MEAN WIDTH OF REGULAR POLYTOPES AND EXPECTED MAXIMA OF CORRELATED GAUSSIAN VARIABLES

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ABSTRACT. An old conjecture states that among all simplices inscribed in the unit sphere the regular one has the maximal mean width. An equivalent formulation is that for any centered Gaussian vector $(\xi_1, \ldots, \xi_n)$ satisfying $\mathbb{E}\xi_1^2 = \cdots = \mathbb{E}\xi_n^2 = 1$ one has

$$\mathbb{E}\max\{\xi_1, \ldots, \xi_n\} \leq \sqrt{\frac{n}{n-1}}\, \mathbb{E}\max\{\eta_1, \ldots, \eta_n\},$$

where $\eta_1, \eta_2, \ldots$, are independent standard Gaussian variables. Using this probabilistic interpretation we derive an *asymptotic* version of the conjecture. We also show that the mean width of the regular simplex with $2n$ vertices is remarkably close to the mean width of the regular crosspolytope with the same number of vertices. Interpreted probabilistically, our result states that

$$1 \leq \frac{\mathbb{E}\max\{|\eta_1|, \ldots, |\eta_n|\}}{\mathbb{E}\max\{\eta_1, \ldots, \eta_{2n}\}} \leq \min\left\{\sqrt{\frac{2n}{2n-1}}, 1 + \frac{C}{n \log n}\right\},$$

where $C > 0$ is an absolute constant. We also compute the higher moments of the projection length W of the regular cube, simplex and crosspolytope onto a line with random direction, thus proving several formulas conjectured by S. Finch. Finally, we prove distributional limit theorems for the length of random projection as the dimension goes to ∞. In the case of the n-dimensional unit cube Q_n, we prove that

$$W_{Q_n} - \sqrt{\frac{2n}{\pi}} \xrightarrow[n \to \infty]{d} \mathcal{N}\left(0, \frac{\pi - 3}{\pi}\right),$$

whereas for the simplex and the crosspolytope the limiting distributions are related to the Gumbel double exponential law.

1. CONJECTURE ON THE MEAN WIDTH

1.1. **Introduction.** The mean width of a compact convex body $K \subset \mathbb{R}^n$ is the expected length of a projection of this body onto a line with uniformly chosen, random direction. That is, the mean width equals $\mathbb{E}[W_K]$, where

$$W_K = \sup_{t \in K}\langle U, t\rangle - \inf_{t \in K}\langle U, t\rangle,$$

and U is uniformly distributed on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

How should $n + 1$ points be arranged on the $(n - 1)$-dimensional unit sphere so as to maximize the mean width of their convex hull? An old conjecture states (see [15, Section 9.10.2]) that the arrangement must be *regular*.

The mean width is just a multiple of the first intrinsic volume V_1, namely

$$(1) \qquad V_1(K) = \sqrt{\pi} \, \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \, \mathbb{E}\,[W_K];$$

see [20, p. 210]. The first intrinsic volume has the advantage of not depending on the dimension of the surrounding space. Hence the conjecture can be formulated as follows:

$$(2) \qquad \sup_{x_1,\dots,x_{n+1} \in \mathbb{S}^{n-1}} V_1(\mathrm{conv}(x_1,\dots,x_{n+1})) = V_1(T_n),$$

where T_n is a regular simplex with $n+1$ vertices inscribed in the sphere $\mathbb{S}^{n-1}$, and conv denotes the convex hull.

This question is surprisingly hard. Several authors [14, 3, 4, 22] assumed the existence of a proof, but the problem is still open. Besides very natural formulation in Convex Geometry this problem is very important in Information Theory, as it is closely related to the the long-standing simplex code conjecture [9].

1.2. **Probabilistic statement.** The conjecture can be reformulated in terms of Gaussian processes in the following way. Throughout the paper, $\eta = (\eta_1,\dots,\eta_n)$ denotes a standard Gaussian vector in $\mathbb{R}^n$. Consider a compact set $K \subset \mathbb{R}^n$. Using the fact that the norm and the direction of η are independent, it is not difficult to derive Sudakov's formula

$$(3) \qquad V_1(\mathrm{conv}(K)) = \sqrt{2\pi} \, \mathbb{E} \sup_{x \in K} \langle \eta, x \rangle$$

(see [21] for details and for a generalization to the infinite-dimensional case, or Theorem 3.1 in the present paper for a more general result). This probabilistic interpretation of the first intrinsic volume allows to reformulate the conjecture as follows.

Proposition 1.1. *For every integer $n \geq 2$ the following two statements are equivalent:*

  (i) *One has*

$$(4) \qquad \sup_{x_1,\dots,x_n \in \mathbb{S}^{n-2}} V_1(\mathrm{conv}(x_1,\dots,x_n)) = V_1(T_{n-1}),$$

  and the equality is attained iff $x_1,\dots,x_n$ are vertices of a regular simplex.
  (ii) *For every centered Gaussian vector $(\xi_1,\dots,\xi_n)$ satisfying*

$$(5) \qquad \mathbb{E}\xi_1^2 = \dots = \mathbb{E}\xi_n^2 = 1,$$

  one has

$$(6) \qquad \mathbb{E} \max\{\xi_1,\dots,\xi_n\} \leq \sqrt{\frac{n}{n-1}} \, \mathbb{E} \max\{\eta_1,\dots,\eta_n\},$$

  and the equality is attained iff $\mathbb{E}\,[\xi_i\xi_j] = -1/(n-1)$ for all $i \neq j$.

Proof. First of all note that

$$(7) \qquad \sup_{x_1,\dots,x_n \in \mathbb{S}^{n-2}} V_1(\mathrm{conv}(x_1,\dots,x_n)) = \sup_{y_1,\dots,y_n \in \mathbb{S}^{n-1}} V_1(\mathrm{conv}(y_1,\dots,y_n))$$

because there is an $(n-1)$-dimensional affine subspace (and hence, an $(n-2)$-dimensional sphere of radius at most 1) containing $y_1, \ldots, y_n$. Therefore, we can restate (i) as follows:

$$(8) \qquad \sup_{y_1, \ldots, y_n \in \mathbb{S}^{n-1}} V_1(\operatorname{conv}(y_1, \ldots, y_n)) = V_1(T_{n-1}),$$

and the equality is attained iff $y_1, \ldots, y_n$ are vertices of a regular simplex centered at the origin. Let $\{e_1, \ldots, e_n\}$ be a standard orthonormal basis in $\mathbb{R}^n$. As a realization of such a simplex we can take the convex hull of the points

$$v_i := \sqrt{\frac{n}{n-1}} \left(e_i - \frac{e_1 + \ldots + e_n}{n} \right), \quad i = 1, \ldots, n.$$

