

A geometric construction of generalized quiver Hecke algebras

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Abstract

We provide a common generalization of the Springer map and quiver-graded Springer map due to Lusztig, called generalized quiver-graded Springer map associated to generalized quiver representations introduced by Derksen and Weyman. Following Chriss and Ginzburg for any equivariant projective map $\pi: E \rightarrow V$, there is an algebra structure on the equivariant Borel-Moore homology of $Z = E \times_V E$, we call it the Steinberg algebra (of π)¹. For the classical Springer map, it is a skew group ring of the Weyl group operating on a polynomial ring (having the Weyl group ring as degree zero part) and for the quiver-graded Springer map, it is (one graded part of) the quiver Hecke algebra, this is due to Varagnolo and Vasserot. As our main result we calculate (under mild assumptions) generators and relations for Steinberg algebra of the generalized quiver-graded Springer map using the methods of Varagnolo and Vasserot. In the end, we discuss examples. As a third example we define the symplectic symmetric quiver-graded Springer map and give for quivers without loops the generators and relations of the associated algebra. We end with an overview table of known examples.

1 Introduction

Any equivariant projective map $\pi: E \rightarrow V$ of varieties gives rise to an algebra structure on the equivariant Borel-Moore homology of the cartesian product $Z = E \times_V E$, defined by Chriss and Ginzburg, [CG97], chapter 2. We call this algebra the **Steinberg algebra** (associated to π). If E is smooth, the Steinberg algebra is naturally graded. Our aim is to study Steinberg algebras in a set-up which is general enough to include the later mentioned examples but specific enough to be able to calculate them explicitly.

In Chriss and Ginzburg's version of Springer theory (loc. cit.), this construction is the main ingredient to prove the Springer correspondence (a geometric realization of the Weyl group as the degree zero part of the Steinberg algebra of the Springer map together with its simple modules as isotypic parts of the homology groups of the fibres of the Springer map). A K-theoretic version of it realizes the affine Hecke algebra and its simple modules, this is called Deligne-Langlands correspondence.

Independently, Lusztig constructed (see e.g. [Lus91] or subsection 4.2), a map associated to a quiver and a dimension vector, which we call quiver-graded Springer map following Reineke in [Rei03]. Lusztig showed that the direct summands of the pushforward of the constant sheaf under all quiver-graded Springer maps have a monoidal structure, which categorifies the negative half of the quantum group (associated to the quiver). Later, Khovanov and Lauda gave a second categorification in terms of graded modules over the quiver Hecke algebra (or KLR-algebra) and conjectured that the Steinberg algebras of the quiver-graded Springer map gives the graded parts of the quiver Hecke algebra, which implies that the two categorifications are equivalent (see [KL09]). This has been proven by Varagnolo and Vasserot and independently by Rouquier (see [VV11b], [Rou11]).

Following Chriss and Ginzburg, other Steinberg algebras have been studied in exotic Springer theory, compare the work of Kato, Achar and Henderson (see [Kat09], [Kat11], [AH08]). Following Lusztig's construction, Stroppel and Webster introduced quiver Schur algebras (see [SW11]) and KLR-algebras associated to Borcherds Cartan data have been geometrically constructed by Kang, Kashiwara and Park in [KKP13]. You find more details on the background in the section 4 on examples.

Note that the Springer map as well as the quiver-graded Springer map are instances of collapsings of homogeneous vector bundles E over complete flag varieties G/B with G a reductive group and B a Borel subgroup (or unions of those). The notion of a collapsing goes back to Kempf [Kem76]², cp. definition 1. In this case the Steinberg algebra has a natural basis parametrized by the torus fixed points of the flag varieties G/B because $Z = E \times_V E$ is a generalized cellular fibration (see lemma 2.0.1). Varagnolo and Vasserot observed, assuming no nonzero torus fixed points in the image of π , that the operation of the Steinberg algebra on the equivariant cohomology of E is faithful. Also, we know that $H_*^G(E)$ is just a (direct sum of) polynomial ring(s).

But in general, for the Steinberg algebra of such a collapsing map, we have no methods to calculate the natural basis elements as operators on polynomial rings. Therefore, we specialize to a situation where Varagnolo and Vasserot's method of calculating the product on the torus fixed points is still available (using parts of the Goresky-Kottwitz-McPherson localization theory). The main trick is to use a relative situation where you can embed into a Springer map where these products are known. This means we replace the adjoint representation

¹Chriss and Ginzburg call Z a Steinberg variety, if π is the Springer map. So we always say Steinberg algebra as a shortage for equivariant Borel-Moore homology (with complex coefficients) of the Steinberg variety with the Chriss-Ginzburg algebra structure

²We exclude Kempf's assumption that the fibre of the homogeneous bundle is a completely reducible representation.

of the Lie algebra by Derksen and Weyman's generalized quiver representations (see [DW02], and subsection 2.1), here: Given $\mathbb{G}, \mathbb{B}, T$ a complex reductive group with a Borel subgroup and maximal torus of rank n and let (\mathbb{W}, \mathbb{S}) its associated Weyl group. For a subgroup $H \subset T$, we denote $G := C_G(H)^0, B = \mathbb{B} \cap G$ and W the Weyl group associated to (G, T) . Then, one has

$$(\mathbb{G}/\mathbb{B})^H = \bigsqcup_I G/B, \quad I = W \setminus \mathbb{W}$$

and $H_*^G(\bigsqcup_I G/B)$ can be identified with the induced representation $\text{ind}_W^{\mathbb{W}} \mathbb{C}[x_1, \dots, x_n] = \bigoplus_{i \in I} \mathbb{C}[x_i(1), \dots, x_i(n)]$, $I = W \setminus \mathbb{W}$. In particular, if E is a union of vector bundles over G/B indexed by $i \in I$, we get the cohomology ring $H_*^G(E) = H_*^G(\bigsqcup_I G/B)$ is the induced \mathbb{W} -representation from before (see later subsection 3.8).

Our motivation to give a common generalization of Varagnolo and Vasserot's method to general connected reductive groups was to clarify the situation by giving a very detailed account and give a playground for a common investigation of these Hecke-like algebras.

Our main result is a calculation of the Steinberg algebras for a collapsing of homogeneous vector bundles associated to Derksen and Weyman's generalized quiver representation in terms of generators and relations (under some extra assumptions, see theorem 3.12). In fact, as \mathbb{C} -algebras they always have three types of generators $1_i, z_i(t), \sigma_i(s)$: where 1_i are idempotents corresponding to the disjoint union of homogeneous bundles you started with, $z_i(t)$ are the generators of the polynomial rings given by the equivariant cohomology of E and $\sigma_i(s), s \in \mathbb{W}$ are simple reflection coming from the cellular fibration mentioned before. More precisely:

Theorem 1.1. *Let $(\mathbb{G}, \mathbb{B}, T), (\mathbb{W}, \mathbb{S})$ and $(G, B), W$ be as before. We choose representatives $x_i \in \mathbb{W}$ of the cosets $W \setminus \mathbb{W}$ and a G -subrepresentation V of a direct sum $\text{Lie}(\mathbb{G})^{\oplus r}$, we get $F_i = V \cap x_i(\text{Lie } \mathbb{U})^{\oplus r}, i \in I$ where $\mathbb{U} \subset \mathbb{B}$ is the unipotent radical, this is a $B_i = x_i \mathbb{B} \cap G$ -subrepresentation of V . Then the collapsing map*

$$\pi: E := \bigsqcup_{i \in I} (G \times^{B_i} F_i) \rightarrow V, [g, f] \mapsto gf$$

is our **generalized quiver-graded Springer map**. The Steinberg algebra of π is our **generalized quiver Hecke algebra**³. It is the subalgebra of $\text{End}_{\mathbb{C}}(H_*^G(G/B))$ given by generators

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n, i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

with $1_i, z_i(t)$ the obvious operators and for f in the k -th summand $\sigma_i(s)(f) = 0$ unless $k = is$ and then $\sigma_i(s)(f) = \alpha_s^{h_i(s)} \frac{s(f) - f}{\alpha_s}$ if $i = is$ or $\sigma_i(s)(f) = \alpha_s^{h_i(s)} s(f)$ if $i \neq is$, where α_s is the positive root for $s \in \mathbb{S}$ seen under $\text{Lie}(T)^* \subset \mathbb{C}[x_1, \dots, x_n]$ as linear polynomial and $h_i(s) := \#\{k \in \{1, \dots, r\} \mid x_i(\alpha_s) \in \Phi_{V^{(k)}}\}$, $V = \bigoplus_k V^{(k)}, V^{(k)} \subset \text{Lie } \mathbb{G}$ and $\Phi_{V^{(k)}}$ is the set of T -weights of $V^{(k)}$.

Under some limitations for the choices of V (such that the $h_i(s)$ are bounded), we calculate generators and relations for the generalized quiver Hecke algebras, see theorem 3.12.

The text is structured as follows: We established the basic properties of Steinberg algebras of collapsings of homogeneous bundles and recall the set-up for generalized quiver representations in section 2. In the third section, we give our main result. In the last section, we come back to the examples mentioned in the beginning.

2 Collapsings of homogeneous bundles and generalized quiver representations

Roughly, following the introduction of Chriss and Ginzburg's book ([CG97])⁴,

To introduce this construction, recall for any algebraic group G and closed subgroup P (over \mathbb{C}) we call the principal bundles $G \rightarrow G/P$ **homogeneous**. For any P -variety F given we have the **associated bundle** defined by the quotient

$$G \times^P F := G \times F / \sim, \quad (g, f) \sim (g', f'): \iff \text{there is } p \in P: (g, f) = (g'p, p^{-1}f')$$

and $G \times^P F \rightarrow G/P, (g, f) \mapsto gP$. Given a representation $\rho: P \rightarrow \mathbf{GL}(F)$, i.e. a morphism of algebraic groups, we call associated bundles of the form $G \times^P F \rightarrow G/P$ **homogeneous vector bundles**.

³technically speaking it should be called generalizing a graded part of the quiver Hecke algebra, the horizontal map of the quiver Hecke algebra is a type A phenomenon which does not exist for general reductive groups.

⁴We take a more general approach, what usually is considered as Springer theory you find in the example *classical Springer theory*. Nevertheless, our approach is still only a special case of [CG97], chapter 8.

Definition 1. Our geometric construction is given by the following: Given $(G, P_i, V, F_i)_{i \in I}$, where I is a finite set, with

- (*) G a connected reductive group with parabolic subgroups P_i .
- We also assume there exists a maximal torus $T \subset G$ which is contained in every P_i .
- (*) V a finite dimensional G -representation, $F_i \subset V$ a P_i -subrepresentation of V , $i \in I$.

Let $E_i := G \times^{P_i} F_i, i \in I$

$$\begin{array}{ccc}
 & E := \bigsqcup_{i \in I} E_i & \\
 \pi \swarrow & & \searrow \mu \\
 V & & \bigsqcup_{i \in I} G/P_i
 \end{array}
 \qquad
 \begin{array}{ccc}
 & [(g, f_i)] & \\
 \swarrow & & \searrow \\
 gf_i & & gP_i
 \end{array}$$

Then, $E \rightarrow V \times \bigsqcup_{i \in I} G/P_i, [(g, f_i)] \mapsto (gf_i, gP_i)$ is a closed embedding (see [Slo80b], p.25,26), it follows that π is projective, we call it a **collapsing of a homogeneous vector bundle** or sometimes even a Springer map and its fibres **Springer fibres**. Via restriction of $E \rightarrow V \times \bigsqcup_{i \in I} G/P_i$ to $\pi^{-1}(x) \rightarrow \{x\} \times \bigsqcup_{i \in I} G/P_i$ one sees that all Springer fibres are via μ closed subschemes of $\bigsqcup_{i \in I} G/P_i$.

We also have another induced roof-diagramm

$$\begin{array}{ccc}
 & Z := E \times_V E & \\
 p \swarrow & & \searrow m \\
 V & & (\bigsqcup_{i \in I} G/P_i) \times (\bigsqcup_{i \in I} G/P_i)
 \end{array}$$

with $p: E \times_V E \xrightarrow{pr_E} E \xrightarrow{\pi} V$ projective and $m: E \times_V E \xrightarrow{(pr_E, pr_E)} E \times E \xrightarrow{\mu \times \mu} (\bigsqcup_{i \in I} G/P_i) \times (\bigsqcup_{i \in I} G/P_i)$. Observe, by definition

$$Z = \bigsqcup_{i, j \in I} Z_{i, j}, \quad Z_{i, j} = E_i \times_V E_j.$$

We call the scheme Z **Steinberg variety** (even though as a scheme Z might be neither reduced nor irreducible).

If all parabolic groups P_i are Borel groups, the Steinberg variety Z is an iterated cellular fibration over $\bigsqcup_{i \in I} G/P_i$ (for an appropriate definition of iterated cellular fibration), for the precise statement see the next lemma. We choose a (co-)homology theory which can be calculated for spaces with cellular fibration property and which has a localization to the T -fixpoint theory: Let $H_*^A, A \in \{pt, T, G\}$ be (**A -equivariant**) **Borel-Moore homology** with complex coefficients. There is a natural product $*$ on $H_*^A(Z)$ called convolution product constructed by Chriss and Ginzburg in [CG97].

$$\begin{aligned}
 * : H_*^A(Z) \times H_*^A(Z) &\rightarrow H_*^A(Z) \\
 (c_{1,2}, c_{2,3}) &\mapsto c_{1,2} * c_{2,3} := (q_{1,3})_*(p_{1,2}^*(c_{1,2}) \cap p_{2,3}^*(c_{2,3}))
 \end{aligned}$$

where $\cap: H_p^A(X) \times H_q^A(Y) \rightarrow H_{p+q-2d}^A(X \cap Y)$ is the intersection pairing which is induced by the \cup -product in relative singular cohomology for $X, Y \subset M$ two A -equivariant closed subsets of a d -dimensional complex manifold M (cp. [CG97], p.98, (2.6.16)) and where $p_{a,b}: E \times E \times E \rightarrow E \times E$ is the projection on the a, b -th factors, $q_{a,b}$ is the restriction of $p_{a,b}$ to $E \times_V E \times_V E$. One has

$$H_p^A(Z_{i,j}) * H_q^A(Z_{k,\ell}) \subset \delta_{j,k} H_{p+q-e_k}^A(Z_{i,\ell}), \quad e_k = \dim_{\mathbb{C}} E_k.$$

If we set

$$H_{[p]}^A(Z) := \bigoplus_{i, j \in I} H_{e_i + e_j - p}^A(Z_{i, j})$$

then $H_{[*]}^A(Z)$ is a graded $H_A^*(pt)$ -algebra. We call $(H_*^A(Z), *)$ the (A -equivariant) **Steinberg algebra** for $(G, P_i, V, F_i)_{i \in I}$.

Convolution Modules (recall from [CG97], section 2.7).

Given three sets M_1, M_2, M_3 and two subsets $S_{1,2} \subset M_1 \times M_2$, $S_{2,3} \subset M_2 \times M_3$ the set-theoretic convolution is defined as

$$S_{1,2} \circ S_{2,3} := \{(m_1, m_3) \mid \exists m_2 \in M_2 : (m_1, m_2) \in S_{1,2}, (m_2, m_3) \in S_{2,3}\} \subset M_1 \times M_3.$$

Now, let $S_{i,j} \subset M_i \times M_j$ be A -equivariant locally closed subsets of smooth complex A -varieties, let $p_{i,j} : M_1 \times M_2 \times M_3 \rightarrow M_i \times M_j$ be projection on the (i, j) -th factors and assume $q_{1,3} := p_{1,3}|_{p_{12}^{-1}(S_{1,2}) \cap p_{2,3}^{-1}(S_{2,3})}$ is proper.

Then we get a map

$$\begin{aligned} * : H_p^A(S_{1,2}) \times H_q^A(S_{2,3}) &\rightarrow H_{p+q-2 \dim_{\mathbb{C}} M_2}^A(S_{1,2} \circ S_{2,3}) \\ c_{1,2} * c_{2,3} &:= (q_{1,3})_*(p_{1,2}^* c_{1,2} \cap p_{2,3}^* c_{2,3}). \end{aligned}$$

This way we defined the algebra structure on the Steinberg algebra, but it also gives a left module structure on $H_*^A(S)$ for any A -variety S with $Z \circ S = S$ and a right module structure when $S \circ Z = S$.

- (a) $M_1 = M_2 = M_3 = E$, embedd $Z = E \times_V E \subset E \times E$, $E = E \times pt \subset E \times E$, then one has $Z \circ E = E$. If we regrade the Borel-Moore homology (and the Poincare dual A -equivariant cohomology) of E as follows

$$H_{[p]}^A(E) := \bigoplus_{i \in I} H_{e_i - p}^A(E_i) \quad (= \bigoplus_{i \in I} H_A^{e_i + p}(E_i) =: H_A^{[p]}(E))$$

then $H_{[*]}^A(E)$ and $H^{[*]}(E)$ carry the structure of a graded left $H_{[*]}^A(Z)$ -module.

- (b) $M_1 = M_2 = M_3 = E$, embedd $E \subset E \times E$ diagonally, then $E \circ E = E$, one has $H_{[*]}^A(E) = H_A^*(E)$ as graded algebras where $H_{(p)}^A(E) := \bigoplus_i H_{2e_i - p}^A(E_i)$ and the ring structure on the cohomology is given by the cup product. If we take now $Z = E \times_V E \subset E \times E$ then $E \circ Z = Z$ and we get a structure as graded left $H_A^*(E)$ -module on $H_{[*]}^A(Z)$.

- (c) $M_1 = M_2 = M_3 = E$, $A = pt$ embedd $Z = E \times_V E \subset E \times E$, $\pi^{-1}(x) = \pi^{-1}(x) \times pt \subset E \times E$, then one has $Z \circ \pi^{-1}(x) = E$. If we regrade the Borel-Moore homology and singular cohomology of $\pi^{-1}(x)$ as follows

$$H_{[p]}(\pi^{-1}(x)) := \bigoplus_{i \in I} H_{e_i - p}(\pi_i^{-1}(x)) \quad H^{[p]}(\pi^{-1}(x)) := \bigoplus_{i \in I} H^{e_i + p}(\pi_i^{-1}(x))$$

then $H_{[*]}(\pi^{-1}(x))$ and $H^{[*]}(\pi^{-1}(x))$ are graded left $H_{[*]}(Z)$ -module.

We call these the **Springer fibre modules**.

Similarly in all examples one can obtain right module structure (the easy swaps are left to the reader).

2.0.1 Cellular fibration

We set $\widetilde{W} := \bigsqcup_{i,j \in I} W_{i,j}$ with $W_{i,j} := W_i \setminus W/W_j$ where W is the Weyl group for (G, T) and $W_i \subset W$ is the Weyl group for (L_i, T) with $L_i \subset P_i$ is the Levi subgroup. We will fix representatives $w \in G$ for all elements $w \in \widetilde{W}$. If all P_i are Borel subgroups, then \widetilde{W} can be identified with the set of T -fixed points $(\bigsqcup_{i \in I} G/P_i)^T$. Let $C_w = G \cdot (eP_i, wP_j)$ be the G -orbit in $G/P_i \times G/P_j$ corresponding to $w \in W_{i,j}$.

Lemma 1. (1) $p : C_w \subset G/P_i \times G/P_j \xrightarrow{pr_1} G/P_i$ is G -equivariant, locally trivial with fibre $p^{-1}(eP_i) = P_i w P_j / P_j$.

- (2) $P_i w P_j / P_j$ admits a cell decomposition into affine spaces via Schubert cells $x B_j x^{-1} v P_j / P_j$, $v \in W_i$ (and for a fixed $x \in W$ such that ${}^x P_j = P_i$, $B_j \subset P_j$ the Borel subgroup). In particular, $H_{\text{odd}}(P_i w P_j / P_j) = 0$ and

$$H_*(P_i w P_j / P_j) = \bigoplus_{v \in W_i} \mathbb{C} b_{i,j}(v), \quad b_{i,j}(v) := [x B_j x^{-1} v P_j / P_j].$$

One has $\deg b_{i,j}(v) = 2\ell_{i,j}(v)$ where $\ell_{i,j}(v)$ is the length of a minimal coset representative in W for $x^{-1} v W_j \in W/W_j$.

- (3) For $A \in \{pt, T, G\}$ one has $H_{\text{odd}}^A(C_w) = 0$ and since G/P_i is simply connected

$$H_n^A(C_w) = \bigoplus_{p+q=n} H_A^p(G/P_i) \otimes H_q(P_i w P_j / P_j), \quad H_*(C_w) = \bigoplus_{u \in W/W_i, v \in W_i} \mathbb{C} b_i(u) \otimes b_{i,j}(v),$$

where $b_i(u) = [B_i u P_i / P_i]^*$ is of degree $2 \dim_{\mathbb{C}} G/P_i - 2\ell_i(u)$ with $\ell_i(u)$ is the length of a minimal coset representative for $u \in W/W_i$ and $b_{i,j}(v)$ as in (2).

proof: Is left out because it is standard techniques.

This implies the following properties for the homology of Z .

Lemma 2. (1) Z has a filtration by closed G -invariant subvarieties such that the successive complements are $Z_w := m^{-1}(C_w)$, $w \in \widetilde{W}$ and the restriction of m to Z_w is a vector bundle over C_w of rank d_w (as complex vector bundle). Furthermore,

$$\begin{aligned} H_n^A(Z) &= \bigoplus_{w \in \widetilde{W}} H_n^A(Z_w) = \bigoplus_{w \in \widetilde{W}} H_{n-2d_w}^A(C_w), \\ H_n(Z) &= \bigoplus_{i,j \in I} \bigoplus_{w \in W_{i,j}} \bigoplus_{u \in W/W_i, v \in W_i} \mathbb{C} b_i(u) \otimes b_{i,j}(v) \end{aligned}$$

where the last direct sum goes over the u, v with the property $2 \dim G/P_i - 2\ell_i(u) + 2\ell_{i,j}(v) = n - 2d_w$.

(2) $H_{\text{odd}}(Z) = 0$, $H^{\text{odd}}(Z) = 0$.

(3) Z is equivariantly formal (for T and G , for Borel-Moore homology and cohomology).

In particular, for $A \in \{T, G\}$ the following forgetful maps $H_*^A(Z) \rightarrow H_*(Z)$ and $H_A^*(Z) \rightarrow H^*(Z)$ are surjective algebra homomorphisms. It even holds the stronger isomorphism of \mathbb{C} -algebras

$$\begin{aligned} H_*(Z) &= H_*^A(Z) / H_{<0}^A(pt) H_*^A(Z) \\ H^*(Z) &= H_A^*(Z) / H_A^{>0}(pt) H_A^*(Z) \end{aligned}$$

As a consequence we get the following isomorphisms.

$$\begin{aligned} 1) \quad H_*^A(Z) &= H_*(Z) \otimes_{\mathbb{C}} H_*^A(pt) \quad \text{of } H_*^A(pt)\text{-modules} \\ 2) \quad H_A^*(Z) &= H^*(Z) \otimes_{\mathbb{C}} H_A^*(pt) \quad \text{of } H_A^*(pt)\text{-modules} \end{aligned}$$

We can see that $H_{[*]}^A(Z)$ has finite dimensional graded pieces and the graded pieces are bounded from below in negative degrees.

2.1 Reminder on generalized quiver representations

This is a short reminder of Derksen and Weyman's generalized quiver representations from [DW02].

