint probability density function equals

$$p = 2^{-n-1}\mathbb{1}_{[-1,1]^{n+1}}.$$

Thus it follows from (17) and (16) that to finish the proof, it is enough to show that for this specific p the right-hand side of (18) is equal to

$$\int_\Omega \psi(z)\,\nu(dz),$$

where ψ is defined in (3).

Indeed, the integrand in (3) can be transformed as follows:

$$\left|\sum_{k=1}^{n-1} t_k \left((k+1)z^k - \frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z}\right)\right|^2 = \frac{1}{r^2}\left|\sum_{k=1}^{n-1} t_k \left((k+1)z^{k+1} - z\frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z}\right)\right|^2$$

$$= \left|\sum_{k=1}^{n-1} t_k r^k \left(\left[(k+1)\cos(k+1)\alpha - \cos\alpha\,\frac{\sin(k+1)\alpha}{\sin\alpha}\right] + i\left[k\sin(k+1)\alpha\right]\right)\right|^2,$$

and the functions that define the region D_z can be transformed as follows:

$$\left|\sum_{k=1}^{n-1} t_k \left(z^{k+1} - z\frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z}\right)\right| = \left|\sum_{k=1}^{n-1} t_k \left(\operatorname{Re} z^{k+1} - \operatorname{Re} z\frac{\operatorname{Im} z^{k+1}}{\operatorname{Im} z}\right)\right|$$

$$= \left|\frac{1}{\sin\alpha}\sum_{k=1}^{n-1} t_k r^{k+1}\left(\sin\alpha\cos(k+1)\alpha - \cos\alpha\sin(k+1)\alpha\right)\right|$$

$$= \left|\frac{1}{\sin\alpha}\sum_{k=1}^{n-1} t_k r^{k+1}\sin k\alpha\right|,$$

and

$$\left|\frac{1}{\operatorname{Im} z}\sum_{k=1}^{n-1} t_k \operatorname{Im} z^{k+1}\right| = \left|\frac{1}{\sin\alpha}\sum_{k=1}^{n-1} t_k r^k \sin(k+1)\alpha\right|.$$

The proof follows.

Acknowledgments. We are grateful to Vasili Bernik for his valuable comments. We also wish to thank Natalia Budarina and Hanna Husakova for helpful discussions.

References

[1] P. Bachmann. *Die analytische Zahlentheorie*, volume 2. BG Teubner, Leipzig, 1894.

[2] A. Baker and W. Schmidt. Diophantine approximation and Hausdorff dimension. *Proc. London Math. Soc.*, 3(1):1–11, 1970.

[3] V. Beresnevich. On approximation of real numbers by real algebraic numbers. *Acta Arith.*, 90(2):97–112, 1999.

[4] V. Beresnevich, V. Bernik, and F. Götze. The distribution of close conjugate algebraic numbers. *Compos. Math.*, 146(5):1165–1179, 2010.

[5] V. Bernik and D. Vasil'ev. A Khinchin-type theorem for integer-valued polynomials of a complex variable. *Tr. Inst. Mat. NAN Belarusi*, 3:10–20, 1999. (In Russian).

[6] H. Brown and K. Mahler. A generalization of Farey sequences: Some exploration via the computer. *J. Number Theory*, 3(3):364–370, 1971.

[7] N. Budarina and F. Götze. Distance between conjugate algebraic numbers in clusters. *Math. Notes*, 94(5–6):816–819, 2013.

[8] Y. Bugeaud. *Approximation by algebraic numbers*, volume 160. Cambridge University Press, 2004.

[9] Y. Bugeaud and A. Dujella. Root separation for irreducible integer polynomials. *Bull. Lond. Math. Soc.*, 43(6):1239–1244, 2011.

[10] Y. Bugeaud and A. Dujella. Root separation for reducible integer polynomials. *Acta Arith.*, 162(4):393–403, 2014.

[11] Y. Bugeaud and M. Mignotte. On the distance between roots of integer polynomials. *Proc. Edinb. Math. Soc. (2)*, 47(3):553–556, 2004.