To see this, note that the $(n-1)$-dimensional regular simplex

$$S_{n-1} := \operatorname{conv}(e_1, \ldots, e_n)$$

can be inscribed in an $(n-2)$-dimensional sphere of radius $\sqrt{(n-1)/n}$ centered at $(e_1 + \ldots + e_n)/n$. It follows from (3) applied to $K = S_{n-1}$ that

$$(9) \qquad V_1(T_{n-1}) = \sqrt{\frac{n}{n-1}} V_1(S_{n-1}) = \sqrt{2\pi} \sqrt{\frac{n}{n-1}} \, \mathbb{E} \, \max\{\eta_1, \ldots, \eta_n\}.$$

To any points $y_1, \ldots, y_n \in \mathbb{S}^{n-1}$ we associate a centered Gaussian vector $(\xi_1, \ldots, \xi_n)$ such that $\mathbb{E}\xi_1^2 = \cdots = \mathbb{E}\xi_n^2 = 1$ via

$$\xi_1 := \langle \eta, y_1 \rangle, \ldots, \xi_n := \langle \eta, y_n \rangle.$$

If we agree to identify two Gaussian vectors if they have the same distribution and two tuples $(y_1, \ldots, y_n)$ and $(y_1', \ldots, y_n')$ if $\langle y_i, y_j \rangle = \langle y_i', y_j' \rangle$ for all $i, j \in \{1, \ldots, n\}$, then this correspondence becomes one-to-one because $\operatorname{Cov}(\xi_i, \xi_j) = \langle y_i, y_j \rangle$. It follows from (3) that

$$\sqrt{2\pi} \, \mathbb{E} \, \max\{\xi_1, \ldots, \xi_n\} = V_1(\operatorname{conv}(y_1, \ldots, y_n)).$$

The Gaussian vector corresponding to the points $v_1, \ldots, v_n$ satisfies

$$\mathbb{E}\left[\xi_i \xi_j\right] = \langle v_i, v_j \rangle = -1/(n-1), \quad i \neq j.$$

Taken together, the above considerations show the equivalence of (i) and (ii). $\qquad \square$

1.3. Asymptotic version of the conjecture.
We now show that (2) holds *asymptotically*.

Theorem 1.2. *For some absolute constant $C > 0$ and all $n \in \mathbb{N}$,*

$$V_1(T_n) \leq \sup_{x_1, \ldots, x_{n+1} \in \mathbb{S}^{n-1}} V_1(\operatorname{conv}(x_1, \ldots, x_{n+1})) \leq V_1(T_n) \left(1 + C \frac{\log \log n}{\log n} \right).$$

Proof. The first inequality is trivial because we can take $x_1, \ldots, x_{n+1}$ to be the vertices of T_n. Replacing n by $n-1$ and using (7) we can restate that second inequality as follows: For all $n \geq 2$,

$$\sup_{x_1, \ldots, x_n \in \mathbb{S}^{n-1}} V_1(\operatorname{conv}(x_1, \ldots, x_n)) \leq V_1(T_{n-1}) \left(1 + C \frac{\log \log n}{\log n} \right)$$

Fix $x_1, \ldots, x_n \in \mathbb{S}^{n-1}$. For $k = 1, \ldots, n$ define Gaussian random variables $\xi_k := \langle x_k, \eta \rangle$ and note that ξ_k has zero mean and unit variance. It is known (see, e.g., [8, p. 138]) that

$$(10) \qquad \mathbb{E} \, \max\{\xi_1, \ldots, \xi_n\} \leq \sqrt{2 \log n}.$$

We provide a proof for the sake of completeness. For $t > 0$ one has

$$\exp\left(t\,\mathbb{E}\,\max\{\xi_1,\ldots,\xi_n\}\right) \leq \mathbb{E}\,\exp\left(t\max\{\xi_1,\ldots,\xi_n\}\right)$$

$$= \mathbb{E}\,\max\{e^{t\xi_1},\ldots,e^{t\xi_n}\} \leq \sum_{k=1}^{n}\mathbb{E}\,e^{t\xi_k} = ne^{t^2/2}.$$

Letting $t = \sqrt{2\log n}$ yields (10).

On the other hand, it is well-known in the theory of extreme values, see [17, Theorem 1.5.3 on p. 14] and [19], that

$$(11) \qquad \mathbb{E}\,\max\{\eta_1,\ldots,\eta_n\} = \sqrt{2\log n} - O\left(\frac{\log\log n}{\sqrt{2\log n}}\right), \quad n \to \infty.$$

Using (3) and (10), we obtain

$$V_1(\text{conv}(x_1,\ldots,x_n)) = \sqrt{2\pi}\,\mathbb{E}\,\max\{\xi_1,\ldots,\xi_n\} \leq \sqrt{4\pi\log n}.$$

Combining this with (9) and (11) gives

$$V_1(\text{conv}(x_1,\ldots,x_n)) \leq V_1(T_{n-1}) \cdot \sqrt{\frac{n-1}{n}}\left(1 - O\left(\frac{\log\log n}{\log n}\right)\right)^{-1}$$

$$= V_1(T_{n-1}) \cdot \left(1 + O\left(\frac{\log\log n}{\log n}\right)\right),$$

as $n \to \infty$. This proves the claim. $\qquad\square$

2. Regular simplex and regular crosspolytope

In this section we compare the mean width of the regular simplex T_{2n-1} to the mean width of the regular n-dimensional crosspolytope defined by

$$C_n = \text{conv}(\pm e_1,\ldots,\pm e_n).$$

Note that both T_{2n-1} and C_n (which can be considered as a degenerate simplex) have $2n$ vertices and can be inscribed in $\mathbb{S}^{2n-2}$. We will show that conjecture (2) is true in this special case, that is $V_1(C_n) \leq V_1(T_{2n-1})$. Moreover, we will prove a lower bound which shows that the mean width of T_{2n-1} is remarkably close to the mean width of C_n.

2.1. **Mean width and extreme values.** It follows from Sudakov's formula (3), see also (9), that

$$(12) \qquad V_1(T_{n-1}) = \sqrt{2\pi}\sqrt{\frac{n}{n-1}}\,\mathbb{E}\,\max\{\eta_1,\ldots,\eta_n\} = \sqrt{\frac{n}{n-1}}V_1(S_{n-1}),$$

$$(13) \qquad V_1(C_n) = \sqrt{2\pi}\,\mathbb{E}\,\max\{\pm\eta_1,\ldots,\pm\eta_n\} = \sqrt{2\pi}\,\mathbb{E}\,\max\{|\eta_1|,\ldots,|\eta_n|\},$$

where we recall that $S_{n-1} = \text{conv}(e_1,\ldots,e_n)$. It is well-known in the theory of extreme values [17, Theorem 1.5.3 on p. 14] that

$$(14) \qquad \lim_{n\to\infty}\mathbb{P}\left[\max\{\eta_1,\ldots,\eta_n\} \leq u_n + \frac{x}{\sqrt{2\log n}}\right] = e^{-e^{-x}},$$

$$(15) \qquad \lim_{n\to\infty}\mathbb{P}\left[\max\{|\eta_1|,\ldots,|\eta_n|\} \leq u_{2n} + \frac{x}{\sqrt{2\log n}}\right] = e^{-e^{-x}},$$

where u_n is any sequence satisfying $\sqrt{2\pi}u_n e^{u_n^2/2} \sim n$, for example[1]

$$(16) \qquad u_n = \sqrt{2\log n} - \frac{\frac{1}{2}\log\log n + \log(2\sqrt{\pi})}{\sqrt{2\log n}}.$$