Definition 2. A *generalized quiver with dimension vector* is a triple (\mathbb{G}, G, V) where \mathbb{G} is a reductive group, G is a centralizer of a Zariski closed abelian reductive subgroup H of \mathbb{G} , i.e.

$$G = C_{\mathbb{G}}(H) = \{g \in \mathbb{G} \mid ghg^{-1} = h \quad \forall h \in H\}$$

(then G is also reductive, see lemma below) and V is a representation of G which decomposes into irreducible representations which also appear in $\mathcal{G} := \text{Lie}(\mathbb{G})$ seen as an G -module.

A *generalized quiver representation* is a quadruple (\mathbb{G}, G, V, Gv) where (\mathbb{G}, G, V) is a generalized quiver with dimension vector, v in V and Gv is the G -orbit.

Remark. Any such reductive abelian group is of the form $H = A \times S$ with A finite abelian and S a torus, this implies that there exists finitely many elements h_1, \dots, h_m such that $C_{\mathbb{G}}(H) = \bigcap_{i=1}^m C_{\mathbb{G}}(h_i)$, see for example Humphreys' book [Hum75], Prop. in 16.4, p.107.

We would like to work with the associated Coxeter systems, therefore it is sensible to assume \mathbb{G} connected and replace G by its identity component G^o . There is the following proposition

Proposition 1. Let \mathbb{G} be a connected reductive group and $H \subset \mathbb{G}$ an abelian group which lies in a maximal torus. We set $G := C_G(H)^o = (\bigcap_{i=1}^m C_{\mathbb{G}}(h_i))^o$. Then one has

(1) For any maximal torus $T \subset \mathbb{G}$, the following three conditions are equivalent:

- (i) $T \subset G$.
- (ii) $H \subset T$.
- (iii) $\{h_1, \dots, h_m\} \subset T$.

(2) G is a reductive group.

(3) If $\underline{\Phi}$ is the set of roots of \mathbb{G} with respect to a maximal torus T with $H \subset T$, then $\Phi := \{\alpha \in \underline{\Phi} \mid \alpha(h) = 1 \ \forall h \in H\}$ is the set of roots for G with respect to T , its Weyl group is $\langle s_\alpha \mid \alpha \in \Phi \rangle$ and for all $\alpha \in \Phi$ the weight spaces are equal $\mathfrak{g}_\alpha = \mathcal{G}_\alpha$ (and 1-dimensional \mathbb{C} -vector spaces).

(4) There is a surjection

$$\begin{aligned} \{\mathbb{B} \subset \mathbb{G} \mid \mathbb{B} \text{ Borel subgroup, } H \subset \mathbb{B}\} &\rightarrow \{B \subset G \mid B \text{ Borel subgroup}\} \\ \mathbb{B} &\mapsto \mathbb{B} \cap G \end{aligned}$$

If $\underline{\Phi}^+$ is the set of positive roots with respect to $(\mathbb{G}, \mathbb{B}, T)$ with $H \subset T$, then $\Phi^+ := \Phi \cap \underline{\Phi}^+$ is the set of positive roots for $(G, G \cap \mathbb{B}, T)$.

proof: Ad (1): This is easy to prove directly.

(2)-(4) are proven if $G = C_{\mathbb{G}}(h)^o$ for one semisimple element $h \in \mathbb{G}$ in Carters book [Car85], section 3.5. p.92-93. In general $G = (\bigcap_{i=1}^m C_{\mathbb{G}}(h_i)^o)^o$ for certain $h_i \in H, 1 \leq i \leq m$. The result follows via induction on m . Set $\mathbb{G}_1 := C_G(h_1)^o$. One has $G = (\bigcap_{i=2}^m C_{\mathbb{G}_1}(h_i)^o)^o = C_{\mathbb{G}_1}(H)^o \subset \mathbb{G}_1$ and \mathbb{G}_1 is a connected reductive group. By induction hypothesis, all statements are true for (G, \mathbb{G}_1) , so in particular G is a reductive group. The other statements are then obvious. \square

We need to study the relationship of the Coxeter systems (W, S) and (\mathbb{W}, \mathbb{S}) corresponding to (\mathbb{G}, T) and (G, T) .

Lemma 3. One has $G \cap \mathbb{W} = W$ and $W \cap \mathbb{S} \subset S$. Let l_S be the length function with respect to (W, S) and $l_{\mathbb{S}}$ be the length function with respect to (\mathbb{W}, \mathbb{S}) . For every $w \in W$ one has $l_S(w) \leq l_{\mathbb{S}}(w)$.

proof: $N_{\mathbb{G}}(T) \cap G = N_G(T)$ implies $G \cap \mathbb{W} = W$. The inclusion $\Phi^+ \cap s(-\Phi^+) \subset \underline{\Phi}^+ \cap s(-\underline{\Phi}^+)$ for any $s \in \mathbb{S}$ implies $W \cap \mathbb{S} \subset S$.

Let $w = t_1 \cdots t_r \in W, t_i \in S$ reduced expression and assume $l_S(w) < r$. It must be possible in \mathbb{W} to write w as a subword of $t_1 \cdots \hat{t}_i \cdots t_r$ for some $i \in \{1, \dots, r\}$. But then $r = l_S(w) \leq l_{\mathbb{S}}(t_1 \cdots \hat{t}_i \cdots t_r) < r$. \square

Lemma 4. Let $(\mathbb{G}, \mathbb{B}, T), (G, B, T)$ and $(\mathbb{W}, \mathbb{S}), (W, S)$ as before. Then $G \cap {}^x \mathbb{B}, x \in \mathbb{W}$ is a Borel subgroup of G . Let $s \in \mathbb{S}$, then one has

- (1) If $Wxs \neq Wx$ then $G \cap {}^{xs} \mathbb{B} = G \cap {}^x \mathbb{B}$.
- (2) If $Wxs = Wx$, then ${}^x s \in W$ and $G \cap {}^{xs} \mathbb{B} = {}^{xs}[G \cap {}^x \mathbb{B}]$.

This gives an algorithm to find for any $x \in \mathbb{W}$ a $z \in W$ such that $G \cap {}^x \mathbb{B} = {}^z[G \cap \mathbb{B}]$.

proof: Let $s \in \mathbb{S}, {}^x s \notin W$, then $\pm x(\alpha_s) \notin \Phi$ and this implies

$$\Phi \cap xs(\Phi) = \Phi \cap [x(\Phi) \setminus \{x(\alpha_s)\} \cup \{-x(\alpha_s)\}] = \Phi \cap x(\Phi).$$

Therefore, the Lie algebras of the Borel groups $G \cap {}^x \mathbb{B}$ and $G \cap {}^{xs} \mathbb{B}$ have the same weights for T , this proves they are equal. \square

Remark. In the setup of the beginning, we can always find unique representatives $x_i \in \mathbb{W}, i \in I$ for the elements in $W \setminus \mathbb{W}$ which fulfill

$$B_i = G \cap {}^{x_i} \mathbb{B} = G \cap \mathbb{B} = B.$$

This follows because for every $i \in I$ there is a bijection

$$\begin{aligned} Wx_i &\rightarrow \{\text{Borel subgroups of } G \text{ containing } T\} \\ vx_i &\mapsto {}^v[G \cap {}^{x_i} \mathbb{B}] \end{aligned}$$

Then, there exists a unique $v \in W$ such that ${}^v[G \cap {}^{x_i} \mathbb{B}] = G \cap \mathbb{B}$, replace x_i by vx_i as a representative for Wx_i . We will call these representatives **minimal coset representatives**⁵. Observe for $is \neq i$ one has $x_{is} = x_i s$ by lemma 4, (b), (2).

But since the images of $G/B_i, i \in I$ inside \mathbb{G}/\mathbb{B} are disjoint, we prefer not to identify all $B_i, i \in I$.

⁵if G is a Levi-group in \mathbb{G} they are the minimal coset representatives, in this more general situation the notion is not defined.

3 Generalized quiver Hecke algebras

Notational conventions: We fix the ground field for all algebraic varieties and Lie algebras to be \mathbb{C} . If we denote an algebraic group by double letters like $\mathbb{G}, \mathbb{B}, \mathbb{U}$ we take the calligraphic letters for the Lie algebras $\mathcal{G}, \mathcal{B}, \mathcal{U}$ respectively. If we denote an algebraic group by roman letters like G, B, U we take the small fraktur letters for the Lie algebras, $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}$ respectively.

If we have a subgroup $P \subset G$ of a group and an element $g \in G$ we write ${}^gP := gPg^{-1}$ for the conjugate subgroup.

3.1 Generalized quiver-graded Springer map and the generalized quiver Hecke algebra

We define a generalized quiver-graded Springer theory for generalized quiver representations in the sense of Derksen and Weymann. Given $(\mathbb{G}, \mathbb{B}, \mathcal{U}, H, V)$ (and some not mentioned $H \subset T \subset \mathbb{B}$) with

- * \mathbb{G} is a connected reductive group, $H \subset T$ is a subgroup of a maximal torus in \mathbb{G} , we set $G = C_{\mathbb{G}}(H)^\circ$ (then G is also reductive with $T \subset G$ is a maximal torus in G).
- * $T \subset \mathbb{B} \subset \mathbb{G}$ a Borel subgroup, then $B := \mathbb{B} \cap G$ is a Borel subgroup of G , We write (\mathbb{W}, \mathbb{S}) for the Coxeter system associated with $(\mathbb{G}, \mathbb{B}, T)$ and (W, S) for the one associated to (G, B, T) . Observe, that $W \subset \mathbb{W}$.
- * We call a \mathbb{B} -subrepresentation $\mathcal{U}' \subset \mathcal{G} = \text{Lie}(\mathbb{G})$ (of the adjoint representation which we denote by $(g, x) \mapsto {}^g x, g \in \mathbb{G}, x \in \mathcal{G}$) *suitable* if $(\mathcal{U}')^T = \{0\}$ and $\mathcal{U}' \cap {}^s \mathcal{U}'$ is \mathbb{B} -stable for all $s \in \mathbb{S}$. Let $\mathcal{U} = \bigoplus_{k=1}^r \mathcal{U}^{(k)}$ a \mathbb{B} -representation with each $\mathcal{U}^{(k)}$ is suitable. We call $I := W \setminus \mathbb{W}$ the set of **complete dimension filtrations**. Let $\{x_i \in \mathbb{W} \mid i \in I\}$ be a complete representing system of the cosets in I . We lift every element of the Weyl groups \mathbb{W} (and W) to elements in \mathbb{G} (and G) and denote the lifts by the same letter. For every $i \in I$ we set

$$B_i := {}^{x_i} \mathbb{B} \cap G,$$

this is a Borel subgroup of \mathbb{G} , see lemma 4.

- * $V = \bigoplus_{k=1}^r V^{(k)}$ with $V^{(k)} \subset \mathcal{G}$ is a G -subrepresentation.
 $F_i = \bigoplus_{k=1}^r F_i^{(k)}$ with $F_i^{(k)} := V^{(k)} \cap {}^{x_i} \mathcal{U}^{(k)}$ is a B_i -subrepresentation of $V^{(k)}$.

Then, we get a tuple $(G, B_i, V, F_i)_{i \in I}$ as in definition 1. Recall the notation, we have the union of the associated fibre bundles $\mu: E = \bigsqcup_{i \in I} \underbrace{G \times^{B_i} F_i}_{E_i} \rightarrow \bigsqcup_{i \in I} G/B_i = (\mathbb{G}/\mathbb{B})^H \subset \mathbb{G}/\mathbb{B}$ where the H -operation is multiplication

from the left. We have the Springer map $\pi: E \rightarrow V, [g, f] \mapsto gf$ and we have the Steinberg variety $Z = E \times_V E = \bigsqcup_{i, j \in I} Z_{i, j}$ with its two maps $V \xleftarrow{p} Z \xrightarrow{m} \mathbb{G}/\mathbb{B} \times \mathbb{G}/\mathbb{B}$. For each $i, j \in I$ we write $\mu_i = \mu|_{E_i}, \pi_i = \pi|_{E_i}, m_{ij} = m|_{Z_{ij}}, p_{ij} = p|_{Z_{ij}}$ for the restrictions.

The equivariant Borel-Moore homology of a Steinberg variety with complex coefficients together with the convolution operation defines a graded \mathbb{C} -algebra

$$\mathcal{Z}_G := H_*^G(Z),$$

which we call (G -equivariant) Steinberg algebra. The aim of this section is to describe \mathcal{Z}_G in terms of generators and relations. If we set

$$H_{[p]}^G(Z) := \bigoplus_{i, j \in I} H_{e_i + e_j - p}^G(Z_{i, j}), \quad e_i = \dim_{\mathbb{C}} E_i$$

then $H_{[*]}^G(Z)$ is a graded $H_G^*(pt)$ -algebra. We denote the right \mathbb{W} -operation on $I = W \setminus \mathbb{W}$ by $(i, w) \mapsto iw, i \in I, w \in \mathbb{W}$.

Theorem 3.1. *Let $\mathcal{E}_i = \mathbb{C}[t] = \mathbb{C}[x_i(1), \dots, x_i(n)], i \in I$. Then $\mathcal{Z}_G \subset \text{End}_{\mathbb{C}[t]^W \text{-mod}}(\bigoplus_{i \in I} \mathcal{E}_i)$ is the graded \mathbb{C} -subalgebra generated by*

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = \text{rk}(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

defined for $k \in I, f \in \mathcal{E}_k$ as $1_i(f) = \delta_{ik}(f), z_i(t)(f) = \delta_{ik}(f)$ and

$$\sigma_i(s)(f) := \begin{cases} q_i(s) \frac{s(f) - f}{\alpha_s}, & (\in \mathcal{E}_i) \text{ if } i = is = k, \\ q_i(s) s(f) & (\in \mathcal{E}_i) \text{ if } i \neq is = k, \\ 0, & \text{else,} \end{cases} \quad q_i(s) := \prod_{\substack{\alpha \in \Phi_U, s(\alpha) \notin \Phi_U, \\ x_i(\alpha) \in \Phi_V}} \alpha \in \mathcal{E}_i$$

with $\Phi_{\mathcal{U}} = \bigsqcup_k \Phi_{\mathcal{U}^{(k)}}$, $\Phi_{\mathcal{U}^{(k)}} \subset \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C}) \subset \mathbb{C}[t]$ is the set of T -weights for $\mathcal{U}^{(k)}$ and $\Phi_V = \bigsqcup_k \Phi_{V^{(k)}}$, $\Phi_{V^{(k)}} \subset \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ is the set of T -weights for $V^{(k)}$.

Furthermore, one has

$$\deg 1_i = 0, \quad \deg z_i(k) = 2, \quad \deg \sigma_i(s) = \begin{cases} 2(\deg q_i(s)) - 2, & \text{if } is = i \\ 2 \deg q_i(s), & \text{if } is \neq i \end{cases}$$

where $\deg q_i(s)$ refers to the degree as homogeneous polynomial in $\mathbb{C}[t]$.

For $\mathcal{U} = \text{Lie}(\mathbb{U})^{\oplus r}$, where $\mathbb{U} \subset \mathbb{B}$ is the unipotent radical, one has $q_i(s) = \alpha_s^{h_i(s)}$, where $h_i(s) := \#\{k \in \{1, \dots, r\} \mid x_i(\alpha_s) \in \Phi_{V^{(k)}}\}$, $V = \bigoplus_k V^{(k)}$.

If $Wx_i \neq Wx_i s$ then $h_i(s) = \#\{k \mid V^{(k)} \subset \mathcal{R}, x_i(\alpha_s) \in \Phi_{V^{(k)}}\}$. We say that this number counts arrows.

If $Wx_i = Wx_i s$, then $h_i(s) = \#\{k \mid V^{(k)} \subset \mathfrak{g}, x_i(\alpha_s) \in \Phi_{V^{(k)}}\}$. We say that this number counts loops.

For $\mathcal{U} = \text{Lie}(\mathbb{U})^{\oplus r}$ for $\mathbb{U} \subset \mathbb{B}$ the unipotent radical we have the following result which generalizes the part- d subalgebras of KLR algebras to arbitrary connected reductive groups and allowing quivers with loops. In that case, we call the Steinberg algebra \mathcal{Z}_G **generalized quiver Hecke algebra**.

It can be described by the following generators and relations. For a reduced expression $w = s_1 s_2 \cdots s_k$ we set

$$\sigma_i(s_1 s_2 \cdots s_k) := \sigma_i(s_1) \sigma_{i s_1}(s_2) \cdots \sigma_{i s_1 s_2 \cdots s_{k-1}}(s_k)$$

If it is understood that the definition depends on a particular choice of a reduced expression for w , we write $\sigma_i(w) := \sigma_i(s_1 s_2 \cdots s_k)$. Furthermore, we consider

$$\Phi: \bigoplus_{i \in I} \mathbb{C}[x_i(1), \dots, x_i(n)] \cong \bigoplus_{i \in I} \mathbb{C}[z_i(1), \dots, z_i(n)], \quad x_i(t) \mapsto z_i(t)$$

as the left \mathbb{W} -module $\text{Ind}_W^{\mathbb{W}} \mathbb{C}[t]$, we fix the polynomials

$$c_i(s, t) := \Phi(\sigma_i(s)(x_i(t))) \in \bigoplus_{i \in I} \mathbb{C}[z_i(1), \dots, z_i(n)], \quad i \in I, \quad 1 \leq t \leq n, \quad s \in \mathbb{S}.$$

If we add in some cases limits to the counting numbers $h_i(s)$, we can describe the generators and relations fully.

Theorem 3.2. *Let $\mathbb{S} \subset \mathbb{W} = \text{Weyl}(\mathbb{G}, T)$ be the simple reflections. Under the following assumption for the data $(\mathbb{G}, \mathbb{B}, \mathcal{U} = (\text{Lie}(\mathbb{U}))^{\oplus r}, H, V)$. We assume for any $s, t \in \mathbb{S}, i \in I$ the following limits for the counting numbers $h_i(s), h_i(t)$ (see previous theorem)*

(B2) *If the root system spanned by α_s, α_t is of type B_2 (or $stst = tsts$ is the minimal relation), then for every $i \in I$ such that $is = i = it$ one has $h_i(s), h_i(t) \in \{0, 1, 2\}$.*

(G2) *If the root system spanned by α_s, α_t is of type G_2 (or $ststst = tststs$ is the minimal relation), then for every $i \in I$ such that $is = i = it$ one has $h_i(s) = 0 = h_i(t)$.*

Then the generalized quiver Hecke algebra for $(\mathbb{G}, \mathbb{B}, \mathcal{U} = (\text{Lie}(\mathbb{U}))^{\oplus r}, H, V)$ is the graded \mathbb{C} -algebra with generators

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = \text{rk}(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

in degrees

$$\deg 1_i = 0, \quad \deg z_i(k) = 2, \quad \deg \sigma_i(s) = \begin{cases} 2h_i(s) - 2, & \text{if } is = i \\ 2h_i(s), & \text{if } is \neq i \end{cases}$$

and relations

(1) (orthogonal idempotents) $1_i 1_j = \delta_{i,j} 1_i, \quad 1_i z_i(t) 1_i = z_i(t), \quad 1_i \sigma_i(s) 1_{is} = \sigma_i(s)$

(2) (polynomial subalgebras) $z_i(t) z_i(t') = z_i(t') z_i(t)$

(3) (relation implied by $s^2 = 1$)

$$\sigma_i(s) \sigma_{is}(s) = \begin{cases} 0 & , \text{if } is = i, h_i(s) \text{ is even} \\ -2\alpha_s^{h_i(s)-1} \sigma_i(s) & , \text{if } is = i, h_i(s) \text{ is odd} \\ (-1)^{h_{is}(s)} \alpha_s^{h_i(s)+h_{is}(s)} & , \text{if } is \neq i \end{cases}$$

(4) (straightening rule)

$$\sigma_i(s)z_i(t) - s(z_i(t))\sigma_i(s) = \begin{cases} c_i(s, t), & , \text{ if } is = i \\ 0 & , \text{ if } is \neq i. \end{cases}$$

(5) (braid relations) Let $s, t \in \mathbb{S}, st = ts$, then $\sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s)$

Let $s, t \in \mathbb{S}$ not commuting such that $x := sts \cdots = tst \cdots$ minimally, $i \in I$. There exists explicit polynomials $(Q_w)_{w < x}$ in $\alpha_s, \alpha_t \in \mathbb{C}[\mathfrak{t}]$ such that $\sigma_i(sts \cdots) - \sigma_i(tst \cdots) = \sum_{w < x} Q_w \sigma_i(w)$. Observe that for $w < x$ there exists just one reduced expression.

The polynomials Q_w can be made explicit, the interested reader can find them in the proof in the end of this section, see subsection 3.12.

3.2 The equivariant cohomology of flag varieties

The (co)-homology rings of a point. Let G be reductive group, $T \subset P \subset G$ with P a parabolic subgroup and T a maximal torus, we write W for the Weyl group associated to (G, T) and $X(T) = \text{Hom}_{Gr}(T, \mathbb{C}^*)$ for the group of characters. Let ET be a contractible topological space with a free T -operation from the right.

(1) For every character $\lambda \in X(T)$ denote by

$$S_\lambda := ET \times^T \mathbb{C}_\lambda$$

the associated T -equivariant line bundle over $BT := ET/T$ to the T -representation \mathbb{C}_λ which is \mathbb{C} with the operation $t \cdot c := \lambda(t)c$. The first chern class defines a homomorphism of abelian groups

$$c: X(T) \rightarrow H^2(BT), \quad \lambda \mapsto c_1(S_\lambda).$$

Let $\text{Sym}_{\mathbb{C}}(X(T))$ be the symmetric algebra with complex coefficients generated by $X(T)$, it can be identified with the ring of regular function $\mathbb{C}[\mathfrak{t}]$ on $\mathfrak{t} = \text{Lie}(T)$ (with doubled degrees), where $X(T) \otimes_{\mathbb{Z}} \mathbb{C}$ is mapped via taking the differential (of elements in $X(T)$) to $\mathfrak{t}^* = \text{Hom}_{\mathbb{C}\text{-lin}}(\mathfrak{t}, \mathbb{C}) \subset \mathbb{C}[\mathfrak{t}]$ (both are the degree 2 elements).

The previous map extends to an isomorphism of graded \mathbb{C} -algebras

$$\mathbb{C}[\mathfrak{t}] \rightarrow H_T^*(pt) = H^*(BT)$$

In fact this is a W -linear isomorphism where the W -operation on $\mathbb{C}[\mathfrak{t}]$ is given by, $(w, f) \mapsto w(f), w \in W, f \in \mathbb{C}[\mathfrak{t}]$ with

$$w(f): \mathfrak{t} \rightarrow \mathbb{C}, t \mapsto f(w^{-1}tw),$$

see [Tu03]. We can choose ET such that it also has a free G -operation from the right (i.e. $ET := EG$), then $BT = ET/T$ has an induced Weyl group action from the right given by $xT \cdot w := xwT, w \in W, x \in ET$. The pullbacks of this group operation induce a left W -operation on $H_T^*(pt)$.

(2) $H_T^*(pt) = H_{-*}^T(pt)$ (Poincare duality), $H_G^*(pt) = (H_T^*(pt))^W = (H_{-*}^T(pt))^W = H_{-*}^G(pt)$ (Splitting principle).

The cohomology rings of homogeneous vector bundles over G/P . Let G be reductive group, $T \subset B \subset P \subset G$ with B a Borel subgroup, P parabolic and T a maximal torus.