[12] Y. Bugeaud and M. Mignotte. Polynomial root separation. *Int. J. Number Theory*, 6(3):587–602, 2010.

[13] S.-J. Chern and J. Vaaler. The distribution of values of Mahler's measure. *J. Reine Angew. Math.*, (540):1–47, 2001.

[14] H. Davenport. On a principle of Lipschitz. *J. London Math. Soc*, 26:179–183, 1951.

[15] P. Erdős and P. Turán. On the distribution of roots of polynomials. *Ann. Math.*, 51(1):105–119, 1950.

[16] J.-H. Evertse. Distances between the conjugates of an algebraic number. *Publ. Math. Debrecen*, 65(3–4):323–340, 2004.

[17] F. Götze, D. Kaliada, and D. Zaporozhets. Correlation functions of real zeros of random polynomials. *Preprint, arXiv:1510.00025*, 2015.

[18] F. Götze, D. Kaliada, and D. Zaporozhets. Correlations between real conjugate algebraic numbers. *Preprint, arXiv:1510.00536*, 2015.

[19] F. Götze and D. Zaporozhets. Discriminant and root separation of integral polynomials. *Preprint, arXiv:1407.6388*, 2014.

[20] I. Ibragimov and D. Zaporozhets. On distribution of zeros of random polynomials in complex plane. *Prokhorov and Contemporary Probability Theory. Springer Proceedings in Mathematics & Statistics*, 33:303–323, 2013.

[21] D. Kaliada. Distribution of real algebraic numbers of the second degree. *Vestsi NAN Belarusi*, (3):54–63, 2013. (In Russian).

[22] D. Kaliada. On the density function of the distribution of real algebraic numbers. *Preprint, arXiv:1405.1627*, 2014.

[23] D. Koleda. On the asymptotic distribution of algebraic numbers with growing naive height. *Chebyshevskii Sb.*, 16(1):191–204, 2015. (In Russian).

[24] S. Lang. *Fundamentals of Diophantine geometry*. Springer, Heidelberg, 1983.

[25] D. Masser and J. Vaaler. Counting algebraic numbers with large height II. *Trans. Am. Math. Soc.*, 359(1):427–445, 2007.

[26] D. Masser and J. Vaaler. Counting algebraic numbers with large height I. In *Diophantine approximation*, volume 16, pages 237–243. Springer, 2008.

[27] I. Pritsker. Distribution of algebraic numbers. *J. Reine Angew. Math.*, (657):57–80, 2011.

[28] H. Rademacher. *Lectures on elementary number theory*. Huntington, 1977.

[29] O. Ramaré. From explicit estimates for primes to explicit estimates for the Möbius function. *Acta Arith.*, 157(4):365–379, 2013.

[30] D. Shparo and M. Shur. On the distribution of roots of random polynomials. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, 3:40–43, 1962. (In Russian).
[31] M. Skriganov. Ergodic theory on SL(n), diophantine approximations and anomalies in the lattice point problem. *Invent. Math.*, 132(1):1–72, 1998.
[32] K. Soundararajan. Partial sums of the Möbius function. *J. reine angew. Math.*, (631):141–152, 2009.
[33] B. L. van der Waerden. Die Seltenheit der reduziblen Gleichungen und der Gleichungen mit Affekt. *Monatsh. Math. Phys.*, 43(1):133–147, 1936.
[34] D. Zaporozhets. On distribution of the number of real zeros of a random polynomial. *Zap. Nauchn. Sem. POMI*, 320:69–79, 2004. English translation: J. Math. Sci.

Friedrich Götze, Department of Mathematics, Bielefeld University, P. O. Box 10 01 31, 33501 Bielefeld, Germany
E-mail address: goetze@math.uni-bielefeld.de

Dzianis Kaliada, Institute of Mathematics, National Academy of Sciences of Belarus, 220072 Minsk, Belarus
E-mail address: koledad@rambler.ru

Dmitry Zaporozhets, St. Petersburg Department of Steklov Institute of Mathematics, Fontanka 27, 191011 St. Petersburg, Russia
E-mail address: zap1979@gmail.com