Note that (15) (together with (14)) expresses the fact that the minimum and the maximum of $\eta_1, \ldots, \eta_n$ become asymptotically independent; see [17, Theorem 1.8.3 on p. 28]. Taking the expectation (which is justified by [19]) and noting that the expectation of the Gumbel distribution on the right-hand side of (14) and (15) is the Euler constant γ, we obtain the large n asymptotics

$$(17) \qquad V_1(T_{n-1}) = \sqrt{2\pi}\left(u_n + \frac{\gamma + o(1)}{\sqrt{2\log n}}\right), \quad n \to \infty,$$

$$(18) \qquad V_1(C_n) = \sqrt{2\pi}\left(u_{2n} + \frac{\gamma + o(1)}{\sqrt{2\log n}}\right), \quad n \to \infty.$$

These formulas are known; see [2], [12, p. 5], [11, p. 8].

2.2. **Comparing $V_1(T_{2n-1})$ and $V_1(C_n)$.** We are going to show that distance between $V_1(T_{2n-1})$ and $V_1(C_n)$ is in fact much smaller than the bound $o(1/\sqrt{2\log n})$ implied by (17) and (18). First we state the corresponding probabilistic result.

Theorem 2.1. *If $\eta_1, \ldots, \eta_{2n}$ are independent standard Gaussian variables, then*

$$\mathbb{E}\max\{\eta_1, \ldots, \eta_{2n}\} \le \mathbb{E}\max\{|\eta_1|, \ldots, |\eta_n|\} \le \sqrt{\frac{2n}{2n-1}}\,\mathbb{E}\max\{\eta_1, \ldots, \eta_{2n}\}.$$

The left hand-side inequality immediately follows from Slepian's lemma [17, Corollary 4.2.3 on p. 84] because the random vector $(\eta_1, \ldots, \eta_{2n})$ is uncorrelated, whereas the off-diagonal correlations of $(\eta_1, -\eta_1, \ldots, \eta_n, -\eta_n)$ are non-positive. The proof of the second estimate will be given in Section 4. Theorem 2.1 together with (12) and (13) implies the following

Corollary 2.2. *For every $n \in \mathbb{N}$,*

$$\sqrt{\frac{2n-1}{2n}}V_1(T_{2n-1}) \le V_1(C_n) \le V_1(T_{2n-1}).$$

We now provide a bound which is asymptotically sharper. Its proof will be given in Section 5.

Theorem 2.3. *Let $\eta_1, \eta_2, \ldots,$ be independent standard Gaussian variables. Then, as $n \to \infty$, one has*

$$\mathbb{E}\max\{|\eta_1|, \ldots, |\eta_n|\} = \left(1 + \frac{1 + o(1)}{8n\log n}\right)\mathbb{E}\max\{\eta_1, \ldots, \eta_{2n}\}.$$

Combining Theorem 2.3 with (12) and (13) yields the following

Corollary 2.4. *As $n \to \infty$,*

$$V_1(C_n) = V_1(T_{2n-1})\left(1 - \frac{1 + o(1)}{4n}\right), \quad V_1(C_n) = V_1(S_{2n-1})\left(1 + \frac{1 + o(1)}{8n\log n}\right).$$

It is possible to obtain further asymptotic terms in (17) and (18), (see, e.g., [17, Eq. (2.4.8) on p. 39]) but it seems that none of these expansions can correctly capture the very small difference between the expectations in Theorems 2.1 and 2.3.

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[1]$a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

3. HIGHER MOMENTS AND LIMITING DISTRIBUTION OF THE RANDOM WIDTH

3.1. **Sudakov's formula for higher moments.**

Given a convex compact set $K \subset \mathbb{R}^n$ we denote by W_K the length of the projection of K onto a uniformly chosen direction, that is

$$(19) \qquad W_K = \sup_{t \in K} \langle U, t \rangle - \inf_{t \in K} \langle U, t \rangle,$$

where U has uniform distribution on the sphere $\mathbb{S}^{n-1}$. In this section we study the higher moments of the random variable W_K.

Recall that $\eta = (\eta_1, \ldots, \eta_n)$ denotes a random vector having standard normal distribution on $\mathbb{R}^n$. The *isonormal Gaussian process* is defined by

$$\Xi(t) = \langle \eta, t \rangle, \quad t \in \mathbb{R}^n.$$

It is characterized by

$$(20) \qquad \mathbb{E}\left[\Xi(t)\right] = 0, \quad \mathbb{E}\left[\Xi(t)\Xi(s)\right] = \langle t, s \rangle, \quad t, s \in \mathbb{R}^n.$$

For a compact set $K \subset \mathbb{R}^n$ define the range of Ξ over K to be

$$\operatorname*{Range}_{t \in K} \Xi(t) = \sup_{t \in K} \Xi(t) - \inf_{t \in K} \Xi(t).$$

The next theorem generalizes Sudakov's formula (3) to higher moments.

Theorem 3.1. *If the set $K \subset \mathbb{R}^n$ is convex and compact, then*

$$(21) \qquad \mathbb{E}[W_K^k] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbb{E}\left[\left(\operatorname*{Range}_{t \in K} \Xi(t)\right)^k\right].$$

If, moreover, the set K is symmetric with respect to the origin, then

$$(22) \qquad \mathbb{E}[W_K^k] = 2^{\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbb{E}\left[\left(\sup_{t \in K} \Xi(t)\right)^k\right].$$

Proof. The standard Gaussian vector η in $\mathbb{R}^n$ can be written as

$$\eta \stackrel{d}{=} RU,$$

where U and R^2 are such that

(1) U is a random vector with uniform distribution on the unit sphere in $\mathbb{R}^n$;
(2) R^2 is a random variable having χ^2-distribution with n degrees of freedom;
(3) U and R^2 are independent.