(1) For $\lambda \in X(T)$ we denote by $L_\lambda := G \times^B \mathbb{C}_\lambda$ the associated line bundle to the B -representation \mathbb{C}_λ given by the trivial representation when restricted to the unipotent radical and λ when restricted to T . Let $\mu: E \rightarrow G/B$ be a G -equivariant vector bundle. Then, $\mu^*(L_\lambda)$ is a line bundle on E and

$$K_\lambda := EG \times^G \mu^*(L_\lambda) \rightarrow EG \times^G E$$

is a line bundle over $EG \times^G E$. There is an isomorphism of graded \mathbb{C} -algebras

$$\begin{aligned} \mathbb{C}[\mathfrak{t}] &\rightarrow H_G^*(E) = H^*(EG \times^G E) \\ X(T) \ni \lambda &\mapsto c_1(K_\lambda). \end{aligned}$$

with $\deg \lambda = 2$ for $\lambda \in X(T)$.

(By definition, equivariant chern classes are defined as $c_1^G(\mu^*L_\lambda) := c_1(K_\lambda)$).

(2) Let $\mu: E \rightarrow G/P$ be a G -equivariant vector bundle, then there is an isomorphism of graded \mathbb{C} -algebras

$$H_G^*(E) \rightarrow (H_T^*(pt))^{W_L}.$$

The proofs of these statements:

ad (1) Arabia proved that $H_G^*(G/B) \cong H_T^*(pt)$ as graded \mathbb{C} -algebras (cp. [Ara85]), the composition with the isomorphism from the previous lemma gives an isomorphism

$$c: \mathbb{C}[t] \rightarrow H_G^*(G/B), : \lambda \mapsto c_1(EG \times^G L_\lambda) =: c_1^G(L_\lambda)$$

Now, we show that for a vector bundle $\mu: E \rightarrow G/P$ with $P \subset G$ parabolic, the induced pullback map

$$\mu^*: H_G^*(G/P) \rightarrow H_G^*(E), \quad c_1^G(L_\lambda) \mapsto c_1^G(\mu^*L_\lambda)$$

is an isomorphism of graded $H_G^*(pt)$ -algebras. We already know that it is a morphism of graded $H_G^*(pt)$ -algebras, to see it is an isomorphism, apply the definition and Poincare duality to get a commutative diagram

$$\begin{array}{ccc} H_G^k(G/P) & \xrightarrow{\mu^*} & H_G^k(E) \\ \cong \downarrow & & \cong \downarrow \\ H_{2 \dim G/P - k}^G(G/P) & \xrightarrow{\mu^*} & H_{2 \dim E - k}^G(E) \end{array}$$

the lower morphism μ^* is the pullback morphism which gives the Thom isomorphism, therefore the upper μ^* is also an isomorphism.

ad (2) By the last proof, we already know $H_G^*(E) \cong H_G^*(G/P)$. Then apply the isomorphism of Arabia see [Ara85], this gives $H_G^*(G/P) \cong H_P^*(pt)$. Now, P homotopy-retracts on its Levy subgroup L , this implies $H_P^*(pt) = H_L^*(pt)$, together with the (2) in the previous lemma we are done. □

The cohomology ring of the flag variety as subalgebra of the Steinberg algebra. Let G be reductive group, $T \subset P \subset G$ with P parabolic and T a maximal torus. Let V be a G -representation and $F \subset V$ be a P -subrepresentation, let $E := G \times^P F$ and $Z := E \times_V E$ be the associated Steinberg variety. The diagonal morphism $E \rightarrow E \times E$ factorizes over Z and induces an isomorphism $E \rightarrow Z_e$ which induces an isomorphism of algebras

$$H_G^*(G/P) \rightarrow H_{2 \dim E - *}^G(Z_e),$$

recall that the convolution product on $H_*^G(Z_e)$ maps degrees $(i, j) \mapsto i + j - 2 \dim E$. For the proof of the statement: Obviously you have an isomorphism

$$H_G^*(G/B) \xrightarrow{\mu^*} H_G^*(E) \cong H_G^*(Z_e) \rightarrow H_{2 \dim E - *}^G(Z_e)$$

where the last isomorphism is Poincare duality. But we need to see that this is a morphism of algebras where $H_*^G(Z_e)$ is the convolution algebra with respect to the embedding $Z_e \cong E \xrightarrow{\text{diag}} E \times E$. This follows from [CG97], Example 2.7.10 and section 2.6.15. □

We observe that the algebra $\mathbb{C}[t]$ with generators $t \in \mathfrak{t}^*$ in degree 2 plays three different roles in the last lemmata. It is the T -equivariant cohomology of a point, it is the G -equivariant cohomology of a complete flag variety G/B , it can be found as the subalgebra $H_*^G(Z_e) \subset H_*^G(Z)$.

3.3 Notation for the fixed points

Torus fixed points. Let $T \subset P \subset G$ be reductive group with a parabolic subgroup P and a maximal torus T . Let W be the Weyl group associated to (G, T) and $\text{Stab}(P) := \{w \in W \mid wPw^{-1} = P\}$. For $w = x \text{Stab}(P) \in W/\text{Stab}(P)$ we set $wP := xP \in G/P$. Then, one has $(G/P)^T = \{wP \in G/P \mid w \in W/\text{Stab}(P)\}$. See e.g. [Här99], satz 2.12, page 13.

Let $P_1, P_2 \subset G$ be a reductive group with two parabolic subgroup, $F_1, F_2 \subset V$ a G -representation with a P_1 and P_2 -subrepresentation. Assume $(GF_i)^T = \{0\}$. We write

($E_i = G \times^{P_i} F_i, \mu_i: E_i \rightarrow G/P_i, \pi: E_i \rightarrow V$) for the associated Springer triple and $Z := E_1 \times_V E_2, m: Z \rightarrow (G/P_1) \times (G/P_2)$ for the Steinberg variety. Then, there are induced a bijections $\mu_i^T: E_i^T \rightarrow (G/P_i)^T, m^T: Z^T \rightarrow (G/P_1)^T \times (G/P_2)^T$. More explicit we have

$$\begin{aligned} E_i^T &= \{\phi_w := (0, wP_i) \in V \times G/P_i \mid w \in W/\text{Stab}(P_i)\} \subset E_i \\ Z^T &= \{\phi_{x,y} := (0, xP_1, yP_2) \in V \times G/P_1 \times G/P_2 \mid x \in W/\text{Stab}(P_1), y \in W/\text{Stab}(P_2)\} \\ &\subset Z. \end{aligned}$$

Furthermore, for any $w \in W/\text{Stab}(P_2)$ let $Z^w := m^{-1}(G \cdot (P_1, wP_2))$ and $m_w := m|_{Z^w}: Z^w \rightarrow G \cdot (P_1, wP_2)$ be the induced map. There is an induced Bruhat order \leq on $W/\text{Stab}(P_2)$ by taking the Bruhat order of minimal length representatives.

$$\begin{aligned} (Z^w)^T &= \{\phi_{x,xw} = (0, xP_1, xwP_2) \in V \times G/P_1 \times G/P_2 \mid x \in W\} \\ \overline{Z^w}^T &= \{\phi_{x,xv} \mid x \in W, v \leq w\} = \bigcup_{v \leq w} (Z^v)^T \end{aligned}$$

There is a bijection $W/(\text{Stab}(P_1) \cap {}^w\text{Stab}(P_2)) \rightarrow (Z^w)^T, x \mapsto \phi_{x,xw}$.

For the proof: Obviously, one has $E_i^T \subset V^T \times (G/P_i)^T = \{0\} \times (G/P_i)^T$. But we also have a zero section s of the vector bundle $\pi: E_i \rightarrow G/P_i$ which gives the closed embedding $G/P_i \rightarrow E_i \subset V \times (G/P_i), gP_i \mapsto (0, gP_i)$. One has $Z^T \subset V^T \times (G/P_1)^T \times (G/P_2)^T = \{0\} \times (G/P_1)^T \times (G/P_2)^T$. But using the description of $Z = \{(v, gP_1, hP_2) \in V \times G/P_1 \times G/P_2 \mid (v, gP_1) \in E_1, (v, hP_2) \in E_2\}$, we see that $\{0\} \times (G/P_1)^T \times (G/P_2)^T \subset Z$ and these are obviously T -fixed points.

We have $(Z^w)^T \subset Z^w \cap Z^T = \{\phi_{x,xw} \mid x \in W\}$ and one can see the other inclusion, too. Also, we have $\overline{Z^w}^T \subset (\bigcup_{v \leq w} Z^v)^T = \bigcup_{v \leq w} (Z^v)^T$. Consider the closed embedding

$$s: G/P_1 \times G/P_2 \rightarrow Z, \quad (gP_1, hP_2) \mapsto (0, gP_1, hP_2).$$

Clearly $s(G(P_1, wP_2)) \subset Z^w \subset \overline{Z^w}$, but since s is a closed embedding we have

$$\bigcup_{v \leq w} (Z^v)^T \subset \overline{s(G(P_1, wP_2))} = \overline{s(G(P_1, wP_2))} \subset \overline{Z^w}$$

which yields the other inclusion.

notation. Now, in the set-up of this section, observe, that $(\bigsqcup_{i \in I} G_i/B_i)^T = ((\mathbb{G}/\mathbb{B})^H)^T = (\mathbb{G}/\mathbb{B})^T$, and $(\mathbb{G}/\mathbb{B})^T = \{w\mathbb{B} \mid w \in \mathbb{W}\}$. For any $w \in \mathbb{W}$ there exists a unique $i \in I$ such that $x^i := wx_i^{-1} \in W$, this implies $w\mathbb{B} = x^i(x_i\mathbb{B}) = \iota_i(x^i B_i) \in (G/B_i)^T$. Therefore, we write

$$\begin{aligned} \left(\bigsqcup_{i \in I} G_i/B_i\right)^T &= (\mathbb{G}/\mathbb{B})^T = \bigsqcup_{i \in I} \{wx_i\mathbb{B} \mid w \in W \subset \mathbb{W}\} \\ E^T &= \{\phi_{wx_i} = (0, wx_i\mathbb{B}) \mid i \in I, w \in W\} \\ Z^T &= \bigsqcup_{i,j \in I} \{\phi_{wx_i, vx_j} = (0, wx_i\mathbb{B}, vx_j\mathbb{B}) \mid w \in W, v \in W\}. \end{aligned}$$

Let $w, v \in \mathbb{W}$ and $i, j \in I$ such that $w^i := wx_i^{-1} \in W, v^j := vx_j^{-1} \in W$. We then set $\phi_w := \phi_{w^i x_i}, \phi_{w,v} = \phi_{w^i x_i, v^j x_j}$.

As we have bijections $(0, wx_i\mathbb{B}) \mapsto wx_i\mathbb{B}, (0, wx_i\mathbb{B}, vx_j\mathbb{B}) \mapsto (wx_i\mathbb{B}, vx_j\mathbb{B})$ between E^T and $(\mathbb{G}/\mathbb{B})^T, Z^T$ and $(\mathbb{G}/\mathbb{B} \times \mathbb{G}/\mathbb{B})^T$, we denote the T -fixed by the same symbols.

3.3.1 The fibres over the fixpoints.

Remember, by definition we have $F_i = \mu^{-1}(\phi_{x_i})$. For any $w = w^i x_i \in \mathbb{W}, w^i \in W$ We set

$$F_w := \mu^{-1}(\phi_w) = \mu_i^{-1}(\phi_{w^i x_i}) = w^i F_i = \bigoplus_{k=1}^r V^{(k)} \cap w\mathcal{U}^{(k)}$$

and for $x \in \mathbb{W}$

$$\begin{aligned} F_{x,xw} &:= m^{-1}(\phi_{x,xw}) = F_x \cap F_{xw} \\ &= \bigoplus_{k=1}^r V^{(k)} \cap x[\mathcal{U}^{(k)} \cap w\mathcal{U}^{(k)}] \end{aligned}$$

Now assume $\mathcal{U} = \text{Lie}(\mathbb{U})^{\oplus r}$. We choose $V = \bigoplus_{k=1}^t V^{(k)} \oplus \bigoplus_{k=t+1}^r V^{(k)}$ with $V^{(k)} \subset \mathcal{R}, 1 \leq k \leq t, V^{(k)} = \mathfrak{g}^{(k)}$ with $\mathfrak{g}^{(k)} \subset \mathfrak{g}$ is a direct summand, $t+1 \leq k \leq r$. The fibres look like

$$F_w = \bigoplus_{k=1}^t V^{(k)} \cap {}^w \text{Lie}(\mathbb{U}) \oplus \bigoplus_{k=t+1}^r V^{(k)} \cap {}^w \mathfrak{u}^{(k)}$$

where $\mathfrak{u}^{(k)}$ is the Lie subalgebra spanned by the weights > 0 in $\mathfrak{g}^{(k)}$.

$$F_{x,xw} = \bigoplus_{k=1}^t V_k \cap {}^x [(\text{Lie}(\mathbb{U})) \cap {}^w (\text{Lie}(\mathbb{U}))] \oplus \bigoplus_{k=t+1}^r V_k \cap {}^x [\mathfrak{u}^{(k)} \cap {}^w \mathfrak{u}^{(k)}]$$

Lemma 5. *Let $\mathcal{U} = \text{Lie}(\mathbb{U})^{\oplus r}$ and $x \in \mathbb{W}, s \in \mathbb{S}$ we set $h_{\overline{x}}(s) := \#\{k \in \{1, \dots, r\} \mid x(\alpha_s) \in \Phi_{V^{(k)}}\}$ where $V = \bigoplus_{k=1}^r V^{(k)}$ and $\Phi_{V^{(k)}} \subset \Phi$ are the T -weights of $V^{(k)}$. If $x = x^i x_i$ with $x^i \in W$, then $h_{\overline{x}}(s) = h_{\overline{x^i}}(s) =: h_i(s)$. One has*

$$F_{x_i}/F_{x_i, x_i s} = (\mathcal{G}_{x_i(\alpha_s)})^{\oplus h_i(s)}.$$

- (1) If $x s \notin W$ then $h_i(s) = \#\{k \mid V^{(k)} \subset \mathcal{R}, x_i(\alpha_s) \in \Phi_{V^{(k)}}\}$.
(2) If $x s \in W$, then $h_i(s) = \#\{k \mid V^{(k)} \subset \mathfrak{g}, x_i(\alpha_s) \in \Phi_{V^{(k)}}\}$.

proof: Without loss of generality $V \subset \mathcal{G}, \mathcal{U} = \text{Lie}(\mathbb{U})$, set $x := x_i$, we have a short exact sequence

$$0 \rightarrow V \cap {}^x [\mathcal{U} \cap {}^s \mathcal{U}] \rightarrow V \cap {}^x \mathcal{U} \rightarrow V \cap \mathcal{G}_{x(\alpha_s)} \rightarrow 0$$

Now, $V \cap \mathcal{G}_{x(\alpha_s)} = 0$ if and only if $x(\alpha_s) \notin \Phi_V$.

(1) If $x s \notin W$ then $x(\alpha_s) \notin \Phi$ where Φ are the T -weights of \mathfrak{g} .

That means, if $V \subset \mathfrak{g}$ we get $h_i(s) = 0$. (2) If $x s \in W$, then $x(\alpha_s) \in \Phi$. This means, if $V \subset \mathcal{R}$ we get $h_i(s) = 0$.

□

3.4 Relative position stratification

3.4.1 In the flag varieties

Let $w \in \mathbb{W}, i, j \in I$. We define

$$\begin{aligned} C^w &:= \mathbb{G}\phi_{e,w} \cap \left(\bigsqcup_{i \in I} G/B_i \times \bigsqcup_{i \in I} G/B_i \right), & C^{\leq w} &:= \overline{\mathbb{G}\phi_{e,w}} \cap \left(\bigsqcup_{i \in I} G/B_i \times \bigsqcup_{i \in I} G/B_i \right) \\ C_{i,j}^w &:= C^w \cap (G/B_i \times G/B_j), & C_{i,j}^{\leq w} &:= C^{\leq w} \cap (G/B_i \times G/B_j) \end{aligned}$$

One has $C_{i,j}^w \cong G/(B_i \cap {}^w B_j \cap G)$. The following lemma is due to Varagnolo and Vasserot, see [VV11b], lemma 2.6 They state it for \mathbf{GL}_n but their proof works for connected reductive groups.

Lemma 6. ([VV11b], Lemma 2.6) *Let $s \in \mathbb{S}, i, j \in I$.*

- (1) $C^{\leq s}$ is smooth, it equals $C^s \cup C^e$.
(2) $C_{i,j}^{\leq s} = \emptyset$ unless $Wx_j \in \{Wx_i, Wx_i s\}$.
(3) Assume that $Wx_i \neq Wx_i s$ and let $j \in I$ such that $x_i s x_j^{-1} \in W$, then one has

$$\iota_i(G/B_i) \neq \iota_j(G/B_j), \quad C_{i,j}^{\leq s} = C_{i,j}^s, \quad C_{i,i}^{\leq s} = C_{i,i}^e$$

and $G \cap {}^{x_i} [\mathbb{B} \cap {}^s \mathbb{B}] = G \cap {}^{x_i} \mathbb{B}, C_{i,j}^s = G/(G \cap {}^{x_i} \mathbb{B})$.

- (4) Assume that $Wx_i = Wx_i s W_J = Wx_j$, then one has $i = j$, in particular

$$\iota_i(G/B_i) = \iota_j(G/B_j), \quad C_{i,j}^w = C_{i,i}^w, \quad \text{for all } w$$

and the first equality implies $({}^{x_i} \mathbb{B}) \cap G \neq ({}^{x_i s} \mathbb{B}) \cap G$, there is an isomorphism of G -varieties

$$G \times^{P_i} (({}^{x_i} \mathbb{P}_{J \cup \{s\}} \cap G)/P_i) \rightarrow C_{i,i}^{\leq s}, \quad (g, hP_i) \mapsto (gP_i, ghP_i).$$

3.4.2 In the Steinberg variety

Let $w \in \mathbb{W}$, $i, j \in I$, recall that we have a map $m: Z \rightarrow \mathbb{G}/\mathbb{B}$.

$$\begin{aligned} Z_{i,j}^w &:= m_{i,j}^{-1}(C_{i,j}^w), & Z^w &:= \bigsqcup_{i,j \in I} Z_{i,j}^w \\ Z_{i,j}^{\leq w} &:= \bigcup_{v \leq w, v \in \mathbb{W}} Z_{i,j}^v, & Z^{\leq w} &= \bigcup_{v \leq w, v \in \mathbb{W}^J} Z^v \end{aligned}$$

The next lemma is implicit in the article of [VV11b], we give it as an intermediate step.

Lemma 7. (a) If $C_{i,j}^w \neq \emptyset$, the restriction $m_{i,j}: Z_{i,j}^w \rightarrow C_{i,j}^w$ is a vector bundle with fibres isomorphic to $F_i \cap {}^{x_i w x_j^{-1}} F_j$, it induces a bijection on T -fixed points. In particular, all nonempty $Z_{i,j}^w$ are smooth.

(b) For any $s \in \mathbb{S}$ the restriction $m: \overline{Z}^s \rightarrow C^{\leq s}$ is a vector bundle over its image, in particular \overline{Z}^s is smooth. More precisely, it is a disjoint union $\overline{Z}_{i,j}^s \rightarrow C_{i,j}^{\leq s}$ with

- (1) $\overline{Z}_{i,j}^s \neq \emptyset$ implies $Wx_j \mathbb{W}_J = Wx_i s \mathbb{W}_J$.
- (2) Assume that $Wx_i \mathbb{W}_J \neq Wx_i s \mathbb{W}_J$, then $\overline{Z}_{i,j}^s = Z_{i,j}^s$ and $\overline{Z}_{i,i}^s = \emptyset$.
- (3) Assume that $Wx_i \mathbb{W}_J = Wx_i s \mathbb{W}_J$, then one has $\overline{Z}_{i,i}^s \rightarrow C_{i,i}^{\leq s}$ is a vector bundle.

proof:

- (a) As $C_{i,j}^w$ is a diagonal G -orbit in $G/B_i \times G/B_j$, it is a homogeneous space and the statement easily follows from a wellknown lemma, cp. [Slo80b], p.26, lemma 4.
- (b) (1) If $\overline{Z}_{i,j}^s \neq \emptyset$, then $C_{i,j}^s \neq \emptyset$ and by the proof of the previous lemma 6, (2), the claim follows.
- (2) If $Wx_i \neq Wx_i s$, then by lemma 6, (3), $C_{i,j}^{\leq s} = C_{i,j}^s$ is already closed, therefore $Z_{i,j}^s$ is closed as well. Also, $C_{i,i}^{\leq s} = C_{i,i}^e$ is already closed, therefore $Z_{i,i}^e$ is closed as well.
- (3) If $Wx_i = Wx_i s$, then $C_{i,i}^{\leq s}$ is the closure of the G -orbit $C_{i,i}^s$ and by lemma 6, (4) we have $G \times^{B_i} (({}^{x_i} \mathbb{B} \cap G)/B_i) \rightarrow C_{i,i}^{\leq s}$, $(g, hB_i) \mapsto (gB_i, ghB_i)$ is an isomorphism. We set $X :=$

$$\begin{aligned} X := \{ & (gf, gB_i, ghB_i) \in G(F_i \cap {}^{x_i s} F_i) \times G/B_i \times G/B_i \mid \\ & g \in G, f \in F_i \cap {}^{x_i s x_i^{-1}} F_i, h \in {}^{x_i} \mathbb{B} \cap G \} \end{aligned}$$

and we claim $\overline{Z}_{i,i}^s = X$. First, observe that $X \subset Z_{i,i}$ because $gf = gh(h^{-1}f)$ with $h^{-1}f \in F_i \cap {}^{x_i s x_i^{-1}} F_i$. One can easily check the following steps.

- (*) $X \rightarrow C_{i,i}^{\leq s}$ is a vector bundle with fibre over $F_i \cap {}^{x_i s x_i^{-1}} F_i$. In particular, we get that X is smooth irreducible and $\dim X = \dim Z_{i,i}^s$.
- (*) $Z_{i,i}^s \subset X$.
- (*) X is closed in $Z_{i,i}$ because we can write it as $X = p^{-1}(G(F_i \cap {}^{x_i s x_i^{-1}} F_i)) \cap m^{-1}(C_{i,i}^{\leq s})$. Since $F_i \cap {}^{x_i s x_i^{-1}} F_i$ is (by definition) B_i -stable, we get $G(F_i \cap {}^{x_i s x_i^{-1}} F_i)$ is closed in V . This implies X is closed.

3.5 Convolution operation on the equivariant Borel-Moore homology of the Steinberg variety

Let $A \in \{pt, T, G\}$ with $T \subset G$ as before the maximal torus from the construction data and let $\mathcal{Z}_A := H_*^A(Z)$ the A -equivariant Steinberg algebra and $\mathcal{E}_A := H_A^*(E)$ the A -equivariant cohomology algebra (wrt the cup product) of the variety E . Recall from the the Chriss and Ginzberg convolution operation that \mathcal{Z}_A is a graded \mathcal{E}_A -algebra, we denote this operation by \diamond , and also that \mathcal{E}_A is a graded \mathcal{Z}_A -algebra, we denote this operation by $*$. From [VV11a], section 5, p.606, we know that the operation of \mathcal{Z}_G on \mathcal{E}_G is faithful, i.e. we get an injective \mathbb{C} -algebra homomorphism $\mathcal{Z}_G \hookrightarrow \text{End}(\mathcal{E}_G)$. Recall the cellular fibration property from subsection ???: We choose a total order \leq refining Bruhat order on \mathbb{W} . For each $i, j \in I$ we get a filtration into closed G -stable subsets of $Z_{i,j}$ by setting $Z_{i,j}^{\leq w} := \bigcup_{v \leq w} Z_{i,j}^v$, $w \in \mathbb{W}$. Via the first projection $pr_1: C_{i,j}^v \rightarrow G/B_i$ is a G -equivariant vector bundle with fibre $B_i v B_j / \overline{B}_j$, we call its (complex) dimension $d_{i,j}^v$, also $Z_{i,j}^v \rightarrow C_{i,j}^v$ is a G -equivariant vector

bundle, we define the complex fibre dimension $f_{i,j}^v$. By the G -equivariant Thom isomorphism (applied twice) we get

$$H_m^G(Z_{i,j}^v) = H_{m-2d_{i,j}^v-2f_{i,j}^v}^G(G/B_i).$$

In particular, it is zero when m is odd and $H_*^G(Z_{i,j}^v)$ is a free $H_*^G(pt)$ -module with basis b_x , $x \in W$, $\deg b_x = 2 \dim(B_i x B_i)/B_i + 2d_{i,j}^v + 2f_{i,j}^v$.

Using the long exact localization sequence in G -equivariant Borel-Moore homology for every $v \in \mathbb{W}$, we see that $Z_{i,j}^v$ is open in $Z_{i,j}^{\leq v}$ with an closed complement $Z_{i,j}^{<v}$. We conclude inductively using the Thom isomorphism that $H_{\text{odd}}^G(Z_{i,j}^{\leq v}) = 0$ and that $H_*^G(Z_{i,j}^{\leq w}) = \bigoplus_{v \leq w} H_*^G(Z_{i,j}^v)$. We observe, that $\#\{w \in \mathbb{W} \mid Z_{i,j}^w \neq \emptyset\} = \#W$ for every $i, j \in I$. It follows that $H_*^G(Z_{i,j})$ is a free $H_*^G(pt)$ -module of rank $\#(W \times W)$, and that every $H_*^G(Z_{i,j}^{\leq v}) \xrightarrow{i_*} H_*^G(Z_{i,j})$ is injective. We can strengthen this result to the following lemma.

Lemma 8. *Let \leq be a total order refining Bruhat order on \mathbb{W} . For any $w \in \mathbb{W}$ set $Z^{\leq w} := m^{-1}(\bigcup_{v \leq w} C^v) = \bigcup_{v \leq w} Z^v$. The closed embedding $i: Z^{\leq w} \rightarrow Z$ gives rise to an injective morphism of $H_G^*(E)$ -modules $i_*: \mathcal{Z}_G^{\leq w} := H_G^*(Z^{\leq w}) \rightarrow \mathcal{Z}_G$. We identify in the following $\mathcal{Z}_G^{\leq w}$ with its image in \mathcal{Z}_G . For all $v \in W$ we have*

$$\begin{aligned} \mathcal{Z}_G^{\leq w} &= \bigoplus_{v \leq w} \mathcal{E}_G \diamond [\overline{Z^v}] && \text{as } \mathcal{E}_G\text{-module} \\ 1_i * \mathcal{Z}_G^{\leq w} * 1_j &= \bigoplus_{v \leq w} \mathcal{E}_i \diamond [\overline{Z_{i,j}^v}] && \text{as } \mathcal{E}_i\text{-module} \end{aligned}$$

where $\mathcal{E}_i = H_G^*(E_i)$. Each $[\overline{Z^v}]$ is nonzero (and not necessarily a homogeneous element). In particular, \mathcal{Z}_G (as ungraded module) is a free left \mathcal{E}_G -module of rank $\#\mathbb{W}$.

proof: Now first observe that set-theoretically we have $E \circ Z^v = Z^v$ (where we use the diagonal embedding for E again). This implies that the direct sum decomposition $H_*^G(Z) = \bigoplus_{v \in \mathbb{W}} H_*^G(Z^v)$ is already a decomposition of $H_G^*(E)$ -modules.

Now we know that we have by the Thom-isomorphism algebra isomorphisms

$$H_G^*(E) \cong H_G^*\left(\bigsqcup_{i \in I} G/B_i\right) \cong H_G^*(Z^v),$$

using that $\#\{(i, j) \mid Z_{i,j}^v \neq \emptyset\} = \#I$. Now, Poincare duality is given by $H_G^q(Z_{i,j}^v) \rightarrow H_{2 \dim Z_{i,j}^v - q}^G(Z_{i,j}^v)$, $\alpha \mapsto \alpha \cdot [Z_{i,j}^v]$ the composition gives

$$H_G^p(E_i) \rightarrow H_{2 \dim Z_{i,i}^v - q}^G(Z_{i,i}^v), \quad c \mapsto c \cdot [Z_{i,i}^v].$$

□

Lemma 9. *For each $x, y \in \mathbb{W}$ with $l(x) + l(y) = l(xy)$ we have*

$$\mathcal{Z}_G^{\leq x} * \mathcal{Z}_G^{\leq y} \subset \mathcal{Z}_G^{\leq xy}$$

proof: By definition of the convolution product, it is enough to check that for all $w \leq x, v \leq y$ one has for the set theoretic convolution product

$$Z_{i,j}^w \circ Z_{j',k}^v \subset \begin{cases} \emptyset, & j \neq j' \\ Z_{i,k}^{\leq xy}, & j = j' \end{cases}$$

for $i, j, j', k \in I$, because by definition $Z^{\leq x} \circ Z^{\leq y} = \bigcup_{w \leq x, v \leq y} Z^w \circ Z^v$. Now, the case $j \neq j'$ follows directly from the definition. Let $j = j'$. Let $\mathbb{C}^w := \mathbb{G}(\mathbb{B}, w\mathbb{B}) \subset \mathbb{G}/\mathbb{B} \times \mathbb{G}/\mathbb{B}$. According to Hinrich, Joseph [HJ05], 4.3 it holds $\mathbb{C}^w \circ \mathbb{C}^v \subset \mathbb{C}^{wv}$ for all $v, w \in \mathbb{W}$. Now, we can adapt this argument to prove that $C_{i,j}^w \circ C_{j,k}^v \subset C_{j,k}^{wv}$ as follows:

Since $C_{i,j}^w \neq \emptyset, C_{j,k}^v \neq \emptyset$ we have that $w_0 = x_i w x_j^{-1} \in W, v_0 = x_j v x_k^{-1} \in W$ and $C_{i,j}^w = G(B_i, w_0 B_j), C_{j,k}^v = G(B_j, v_0 B_k)$. We pick $M_1 = G/B_i, M_2 = G/B_j, M_3 = G/B_k$ for the convolution and get

$$p_{13}(p_{12}^{-1} C_{i,j}^w \cap p_{23}^{-1} C_{j,k}^v) = \{g(B_i, w_0 b v_0 B_k) \mid g \in G, b \in B_j\}.$$

Now since the length are adding one finds $B_i w_0 B_j v_0 B_k = B_i (w_0 v_0) B_k$, as follows

$$\begin{aligned} w_0 B_j v_0 B_k &= x_i [w (x_j^{-1} G \cap \mathbb{B}) v (x_k^{-1} G \cap \mathbb{B})] x_k^{-1} \\ &\subset x_i [w \mathbb{B} v \mathbb{B}] x_k \cap G \subset x_i [\mathbb{B} w v \mathbb{B}] x_k^{-1} \cap G \\ &= [x_i \mathbb{B} (x_i w v x_k^{-1})^{x_k} \mathbb{B}] \cap G = B_i w_0 v_0 B_k \end{aligned}$$

For the last equality, clearly $B_i w_0 v_0 B_k \subset [x_i \mathbb{B} (x_i w v x_k^{-1})^{x_k} \mathbb{B}] \cap G$. Assume

$$[x_i \mathbb{B} (x_i w v x_k^{-1})^{x_k} \mathbb{B}] \cap G = \bigcup B_i t B_k$$

for certain $t \in W$, then clearly $B_i t B_k \subset [x_i \mathbb{B} (x_i w v x_k^{-1})^{x_k} \mathbb{B}] \cap G \cap [x_i \mathbb{B} t^{x_k} \mathbb{B}] \cap G$ as this intersection is empty if $t \neq (x_i w v x_k^{-1})$, the last equality follows.

Then using $Z_{i,j}^w = \{g(f_i = w_0 f_j, B_i, w_0 B_j) \in V \times G/B_i \times G/B_j \mid g \in G, f_i \in F_i, f_j \in F_j\}$ one concludes by definition that $Z_{i,j}^w \circ Z_{j,k}^v \subset Z_{i,k}^{wv}$ \square

We have the following corollary whose proof we have to delay until we have introduced the localization to the T -fixed point.

Corollary 3.1. For $s \in \mathbb{S}, w \in \mathbb{W}$ with $l(sw) = l(w) + 1$,

$$[\overline{Z^s}] * [\overline{Z^w}] = [\overline{Z^{sw}}] \text{ in } \mathcal{Z}_G^{\leq sw} / \mathcal{Z}_G^{< sw}.$$

Since $[\overline{Z^v}] = \sum_{s,t \in I} [\overline{Z_{s,t}^v}]$ for all $v \in \mathbb{W}$, this is equivalent to $i, j, l, k \in I$ we have

$$[\overline{Z_{i,j}^s}] * [\overline{Z_{l,k}^w}] = \delta_{l,j} [\overline{Z_{i,k}^{sw}}] \text{ in } \mathcal{Z}_G^{\leq sw} / \mathcal{Z}_G^{< sw}.$$

3.6 Computation of some Euler classes

Definition 3. (Euler class) Let T be a torus and $\mathfrak{t} := \text{Lie}(T)$. Let M be a finite dimensional complex \mathfrak{t} -representation. Then, we have a weight space decomposition

$$M = \bigoplus_{\alpha \in \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})} M_{\alpha}, \quad M_{\alpha} = \{m \in M \mid tm = \alpha(t)m\}.$$

We define

$$\text{eu}(M) := \prod_{\alpha \in \text{Hom}(\mathfrak{t}, \mathbb{C})} \alpha^{\dim M_{\alpha}} \in \mathbb{C}[\mathfrak{t}] = H_T^*(pt)$$

For a T -variety X and a T -fixed point $x \in X$, we define the **Euler class** of $x \in X$ to be

$$\text{eu}(X, x) := \text{eu}(T_x X),$$

where the \mathfrak{t} -operation on the tangent space $T_x X$ is the differential of the natural T -action.

Observe, that $\text{eu}(T_x^* X) = (-1)^{\dim T_x X} \text{eu}(T_x X)$.

Recall from an earlier section the notation $Z^w := m^{-1}(C^w)$. We are particularly interested in the following Euler classes, let $w = w^k x_k, x = x^i x_i, y = y^j x_j \in \mathbb{W}, w^k, x^i, y^j \in W$

$$\begin{aligned} \Lambda_w &:= \text{eu}(E, \phi_w) = \text{eu}(T_{\phi_w^k x_k} E_k), & \in H_T^*(pt) \\ \text{eu}(\overline{Z^w}, \phi_{x,y}) &= (\text{eu}(T_{\phi_{x^i x_i, y^j x_j} \overline{Z_{ij}^w}})^{-1}), & \in K := \text{Quot}(H_T^*(pt)) \end{aligned}$$

Remember $F_w := \mu^{-1}(\phi_w) = \mu_k^{-1}(\phi_{w^k x_k}) = w^k F_k, F_{x,y} := m^{-1}(\phi_{x,y}) = x^i F_i \cap y^j F_j = F_x \cap F_y$. In particular, we can see them as \mathfrak{t} -representations. We also consider the following \mathfrak{t} -representations

$$\begin{aligned} \mathfrak{n}_w &:= T_{w^k P_k} G/P_k = \mathfrak{g} \cap {}^w \mathcal{U}^- = w^k [\mathfrak{g} \cap x_k \mathcal{U}^-] \\ \mathfrak{n}_{x,y} &:= \frac{\mathfrak{n}_x}{\mathfrak{n}_x \cap \mathfrak{n}_y} = \mathfrak{g} \cap \frac{x \mathcal{U}^-}{x \mathcal{U}^- \cap y \mathcal{U}^-} \end{aligned}$$

where $\mathcal{U}^- := \text{Lie}(\mathbb{U}^-)$ with $\mathbb{U}^- \subset \mathbb{B}^- := {}^{w_0} \mathbb{B}$ is the unipotent radical where $w_0 \in \mathbb{W}$ is the longest element. The following properties can easily be seen.

(1) $\mathfrak{n}_x = \prod_{\alpha \in \Phi \cap x^{-1} \Phi^-} \alpha$.

- (2) If $s \in \mathbb{S}$, $x \in \mathbb{W}$ such that ${}^x s \in W$, then $\text{eu}(\mathbf{n}_x) = -\text{eu}(\mathbf{n}_{xs})$, $\text{eu}(\mathbf{m}_{x,xs}) = -\text{eu}(\mathbf{m}_{xs,x}) = x(\alpha_s)$
(3) If $s \in \mathbb{S}$, $x \in \mathbb{W}$ such that ${}^x s \notin W$, then $\mathbf{n}_x = \mathbf{n}_{xs}$, $\text{eu}(\mathbf{m}_{x,xs}) = \text{eu}(\mathbf{m}_{xs,x}) = 0$.

Furthermore, for $s \in S$, $x \in \mathbb{W}$, $i \in I$ we write set as a shortage

$$\begin{aligned} Q_x(s) &:= \text{eu}(F_x/F_{x,xs}), \\ Q_i(s) &:= Q_{x_i}(s), \\ q_i(s) &:= \prod_{\alpha \in \Phi_{\mathcal{U},s}(\alpha) \notin \Phi_{\mathcal{U},x_i}(\alpha) \in \Phi_{\mathcal{V}}} \alpha. \end{aligned}$$

for $x = x^i x_i$ with $x^i \in W$ one has $Q_x(s) = x^i(Q_i(s))$, $Q_i(s) = x_i(q_i(s))$, i.e.

$$Q_x(s) = x(q_i(s))$$

Lemma 10. (1) For $w \in \mathbb{W}$ one has $\Lambda_w = \text{eu}(F_w \oplus \mathbf{n}_w)$
(2) If $s \in \mathbb{S}$, $x \in \mathbb{W}$, $\alpha_s \in \Phi^+$ with $s(\alpha_s) = -\alpha_s$ and ${}^x s \in W$ one has

$$\begin{aligned} \text{eu}(\overline{Z^s}, \phi_{x,xs}) &= \text{eu}(F_{x,xs} \oplus \mathbf{n}_x \oplus \mathbf{m}_{x,xs}) = x(\alpha_s) Q_x(s)^{-1} \Lambda_x \\ \text{eu}(\overline{Z^s}, \phi_{x,x}) &= \text{eu}(F_{x,xs} \oplus \mathbf{n}_x \oplus \mathbf{m}_{xs,x}) = -\text{eu}(\overline{Z^s}, \phi_{x,xs}). \end{aligned}$$

- (3) If $s \in \mathbb{S}$, $x \in \mathbb{W}$ and ${}^x s \notin W$ one has $\text{eu}(\overline{Z^s}, \phi_{x,xs}) = \text{eu}(F_{x,xs} \oplus \mathbf{n}_x) = Q_x(s)^{-1} \Lambda_x$.
(4) Let $x, w \in \mathbb{W}$. Then, one has $\text{eu}(\overline{Z^w}, \phi_{x,xw}) = \text{eu}(F_{x,xw} \oplus \mathbf{n}_x \oplus \mathbf{m}_{x,xw})$

proof:

- (1) We know $\mu_k: E_k \rightarrow G/B_k$, $B_k = G \cap {}^{x_k} \mathbb{B}$ is a vector bundle, therefore we have a short exact sequence of tangent spaces

$$0 \rightarrow T_{\phi_w} \mu_k^{-1}(w^k B_k) \rightarrow T_{\phi_w} E_k \rightarrow T_{w^k B_k} G/B_k \rightarrow 0$$

which is a split sequence of T -representations implying the first statement.

ad (3,2) Let $i, j \in I_J$ such that $x^i := x x_i^{-1}$, $y^j := x s x_j^{-1} \in W$.

- (2) If ${}^x s \in W$ we have that $i = j$ and $\overline{Z_{i,i}^s} \rightarrow C_{i,i}^{\leq s} \cong G \times^{B_i} (G \cap {}^{x_i} \mathbb{P}_{\{s\}}) / B_i$ is a vector bundle. For $x' \in \{x, xs\}$ we have a short exact sequence on tangent spaces

$$0 \rightarrow F_{x,xs} \rightarrow T_{\phi_{x,x'}} \overline{Z_{i,i}^s} \rightarrow T_{\phi_{x,x'}} C_{i,i}^{\leq s} \rightarrow 0$$

Using the isomorphism $G \times^{B_i} [({}^{x_i} \mathbb{P}_{\{s\}} \cap G) / B_i] \rightarrow C_{i,i}^{\leq s}$, $(g, hB_i) \mapsto (gB_i, ghB_i)$ we get

$$\begin{aligned} \text{eu}(T_{\phi_{x,x'}} C_{i,i}^{\leq s}) &= \\ &\begin{cases} \text{eu}(T_{\overline{({}^{x_i}, B_i)}} G \times^{B_i} [({}^{x_i} \mathbb{P}_{\{s\}} \cap G) / B_i]) = \text{eu}(\mathbf{n}_x) \cdot \text{eu}(\mathbf{m}_{xs,x}), & x' = x \\ \text{eu}(T_{\overline{({}^{x_i}, x_i s B_i)}} G \times^{B_i} [({}^{x_i} \mathbb{P}_{\{s\}} \cap G) / B_i]) = \text{eu}(\mathbf{n}_x) \cdot \text{eu}(\mathbf{m}_{xs,x}), & x' = xs \end{cases} \end{aligned}$$

It follows $\text{eu}(\overline{Z^s}, \phi_{x,x}) = \text{eu}(F_{x,xs}) \cdot \text{eu}(\mathbf{n}_x) \cdot \text{eu}(\mathbf{m}_{xs,x})$ and $\text{eu}(\overline{Z^s}, \phi_{x,xs}) = \text{eu}(F_{x,xs} \oplus \mathbf{n}_x \oplus \mathbf{m}_{x,xs})$.

- (3) If ${}^x s \notin W$ we get $i \neq j$ and $Z_{i,j}^s$ is closed and a vector bundle over $C_{i,j}^s = G / (G \cap {}^x \mathbb{B})$, we get a short exact sequence on tangent spaces

$$0 \rightarrow F_{x,xs} \rightarrow T_{\phi_{x,xs}} Z_{i,j}^s \rightarrow T_{\phi_{x,xs}} C_{i,j}^s \rightarrow 0.$$

We obtain $\text{eu}(\overline{Z^s}, \phi_{x,xs}) = \text{eu}(F_{x,xs}) \text{eu}(\mathbf{n}_x)$.

- (4) Pick $i, j \in I$ such that $x \in W x_i$, $xw \in W x_j$. We have the short exact sequence

$$0 \rightarrow F_{x,xw} \rightarrow T_{\phi_{x,xw}} \overline{Z_{i,j}^w} \rightarrow T_{\phi_{x,xw}} C_{i,j}^w \rightarrow 0$$

Then, recall the isomorphism

$$\begin{aligned} C_{i,j}^w &= G \phi_{x,xw} \rightarrow G / (G \cap {}^x \mathbb{B} \cap {}^{xw} \mathbb{B}) \\ \phi_{x,xw} &\mapsto \bar{e} := e(G \cap {}^x \mathbb{B} \cap {}^{xw} \mathbb{B}) \end{aligned}$$

Again we have a short exact sequence

$$0 \rightarrow T_{\bar{e}}(G \cap {}^x \mathbb{B}) / (G \cap {}^x \mathbb{B} \cap {}^{xw} \mathbb{B}) \rightarrow T_{\bar{e}} G / (G \cap {}^x \mathbb{B} \cap {}^{xw} \mathbb{B}) \rightarrow T_{\bar{e}} G / (G \cap {}^x \mathbb{B}) \rightarrow 0$$

Together it implies $\text{eu}(\overline{Z_{i,j}^w}, \phi_{x,xw}) = \text{eu}(F_{x,xw}) \text{eu}(\mathbf{n}_x / (\mathbf{n}_x \cap \mathbf{n}_{xw})) \text{eu}(\mathbf{n}_x)$.

□

Corollary 3.2. *Let $\mathcal{U} = \text{Lie}(\mathbb{U})^{\oplus r}$, one has*

(1) *If $s \in \mathbb{S}, x \in \mathbb{W}$ and $xs \in W$, then $h_{\bar{x}}(s) = h_{\overline{xs}}(s)$ and*

$$\begin{aligned}\Lambda_x &= (-1)^{1+h_{\overline{xs}}(s)} \Lambda_{xs} \\ \text{eu}(\overline{Z^s}, \phi_{x,xs}) &= (x(\alpha_s))^{1-h_{\bar{x}}(s)} \Lambda_x\end{aligned}$$

(2) *If $s \in \mathbb{S}, x \in \mathbb{W}$ and $xs \notin W$*

$$\text{eu}(\overline{Z^s}, \phi_{x,xs}) = x(\alpha_s)^{-h_{\bar{x}}(s)} \Lambda_x$$

proof: This follows from $q_x(s) = x(\alpha_s)^{h_{\bar{x}}(s)}$ and if $xs \in W$ we have that $i = j$ and $h_{\bar{x}}(s) = h_{\overline{xs}}(s)$. Therefore we get

$$\begin{aligned}\text{eu}(F_x) &= x(\alpha_s)^{h_{\bar{x}}(s)} \text{eu}(F_{x,xs}) \\ &= (-1)^{h_{\overline{xs}}(s)} (xs(\alpha_s))^{h_{\overline{xs}}(s)} \text{eu}(F_{xs,x}) \\ &= (-1)^{h_{\overline{xs}}(s)} \text{eu}(F_{xs})\end{aligned}$$

Using that $\text{eu}(\mathfrak{n}_x) = -\text{eu}(\mathfrak{n}_{xs})$ we obtain $\Lambda_x = (-1)^{1+h_{\overline{xs}}(s)} \Lambda_{xs}$ □

3.7 Localization to the torus fixed points

Now, we come to the application of localization to T -fixed points. We remind the reader that Z is a cellular fibration and E is smooth, therefore in both cases the odd ordinary (=singular) cohomology groups vanish for Z and E . This implies in particular that E, Z are *equivariantly formal*, which is (in the case of finitely T -fixed points) equivalent to \mathcal{Z}_G and \mathcal{E}_G are free modules over $H_G^*(pt)$.

If we denote by K the quotient field of $H_G^*(pt)$ and for any T -variety X

$$H_*^T(X) \rightarrow \mathcal{H}_*(X) := H_*^T(X) \otimes_{H_T^*(pt)} K, \quad \alpha \mapsto \alpha \otimes 1.$$

Lemma 11. (1)

$$\mathcal{H}_*(E) = \bigoplus_{w \in \mathbb{W}} K\psi_w, \quad \mathcal{H}_*(Z) = \bigoplus_{x,y \in \mathbb{W}} K\psi_{x,y}$$

where $\psi_w = [\{\phi_w\}] \otimes 1, \psi_{x,y} = [\{\phi_{x,y}\}] \otimes 1$.