It follows that we have the distributional equality

$$(23) \qquad \operatorname*{Range}_{t \in K} \Xi(t) = \sup_{t \in K} \langle \eta, t \rangle - \inf_{t \in K} \langle \eta, t \rangle \stackrel{d}{=} \sup_{t \in K} \langle RU, t \rangle - \inf_{t \in K} \langle RU, t \rangle = RW_K.$$

Taking k-th moments of both parts and noting that R and W_K are independent, we obtain that

$$\mathbb{E}\left[\left(\operatorname*{Range}_{t \in K} \Xi(t)\right)^k\right] = \mathbb{E}[R^k]\,\mathbb{E}[W_K^k].$$

The moments $\mathbb{E}[R^k]$ of the χ^2-distribution are known. Inserting the value of the moment, we obtain (21) (which holds without the symmetry assumption on K). If the set K is symmetric with respect to the origin, then $\operatorname{Range}_{t \in K} \Xi(t) = 2 \sup_{t \in K} \Xi(t)$ and we obtain (22). $\qquad \square$

Remark 3.2. In particular, taking $k = 1$ in Theorem 3.1 and noting that the first intrinsic volume is related to the mean width $\mathbb{E}[W_K]$ by (1), we recover from (21) Sudakov's [21] formula

$$(24) \qquad V_1(K) = \sqrt{\frac{\pi}{2}}\, \mathbb{E}\left[\operatorname*{Range}_{t \in K} \Xi(t)\right] = \sqrt{2\pi}\, \mathbb{E}\left[\sup_{t \in K} \Xi(t)\right].$$

Note that the symmetry assumption on K is not needed in the derivation of (24) because in the last equality we used only that $\sup_{t \in K} \Xi(t)$ has the same distribution as $-\inf_{t \in K} \Xi(t)$.

3.2. **Applications to regular polytopes.** Theorem 3.1 can be used to prove several conjectures on projections of regular polytopes which are due to Finch [11, 12, 13].

Example 3.3. Let $Q_n = [-\frac{1}{2}, +\frac{1}{2}]^n$ be the n-dimensional cube of unit volume. It is easy to see that $\operatorname*{Range}_{t \in Q_n} \Xi(t) = \sum_{i=1}^n |\eta_i|$. Therefore, by (21),

$$(25) \qquad \mathbb{E}[W_{Q_n}^k] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbb{E}\left[\left(\sum_{i=1}^n |\eta_i|\right)^k\right].$$

In particular, taking $k = 1$ and noting that $\mathbb{E}|\eta_1| = \sqrt{\frac{2}{\pi}}$ we obtain that the mean width is

$$\mathbb{E}[W_{Q_n}] = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)}\, n\, \mathbb{E}|\eta_1| = \frac{n\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)},$$

or, equivalently, $V_1(Q_n) = n$, which is well known. The second moment of the projection length is given by

$$\mathbb{E}[W_{Q_n}^2] = \frac{1}{n}\mathbb{E}\left[(|\eta_1| + \ldots + |\eta_n|)^2\right] = \mathbb{E}|\eta_1^2| + (n-1)\mathbb{E}|\eta_1\eta_2| = 1 + \frac{2}{\pi}(n-1),$$

where we have used that $\mathbb{E}|\eta_1^2| = 1$ and $\mathbb{E}|\eta_1\eta_2| = (\mathbb{E}|\eta_1|)^2 = \frac{2}{\pi}$. This formula has been conjectured by Finch [12, p. 9] who established it for $n = 2, 3$ by purely geometric arguments [13]. Using (25) it is possible to compute more moments of W_{Q_n}, for example

$$\mathbb{E}[W_{Q_n}^3] = \frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+3}{2}\right)} \pi^{-\frac{3}{2}} n \left(2n^2 + (3\pi - 6)n + 4 - \pi\right),$$

$$\mathbb{E}[W_{Q_n}^4] = \frac{1}{(n+2)\pi^2}\left(4n^3 + (12\pi - 24)n^2 + (44 - 20\pi + 3\pi^2)n + 8\pi - 24\right),$$

where we have used that $\mathbb{E}|\eta_1| = \sqrt{\frac{2}{\pi}}$, $\mathbb{E}|\eta_1^2| = 1$, $\mathbb{E}|\eta_1^3| = 2\sqrt{\frac{2}{\pi}}$, $\mathbb{E}|\eta_1^4| = 3$.

Example 3.4. For the regular crosspolytope $C_n = \operatorname{conv}(\pm e_1, \ldots, \pm e_n)$ we have $\sup_{t \in C_n} \Xi(t) = \max\{|\eta_1|, \ldots, |\eta_n|\}$ and therefore Theorem 3.1 yields

$$\mathbb{E}[W_{C_n}^k] = 2^{\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbb{E}\left[\left(\max_{1 \le i \le n} |\eta_i|\right)^k\right], \quad k \in \mathbb{N}.$$

For $k = 2$, this formula was conjectured by Finch in [11, p. 3]; see also [12].

Example 3.5. For the regular $(n-1)$-dimensional simplex $S_{n-1} = \text{conv}(e_1, \ldots, e_n) \subset \mathbb{R}^n$, Theorem 3.1 yields

$$\mathbb{E}[W^k_{S_{n-1}}] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \mathbb{E}\left[\left(\max_{1 \leq i \leq n} \eta_i - \min_{1 \leq i \leq n} \eta_i\right)^k\right].$$

Note that in this formula, S_{n-1} is projected onto a random direction in $\mathbb{R}^n$, even though S_{n-1} is $(n-1)$-dimensional.

It is more natural to state the corresponding formula for T_{n-1} (which is a regular simplex with n vertices inscribed in $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$) projected onto a random direction in $\mathbb{R}^{n-1}$. As a realization of T_{n-1} we take the points

$$v_i := \sqrt{\frac{n}{n-1}}\left(e_i - \frac{e_1 + \ldots + e_n}{n}\right), \quad i = 1, \ldots, n,$$

in the hyperplane $L := \{x_1 + \ldots + x_n = 0\} \subset \mathbb{R}^n$ (which can be identified with $\mathbb{R}^{n-1}$). By (20), the isonormal process Ξ on L satisfies

$$(\Xi(v_i))_{i=1,\ldots,n} \overset{d}{=} \sqrt{\frac{n}{n-1}}\left(\eta_i - \frac{\eta_1 + \ldots + \eta_n}{n}\right)_{i=1,\ldots,n},$$

so that for its range on T_{n-1} we have

$$\underset{t \in T_{n-1}}{\text{Range}\,\Xi(t)} \overset{d}{=} \sqrt{\frac{n}{n-1}}\left(\max_{1 \leq i \leq n} \eta_i - \min_{1 \leq i \leq n} \eta_i\right).$$

Therefore, for the projection length of T_{n-1} onto a uniformly chosen random direction in the hyperplane L we obtain

$$\mathbb{E}[W^k_{T_{n-1}}] = 2^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1+k}{2}\right)} \left(\frac{n}{n-1}\right)^{k/2} \mathbb{E}\left[\left(\max_{1 \leq i \leq n} \eta_i - \min_{1 \leq i \leq n} \eta_i\right)^k\right].$$

For $k = 2$, this formula was conjectured by Finch [12, p. 4] who verified it for small values of n.

3.3. **Limit distribution for the random width.** What is the asymptotic distribution of the projection length of a high-dimensional regular polytope onto a random line? The next two theorems answer this question. The proofs are postponed to Section 6.

Theorem 3.6. *The random width of the cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ satisfies the following central limit theorem:*

$$W_{Q_n} - \sqrt{\frac{2n}{\pi}} \xrightarrow[n \to \infty]{d} \mathcal{N}\left(0, \frac{\pi - 3}{\pi}\right).$$

Theorem 3.7. *For the random width of the simplex $S_{n-1} = \text{conv}(e_1, \ldots, e_n)$ and the crosspolytope $C_n = \text{conv}(\pm e_1, \ldots, \pm e_n)$ we have*

$$(26) \qquad \sqrt{2n \log n}\left(W_{S_{n-1}} - \frac{2u_n}{\sqrt{n}}\right) \xrightarrow[n \to \infty]{d} G_1 + G_2,$$

$$(27) \qquad \sqrt{2n \log n}\left(W_{C_n} - \frac{2u_{2n}}{\sqrt{n}}\right) \xrightarrow[n \to \infty]{d} 2G_1,$$

where G_1, G_2 are independent random variables with the Gumbel double exponential distribution $\mathbb{P}[G_1 \leq x] = \mathbb{P}[G_2 \leq x] = e^{-e^{-x}}$, $x \in \mathbb{R}$.

Remark 3.8. It is easy to check that the density of $G_1 + G_2$ equals $2\mathrm{e}^{-x}K_0(2\mathrm{e}^{-x/2})$, $x \in \mathbb{R}$, where

$$K_0(z) = \int_0^\infty \mathrm{e}^{-z\cosh t}\mathrm{d}t, \quad z > 0,$$

is the modified Bessel function of the second kind.

4. Proof of Theorem 2.1

As already mentioned, the first estimate in Theorem 2.1 is a consequence of the Slepian lemma. Therefore, we concentrate on proving the inequality

$$\mathbb{E}\max\{|\eta_1|,\ldots,|\eta_n|\} \le \sqrt{\frac{2n}{2n-1}}\,\mathbb{E}\max\{\eta_1,\ldots,\eta_{2n}\}.$$

The idea of the proof of goes back to the work of Chatterjee (see [7] or [1, p. 50]). For $t \in [0,1]$ consider a centered Gaussian vector

$$\xi(t) = (\xi_1(t),\ldots,\xi_{2n}(t))$$

with correlations defined by

$$\mathbb{E}\left[\xi_i^2(t)\right] = \frac{2n}{t+2n-1}, \quad i = 1,\ldots,2n,$$

$$\mathbb{E}\left[\xi_{2i-1}(t)\xi_{2i}(t)\right] = -\frac{2nt}{t+2n-1}, \quad i = 1,\ldots,n,$$

and $\mathbb{E}\left[\xi_i(t)\xi_j(t)\right] = 0$ otherwise. We have

$$\xi(0) \overset{d}{=} \sqrt{\frac{2n}{2n-1}}\,(\eta_1,\ldots,\eta_{2n}), \quad \xi(1) \overset{d}{=} (\eta_1,-\eta_1,\eta_2,-\eta_2,\ldots,\eta_n,-\eta_n).$$

Hence it is sufficient to show that the function

$$\varphi(t) := \mathbb{E}\max\{\xi_1(t),\ldots,\xi_{2n}(t)\}$$

is non-increasing on $[0,1]$. Consider the function

$$F_\beta(x_1,\ldots,x_{2n}) := \frac{1}{\beta}\log\left(\sum_{i=1}^{2n}\mathrm{e}^{\beta x_i}\right).$$

It is immediate that

$$\max\{x_1,\ldots,x_{2n}\} \le F_\beta(x_1,\ldots,x_{2n}) \le \frac{\log 2n}{\beta} + \max\{x_1,\ldots,x_{2n}\}.$$

Therefore we only need to show that for any $\beta > 0$ the function

$$\varphi_\beta(t) := \mathbb{E}\,F_\beta(\xi(t))$$

is non-increasing on $[0,1]$.

In what follows, $\mathbf{x}$ stands for $(x_1,\ldots,x_{2n})$. Set $\sigma_{ij}(t) := \mathbb{E}\left[\xi_i(t)\xi_j(t)\right]$ and let us denote by $f(t,\mathbf{x})$ the probability density function of $\xi(t)$. It is a well-known property of f that

$$\frac{\partial f}{\partial \sigma_{ii}} = \frac{1}{2}\frac{\partial^2 f}{\partial x_i^2}, \qquad \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i \ne j.$$

Therefore,

$$\frac{\partial \varphi_\beta}{\partial t} = \int_{\mathbb{R}^{2n}} F_\beta(\mathbf{x})\frac{\partial f}{\partial t}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{2n}} F_\beta(\mathbf{x})\sum_{i,j=1}^{2n}\frac{\partial f}{\partial \sigma_{ij}}(\mathbf{x})\frac{\partial \sigma_{ij}}{\partial t}\,\mathrm{d}\mathbf{x}.$$

We have

$$\frac{\partial \sigma_{ii}}{\partial t} = -\frac{2n}{(t + 2n - 1)^2}, \quad i = 1, \ldots, 2n,$$

$$\frac{\partial \sigma_{2i-1,2i}}{\partial t} = -\frac{2n(2n - 1)}{(t + 2n - 1)^2}, \quad i = 1, \ldots, n,$$

and $\partial \sigma_{ij} / \partial t = 0$ otherwise. Thus we obtain

$$\frac{\partial \varphi_\beta}{\partial t} = -\frac{4n}{(t + 2n - 1)^2} \int_{\mathbb{R}^{2n}} F_\beta(\mathbf{x}) \sum_{i=1}^{n} \left[\frac{\partial^2 f}{\partial x_{2i-1}^2}(\mathbf{x}) + 2(2n - 1)\frac{\partial^2 f}{\partial x_{2i-1}\partial x_{2i}}(\mathbf{x}) + \frac{\partial^2 f}{\partial x_{2i}^2}(\mathbf{x}) \right] d\mathbf{x}$$

$$= -\frac{4n}{(t + 2n - 1)^2} \sum_{i=1}^{n} \int_{\mathbb{R}^{2n}} f(\mathbf{x}) \left[\frac{\partial^2 F_\beta}{\partial x_{2i-1}^2}(\mathbf{x}) + 2(2n - 1)\frac{\partial^2 F_\beta}{\partial x_{2i-1}\partial x_{2i}}(\mathbf{x}) + \frac{\partial^2 F_\beta}{\partial x_{2i}^2}(\mathbf{x}) \right] d\mathbf{x}$$

$$= -\frac{4n}{(t + 2n - 1)^2} \sum_{i=1}^{n} \mathbb{E} \left[\frac{\partial^2 F_\beta}{\partial x_{2i-1}^2}(\mathbf{x}) + 2(2n - 1)\frac{\partial^2 F_\beta}{\partial x_{2i-1}\partial x_{2i}}(\mathbf{x}) + \frac{\partial^2 F_\beta}{\partial x_{2i}^2}(\mathbf{x}) \right].$$