(2) *For every $i \in I, w \in Wx_i$ we have a map $w \cdot: \mathcal{E}_i := H_G^*(E_i) \rightarrow \mathbb{C}[\mathfrak{t}]$, via taking the forgetful map composed with the pullback map under the closed embedding $i_w: \{\phi_w\} \rightarrow E_i$*

$$\mathcal{E}_i = H_G^*(E_i) \rightarrow H_T^*(E_i) \xrightarrow{i_w^*} H_T^*(pt) = \mathbb{C}[\mathfrak{t}],$$

we denote the map by $f \mapsto w(f), f \in \mathcal{E}_i, w \in \mathbb{W}$. Furthermore, composing the forgetful map with the map from before we get an injective algebra homomorphism

$$\begin{aligned}\Theta_i: \mathcal{E}_i &\rightarrow H_T^*(E_i) \hookrightarrow H_T^*(E_i) \otimes K \cong \bigoplus_{w \in Wx_i} K\psi_w \\ c &\longmapsto \sum_{w \in Wx_i} w(c) \Lambda_w^{-1} \psi_w.\end{aligned}$$

We set $\Theta = \bigoplus_{i \in I} \Theta_i: \mathcal{E}_G \rightarrow \bigoplus_{w \in \mathbb{W}} K\psi_w$.

proof:

- (1) This is GKM-localization theorem for T -equivariant cohomology, for a source also mentioning the GKM-theorem for T -equivariant Borel-Moore homology see for example [Bri00], Lemma 1.
- (2) This is [EG98], Thm 2, using the equivariant cycle class map to identify T -equivariant Borel-Moore homology of E with the T -equivariant Chow ring.

□

3.8 The \mathbb{W} -operation on \mathcal{E}_G :

Recall that the ring of regular functions $\mathbb{C}[\mathfrak{t}]$ on $\mathfrak{t} = \text{Lie}(T)$ is a left W -module and a left \mathbb{W} -module with respect to $w \cdot f(t) = f(w^{-1}tw)$, $w \in \mathbb{W}(\supset W)$. The from W to \mathbb{W} induced representation is given by

$$\text{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}] = \bigoplus_{i \in I} x_i^{-1} \mathbb{C}[\mathfrak{t}],$$

for $w \in \mathbb{W}, i \in I$ the operation of w on $x_i^{-1} \mathbb{C}[\mathfrak{t}]$ is given by

$$\begin{aligned} x_i^{-1} \mathbb{C}[\mathfrak{t}] &\rightarrow x_{iw^{-1}}^{-1} \mathbb{C}[\mathfrak{t}] \\ x_i^{-1} f &\mapsto wx_i^{-1} f \end{aligned}$$

where we use that $wx_i^{-1}W = x_{iw^{-1}}^{-1}W$.

Now, we identify $\mathcal{E}_G = \bigoplus_{i \in I} \mathcal{E}_i$ with the left \mathbb{W} -module $\text{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}]$ via $\mathcal{E}_i = x_i^{-1} \mathbb{C}[\mathfrak{t}]$.

Furthermore, we have the (left) \mathbb{W} -representation on $\bigoplus_{x \in \mathbb{W}} K(\Lambda_x^{-1} \psi_x)$ defined via

$$w(k(\Lambda_x^{-1} \psi_x)) := k(\Lambda_{xw^{-1}}^{-1} \psi_{xw^{-1}}), \quad k \in K, w \in \mathbb{W}.$$

Lemma 12. *The map $\Theta: \mathcal{E}_G \rightarrow \bigoplus_{x \in \mathbb{W}} K(\Lambda_x^{-1} \psi_x)$ is \mathbb{W} -invariant.*

proof: Let $w \in \mathbb{W}$, we claim that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_G & \xrightarrow{\Theta} & \bigoplus_{x \in \mathbb{W}} K(\Lambda_x^{-1} \psi_x) & \begin{array}{ccc} c & \longrightarrow & \sum_{x \in \mathbb{W}} i_x^*(c) \Lambda_x^{-1} \psi_x \\ \downarrow & & \downarrow \\ w \cdot c & \longrightarrow & \sum_{x \in \mathbb{W}} i_{xw}^*(c) \Lambda_x^{-1} \psi_x \end{array} \\ \downarrow w \cdot & & \downarrow w & \\ \mathcal{E}_G & \xrightarrow{\Theta} & \bigoplus_{x \in \mathbb{W}} K(\Lambda_x^{-1} \psi_x) & \end{array}$$

We need to see $i_{xw}^*(w \cdot c) = i_x^*(c)$. Let $xw \in Wx_i, x \in Wx_{iw^{-1}}$. This means that the diagram

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{w \cdot} & \mathcal{E}_{iw^{-1}} \\ & \searrow i_{xw}^* & \swarrow i_x^* \\ & & H_T^*(pt) \end{array}$$

is commutative. But it identifies with

$$\begin{array}{ccc} x_i^{-1} \mathbb{C}[\mathfrak{t}] & \xrightarrow{w \cdot} & x_{iw^{-1}}^{-1} \mathbb{C}[\mathfrak{t}] & \begin{array}{ccc} x_i^{-1} f & \longrightarrow & wx_i^{-1} f \\ & \searrow & \swarrow \\ & & wx_i^{-1} f \end{array} \\ & \searrow xw \cdot & \swarrow x \cdot & \\ & & \mathbb{C}[\mathfrak{t}] & \end{array}$$

The diagram is commutative. □

Remark. From now on, we use the following description of the \mathbb{W} -operation on \mathcal{E}_G . We set $\mathcal{E}_i = \mathbb{C}[\mathfrak{t}]$, $i \in I$. Let $w \in \mathbb{W}$

$$w(\mathcal{E}_i) = \mathcal{E}_{iw^{-1}}, \quad \mathcal{E}_i = \mathbb{C}[\mathfrak{t}] \ni f \mapsto w \cdot f \in \mathbb{C}[\mathfrak{t}] = \mathcal{E}_{iw^{-1}}.$$

The isomorphism $p := \bigoplus_{i \in I} p_i$ defined by

$$\begin{aligned} p_i: \mathbb{C}[\mathfrak{t}] &\rightarrow x_i^{-1} \mathbb{C}[\mathfrak{t}] \\ f &\mapsto x_i^{-1}(x_i f) \end{aligned}$$

gives the identification with the induced representation $\text{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}]$ which we described before.

3.9 Calculations of some equivariant multiplicities

In some situation one can actually say something on the images of algebraic cycle under the GKM-localization map, recall the

Theorem 3.3. (multiplicity formular, [Bri00], section 3) Let X equivariantly formal T -variety with a finite set of T -fixpoints X^T , by the localization theorem,

$$[X] = \sum_{x \in X^T} \Lambda_x^X[\{x\}] \in H_*^T(X) \otimes K$$

where $\Lambda_x^X \in K$. If X is rationally smooth in x , then $\Lambda_x^X \neq 0$ and $(\Lambda_x^X)^{-1} = eu(X, x) \in H_{2n}^T(X)$, $n = \dim_{\mathbb{C}}(X)$.

Remark. One has for any $w \in \mathbb{W}$

$$[Z^w] = \sum_{i,j \in I} [Z_{i,j}^w].$$

Especially $1 = [Z^e] = \sum_{i \in I} [Z_{i,i}^e]$ is the unit and $1_i = [Z_{i,i}^e]$ are idempotent elements, $1_i * 1_j = 0$ for $i \neq j$, $[Z_{i,j}] = 1_i * [Z] * 1_j$. In particular, for $s \in \mathbb{S}$ by lemma 11, we have

$$[Z^s] = \sum_{i \in I: is=i} [Z_{i,i}^s] + \sum_{i \in I: is \neq i} [Z_{i,is}^s].$$

By the multiplicity formula we have

$$[Z_{i,is}^s] = \begin{cases} \sum_{x \in W} \Lambda_{xx_i, xx_i s}^s \psi_{xx_i, xx_i s} + \Lambda_{xx_i, xx_i}^s \psi_{xx_i, xx_i} & , \text{ if } i = is \\ \sum_{x \in W} \Lambda_{xx_i, xx_i s}^s \psi_{xx_i, xx_i s} & , \text{ if } is \neq i \end{cases}$$

with $\Lambda_{y,z}^s = (eu(\overline{Z_{i,j}^s}, \phi_{y,z}))^{-1}$, for all $y, z \in \mathbb{W}$ as above

$$[Z_{i,j}^w] = \begin{cases} \sum_{x \in W} \Lambda_{xx_i, xx_i w}^w \psi_{xx_i, xx_i w} + \sum_{v < w} \Lambda_{xx_i, xx_i v}^w \psi_{xx_i, xx_i v} & , \text{ if } iw = j \\ 0 & , \text{ if } iw \neq j \end{cases}$$

with $\Lambda_{xx_i, xx_i w}^w = (eu(\overline{Z_{i,iw}^w}, \phi_{xx_i, xx_i w}))^{-1}$ for all $x \in W$,

3.10 Convolution on the fixed points

The following key lemma on convolution products of T -fixed points

Lemma 13. For any $w, x, y \in \mathbb{W}$ one has

$$\psi_{x,w} * \psi_w = \Lambda_w \psi_x, \quad \psi_{x,w} * \psi_{w,y} = \Lambda_w \psi_{x,y}$$

proof: We take $M_1 = M_2 = M_3 = E$ and $Z_{1,2} := \{\phi_{x,w} = ((0, xB), (0, wB))\} \subset E \times E$, $Z_{2,3} := \{\phi_{w',y}\} \subset E \times E$, then the set theoretic convolution gives

$$\{\phi_{x,w}\} \circ \{\phi_{w',y}\} = \begin{cases} \{\phi_{x,y}\}, & \text{if } w = w' \\ \emptyset, & \text{if } w \neq w' \end{cases}$$

Similar, take $M_1 = M_2 = E$, $M_3 = pt$, $Z_{12} := \{\phi_{x,w}\}$, $Z_{23} = \phi_{w'} \times pt$, then

$$\{\phi_{x,w}\} \circ \{\phi_w\} = \begin{cases} \{\phi_x\} & \text{if } w = w' \\ \emptyset, & \text{else} \end{cases}$$

To see that we have to multiply with Λ_w , we use the following proposition

Proposition 2. (see [CG97], Prop. 2.6.42, p.109) Let $X_i \subset M$, $i = 1, 2$ be two closed (complex) submanifolds of a (complex) manifold with $X := X_1 \cap X_2$ is smooth and $T_x X_1 \cap T_x X_2 = T_x X$ for all $x \in X$. Then, we have

$$[X_1] \cap [X_2] = e(\mathcal{T}) \cdot [X]$$

where \mathcal{T} is the vector bundle $T_*M/(T_*X_1 + T_*X_2)$ on X and $e(\mathcal{T}) \in H^*(X)$ is the (non-equivariant) Euler class of this vector bundle, $\cap: H_*^{BM}(X_1) \times H_*^{BM}(X_2) \rightarrow H_*^{BM}(X)$ is the intersection pairing (cp. Appendix, or [CG97], 2.6.15) and \cdot on the right hand side stands for the $H^*(X)$ -operation on the Borel-Moore homology (introduced in [CG97], 2.6.40)

Set $E_T := E \times^T ET$, $(\phi_x)_T := \{\phi_x\} \times^T ET (\cong ET/T = BT)$. We apply the proposition for $M = E_T^3$, $X_1 := (\phi_x)_T \times (\phi_w)_T \times E_T$, $X_2 := E_T \times (\phi_w)_T \times (\phi_y)_T$, $X_1 \cap X_2 \cong \{\phi_{x,y}\}_T (\cong BT)$, then $\mathcal{T} = (T_{\phi_w} E) \times^T ET$ and the (non-equivariant) Euler class is the top chern class of this bundle which is the T -equivariant top chern class of the constant bundle $T_{\phi_w} E$ on the point $\{\phi_{x,y}\}$. Since $T_{\phi_w} E = \bigoplus_{\lambda} \mathbb{C}_{\lambda}$ for one-dimensional T -representations \mathbb{C}_{λ} with $t \cdot c := \lambda(t)c$, $t \in T, c \in \mathbb{C} = \mathbb{C}_{\lambda}$. One has

$$c_{top}^T(T_{\phi_w} E) = \prod_{\lambda} c_1^T(\mathbb{C}_{\lambda}) = \prod_{\lambda} \lambda = \Lambda_w.$$

Secondly, apply the proposition with $M = E_T^2 \times (pt)_T$, $X_1 = (\phi_x)_T \times (\phi_w)_T \times (pt)_T$, $X_2 := E_T \times (\phi_w)_T \times (pt)_T$, to see again $e(\mathcal{T}) = \Lambda_w$. \square

Now we can give the missing proof of Corollary 3.1

proof of Corollary 3.1: By the lemma 9 we know that there exists a $c \in \mathcal{E}_G$ such that $[\overline{Z_{i,j}^s}] * [\overline{Z_{j,k}^w}] = c \diamond [\overline{Z_{i,k}^{sw}}]$ in $\mathcal{Z}_G^{\leq sw} / \mathcal{Z}_G^{\leq sw}$. We show that $c = 1$. We pass with the forgetful map to T -equivariant Borel-Moore homology and tensor over $K = \text{Quot}(H_*^T(pt))$ and write $[\overline{Z_{s,t}^x}]$, $x \in \mathbb{W}, s, t \in I$ for the image of the same named elements. Let $i, j, k \in I$ with $x_j w x_k^{-1} \in W$.

$$\begin{aligned} [\overline{Z_{i,j}^s}] * [\overline{Z_{j,k}^w}] &= \left(\sum_{x \in W} \Lambda_{xx_i, xx_i s}^s \psi_{xx_i, xx_i s} + \Lambda_{xx_i, xx_i}^s \psi_{xx_i, xx_i} \right) * \\ &\quad \left(\sum_{x \in W} \Lambda_{xx_j, xx_j w}^w \psi_{xx_j, xx_j w} + \sum_{v < w} \Lambda_{xx_j, xx_j v}^w \psi_{xx_j, xx_j v} \right) \\ &= \sum_{x \in W} \Lambda_{xx_i, xx_i s}^s \Lambda_{xx_i s, xx_i s w}^w \Lambda_{xx_i s} \psi_{xx_i, xx_i s w} + \underbrace{\dots}_{\text{terms in } \mathcal{Z}_G^{\leq sw}} \end{aligned}$$

Now, this has to be equal to $c \sum_{x \in W} \Lambda_{xx_i, xx_i s w} \psi_{xx_i, xx_i s w}$ in $\mathcal{Z}_G^{\leq sw} / \mathcal{Z}_G^{\leq sw}$. Comparing coefficients at x gives

$$\begin{aligned} c &= \frac{\text{eu}(E_j, \phi_{xx_i s}) \text{eu}(\overline{Z_{i,k}^{sw}}, \phi_{xx_i, xx_i s w})}{\text{eu}(\overline{Z_{i,j}^s}, \phi_{xx_i, xx_i s}) \text{eu}(\overline{Z_{j,k}^w}, \phi_{xx_i s, xx_i s w})} \\ &= \frac{\text{eu}(\mathfrak{g} \cap^{xx_i s} \mathcal{U}^- \oplus \mathfrak{g} \cap^{xx_i} \mathcal{U}^- \oplus \mathfrak{g} \cap^{xx_i} (\frac{\mathcal{U}^-}{\mathcal{U}^- \cap^{sw} \mathcal{U}^-}))}{\text{eu}(\mathfrak{g} \cap^{xx_i} \mathcal{U}^- \oplus \mathfrak{g} \cap^{xx_i} (\frac{\mathcal{U}^-}{\mathcal{U}^- \cap \mathcal{U}^-}) \oplus \mathfrak{g} \cap^{xx_i s} \mathcal{U}^- \oplus \mathfrak{g} \cap^{xx_i} (\frac{s \mathcal{U}^-}{s \mathcal{U}^- \cap^{sw} \mathcal{U}^-}))} \\ &\quad \cdot \prod_{l=1}^r \frac{\text{eu}(V^{(l)} \cap^{xx_i} (s \mathcal{U}^{(l)}) \oplus V^{(l)} \cap^{xx_i} (\mathcal{U}^{(l)} \cap^{sw} \mathcal{U}^{(l)}))}{\text{eu}(V^{(l)} \cap^{xx_i} (\mathcal{U}^{(l)} \cap^{s \mathcal{U}^{(l)}}) \oplus V^{(l)} \cap^{xx_i} (s \mathcal{U}^{(l)} \cap^{sw} \mathcal{U}^{(l)}))} \\ &= \frac{\text{eu}(x[\mathfrak{g} \cap^{xx_i} (\frac{\mathcal{U}^-}{\mathcal{U}^- \cap^{sw} \mathcal{U}^-})])}{\text{eu}(x[\mathfrak{g} \cap^{xx_i} (\frac{\mathcal{U}^-}{\mathcal{U}^- \cap \mathcal{U}^-}) \oplus \mathfrak{g} \cap^{xx_i} (\frac{s \mathcal{U}^-}{s \mathcal{U}^- \cap^{sw} \mathcal{U}^-})])} \\ &\quad \cdot \prod_{l=1}^r \frac{\text{eu}(x[V^{(l)} \cap^{xx_i} (\frac{s \mathcal{U}^{(l)}}{\mathcal{U}^{(l)} \cap^{s \mathcal{U}^{(l)}}}) \oplus V^{(l)} \cap^{xx_i} (\frac{sw \mathcal{U}^{(l)}}{s \mathcal{U}^{(l)} \cap^{sw} \mathcal{U}^{(l)}})])}{\text{eu}(x[V^{(l)} \cap^{xx_i} (\frac{sw \mathcal{U}^{(l)}}{\mathcal{U}^{(l)} \cap^{sw} \mathcal{U}^{(l)}})])} \end{aligned}$$

That for each x and each $l \in \{1, \dots, r\}$ the big two fraction in the product are equal to 1 is a consequence of the following lemma. \square

Lemma 14. *Let $T \subset \mathbb{B} \subset \mathbb{G}$ a maximal torus in a Borel subgroup in a reductive group (over \mathbb{C}), $F \subset \text{Lie}(\mathbb{G}) = \mathcal{G}$ a \mathbb{B} -subrepresentation. Let (\mathbb{W}, \mathbb{S}) be the Weyl group for (\mathbb{G}, T) . Let $w \in \mathbb{W}, s \in \mathbb{S}$ such that $l(sw) = l(w) + 1$, then one has for any $x \in \mathbb{W}$*

$$x \left(\frac{{}^s F}{F \cap {}^s F} \oplus {}^s \left(\frac{{}^w F}{F \cap {}^w F} \right) \right) \cong x \left(\frac{{}^{sw} F}{F \cap {}^{sw} F} \right).$$

In particular, this holds also for $F = \mathfrak{u}^-$.

proof: Let $\Phi_F := \{\alpha \in \text{Hom}(\mathfrak{t}, \mathbb{C}) \mid F_{\alpha} \neq 0\} \subset \Phi$, $\Phi^+(y) := \Phi^+ \cap y(\Phi^-)$, $\Phi_F^+(y) := \Phi_F \cap \Phi^+(y)$, $y \in \mathbb{W}$ where Φ, Φ^+, Φ^- are the set of roots (of T on \mathcal{G}), positive roots, negative roots respectively.

The assumption $l(sw) = l(w) + 1$ implies $\Phi_F^+(sw) = s\Phi_F^+(w) \sqcup \Phi_F^+(s)$ and for $\Phi_F^-(y) := -\Phi_F^+(y)$, $\Phi_F(y) := \Phi_F^+(y) \cup \Phi_F^-(y) = \Phi_F \setminus (\Phi_F \cap y\Phi_F)$ one has $\Phi_F(sw) = s\Phi_F(w) \sqcup \Phi_F(s)$ and for any $x \in W$ one has $x\Phi_F(sw) = x(s\Phi_F(w) \sqcup \Phi_F(s))$. Now, the weights of $x(\frac{{}^{sw} F}{F \cap {}^{sw} F})$ are $x\Phi_F(sw)$, the weights of $x(\frac{{}^s F}{F \cap {}^s F} \oplus {}^s(\frac{{}^w F}{F \cap {}^w F}))$ are $x(s\Phi_F(w) \sqcup \Phi_F(s))$. \square

3.11 Generators for \mathcal{Z}_G

Recall, we denote the right \mathbb{W} -operation on $I = W \setminus \mathbb{W}$ by $(i, w) \mapsto iw$, $i \in I, w \in \mathbb{W}$. For $i \in I$ we set $\mathcal{E}_i := H_G^*(E_i) = \mathbb{C}[t] = \mathbb{C}[x_i(1), \dots, x_i(m)]$, we write

$$w(\alpha_s) = w(\alpha_s(x_{iw^{-1}}(1), \dots, x_{iw^{-1}}(m))) \in \mathcal{E}_{iw^{-1}}$$

for the element corresponding to the root $w(\alpha_s)$, $s \in \mathbb{S}, w \in \mathbb{W}$ without mentioning that it depends on $i \in I$. We define a collection of elements in \mathcal{Z}_G

$$\begin{aligned} 1_i &:= [Z_{i,i}^e] \\ z_i(t) &:= x_i(t) \in \mathcal{Z}_G^{\leq e} \subset \mathcal{Z}_G \\ \sigma_i(s) &:= [\overline{Z}_{i,j}^s] \in \mathcal{Z}_G^{\leq s}, \text{ where } is = j \end{aligned}$$

where we use that $\mathcal{E}_i \subset \mathcal{Z}_G^{\leq e} \subset \mathcal{Z}_G$ and the degree of $x_i(t)$ is 2 in $H_{[*]}^G(Z)$ by the definition of the grading. It is also easy to see that $1_i \in H_{[0]}^G(Z)$ because $\deg 1_i = 2e_i - 2 \dim Z_{i,i}^e = 0$. Furthermore, the degree of $\sigma_i(s)$ is

$$e_{is} + e_i - 2 \dim Z_{i,is}^s = \begin{cases} 2 \deg q_i(s) - 2, & \text{if } is = i \\ 2 \deg q_i(s), & \text{if } is \neq i. \end{cases}$$

Recall $\mathcal{Z}_G \hookrightarrow \text{End}(\mathcal{E}_G) = \text{End}(\bigoplus_{i \in I} \mathcal{E}_i)$ from [VV11a], remark after Prop.3.1, p.12. Let us denote by $\widetilde{1}_i, \widetilde{z}_i(t), \widetilde{\sigma}_i(s)$ be the images of $1_i, z_i(t), \sigma_i(s)$.

Proposition 3. *Let $k \in I$, $f \in \mathcal{E}_k$, $\alpha_s \in \underline{\Phi}^+$ be the positive root such that $s(\alpha_s) = -\alpha_s$. One has $\widetilde{1}_i(f) := 1_i * f = \delta_{i,k}(f)$, $\widetilde{z}_i(t)(f) := x_i(t) \diamond f = \delta_{i,k} x_i(t) f$ and*

$$\widetilde{\sigma}_i(s)(f) := \begin{cases} q_i(s) \frac{s(f) - f}{\alpha_s}, & \text{if } i = is = k, \\ q_i(s) s(f) & \text{if } i \neq is = k, \\ 0, & \text{else.} \end{cases}$$

for $\mathcal{U} = \text{Lie}(\mathbb{U})^{\oplus r}$ one has $q_i(s) = \alpha_s^{h_i(s)}$. We write $\delta_s := \frac{s-1}{\alpha_s}$, it is the BGG-operator from [Dem73], i.e. for $is = i, f \in \mathcal{E}_i$, $\sigma_i(s)(f) = q_i(s) \delta_s(f)$.

proof: Consider the following two maps

$$\begin{aligned} \Theta: \mathcal{E}_G &\rightarrow \mathcal{E}_T \rightarrow \mathcal{E}_T \otimes K \rightarrow \bigoplus_{w \in \mathbb{W}} K \psi_w \\ \mathcal{E}_k \ni f &\mapsto \sum_{w \in W x_k} w(f) \Lambda_w^{-1} \psi_w \end{aligned}$$

$$\begin{aligned} C: \bigoplus_{w \in \mathbb{W}} K \psi_w &\rightarrow \bigoplus_{w \in \mathbb{W}} K \psi_w, \quad \psi_w \mapsto [\overline{Z}_{i,is}^s] * \psi_w = \\ &\begin{cases} \left(\sum_{x \in W} \Lambda_{xx_i, xx_i}^s \psi_{xx_i, xx_i} + \Lambda_{xx_i, x^i s x_i}^s \psi_{xx_i, x^i s x_i} \right) * \psi_w & \text{if } w \in W x_i, i = is \\ = \Lambda_{w,w}^s \Lambda_w \psi_w + \Lambda_{ws,w}^s \Lambda_w \psi_{ws}, & \\ \left(\sum_{x \in W} \Lambda_{xx_i, x^i s x_i}^s \psi_{xx_i, x^i s x_i} \right) * \psi_w & \text{if } w \in W x_i s, i \neq is \\ = \Lambda_{ws,w}^s \Lambda_w \psi_{ws}, & \\ 0, & \text{if } w \notin W x_i s \end{cases} \end{aligned}$$

To calculate $[\overline{Z}_{i,is}^s] * f$, $f \in \mathcal{E}_k$ it is enough to calculate $[\overline{Z}_{i,is}^s] * \Theta(f) = C(\Theta(f))$ because Θ is an injective algebra homomorphism.

$$C\Theta(f) = \begin{cases} \delta_{is,k} \sum_{w \in W x_i} [w(f) \Lambda_{w,w}^s + w(sf) \Lambda_{w,ws}^s] \psi_w, & \text{if } i = is \\ \delta_{is,k} \sum_{w \in W x_i} [w(sf) \Lambda_{w,ws}^s] \psi_w, & \text{if } i \neq is \end{cases}$$

Now, recall,

(1) If $i = is = k$

$$\begin{aligned} C\Theta(f) &= \sum_{w \in Wx_i} w[q_i(s) \frac{s(f) - f}{\alpha_s}] \Lambda_w^{-1} \psi_w \\ &= \Theta(q_i(s) \frac{s(f) - f}{\alpha_s}) \end{aligned}$$

Once we identify $\mathcal{E}_k = \mathbb{C}[t]$, $k \in I$, we see that $\sigma_i(s): \mathcal{E}_G \rightarrow \mathcal{E}_G$ is the zero map on the k -th summand, $k \neq i$ and on the i -th summand

$$\begin{aligned} \mathbb{C}[t] &\rightarrow \mathbb{C}[t] \\ f &\mapsto q_i(s) \frac{s(f) - f}{\alpha_s} \end{aligned}$$

(2) If $i \neq is = k$,

$$\begin{aligned} C\Theta(f) &= \sum_{w \in Wx_i} [w(sf) \Lambda_{w,ws}^s] \psi_w \\ &= \Theta(q_i(s)s(f)) \end{aligned}$$

Once we identify $\mathcal{E}_k = \mathbb{C}[t]$, we see that $\sigma_i(s): \mathcal{E}_G \rightarrow \mathcal{E}_G$ is the zero map on the k -th summand, $k \neq is$ and on the is -th summand it is the map

$$\begin{aligned} \mathbb{C}[t] &\rightarrow \mathbb{C}[t] \\ f &\mapsto q_i(s)s(f) \end{aligned}$$

□

Lemma 15. *The algebra \mathcal{Z}_G is generated as \mathbb{C} -algebra by the elements*

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq rk(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I.$$

proof: It follows from the cellular fibration property that \mathcal{Z}_G is generated by $1_i, i \in I, z_i(t), 1 \leq t \leq rk(T), i \in I, [\overline{Z}_{i,j}^w], w \in \mathbb{W}$. By corollary 3.1 it follows that one can restrict to the case $w \in \mathbb{S}$, more precisely as free $H_G^*(E)$ -module it can be generated by

$$\sigma(w) := \sigma(s_1) * \cdots * \sigma(s_t), w \in \mathbb{W}, w = s_1 \cdots s_t \text{ reduced expression, } \sigma(s) := \sum_{i \in I} \sigma_i(s),$$

and this basis has a unitriangular base change to the basis given by the $[\overline{Z}^w]$.

□

3.12 Relations for \mathcal{Z}_G

proof of theorem 3.2. We include the detailed check that the relations hold for the generators of \mathcal{Z}_G : (1), (2) are clear. Let always $f \in \mathbb{C}[t] \cong \mathcal{E}_{is}$. We will use as shortage $\delta_s(f) := \frac{s(f) - f}{\alpha_s}$ and use that these satisfy the usual relations of BGG-operators (cp. [Dem73]).

(3) If $is = i$, then

$$\begin{aligned} \sigma_i(s)\sigma_i(s)(f) &= \alpha_s^{h_i(s)} \delta_s(\alpha_s^{h_i(s)} \delta_s(f)) \\ &= \alpha_s^{h_i(s)} \delta_s(\alpha_s^{h_i(s)}) \delta_s(f) = [(-1)^{h_i(s)} - 1] \alpha_s^{h_i(s)-1} \sigma_i(s)(f). \end{aligned}$$

If $is \neq i$, then

$$\sigma_i(s)(f)\sigma_{is}(s) = \alpha_s^{h_i(s)} s(\alpha_s^{h_{is}(s)}) s(s(f)) = (-1)^{h_{is}(s)} \alpha_s^{h_i(s) + h_{is}(s)} f.$$

(4) (straightening rule)

The case $is \neq i$ is clear by definition. Let $is = i$, then the relation follows directly from the product rule for BGG-operators, which states $\delta_s(xf) = \delta_s(x)f + s(x)\delta_s(f)$, $x, f \in \mathbb{C}[t]$.

(5) (braid relations)

$s, t \in \mathbb{S}, st = ts, f \in \mathbb{C}[t]$, to prove

$$\sigma_i(s)\sigma_{is}(t)(f) = \sigma_i(t)\sigma_{it}(s)(f)$$

we have to consider the following four cases. We use the following:

$$t(\alpha_s) = \alpha_s, s(\alpha_t) = \alpha_t, h_i(s) = h_{it}(s), h_i(t) = h_{is}(t), \delta_s(\alpha_t^{h_i(t)}) = 0 = \delta_t(\alpha_s^{h_i(s)}).$$

1. $is = i, it = i$, use $\delta_s \delta_t = \delta_t \delta_s$

$$\begin{aligned} \sigma_i(t)\sigma_i(s)(f) &= \alpha_t^{h_i(t)} \delta_t(\alpha_s^{h_i(s)} \delta_s(f)) = \alpha_s^{h_i(s)} \alpha_t^{h_i(t)} \delta_s \delta_t(f) \\ &= \alpha_s^{h_i(s)} \alpha_t^{h_i(t)} \delta_t \delta_s(f) \\ &= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)} \delta_t(f)) = \sigma_i(s)\sigma_i(t)(f) \end{aligned}$$

2. $is = i, it \neq i$, use $\delta_s t = t \delta_s$

$$\begin{aligned} \sigma_i(t)\sigma_{it}(s)(f) &= \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)} \delta_s(f)) = \alpha_t^{h_i(t)} \alpha_s^{h_i(s)} t \delta_s(f) \\ &= \alpha_t^{h_{is}(t)} \alpha_s^{h_i(s)} \delta_s(t(f)) \\ &= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)} t(f)) = \sigma_i(s)\sigma_i(t)(f) \end{aligned}$$

3. $is \neq i, it = i$, follows by symmetry from the last case.

4. $is \neq i, it \neq i$.

$$\begin{aligned} \sigma_i(t)\sigma_{it}(s)(f) &= \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)} s(f)) = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)} t(f)) \\ &= \sigma_i(s)\sigma_{is}(t)(f). \end{aligned}$$

Let $st \neq ts$. There are three different possibilities, either

- (A) $sts = tst$ (type A_2)
- (B) $stst = tsts$ (type B_2)
- (C) $ststst = tststs$ (type G_2)

We write $\text{Stab}_i := \{w \in \langle s, t \rangle \mid iw = i\}$. For each case we go through the subgrouplattice to calculate explicitly the polynomials Q_w .

(A) $sts = tst$: $\langle s, t \rangle \cong S_3$, $s(\alpha_t) = t(\alpha_s) = \alpha_s + \alpha_t$. We have five (up to symmetry between s and t) subgroups to consider. Always, one has

$$h_{is}(t) = h_{it}(s), h_{ist}(s) = h_i(t), h_{its}(t) = h_i(s)$$

which implies an equality which we use in all five cases

$$\begin{aligned} \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) &= \alpha_s^{h_i(s)} (\alpha_s + \alpha_t)^{h_{it}(s)} \alpha_t^{h_{ist}(s)} \\ &= \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) \end{aligned}$$

A1. $\text{Stab}_i = \langle s, t \rangle$, this implies $h_i(s) = h_i(t) =: h$ by definition ($x_i(\alpha_s) \in \Phi_{V^{(k)}}$ if and only if $x_{it}(\alpha_s) = x_i(\alpha_s + \alpha_t) = x_{is}(\alpha_t) \in \Phi_{V^{(k)}} \Leftrightarrow x_i(\alpha_t) \in \Phi_{V^{(k)}}$) and as a consequence we get $\alpha^h \delta_s(\alpha_t^h t(\alpha_s^h)) = 0$. This simplifies the equation to

$$\sigma_i(s)\sigma_i(t)\sigma_i(s) - \sigma_i(t)\sigma_i(s)\sigma_i(t) = \delta_s(\alpha_t^h \delta_t(\alpha_s^h))\sigma_i(s) - \delta_t(\alpha_s^h \delta_s(\alpha_t^h))\sigma_i(t)$$

note that $Q_s := \delta_s(\alpha_t^h \delta_t(\alpha_s^h)), Q_t := -\delta_t(\alpha_s^h \delta_s(\alpha_t^h))$ are polynomials in α_s, α_t .

A2. $\text{Stab}_i = \langle s \rangle$ (analogue $\text{Stab}_i = \langle t \rangle$). One has $itst = its$. We use in this case $h_{is}(t) = h_i(t), h_{ist}(s) = h_{it}(s) = h_i(t), h_{its}(t) = h_i(s)$.

$$\begin{aligned} \sigma_i(s)\sigma_i(t)\sigma_{it}(s)(f) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)(f) &= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)} t(\alpha_s^{h_{ist}(s)}) ts(f)) - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts \delta_t(f) \\ &= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)} t(\alpha_s^{h_{ist}(s)})) ts(f) + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) \delta_s(ts(f)) \\ &= -\alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts \delta_t(f) = 0 \end{aligned}$$

Since $st\delta_s = \delta_t st$ and $\delta_s(\alpha_t^h t(\alpha_s^h)) = 0$.

A3. $\text{Stab}_i = \langle sts \rangle$, then $ist = is, its = it$.

$$\begin{aligned} & \sigma_i(s)\sigma_{is}(t)\sigma_{is}(s)(f) - \sigma_i(t)\sigma_{it}(s)\sigma_{it}(t)(f) \\ &= \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)} \delta_t(\alpha_s^{h_{is}(s)} s(f))) - \alpha_t^{h_{is}(s)} t(\alpha_s^{h_{is}(t)} \delta_s(\alpha_t^{h_i(s)} t(f))) \\ &= [\alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s(\delta_t(\alpha_s^{h_{is}(s)}))] - \alpha_t^{h_{is}(s)} t(\alpha_s^{h_{is}(t)}) t(\delta_s(\alpha_t^{h_i(s)}))] \cdot f \end{aligned}$$

using $t\delta_s t = s\delta_t s$.

A4. $\text{Stab}_i = \{1\}$ (and the same for $\text{Stab}_i = \langle st \rangle$)

$$\begin{aligned} & \sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t) \\ &= \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst \\ &= 0 \end{aligned}$$

(B) $stst = tsts$: $\langle s, t \rangle \cong D_4$ (order is 8),

$$\begin{aligned} t(\alpha_s) &= \alpha_s + \alpha_t, & st(\alpha_s) &= \alpha_s + \alpha_t, & tst(\alpha_s) &= \alpha_s \\ s(\alpha_t) &= 2\alpha_s + \alpha_t, & ts(\alpha_t) &= 2\alpha_s + \alpha_t, & sts(\alpha_t) &= \alpha_t. \end{aligned}$$

Here we have to consider ten different cases because D_4 has ten subgroups. It always holds the following

$$h_{itst}(s) = h_i(s), h_{its}(t) = h_{is}(t), h_{it}(s) = h_{ist}(s), h_{ists}(t) = h_i(t)$$

which implies

$$\alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) = \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)})$$

This will be used in all cases, it is particular easy to see that for

$$\text{Stab}_i = \{1\}, \quad \text{Stab}_i = \{1, ts, st, stst\}, \quad \text{Stab}_i = \{1, stst\}$$

we obtain that the difference is zero from the above equality. Let us investigate the other cases. Furthermore, the following is useful to notice

$$\delta_s(t(\alpha_s)^h) = 0, \quad \delta_t(s(\alpha_t)^h) = 0$$

B1. $\text{Stab}_i = \langle s, t \rangle$. We prove the following

$$\begin{aligned} \sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)(f) &= Q_{st}\sigma_i(s)\sigma_i(t) \\ &+ \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}) sts(\alpha_t^{h_i(t)}) \delta_{stst}(f) \\ &+ \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}) \delta_{sts}(\alpha_t^{h_i(t)}) \delta_t \end{aligned}$$

with $Q_{st} = \delta_s(\alpha_t^{h_i(t)}) \delta_t(\alpha_s^{h_i(s)}) + s(\alpha_t^{h_i(t)}) \delta_{st}(\alpha_s^{h_i(s)}) + t(\alpha_s^{h_i(s)}) \delta_{ts}(\alpha_t^{h_i(t)}) = Q_{ts}$ is a polynomial in α_s, α_t . By a long direct calculation (applying the product rule for the δ_s) several times

$$\begin{aligned} \sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)(f) &= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)} \delta_t(\alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}))) \delta_t(f) \\ &+ [\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) s \delta_t(\alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)})) + \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)} \delta_t(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})))] \delta_{st}(f) \\ &+ \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)} t(\alpha_s^{h_i(s)}) ts(\alpha_t^{h_i(t)})) \delta_{tst}(f) \\ &+ \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}) sts(\alpha_t^{h_i(t)}) \delta_{stst}(f) \end{aligned}$$

We have a look at the polynomials occurring in front of the δ_w :

$w = t$: by the product rule

$$\begin{aligned} \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)} \delta_t(\alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}))) &= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)})^2 \delta_t(\alpha_s^{h_i(s)}) \\ &+ \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) \delta_s(\alpha_t^{h_i(t)}) \delta_{st}(\alpha_s^{h_i(s)}) + \alpha_s^{h_i(s)} t(\alpha_s^{h_i(s)}) \delta_s(\alpha_t^{h_i(t)}) \delta_{ts}(\alpha_t^{h_i(t)}) \\ &+ \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}) \delta_{sts}(\alpha_t^{h_i(t)}) \end{aligned}$$

$w = st$:

$$\begin{aligned}
& \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) s \delta_t(\alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)})) \\
& \quad + \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}) \delta_t(\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})) \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) s \delta_s(\alpha_t^{h_i(t)}) s \delta_t(\alpha_s^{h_i(s)}) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}) \delta_{ts}(\alpha_t^{h_i(t)}) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) \delta_s(\alpha_t^{h_i(t)}) \delta_t(\alpha_s^{h_i(s)}) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) \delta_s(s(\alpha_t^{h_i(t)})) s \delta_t(\alpha_s^{h_i(s)}) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)})^2 \delta_{st}(\alpha_s^{h_i(s)}) \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) [\delta_s(\alpha_t^{h_i(t)}) \delta_t(\alpha_s^{h_i(s)}) \\
& \quad + s(\alpha_t^{h_i(t)}) \delta_{st}(\alpha_s^{h_i(s)}) \\
& \quad + t(\alpha_s^{h_i(s)}) \delta_{ts}(\alpha_t^{h_i(t)})] \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) Q_{st}
\end{aligned}$$

using $s(\delta_s(\alpha_t^h)) = \delta_s(\alpha_t^h)$ and $\delta_s(s(\alpha_t^h)) = -\delta_s(\alpha_t^h)$.

$w = tst$:

$$\alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}) t(\alpha_s^{h_i(s)}) ts(\alpha_t^{h_i(t)}) = 0$$

Now, look at $\sigma_i(s)\sigma_i(t)(f) = \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}) \delta_t(f) + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) \delta_{st}(f)$, which implies

$$\alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) Q_{st} \delta_{st}(f) = Q_{st} \sigma_i(s) \sigma_i(t)(f) - \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}) Q_{st} \delta_t(f)$$

replace the previous expression and compare coefficients in front of $\delta_t(f)$ again gives the polynomial

$$\begin{aligned}
& \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}) \delta_t(\alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)})) - \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_i(t)}) Q_{st} = \\
& \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}) \delta_{sts}(\alpha_t^{h_i(t)})
\end{aligned}$$

We conclude

$$\begin{aligned}
& \sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t) - \sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s) = Q_{st}\sigma_i(s)\sigma_i(t) - Q_{st}\sigma_i(t)\sigma_i(s) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_i(s)}) \delta_{sts}(\alpha_t^{h_i(t)}) \delta_t - \alpha_t^{h_i(t)} t(\alpha_s^{h_i(s)}) ts(\alpha_t^{h_i(t)}) \delta_{tst}(\alpha_s^{h_i(s)}) \delta_s
\end{aligned}$$

Since $\delta_{sts}(\alpha_t^h) = 0 = \delta_{tst}(\alpha_s^k)$ for $h, k \in \{0, 1, 2\}$ since the maps $\delta_{sts}, \delta_{tst}$ map polynomials of degree d to polynomials of degree $d-3$ or to zero, the claim follows. In general, if we localize to $\mathbb{C}[\![t][\alpha_t^{-1}, \alpha_s^{-1}]$ we could still have the analogue statement.

B2. $\text{Stab}_i = \langle s \rangle$ (analogue $\text{Stab}_i = \langle t \rangle$) and use $\delta_s(\alpha_t^{h_{is}(t)}) t(\alpha_s^{h_{ist}(s)}) ts(\alpha_t^{h_{ists}(t)}) = 0$ to see

$$\begin{aligned}
& \sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s) \\
& = \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)}) t(\alpha_s^{h_{ist}(s)}) ts(\alpha_t^{h_{ists}(t)}) tst(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)}) tst \delta_s(f) \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) \delta_s tst(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)}) tst \delta_s(f) \\
& = 0
\end{aligned}$$

because $tst \delta_s = \delta_s tst$.

B3. $\text{Stab}_i = \{1, sts\}$ (analogue $\text{Stab}_i = \{1, tst\}$). One has $its = itst, is = ist$. We have

$$\begin{aligned}
& [\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)](f) \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s \delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) st(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts \delta_t(\alpha_s^{h_{itst}(s)}) s(f) \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) [s \delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) t(f) + st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) s \delta_t(st(f))] \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) [ts \delta_t(\alpha_s^{h_{itst}(s)}) t(f) + tst(\alpha_s^{h_{itst}(s)}) ts \delta_t(st(f))] \\
& = [\alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s \delta_t(\alpha_s^{h_{ist}(s)}) - t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts \delta_t(\alpha_s^{h_{itst}(s)})] \sigma_i(t)(f)
\end{aligned}$$

using $s \delta_t st = ts \delta_t s$ and $s \delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) = \alpha_t^{h_i(t)} s \delta_t(\alpha_s^{h_{ist}(s)})$.

B4. $\text{Stab}_i = \{1, s, tst, stst\}$ (analogue $\text{Stab}_i = \{1, t, sts, stst\}$). One has $i = is, it = its, ist = sts, itst = itsts$.

$$\begin{aligned} & [\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)](f) \\ &= \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})t\delta_s(\alpha_t^{h_{ists}(t)})t(f)) \\ & \quad - \alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})t\delta_s(\alpha_t^{h_{its}(t)})t(\alpha_s^{h_{itst}(s)})t\delta_s(f) \\ &= [\alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})t\delta_s(\alpha_t^{h_{ists}(t)}))]f \\ & \quad + [t(\alpha_s^{h_{it}(s)})][s(\alpha_t^{h_{is}(t)})st\delta_s(\alpha_t^{h_{ists}(t)}) - \alpha_t^{h_i(t)}t\delta_s(\alpha_t^{h_{its}(t)})]\sigma_i(s)(f) \end{aligned}$$

using $\delta_s t \delta_s t = t \delta_s t \delta_s$.

This finishes the investigation of the ten possible cases. We also like to remark that in the example in chapter 5 the case B4 only occurs for $\text{Stab}_i = \{1, t, sts, stst\}$, i.e. the other stabilizer never occurs.

(C) $ststst = tststs$: $\langle s, t \rangle \cong D_6$,

$$\begin{aligned} t(\alpha_s) &= \alpha_s + \alpha_t, & st(\alpha_s) &= 2\alpha_s + \alpha_t, & tst(\alpha_s) &= st(\alpha_s), \\ s(\alpha_t) &= 3\alpha_s + \alpha_t, & ts(\alpha_t) &= 3\alpha_s + 2\alpha_t, & sts(\alpha_t) &= ts(\alpha_t). \end{aligned}$$

One has

$$\begin{aligned} h_{itstst}(s) &= h_i(s), & h_{itsts}(t) &= h_{is}(t), & h_{itst}(s) &= h_{ist}(s) \\ h_{its}(t) &= h_{ists}(t), & h_{it}(s) &= h_{istst}(s), & h_i(t) &= h_{iststs}(t). \end{aligned}$$

this implies

$$\begin{aligned} & \alpha_s^{h_i(s)}s(\alpha_t^{h_{is}(t)})st(\alpha_s^{h_{ist}(s)})sts(\alpha_t^{h_{ists}(t)})stst(\alpha_s^{h_{istst}(s)})ststs(\alpha_t^{h_{iststs}(t)}) = \\ & \alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tstst(\alpha_s^{h_{itstst}(s)}) \end{aligned}$$

Now, D_6 has 13 subgroups. In the following cases the above equality directly implies that $\sigma_i(ststst) - \sigma_i(tststs) = 0$:

$$\begin{aligned} \text{Stab}_i &= \{1\}, & \text{Stab}_i &= \{1, tst\}, & \text{Stab}_i &= \{1, sts\}, \\ & & \text{Stab}_i &= \{1, ststst\}, & \text{Stab}_i &= \langle st \rangle = \langle ts \rangle \end{aligned}$$

C1. $\text{Stab}_i = \langle s, t \rangle$. By assumption we have $h_i(s) = 0 = h_i(t)$ in this case, therefore

$$\begin{aligned} & \sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t) - \sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s)\sigma_i(t)\sigma_i(s) = \\ & \delta_s\delta_t\delta_s\delta_t\delta_s\delta_t - \delta_t\delta_s\delta_t\delta_s\delta_t\delta_s = 0 \end{aligned}$$

because that is known for the divided difference operators, cp [Dem73].