It is easy to check that

$$\frac{\partial^2 F_\beta}{\partial x_i^2}(\mathbf{x}) = \beta(p_i(\mathbf{x}) - p_i^2(\mathbf{x})), \qquad \frac{\partial^2 F_\beta}{\partial x_i \partial x_j}(\mathbf{x}) = -\beta p_i(\mathbf{x})p_j(\mathbf{x}), \quad i \neq j,$$

where

$$p_i(\mathbf{x}) := \frac{\partial F_\beta}{\partial x_i}(\mathbf{x}) = \frac{e^{\beta x_i}}{\sum_{i=1}^{2n} e^{\beta x_i}}.$$

Thus,

$$(28) \quad -\frac{(t + 2n - 1)^2}{4n\beta} \cdot \frac{\partial \varphi_\beta}{\partial t} =$$

$$= \sum_{i=1}^{2n} \mathbb{E}\left[p_i(\xi(t))\right] - \sum_{i=1}^{2n} \mathbb{E}\left[p_i^2(\xi(t))\right] - 2(2n - 1)\sum_{i=1}^{n} \mathbb{E}\left[p_{2i-1}(\xi(t))p_{2i}(\xi(t))\right]$$

$$= 1 - \sum_{i=1}^{2n} \mathbb{E}\left[p_i^2(\xi(t))\right] - 2(2n - 1)\sum_{i=1}^{n} \mathbb{E}\left[p_{2i-1}(\xi(t))p_{2i}(\xi(t))\right].$$

For $i = 1, \ldots, n$ the random variables $\xi_{2i-1}(t)$ and $\xi_{2i}(t)$ are non-positively correlated. Therefore it follows from [16] that they are *negatively associated* (see [6] for more details), which implies

$$\mathbb{E}\left[p_{2i-1}(\xi(t))p_{2i}(\xi(t))\right] \leq \mathbb{E}\left[p_i(\xi(t))\right]\mathbb{E}\left[p_j(\xi(t))\right].$$

Hence for all $k \neq l$ we have

$$\mathbb{E}\left[p_{2i-1}(\xi(t))p_{2i}(\xi(t))\right] \leq \mathbb{E}\left[p_k(\xi(t))\right]\mathbb{E}\left[p_l(\xi(t))\right].$$

We thus get

$$\sum_{i=1}^{n} \left(\mathbb{E}\,[p_{2i-1}^2(\xi(t))] + \mathbb{E}\,[p_{2i}^2(\xi(t))] + 2(2n-1)\mathbb{E}\,[p_{2i-1}(\xi(t))p_{2i}(\xi(t))] \right)$$

$$\leq \sum_{i=1}^{n} \left(\mathbb{E}\,[p_{2i-1}^2(\xi(t))] + \sum_{j \neq 2i-1} \mathbb{E}\,[p_{2i-1}(\xi(t))]\mathbb{E}\,[p_j(\xi(t))] + \sum_{j \neq 2i-1} \mathbb{E}\,[p_{2i}(\xi(t))]\mathbb{E}\,[p_j(\xi(t))] \right)$$

$$= \left(\mathbb{E}\left[\sum_{i=1}^{2n} p_i(\xi(t)) \right] \right)^2 = 1.$$

Combining this with (28) yields $\partial\varphi_\beta/\partial t \leq 0$, which completes the proof. $\qquad\square$

5. Proof of Theorem 2.3

Recall that both $A_n := \mathbb{E}\max_{1 \leq i \leq n} |\eta_i|$ and $B_n := \mathbb{E}\max_{1 \leq i \leq 2n} \eta_i$ are asymptotically equivalent to $\sqrt{2\log n}$. Therefore, in order to prove the theorem, we need to show that

$$(29) \qquad \lim_{n \to \infty} 4n\sqrt{2\log n} \left(\mathbb{E}\max_{1 \leq i \leq n} |\eta_i| - \mathbb{E}\max_{1 \leq i \leq 2n} \eta_i \right) = 1.$$

Denote the tail function of the standard normal distribution by

$$\bar{\Phi}(t) = \int_t^\infty e^{-s^2/2}\,\frac{\mathrm{d}s}{\sqrt{2\pi}}.$$

It is well known [1, p. 9] or [8, p. 137] that for $t > 0$ one has

$$(30) \qquad \frac{1}{\sqrt{2\pi}}\left(\frac{1}{t} - \frac{1}{t^3} \right) e^{-t^2/2} \leq \bar{\Phi}(t) \leq \frac{1}{\sqrt{2\pi}\,t} e^{-t^2/2}.$$

The distribution functions of the maxima we are interested in are given by

$$(31) \quad F_n(t) := \mathbb{P}\left[\max_{1 \leq i \leq n} |\eta_i| \leq t \right] = \left(\int_{-t}^t e^{-s^2/2}\,\frac{\mathrm{d}s}{\sqrt{2\pi}} \right)^n = (1 - 2\bar{\Phi}(t))^n, \quad t \geq 0,$$

$$(32) \quad G_n(t) := \mathbb{P}\left[\max_{1 \leq i \leq 2n} \eta_i \leq t \right] = \left(\int_{-\infty}^t e^{-s^2/2}\,\frac{\mathrm{d}s}{\sqrt{2\pi}} \right)^{2n} = (1 - \bar{\Phi}(t))^{2n}, \quad t \in \mathbb{R}.$$

It follows that

$$A_n = \mathbb{E}\max_{1 \leq i \leq n} |\eta_i| = \int_0^\infty (1 - F_n(t))\,\mathrm{d}t,$$

$$B_n = \mathbb{E}\max_{1 \leq i \leq 2n} \eta_i = \int_0^\infty (1 - G_n(t))\,\mathrm{d}t - \int_0^\infty (\bar{\Phi}(t))^{2n}\,\mathrm{d}t.$$

To prove the second equality, note that for $M := \max_{1 \leq i \leq 2n} \eta_i$ we have $M = M_+ - M_-$ with $M_+ = \max(M, 0)$, $M_- = \max(-M, 0)$, and

$$\mathbb{P}\,[M_+ > t] = 1 - G_n(t), \quad t > 0,$$

$$\mathbb{P}\,[M_- > t] = \mathbb{P}\,[M < -t] = (1 - \bar{\Phi}(-t))^{2n} = (\bar{\Phi}(t))^{2n}, \quad t > 0.$$