C2. $\text{Stab}_i = \{1, s\}$ (analogue $\text{Stab}_i = \{1, t\}$). Then, $is = i, itstst = itststs$.

$$\begin{aligned} & [\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{iststs}(t) \\ & \quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_{itsts}(t)\sigma_{itstst}(s)](f) \\ &= \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)})tstst(f)) \\ & \quad - \alpha_t^{h_i(t)}t(\alpha_s^{h_{it}(s)})ts(\alpha_t^{h_{its}(t)})tst(\alpha_s^{h_{itst}(s)})tsts(\alpha_t^{h_{itsts}(t)})tstst(\alpha_s^{h_{itstst}(s)})tstst\delta_s(f) \\ &= \alpha_s^{h_i(s)}\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)}))tstst(f) \\ &= 0 \end{aligned}$$

using $\delta_s t stst = t stst \delta_s$ and

$$\delta_s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)})) = 0$$

because

$$\begin{aligned} & s(\alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(t)})) \\ &= \alpha_t^{h_{is}(t)}t(\alpha_s^{h_{ist}(s)})ts(\alpha_t^{h_{ists}(t)})tst(\alpha_s^{h_{istst}(s)})tsts(\alpha_t^{h_{iststs}(s)}). \end{aligned}$$

C3. $\text{Stab}_i = \{1, tstst\}$ (analogue $\text{Stab}_i = \{1, ststs\}$). Then $its = itst, istst = istst$.

$$\begin{aligned}
& [\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{iststs}(t) \\
& \quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_{itsts}(t)\sigma_{itstst}(s)](f) \\
&= \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) sts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)}) st(f)) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts\delta_t(\alpha_s^{h_{itst}(s)}) s(\alpha_t^{h_{itsts}(t)}) st(\alpha_s^{h_{itstst}(s)}) sts(f)) \\
&= \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) sts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)}) s(f) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) stst(\alpha_s^{h_{istst}(s)}) ststs(\alpha_t^{h_{iststs}(t)}) sts\delta_t st(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts\delta_t(\alpha_s^{h_{itst}(s)}) s(\alpha_t^{h_{itsts}(t)}) st(\alpha_s^{h_{itstst}(s)}) s(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)}) tsts(\alpha_t^{h_{itsts}(t)}) tstst(\alpha_s^{h_{itstst}(s)}) ts\delta_t sts(f) \\
&= [s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) sts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)}) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts\delta_t(\alpha_s^{h_{itst}(s)}) s(\alpha_t^{h_{itsts}(t)})] \sigma_i(s)(f)
\end{aligned}$$

using $ts\delta_t sts = stst\delta_t st$.

C4. $\text{Stab}_i = \{1, s, tstst, ststst\}$ (analogue $\text{Stab}_i = \{1, t, ststs, ststst\}$). Then $is = i, itst = its$. Observe, in this case

$$h_i(t) = h_{it}(t), \text{ and } h_{it}(s) = h_{its}(s)$$

and one has

$$\begin{aligned}
& [\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{iststs}(t) \\
& \quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_i(tsts)\sigma_{itstst}(s)](f) \\
&= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)}) t(\alpha_s^{h_{ist}(s)}) ts(\alpha_t^{h_{ists}(t)}) ts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)}) st(f)) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts\delta_t(\alpha_s^{h_{itst}(s)}) s(\alpha_t^{h_{itsts}(t)}) st(\alpha_s^{h_{itstst}(s)}) st\delta_s(f)) \\
&= \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)}) t(\alpha_s^{h_{ist}(s)}) ts(\alpha_t^{h_{ists}(t)}) ts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)}) \cdot f \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) sts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)}) \delta_s(f) \\
& \quad + \alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)}) t(\alpha_s^{h_{ist}(s)}) ts(\alpha_t^{h_{ists}(t)}) tst(\alpha_s^{h_{istst}(s)}) tsts(\alpha_t^{h_{iststs}(t)}) ts\delta_t st(f)) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) stst(\alpha_s^{h_{istst}(s)}) ststs(\alpha_t^{h_{iststs}(t)}) \\
& \quad \cdot \delta_s ts\delta_t st(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts\delta_t(\alpha_s^{h_{itst}(s)}) s(\alpha_t^{h_{itsts}(t)}) st(\alpha_s^{h_{itstst}(s)}) \delta_s(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)}) tsts(\alpha_t^{h_{itsts}(t)}) tstst(\alpha_s^{h_{itstst}(s)}) \\
& \quad \cdot ts\delta_t st\delta_s(f)) \\
&= [\alpha_s^{h_i(s)} \delta_s(\alpha_t^{h_{is}(t)}) t(\alpha_s^{h_{ist}(s)}) ts(\alpha_t^{h_{ists}(t)}) ts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)})] f \\
& \quad + [s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) sts\delta_t(\alpha_s^{h_{istst}(s)}) s(\alpha_t^{h_{iststs}(t)}) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) ts\delta_t(\alpha_s^{h_{itst}(s)}) s(\alpha_t^{h_{itsts}(t)})] \sigma_i(s)(f)
\end{aligned}$$

using $\delta_s ts\delta_t st = ts\delta_t st\delta_s$.

C5. $\text{Stab}_i = \{1, sts, tst, ststst\}$. Then $is = ist, it = its$. Observe, in this case

$$h_i(s) = h_{is}(s), \text{ and } h_i(t) = h_{it}(t)$$

and and has

$$\begin{aligned}
& [\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t)\sigma_{istst}(s)\sigma_{iststs}(t) \\
& \quad - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s)\sigma_{itsts}(t)\sigma_{itstst}(s)](f) \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s\delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) st(\alpha_s^{h_{istst}(s)}) st\delta_s(\alpha_t^{h_{iststs}(t)}) t(f)) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) t\delta_s(\alpha_t^{h_{its}(t)}) t(\alpha_s^{h_{itst}(s)}) ts(\alpha_t^{h_{itsts}(t)}) ts\delta_t(\alpha_s^{h_{itstst}(s)}) s(f)) \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s\delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) st(\alpha_s^{h_{istst}(s)}) st\delta_s(\alpha_t^{h_{iststs}(t)}) \cdot f \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) stst(\alpha_s^{h_{istst}(s)}) stst\delta_s(\alpha_t^{h_{iststs}(t)}) s\delta_t s(f) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s\delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) st(\alpha_s^{h_{istst}(s)}) sts(\alpha_t^{h_{iststs}(t)}) t\delta_s t(f) \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) stst(\alpha_s^{h_{istst}(s)}) ststst(\alpha_t^{h_{iststs}(t)}) \\
& \quad \cdot s\delta_t st\delta_s t(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) t\delta_s(\alpha_t^{h_{its}(t)}) t(\alpha_s^{h_{itst}(s)}) ts(\alpha_t^{h_{itsts}(t)}) ts\delta_t(\alpha_s^{h_{itstst}(s)}) \cdot f \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)}) tsts(\alpha_t^{h_{itsts}(t)}) tsts\delta_t(\alpha_s^{h_{itstst}(s)}) t\delta_s t(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) t\delta_s(\alpha_t^{h_{its}(t)}) t(\alpha_s^{h_{itst}(s)}) ts(\alpha_t^{h_{itsts}(t)}) tst(\alpha_s^{h_{itstst}(s)}) s\delta_t s(f) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)}) tsts(\alpha_t^{h_{itsts}(t)}) tstsst(\alpha_s^{h_{itstst}(s)}) \\
& \quad \cdot t\delta_s ts\delta_t s(f) \\
& = P_e f + [\alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) sts(\alpha_t^{h_{ists}(t)}) stst(\alpha_s^{h_{istst}(s)}) stst\delta_s(\alpha_t^{h_{iststs}(t)}) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) t\delta_s(\alpha_t^{h_{its}(t)}) t(\alpha_s^{h_{itst}(s)}) ts(\alpha_t^{h_{itsts}(t)}) tst(\alpha_s^{h_{itstst}(s)})] s\delta_t s(f) \\
& \quad + [\alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s\delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) st(\alpha_s^{h_{istst}(s)}) sts(\alpha_t^{h_{iststs}(t)}) \\
& \quad - \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) tst(\alpha_s^{h_{itst}(s)}) tsts(\alpha_t^{h_{itsts}(t)}) tsts\delta_t(\alpha_s^{h_{itstst}(s)})] t\delta_s t(f)
\end{aligned}$$

using $s\delta_t st\delta_s t = t\delta_s ts\delta_t s$ where P_e is a polynomial in α_t, α_s . Then we look at

$$\begin{aligned}
\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)(f) & = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) s\delta_t(\alpha_s^{h_{ist}(s)}) \cdot f \\
& \quad + \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)}) s\delta_t s(f) \\
\sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)(f) & = \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) t\delta_s(\alpha_t^{h_{its}(t)}) \cdot f \\
& \quad + \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)}) t\delta_s t(f)
\end{aligned}$$

and we observe for the coefficient in front of $s\delta_t s(f)$ that it is divisible by

$\alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_{ist}(s)})$ and the one in front of $t\delta_s t$ is divisible by

$\alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_{its}(t)})$. Observe $h_{ist}(s) = h_i(s), h_{its}(t) = h_i(t)$. Use the following simplifications

$$\begin{aligned}
& t\delta_s(\alpha_t^{h_{its}(t)}) t(\alpha_s^{h_{itst}(s)}) ts(\alpha_t^{h_{itsts}(t)}) tst(\alpha_s^{h_{itstst}(s)}) \\
& = t\delta_s(\alpha_t^{h_i(t)}) t(\alpha_s^{h_i(s)}) ts(\alpha_t^{h_{is}(t)}) tst(\alpha_s^{h_i(s)}) \\
& = s(\alpha_t^{h_{is}(t)}) t\delta_s(\alpha_t^{h_i(t)}) t(\alpha_s^{h_i(s)}) st(\alpha_s^{h_i(s)}) \\
& = s(\alpha_t^{h_{is}(t)}) [t\delta_s(\alpha_t^{h_i(t)}) t(\alpha_s^{h_i(s)})] tst(\alpha_s^{h_i(s)}) + ts(\alpha_t^{h_i(t)}) tst(\alpha_s^{h_i(s)}) t\delta_s st(\alpha_s^{h_i(s)}) \\
& = s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_i(s)}) [\alpha_s^{h_i(s)} t\delta_s(\alpha_t^{h_i(t)}) + ts(\alpha_t^{h_i(t)}) t\delta_s t(\alpha_s^{h_i(s)}) \\
& \quad - ts(\alpha_t^{h_i(t)}) t\delta_s t(\alpha_s^{h_i(s)})] \\
& = \alpha_s^{h_i(s)} s(\alpha_t^{h_{is}(t)}) st(\alpha_s^{h_i(s)}) t\delta_s(\alpha_t^{h_i(t)})
\end{aligned}$$

and analogously

$$\begin{aligned}
& s\delta_t(\alpha_s^{h_{ist}(s)}) s(\alpha_t^{h_{ists}(t)}) st(\alpha_s^{h_{istst}(s)}) sts(\alpha_t^{h_{iststs}(t)}) \\
& = s\delta_t(\alpha_s^{h_i(s)}) s(\alpha_t^{h_i(t)}) st(\alpha_s^{h_{it}(s)}) sts(\alpha_t^{h_i(t)}) \\
& = \alpha_t^{h_i(t)} t(\alpha_s^{h_{it}(s)}) ts(\alpha_t^{h_i(t)}) s\delta_t(\alpha_s^{h_i(s)})
\end{aligned}$$

Then a simple substitution gives that the difference above is of the form

$$Q_e f + Q_{sts} \sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)(f) + Q_{tst} \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)$$

for some polynomials Q_e, Q_{sts}, Q_{tst} in α_s, α_t .

Now, let A be the algebra given by generator $\widetilde{1}_i, \widetilde{z}_i(t), \widetilde{\sigma}_i(s)$ subject to relations (1)-(5). Then, by the straightening rule and the braid relation one has that if $w = s_1 \cdots s_k = t_1 \cdots t_k$ are two reduced expressions then

$$\sigma(\widetilde{t_1 \cdots t_k}) \in \sum_{v \leq s_1 \cdots s_k \text{ reduced subword}} \mathcal{E} \diamond \sigma(\widetilde{v}).$$

Therefore, once we have fixed one (any) reduced expression for each for $w \in \mathbb{W}$, one has

$$A = \sum_{w \in \mathbb{W}} \mathcal{E} \diamond \sigma(\widetilde{w}).$$

Since the generators of \mathcal{Z}_G fulfill the relations (1)-(5), we have a surjective algebra homomorphism

$$A \rightarrow \mathcal{Z}_G$$

mapping $\widetilde{1}_i \mapsto 1_i, \widetilde{z}_i(t) \mapsto z_i(t), \widetilde{\sigma}_i(s) \mapsto \sigma_i(s)$. Since $\mathcal{Z}_G = \bigoplus_{w \in \mathbb{W}} \mathcal{E} \diamond \sigma(w)$ and the map is by definition \mathcal{E} -linear it follows that

$$A = \bigoplus_{w \in \mathbb{W}} \mathcal{E} \diamond \sigma(\widetilde{w})$$

and the map is an isomorphism. □

4 Examples

4.1 Classical Springer Theory

This is the case of the following initial data

$$\left[\begin{array}{l} (*) \ G \text{ an arbitrary reductive group,} \\ (*) \ P = B \text{ a Borel subgroup of } G, \text{ denote its Levi decomposition by } B = TU \\ \quad \text{with } T \text{ maximal torus, } U \text{ unipotent.} \\ (*) \ V = \mathfrak{g} \text{ the adjoint representation,} \\ (*) \ F = \mathfrak{n} := \text{Lie}(U). \end{array} \right.$$

It can be obtained as a generalized quiver-graded Springer theory (in the sense of 3.1) by choosing ($\mathbb{G} = G, \mathbb{B} = B, \mathbb{U} = F, \mathbb{H} = T, \mathbb{V} = \text{Lie}(\mathbb{G})$).

We set $\mathcal{N} := G\mathfrak{n}$, i.e. the image of the Springer map, and call it the nilpotent cone. We consider the Springer map as $\pi: E = G \times^B \mathfrak{n} \rightarrow \mathcal{N}$. Explicitly, we can write the Springer map as $E = \{(n, gB) \in \mathcal{N} \times G/B \mid n \in {}^g \mathfrak{b} := \text{Lie}(gBg^{-1})\} \xrightarrow{\pi = pr_1} \mathcal{N}$. Here, π can be identified with the moment map of G , in particular, $E \cong T^*(G/B)$ is the cotangent bundle over G/B and π is a resolution of singularities for \mathcal{N} . But most importantly, this makes the Springer map a symplectic resolution of singularities and one can use symplectic geometry (see for example [CG97]). The Steinberg variety is given by $Z = \{(n, gB, hB) \in \mathcal{N} \times G/B \times G/B \mid n \in {}^g \mathfrak{b} \cap {}^h \mathfrak{b}\}$. Recall, that we had the stratification by relative position $Z^w := m^{-1}(G \cdot (eB, wB)), w \in W$ where W is the Weyl group of G with respect to a maximal torus $T \subset B$. Here $Z^w \rightarrow G \cdot (eB, wB)$ is a vector bundle, $\dim Z^w = \dim G - \dim T = \dim E =: e$, so Z is equi-dimensional of dimension e with irreducible components $\overline{Z^w}, w \in W$. That implies that the top-dimensional Borel-Moore homology group $H_{top}(Z)$ has a \mathbb{C} -vector space basis given by the cycles $[\overline{Z^w}]$. Here in fact, they give a vector space basis of the top-dimensional Borel Moore homology $H_{top}(Z)$ which is the graded degree zero subalgebra of $H_*(Z)$. Here, it even holds $H_{top}(Z) \cong \mathbb{C}W$, $[\overline{Z^s}] \mapsto s-1$. Furthermore, there is a well-known bijection between the simple $\mathbb{C}W$ -modules and the set $\{(x, \chi) \mid x \in \mathcal{N} \text{ rep of } G\text{-orbits}, \chi \in \text{Simp}(C(x)), \text{Hom}_{C(x)}(\chi, H_{top}(\pi^{-1}(x))) \neq 0\}$ where $C(x) = \text{Stab}_G(x)/\text{Stab}_G(x)^0$ is naturally operating on $H_{top}(\pi^{-1}(x))$. This is called **Springer correspondence**.

There are several alternative constructions of the Springer correspondence to the one of Chriss and Ginzberg (in [CG97]), see e.g. [Ara89], section 5 and classically [Spr76],[Spr78], then Kazhdan-Lusztig [KL80], Slodowy [Slo80a], Lusztig [Lus81], Rossmann [Ros91]).

4.2 Quiver-graded Springer Theory

Let Q be a finite quiver with set of vertices Q_0 and set of arrows Q_1 . Let us fix a dimension vector $\underline{d} \in \mathbb{N}_0^{Q_0}$ and a sequence of dimension vectors $\underline{d} := (0 = \underline{d}^0, \dots, \underline{d}^\nu =: \underline{d}), \underline{d}_i^k \leq \underline{d}_i^{k+1}$. Quiver-graded Springer Theory arises

from the following initial data

$$\left[\begin{array}{l} (*) G = \mathbf{Gl}_{\underline{d}} := \prod_{i \in Q_0} \mathbf{Gl}_{d_i}, \\ (*) P = P(\underline{d}) := \prod_{i \in Q_0} P(d_i^\bullet) \text{ where } P(d_i^\bullet) \text{ is the parabolic in } \mathbf{Gl}_{d_i} \text{ fixing a (standard) flag} \\ \quad V_i^\bullet \text{ in } \mathbb{C}^{d_i} \text{ with dimensions given by } d_i^\bullet, \\ (*) V = R_Q(\underline{d}) := \prod_{(i \rightarrow j) \in Q_1} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) \text{ with the operation } (g_i)(M_{i \rightarrow j}) = (g_j M_{i \rightarrow j} g_i^{-1}) \\ \text{is called representation space.} \\ (*) F = F(\underline{d}) := \{(M_{i \rightarrow j}) \in R_Q(\underline{d}) \mid M_{i \rightarrow j}(V_i^k) \subset V_j^k, \ 0 \leq k \leq \nu\} \end{array} \right.$$

Given \underline{d} , $n := \sum_{i \in Q_0} d_i$, set $I := \{\underline{d} = (0 = \underline{d}^0, \dots, \underline{d}^n) \mid \underline{d}^k < \underline{d}^{k+1}, \underline{d}^n = \underline{d}\}$. Then the data $(\mathbf{Gl}_{\underline{d}}, P(\underline{d}), R_Q(\underline{d}), F(\underline{d}))_{\underline{d} \in I}$ can be obtained from the generalized quiver-graded Springer theory with $\mathbb{G} = \mathbf{Gl}_n, \mathbb{B}$ standard Borel, $\mathcal{U} = \text{Lie}(\mathbb{U})^r$ where r the maximal number of arrows between any two vertices, $H \cong \mathbb{G}_m^{\#Q_0}$ is the diagonal torus with the same scalars in the diagonal blocks of sizes d_i , $i \in Q_0$, $V = R_Q(\underline{d})$ embedded as a $\mathbf{Gl}_{\underline{d}} = C_{\mathbb{G}}(H)$ -subrepresentation into $\text{Lie}(\mathbb{G})^r$.

For $\underline{d} \in I$ let $E_{\underline{d}} = \{(M, U^\bullet) \in R_Q(\underline{d}) \times \text{Fl}_{\underline{d}} \mid i \xrightarrow{\forall \alpha} j \in Q_1 : M_\alpha(U_i^k) \subset U_j^k, \ 1 \leq k \leq \nu\}$, then the first projection $\pi : \bigsqcup_{\underline{d} \in I} E_{\underline{d}} \rightarrow R_Q(\underline{d})$ is the quiver-graded Springer map. It has been first defined by Lusztig (cp. e.g. [Lus91]). One has

$$\dim E_{\underline{d}} = \dim \text{Fl}_{\underline{d}} + \dim F(\underline{d}) = \sum_{i \in Q_0} \sum_{k=1}^{\nu-1} d_i^k (d_i^{k+1} - d_i^k) + \sum_{(i \rightarrow j) \in Q_1} \sum_{k=1}^{\nu} (d_i^k - d_i^{k-1}) d_j^k.$$

The $(\mathbf{Gl}_{\underline{d}}$ -equivariant) Steinberg algebra is the quiver Hecke algebra (for Q, \underline{d}). If the quiver Q has no loops, its generators and relations have been calculated by Varagnolo and Vasserot in [VV11b]. They check that this is the same algebra as has been introduced by Khovanov and Lauda in [KL09] (and which was previously conjectured by Khovanov and Lauda to be the Steinberg algebra for quiver-graded Springer theory with complete dimension vectors). Independently, this has been proven by Rouquier in [Rou11].

Theorem 4.1. (quiver Hecke algebra, [VV11b], [Rou11]) *Let Q be a quiver without loops and $\underline{d} \in \mathbb{N}_0^{Q_0}$ be a fixed dimension vector. The $(\mathbf{Gl}_{\underline{d}}$ -equivariant) quiver-graded Steinberg algebra for complete dimension filtrations $R_{\underline{d}}^G := H_*^G(Z)$ for (Q, \underline{d}) is as graded \mathbb{C} -algebra generated by*

$$1_i, i \in I, \quad z_i(k), i \in I, 1 \leq k \leq d, \quad \sigma_i(s), i \in I, s \in \{(1, 2), (2, 3), \dots, (d-1, d)\} =: \mathbb{S},$$

where $d := \sum_{a \in Q_0} d_a$, $I := I_{\underline{d}} := \{(i_1, \dots, i_d) \mid i_k \in Q_0, \sum_{k=1}^d i_k = \underline{d}\}$ and we see $\mathbb{S} \subset S_d$ as permutations of $\{1, \dots, d\}$, we also define

$h_i((\ell, \ell+1)) = h_{i_{\ell+1}, i_\ell} = \#\{\alpha \in Q_1 \mid \alpha : i_{\ell+1} \rightarrow i_\ell\}$ and let

$$\deg 1_i = 0, \quad \deg z_i(k) = 2, \quad \deg \sigma_i((\ell, \ell+1)) = \begin{cases} 2h_i((\ell, \ell+1)) - 2 & , \text{ if } i_\ell = i_{\ell+1} \\ 2h_i((\ell, \ell+1)) & , \text{ if } i_\ell \neq i_{\ell+1} \end{cases}$$

subject to relations

- (1) (orthogonal idempotents) $1_i 1_j = \delta_{i,j} 1_i$, $1_i \sigma_i(s) 1_{is} = \sigma_i(s)$, $1_i z_i(k) 1_i = z_i(k)$
- (2) (polynomial subalgebras) $z_i(k) z_i(k') = z_i(k') z_i(k)$
- (3) For $s = (k, k+1)$, $i = (i_1, \dots, i_d)$ we write $is := (i_1, \dots, i_{k+1}, i_k, \dots, i_d)$ and set $\alpha_s := \alpha_{i,s} := z_i(k) - z_i(k+1)$ if it is clear from the context which i is meant. We denote by $h_i(s) := \#\{\alpha \in Q_1 \mid \alpha : i_k \rightarrow i_{k+1}\}$.

$$\sigma_i(s) \sigma_{is}(s) = \begin{cases} 0, & \text{if } is = i \\ (-1)^{h_{is}(s)} \alpha_s^{h_i(s) + h_{is}(s)}, & \text{if } is \neq i. \end{cases}$$

- (4) (straightening rule) For $s = (\ell, \ell+1)$ we set $s(z_i(k)) = z_i(s(k))$ and

$$\sigma_i(s) z_{is}(k) - s(z_{is}(k)) \sigma_i(s) = \begin{cases} -1_i, & \text{if } is = i, s = (k, k+1) \\ 1_i, & \text{if } is = i, s = (k-1, k) \\ 0, & \text{if } is \neq i. \end{cases}$$

(5) (braid relation) Let $s, t \in \mathbb{S}$, $st = ts$, then $\sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s)$. Let $i \in I$, $s = (k, k+1)$, $t = (k+1, k+2)$. We set $s(\alpha_t) := (z_i(k) - z_i(k+2)) =: t(\alpha_s)$

$$\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t) = \begin{cases} P_{s,t} & \text{if } ist = i, is \neq i, it \neq i \\ 0, & \text{else.} \end{cases}$$

where $P_{s,t} := \frac{\alpha_s^{h_i(s)} \alpha_t^{h_{is}(s)} - (-1)^{h_{is}(s)} \alpha_s^{h_{is}(s)} \alpha_t^{h_i(s)}}{\alpha_s + \alpha_t} - \frac{\alpha_t^{h_i(s)} \alpha_s^{h_{it}(s)} - (-1)^{h_{it}(s)} \alpha_t^{h_i(s)} \alpha_s^{h_{it}(s)}}{\alpha_s + \alpha_t}$ is a polynomial in $z_i(k), z_i(k+1), z_i(k+2)$. We call this the **quiver Hecke algebra** for Q, \underline{d} .