In fact, the second integral in the formula for B_n is negligible. Indeed, noting that $\bar{\Phi}(0) = 1/2$ and using (30) we obtain

$$\int_0^\infty (\bar{\Phi}(t))^{2n}\,\mathrm{d}t \le (\bar{\Phi}(0))^{2n} + \int_1^\infty (\bar{\Phi}(t))^{2n}\,\mathrm{d}t \le 2^{-2n} + \int_1^\infty (2\pi)^{-n}\,t^{-2n}\mathrm{e}^{-t^2 n}\mathrm{d}t$$

$$\le 2^{-2n} + (2\pi\mathrm{e})^{-n} \le 2^{-n}.$$

In view of the above considerations, in order to prove (29) it suffices to show that

$$\lim_{n\to\infty} 4n\sqrt{2\log n}\int_0^\infty (G_n(t) - F_n(t))\,\mathrm{d}t = 1.$$

After a change of variable $t := t_n + \frac{a}{t_n}$, $a \in \mathbb{R}$, where $t_n \sim \sqrt{2\log n}$ is the solution to the equation

(33)
$$\bar{\Phi}(t_n) = \frac{1}{2n},$$

our task reduces to proving that

(34)
$$\lim_{n\to\infty} \int_{-t_n^2}^\infty 4n\left(G_n\left(t_n + \frac{a}{t_n}\right) - F_n\left(t_n + \frac{a}{t_n}\right)\right)\,\mathrm{d}a = 1.$$

First we prove the pointwise convergence of the function under the integral sign. If $a \in \mathbb{R}$ is fixed and $n \to \infty$, then by (30) and (33),

(35)
$$r_n := \bar{\Phi}\left(t_n + \frac{a}{t_n}\right) \sim \frac{1}{\sqrt{2\pi}\,t_n}\mathrm{e}^{-\frac{1}{2}\left(t_n+\frac{a}{t_n}\right)^2} \sim \frac{1}{\sqrt{2\pi}\,t_n}\mathrm{e}^{-\frac{1}{2}t_n^2}\mathrm{e}^{-a} \sim \frac{\mathrm{e}^{-a}}{2n}.$$

Recalling the formulas for F_n and G_n, see (31), (32), we can write

$$F_n\left(t_n + \frac{a}{t_n}\right) = (1 - 2r_n)^n = \mathrm{e}^{n\log(1-2r_n)},$$

$$G_n\left(t_n + \frac{a}{t_n}\right) = (1 - r_n)^{2n} = \mathrm{e}^{2n\log(1-r_n)}.$$

Using (35) and the Taylor series for the logarithm and the exponent, we obtain

$$F_n\left(t_n + \frac{a}{t_n}\right) = \exp\left\{-n\left(2r_n + 2r_n^2 + o\left(\frac{1}{n^2}\right)\right)\right\} = \mathrm{e}^{-2nr_n}\left(1 - 2nr_n^2 + o\left(\frac{1}{n}\right)\right),$$

$$G_n\left(t_n + \frac{a}{t_n}\right) = \exp\left\{-2n\left(r_n + \frac{r_n^2}{2} + o\left(\frac{1}{n^2}\right)\right)\right\} = \mathrm{e}^{-2nr_n}\left(1 - nr_n^2 + o\left(\frac{1}{n}\right)\right).$$

Subtracting both expansions and using (35) twice, we obtain

$$4n\left(G_n\left(t_n + \frac{a}{t_n}\right) - F_n\left(t_n + \frac{a}{t_n}\right)\right) = 4n\mathrm{e}^{-2nr_n}\left(nr_n^2 + o\left(\frac{1}{n}\right)\right) \xrightarrow[n\to\infty]{} \mathrm{e}^{-\mathrm{e}^{-a}}\mathrm{e}^{-2a}.$$

If we allow for a moment interchanging the limit and the integral, the limit in (34) equals

$$\text{LHS of (34)} = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{e}^{-a}}\mathrm{e}^{-2a}\,\mathrm{d}a = \int_0^\infty \mathrm{e}^{-y}y\,\mathrm{d}y = 1,$$

where we used the change of variable $y = \mathrm{e}^{-a}$.

To complete the proof we need to justify the use of the Lebesgue dominated convergence theorem. It suffices to show that for some integrable function $g(a)$,

(36)
$$0 \le n\left(G_n\left(t_n + \frac{a}{t_n}\right) - F_n\left(t_n + \frac{a}{t_n}\right)\right) \le g(a), \quad a > -\frac{1}{4}t_n^2, \quad n \in \mathbb{N},$$

and

$$(37) \qquad \lim_{n \to \infty} n \int_{-t_n^2}^{-t_n^2/4} \left(G_n \left(t_n + \frac{a}{t_n} \right) - F_n \left(t_n + \frac{a}{t_n} \right) \right) \, \mathrm{d}a = 0.$$

The non-negativity of $G_n - F_n$ is a consequence of the inequality $(1-z)^2 \geq 1-2z$; see (31), (32). Now we prove the upper bound in (36). Using the inequality

$$y^n - x^n \leq n(y-x)y^{n-1}$$

for $0 \leq x \leq y$, we obtain that

$$(38) \quad G_n(t) - F_n(t) = (1 - 2\bar{\Phi}(t) + \bar{\Phi}^2(t))^n - (1 - 2\bar{\Phi}(t))^n \leq n\bar{\Phi}^2(t)(1 - \bar{\Phi}(t))^{2n-2}.$$

In the following, $C, C_1, \ldots > 0$ denote absolute constants. Let first $a > -\frac{1}{4}t_n^2$ so that $t_n + \frac{a}{t_n} > \frac{3}{4}t_n$. By (30) and (33),

$$(39) \qquad \bar{\Phi} \left(t_n + \frac{a}{t_n} \right) \leq \frac{C_1}{t_n + \frac{a}{t_n}} \mathrm{e}^{-\frac{1}{2}\left(t_n + \frac{a}{t_n} \right)^2} \leq \frac{4C_1}{3t_n} \mathrm{e}^{-\frac{1}{2}t_n^2} \mathrm{e}^{-a} \leq \frac{C_2}{n} \mathrm{e}^{-a}.$$

On the other hand, if $a \in [-\frac{1}{4}t_n^2, 0]$, then again using (30) and (33) we obtain

$$(40) \qquad \bar{\Phi} \left(t_n + \frac{a}{t_n} \right) \geq \frac{C_1'}{t_n + \frac{a}{t_n}} \mathrm{e}^{-\frac{1}{2}\left(t_n + \frac{a}{t_n} \right)^2} \geq \frac{C_1'}{t_n} \mathrm{e}^{-\frac{1}{2}t_n^2} \mathrm{e}^{-a} \mathrm{e}^{-\frac{a^2}{2t_n^2}} \geq \frac{C_2'}{n} \mathrm{e}^{-\frac{7}{8}a},$$

where in the last estimate we used that $-\frac{a^2}{2t_n^2} \geq \frac{1}{8}a$.