Using the degeneration of the spectral sequence argument from lemma 2 we get

Corollary 4.1. Let Q be a quiver without loops and $\underline{d} \in \mathbb{N}_0^{Q_0}$. The (non-equivariant) Steinberg algebra $R_{\underline{d}} := H_{[*]}(Z)$ is the graded \mathbb{C} -algebra generated by

$$1_i, i \in I, \quad z_i(k), i \in I, 1 \leq k \leq d \quad \sigma_i(s), i \in I, s \in \{(1, 2), (2, 3), \dots, (d-1, d)\}$$

with the same degrees and relations as $R_{\underline{d}}^G$ and the additional relations

$$P(z_i(1), \dots, z_i(d)) = 0, \quad i \in I, \quad P \in \mathbb{C}[x_1, \dots, x_d]^W$$

where $W \subset S_d$ is the Weyl group of $\mathbf{GL}_{\underline{d}}$.

The main result in quiver graded Springer theory is that the graded projective modules over the quiver Hecke algebra(s for all $\underline{d} \in \mathbb{N}_0^{Q_0}$) have a monoidal structure which *categorifies* the negative half of the quantum group associated to the quiver (for the definition see [Lus91]). Here categorifies means that its Grothendieck group carries the structure of a twisted graded Hopf algebra which is after tensoring with $\mathbb{Q}(q)$ isomorphic to this quantum group. The basis of this quantum group obtained by this isomorphism as the image of the classes of the graded projective modules is precisely **Lusztig's canonical basis**. This has been proven by Khovanov and Lauda, and Varagnolo and Vasserot (cp. [KL09], [VV11b]).

4.3 Symplectic quiver-graded Springer theory

This construction works in general for (general) symplectic and (special) orthogonal groups (and products of them) rather analogously to the quiver-graded Springer theory. Partly these constructions overlap with [VV11a]. Before we start we recall some basics about the symplectic group.

The root system of the symplectic group. The group $\mathbf{Sp}_{2n} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{GL}_{2n} \mid {}^tAC = {}^tCA, {}^tDB = {}^tBD, {}^tAD - {}^tCB = E_n \right\}$ has the following maximal (split) torus $T_n := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbf{GL}_n \text{ diagonal} \right\}$. Its Lie algebra is

$$\mathfrak{sp}_{2n} := \text{Lie}(\mathbf{Sp}_{2n}) = \{A \in M_{2n \times 2n} \mid A^t J = -JA\} \rightarrow \{A \in M_{2n \times 2n} \mid A = A^t\} = S^2 \mathbb{C}^{2n}$$

$$A \mapsto JA$$

which maps the adjoint representation on the left hand side to $B \cdot A := BAB^t$ on the right hand side. A general element of the Lie algebra is $\begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix}$ with Y, Z symmetric. If we denote by $\varepsilon_i: T \rightarrow \mathbb{C}^*$ the projection on the i -th diagonal entry $1 \leq i \leq n$, for two map $\lambda, \mu: T \rightarrow \mathbb{C}^*$ we write $\lambda + \mu: T \rightarrow \mathbb{C}^*$, $t \mapsto \lambda(t)\mu(t)$, $-\lambda: T \rightarrow \mathbb{C}^*$, $t \mapsto \lambda(t)^{-1}$, $0: T \rightarrow \mathbb{C}^*$, $t \mapsto 1$, the roots are

$$0, \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j, -\varepsilon_i - \varepsilon_j, 2\varepsilon_i, -2\varepsilon_i, \quad 1 \leq i, j \leq n, i \neq j$$

The root system is of type C_n , the Weyl group is defined as $W = N_{\mathbf{Sp}_{2n}}(T)/T \cong S_n \times (\mathbb{Z}/2\mathbb{Z})^n$, we fix the following set of elements in $N_{\mathbf{Sp}_{2n}}(T)$ whose left cosets generate W :

For $\tau \in S_n$ we write $\tau := \begin{pmatrix} P_\tau & 0 \\ 0 & P_\tau \end{pmatrix}$ with $P_\tau = (e_{\tau(1)}, \dots, e_{\tau(n)}) \in \mathbf{GL}_n$,

for $\sigma_i = (0, \dots, 0, \bar{1}, 0, \dots, 0) \in (\mathbb{Z}/2\mathbb{Z})^n$ we write $\sigma_i = \begin{pmatrix} E_n - E_{ii} & E_{ii} \\ -E_{ii} & E_n - E_{ii} \end{pmatrix}$, $1 \leq i \leq n$.

The positive roots are $0, \varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j, 2\varepsilon_i$, the simple roots are $\varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq n-1, 2\varepsilon_n$. Let $S \subset W$ be the set of reflections defined at the simple roots, it gives a generating set of W . As usually we identify $S = \{(1, 2), \dots, (n-1, n), \sigma_n\} \subset \mathbf{Sp}_{2n}$. The Borel subgroup whose Lie algebra equals the sum of the positive weights is our standard Borel subgroup.

The assumption $h_{a,\sigma} = h_{a,\sigma(a)}$ ensures that we have for every type of arrow in Q_0 an associated indecomposable $G = (G_{|i|})$ -direct summand of \mathfrak{sp} which we used to define the representation space above. To understand this remark look at the schematic picture below.

- (3) Let $\mathbb{W}_n \cong S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ be the Weyl group of \mathbb{G} with respect to the diagonal torus. The embedding gives an inclusion of the Weyl group $W_{|i|}$ of $G_{|i|}$ into \mathbb{W}_n . We fix a bijection

$$W_{|i|} \setminus \mathbb{W}_n \rightarrow \mathbb{I}_{|i|} := \{j \in \mathbb{I} \mid |j| = |i|\}$$

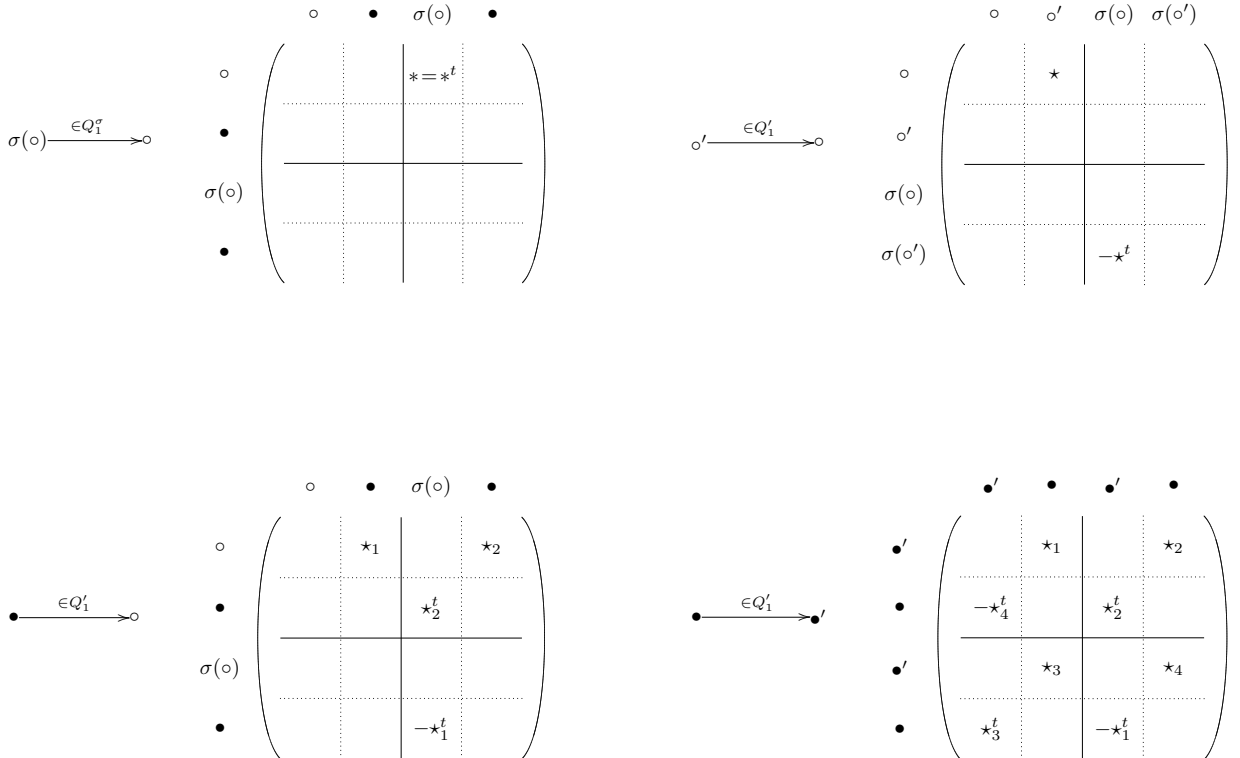
Using the transitive right operation on $\mathbb{I}_{|i|}$ defined as follows:

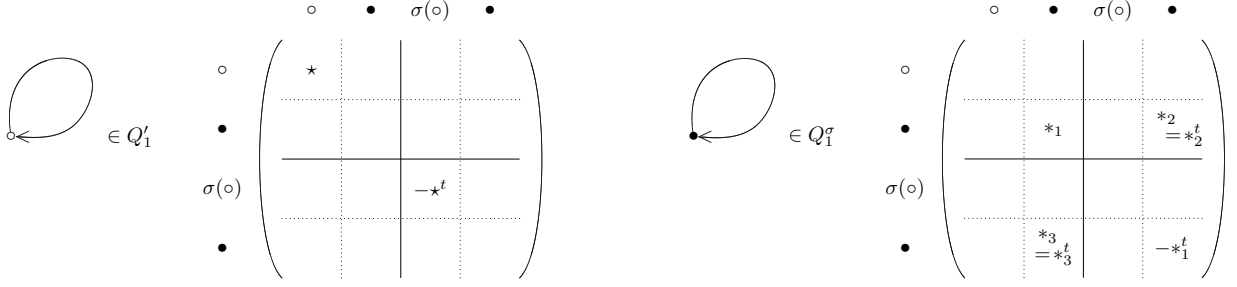
See i as a function $i: \{1, \dots, n\} \sqcup \{1^*, \dots, n^*\} \rightarrow Q_0$ with $\sum_{j=1}^n (i(j) + i(j^*)) = |i|$ with the property $i(j) = v \Leftrightarrow i(j^*) = \sigma(v)$. Then the operation of $w \in S_n$ is given by $iw := i \circ (w \sqcup w)$ and the operation of $(\mathbb{Z}/2\mathbb{Z})^n$ is given by swapping k and k^* , $1 \leq k \leq n$. Then, the stabilizer of every point is isomorphic to $W_{|i|}$. We choose the point which is given by the numbering of $Q_0 := \{k_1, k_2, \dots\}$ which is of the form $i := (k_1, k_1, \dots, k_2, \dots) \in \mathbb{I}_{|i|}$.

Let $\mathbb{B}_n \subset \mathbb{G}_n$ the upper-triangular standard Borel in the symplectic group, $B_{|i|} := G_{|i|} \cap \mathbb{B}_n$. We will choose the unique representatives $x_i \in \mathbb{W}_n, i \in \mathbb{I}_{|i|}$ of the right cosets $W_{|i|} \setminus \mathbb{W}_n$ which satisfy $G_{|i|} \cap x_i \mathbb{B}_n = B_{|i|}$. We set $B_i := B_{|i|}$ as our parabolic subgroup.

- (4) $F_i := V_{|i|} \cap x_i \mathcal{U}_n^\oplus t$ where $\mathcal{U}_n = \text{Lie}(\mathbb{U}_n), \mathbb{U}_n \subset \mathbb{B}_n$ is the unipotent radical, here we use the embedding $V_{|i|} \subset \mathfrak{sp}_{2n}^\oplus, t = \#Q_1 + \#Q'_1$ coming from the matrix block description (see below). There is a different description of F_i in terms of elements of V_i which stabilize a complete isotropic flag (given by x_i applied to the standard flag).

Schematic pictures of the $G_{|i|}$ -subrepresentations of \mathfrak{sp}_{2n} associated to the arrows





As a compromise for readability, I will from now on assume that the quiver Q does not contain any loops.

Theorem 4.2. Let (Q, σ) be a symmetric quiver, Q without loops and fix $|i| \in \Gamma \subset (\mathbb{N}_0^{Q_0})^\sigma$. Set $I := \mathbb{I}_{|i|} = \{j = (j_1, \dots, j_n) \mid j_k \in Q_0, \sum j_k + \sigma(j_k) = |i|\}$, $\mathbb{S} \subset S_n \times (\mathbb{Z}/2\mathbb{Z})^n$ the set of positive roots.

(1) For $i \in I$ we set $\mathcal{E}_i := \mathbb{C}[x_i(1), \dots, x_i(n)]$. We consider $\bigoplus_{i \in I} \mathbb{C}[x_i(1), \dots, x_i(n)]$ as the left $\mathbb{W} := S_n \times (\mathbb{Z}/2\mathbb{Z})^n$ -module $\text{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}_n]$ via $f \in \mathbb{C}[x_i(1), \dots, x_i(n)]$, $w \in \mathbb{W}$ map to $w(f) \in \mathbb{C}[x_{iw^{-1}}(1), \dots, x_{iw^{-1}}(n)]$. In particular, we write $\alpha_s \in \mathcal{E}_i$ for the polynomial corresponding to the simple reflection $s \in \mathbb{S}$ and $w(\alpha_s) \in \mathcal{E}_{iw^{-1}}$, $w \in \mathbb{W}$ without mentioning that it depends on $i \in I$ when this is clear from the context. Then $\mathcal{Z}_{|i|} \subset \text{End}_{\mathbb{C}\text{-lin}}(\bigoplus_{i \in I} \mathcal{E}_i)$ is the \mathbb{C} -subalgebra generated by

$$1_i, z_i(t), \sigma_i(s), \quad i \in I, 1 \leq t \leq n, s \in \mathbb{S}$$

defined by:

Let $k \in I$, $f \in \mathcal{E}_k$. One has $1_i(f) = \delta_{i,k} f$, $z_i(t)(f) = \delta_{i,k} x_i(t) f$ and

$$\sigma_i(s)(f) := \begin{cases} \frac{s(f) - f}{\alpha_s}, & \text{if } i = is = k, \\ \alpha_s^{h_i(s)} s(f) & \text{if } i \neq is = k, \\ 0, & \text{else.} \end{cases}$$

where $h_i(s) = \#\{\alpha \in Q_1^\sigma \sqcup Q_1' \mid \text{for } V := (V_{|i|})_\alpha, x_i(\alpha_s) \in \Phi_V\}$ with Φ_V is the set of T -weights of V . One has for $i = (i_1, \dots, i_n) \in \mathbb{I}$ and $s_\ell = (\ell, \ell + 1) \in S_n$, $e_n = (0, \dots, 0, 1) \in (\mathbb{Z}/2\mathbb{Z})^n$

$$h_i(s_\ell) = h_{i_{\ell+1}, i_\ell}, \quad h_i(e_n) = h_{\sigma(i_n), \sigma}$$

Observe, that there is a natural grading on $\mathcal{Z}_{|i|}$ by

$$\deg 1_i = 0, \deg z_i(n) = 2, \deg \sigma_i(s) = \begin{cases} -2 & , \text{ if } is = i \\ 2h_i(s) & , \text{ if } is \neq i \end{cases}$$

(2) Let W be the Weyl group of $(G_{|i|}, T_{|i|} = T_n)$. We consider $\bigoplus_{i \in I} \mathbb{C}[z_i(1), \dots, z_i(n)]$ as the left $\mathbb{W} := S_n \times (\mathbb{Z}/2\mathbb{Z})^n$ -module $\text{Ind}_W^{\mathbb{W}} \mathbb{C}[\mathfrak{t}_n]$ defined as before. Then $\mathcal{Z}_{|i|}$ is the \mathbb{Z} -graded \mathbb{C} -algebra with generators

$$1_i, i \in I, \quad z_i(t), 1 \leq t \leq n = \text{rk}(T), i \in I, \quad \sigma_i(s), s \in \mathbb{S}, i \in I$$

of the degree as in (1) and relations $1_i 1_j = \delta_{i,j} 1_i$, $1_i z_i(t) 1_i = z_i(t)$, $1_i \sigma_i(s) 1_{is} = \sigma_i(s)$, $z_i(t) z_i(t') = z_i(t') z_i(t)$, $\sigma_i(s) \sigma_{is}(s) = \begin{cases} 0 & , \text{ if } is = i, \\ (-1)^{h_{is}(s)} \alpha_s^{h_i(s) + h_{is}(s)} & , \text{ if } is \neq i \end{cases}$

(*) If $is \neq i$: $\sigma_i(s) z_{is}(t) - s(z_{is}(t) \sigma_i(s)) = 0$

If $is = i$ and $s = s_\ell = (\ell, \ell + 1)$

$$\begin{aligned} \sigma_i(s) z_i(\ell) - z_i(\ell + 1) \sigma_i(s) &= -1_i, \\ \sigma_i(s) z_i(\ell + 1) - z_i(\ell) \sigma_i(s) &= 1_i, \\ \sigma_i(s) z_i(t) - z_i(t) \sigma_i(s) &= 0, \quad \text{if } t \notin \{\ell, \ell + 1\} \end{aligned}$$

If $is = i$ and $s = e_n$

$$\begin{aligned}\sigma_i(s)z_i(n) + z_i(n)\sigma_i(s) &= -1_i \\ \sigma_i(s)z_i(t) - z_i(t)\sigma_i(s) &= 0, \quad \text{if } t \neq n\end{aligned}$$

(*) For $s, t \in \mathbb{S}$, $st = ts$: $\sigma_i(s)\sigma_{is}(t) = \sigma_i(t)\sigma_{it}(s)$

Let $s = s_\ell = (\ell, \ell + 1)$, $t = s_{\ell+1}$

$$\begin{aligned}\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t) \\ = \begin{cases} \alpha_s^{h_i(s)} s(\delta_t(\alpha_s^{h_{is}(s)})) - \alpha_t^{h_{is}(s)} t(\delta_s(\alpha_t^{h_i(s)})), & \text{if } ist = i, is \neq i, it \neq i \\ 0, & \text{else.} \end{cases}\end{aligned}$$

Let $s = (n - 1, n)$, $t = e_n$

$$\begin{aligned}\sigma_i(s)\sigma_{is}(t)\sigma_{ist}(s)\sigma_{ists}(t) - \sigma_i(t)\sigma_{it}(s)\sigma_{its}(t)\sigma_{itst}(s) \\ = \begin{cases} P_t \sigma_i(t), & \text{if } ist = i, is \neq i, it \neq i \\ P_s \sigma_i(s), & \text{if } itst = i, it \neq i, is \neq i \\ R_t \sigma_i(t) + R_e, & \text{if } it = i = ist, is \neq i \\ 0, & \text{else.} \end{cases}\end{aligned}$$

where

$$\begin{aligned}P_t &= \alpha_s^{h_i(s)} s \delta_t(\alpha_s^{h_{ist}(s)}) - t(\alpha_s^{h_{it}(s)}) t s \delta_t(\alpha_s^{h_{itst}(s)}) \\ P_s &= \alpha_t^{h_i(t)} t \delta_s(\alpha_t^{h_{its}(t)}) - s(\alpha_t^{h_{is}(t)}) s t \delta_s(\alpha_t^{h_{ist}(t)}) \\ R_t &= t(\alpha_s^{h_{it}(s)}) t s \delta_t(\alpha_s^{h_{itst}(s)}) - \alpha_s^{h_i(s)} s \delta_t(\alpha_s^{h_{ist}(s)}) \\ R_e &= \delta_t(\alpha_s^{h_{it}(s)}) s \delta_t(\alpha_s^{h_{itst}(s)})\end{aligned}$$

Using the degeneration of the spectral sequence argument from lemma 2 we get

Corollary 4.2. *Let (Q, σ) be a symmetric quiver, Q without loops and fix $|i| \in \Gamma \subset (\mathbb{N}_0^{Q_0})^\sigma$. Then, the (non-equivariant) Steinberg algebra $H_{[*]}(Z)$ is the graded \mathbb{C} -algebra generated by*

$$1_i, i \in I, \quad z_i(k), i \in I, 1 \leq k \leq d \quad \sigma_i(s), i \in I, s \in \{(1, 2), (2, 3), \dots, (n - 1, n), \sigma_n\}$$

with the same degrees and relations as $\mathcal{Z}_{|i|}$ and the additional relations

$$P(z_i(1), \dots, z_i(n)) = 0, \quad i \in I, \quad P \in \mathbb{C}[x_1, \dots, x_n]^{W_{|i|}}$$

For the symmetric quiver-graded Springer theory, our expectation is that the graded projective modules categorify the Hall module (see e.g. Young, [You12]).

Table 1: List of known Steinberg algebras.

	$H_{top}(Z, \mathbb{C})$	$H_*(Z, \mathbb{C})$	$H_*^G(Z, \mathbb{C})$	$K_0^{G \times \mathbb{C}^*}(Z) \otimes_{\mathbb{Z}} \mathbb{C}$
(G, B, \mathfrak{g}, u) classical ST	$\mathbb{C}W$	$\mathbb{C}[t]/I_W \# \mathbb{C}[W]$	$\mathbb{C}[t] \# \mathbb{C}[W]$ degenerate affine Hecke algebra	affine Hecke algebra
$(G, B, \{0\}, \{0\})$ nil ST i.e. $Z = G/B \times G/B$	\mathbb{C}	$\text{End}_{\mathbb{C}\text{-lin}}(H^*(G/B))$	$\text{End}_{H_G^*(pt)}(H_G^*(G/B))$ (affine) nil Hecke algebra	?
quiver-graded ST (complete dim filtrations)	?	Cor 4.1	(graded parts of) quiver Hecke algebra	?
sym. quiver-graded ST (complete dim filtrations)	?	Cor 4.2	see thm 4.2	?

? means unknown to us. Further known examples are:

- (1) There is an exotic Springer map (defined by Kato [Kat09], [Kat11], Achar and Henderson [AH08]). The K -theoretic Steinberg algebra $K_0^{G \times (\mathbb{C}^*)^3}(Z) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the Hecke algebra with unequal parameters of type $C_n^{(1)}$. Also Kato gave an exotic Deligne-Langlands correspondence.
- (2) Quiver-graded Springer maps for the oriented cycle quiver (allowing only nilpotent representations) gives that $H_*^G(Z)$ is isomorphic to the quiver Schur algebra (compare the work of Stroppel and Webster, [SW11].)
- (3) Replacing the usual \mathbf{Gl}_d -operation for the quiver-graded Springer map by a $\mathbf{Gl}_d \times \prod \mathbb{G}_m^{\#Q_1}$ -operation gives Steinberg algebras which realize the graded parts of KLR algebras associated to Borcherds-Cartan data, see Kang, Kashiwara and Park [KKP13].

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