Note that because of $-a \leq \frac{1}{4}t_n^2 \sim \frac{1}{2}\log n$, the right hand-side of (39) can be estimated above by $Cn^{-1/4}$. Using the inequality $1 - x \leq \mathrm{e}^{-\frac{1}{2}x}$ (which is valid for sufficiently small $x > 0$) together with (40), we obtain that for $a \in [-\frac{1}{4}t_n^2, 0]$,

$$\left(1 - \bar{\Phi} \left(t_n + \frac{a}{t_n} \right) \right)^{2n-2} \leq \mathrm{e}^{-(n-1)\frac{C_2'}{n}\mathrm{e}^{-\frac{7}{8}a}} \leq \mathrm{e}^{-C'\mathrm{e}^{-\frac{7}{8}a}}.$$

It follows from this and (38), (39) that for all $a > -\frac{1}{2}t_n^2$,

$$n(G_n - F_n) \left(t_n + \frac{a}{t_n} \right) \leq n^2 \bar{\Phi}^2 \left(t_n + \frac{a}{t_n} \right) \left(1 - \bar{\Phi} \left(t_n + \frac{a}{t_n} \right) \right)^{2n-2}$$

$$\leq C'' \mathrm{e}^{-2a} \mathrm{e}^{-C'\mathrm{e}^{-\frac{7}{8}a}},$$

where in the case $a > 0$ we used the trivial estimate $1 - \bar{\Phi}(t) \leq 1$. The function on the right-hand side is integrable in a, thus completing the proof of (36).

We turn now to (37). Using again (38), the trivial estimate $\bar{\Phi}(t) \leq 1$, and the increasing property of $1 - \bar{\Phi}(t)$, we obtain that

$$I_n := n \int_{-t_n^2}^{-t_n^2/4} (G_n - F_n) \left(t_n + \frac{a}{t_n} \right) \, \mathrm{d}a \leq n^2 t_n^2 \left(1 - \bar{\Phi} \left(\frac{3}{4}t_n^2 \right) \right)^{2n-2}.$$

Recall now that $t_n^2 \sim 2 \log n$ and use (40) which implies that for some $\varepsilon \in (0, 1)$,

$$\bar{\Phi} \left(\frac{3}{4}t_n^2 \right) \geq Cn^{-\varepsilon}, \text{ but } \lim_{n \to \infty} \bar{\Phi} \left(\frac{3}{4}t_n^2 \right) = 0.$$

Again using inequality $1 - x \leq \mathrm{e}^{-\frac{1}{2}x}$ (valid for small $x > 0$), we obtain

$$I_n \leq Cn^2 (\log n) \mathrm{e}^{-Cn^{-\varepsilon}(n-1)} \xrightarrow[n \to \infty]{} 0,$$

thus proving (37). $\qquad \qquad \square$

6. Proofs of Theorems 3.6 and 3.7

Both proofs rely on the observation that a random vector U distributed uniformly on $\mathbb{S}^{n-1}$ can be represented as

$$(41) \qquad U \overset{d}{=} \left(\frac{\eta_1}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}}, \ldots, \frac{\eta_n}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} \right).$$

Proof of Theorem 3.6. It follows from the definition of W_{Q_n}, see (19), and from the central symmetry of the cube that

$$(42) \qquad W_{Q_n} = 2 \sup_{t \in Q_n} \langle t, U \rangle \overset{d}{=} \frac{|\eta_1| + \ldots + |\eta_n|}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}}.$$

Consider a random vector (X_n, Y_n) with

$$(43) \qquad X_n := \frac{|\eta_1| + \ldots + |\eta_n| - n\mu}{\sigma\sqrt{n}}, \quad Y_n := \frac{\eta_1^2 + \ldots + \eta_n^2 - n}{v\sqrt{n}},$$

where

$$(44) \qquad \mu := \mathbb{E}|\eta_1| = \sqrt{2/\pi},$$

$$(45) \qquad \sigma^2 := \operatorname{Var}|\eta_1| = \mathbb{E}[\eta_1^2] - (\mathbb{E}|\eta_1|)^2 = (\pi - 2)/\pi,$$

$$(46) \qquad v^2 := \operatorname{Var}(\eta_1^2) = \mathbb{E}[\eta_1^4] - (\mathbb{E}[\eta_1^2])^2 = 2.$$

Note that $\mathbb{E}X_n = \mathbb{E}Y_n = 0$ and $\operatorname{Var}X_n = \operatorname{Var}Y_n = 1$, while

$$(47) \qquad r := \operatorname{Cov}(X_n, Y_n) = \frac{n \operatorname{Cov}(|\eta_1|, \eta_1^2)}{\sigma v n} = \frac{1}{\sqrt{\pi - 2}},$$

where we used that $\mathbb{E}|\eta_1^3| = 2\sqrt{2/\pi}$. By the bivariate central limit theorem,

$$(48) \qquad (X_n, Y_n) \xrightarrow[n \to \infty]{d} (X, Y),$$

where (X, Y) is a bivariate Gaussian vector with standard margins and covariance r. It follows from (43) that

$$W_{Q_n} \overset{d}{=} \frac{|\eta_1| + \ldots + |\eta_n|}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} = \frac{\mu n + \sigma\sqrt{n}X_n}{\sqrt{n + v\sqrt{n}Y_n}} = \frac{\mu n \left(1 + \frac{\sigma X_n}{\mu\sqrt{n}}\right)}{\sqrt{n}\sqrt{1 + \frac{v Y_n}{\sqrt{n}}}}.$$

Letting $n \to \infty$, expanding into a Taylor series around 0 and noting that $X_n = O_P(1)$, $Y_n = O_P(1)$, we get

$$\frac{|\eta_1| + \ldots + |\eta_n|}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} = \mu\sqrt{n}\left(1 + \frac{1}{\sqrt{n}}\left(\frac{\sigma X_n}{\mu} - \frac{v Y_n}{2}\right) + O_P\left(\frac{1}{n}\right)\right), \quad n \to \infty.$$

It follows that

$$\frac{|\eta_1| + \ldots + |\eta_n|}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} - \mu\sqrt{n} = \sigma X_n - \frac{1}{2}\mu v Y_n + O_P\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.$$

Note that by the bivariate central limit theorem (48), the sequence $\sigma X_n - \frac{1}{2}\mu v Y_n$ has limiting normal distribution with mean zero and variance

$$\operatorname{Var}\left[\sigma X_n - \frac{1}{2}\mu v Y_n\right] = \sigma^2 + \frac{1}{4}\mu^2 v^2 - \sigma\mu v r = \frac{\pi - 3}{\pi},$$

where we used (44), (45), (46), (47). Recalling (42) we obtain

$$W_{Q_n} - \sqrt{\frac{2n}{\pi}} \overset{d}{=} \frac{|\eta_1| + \ldots + |\eta_n|}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} - \mu\sqrt{n} \xrightarrow[n \to \infty]{d} \mathcal{N}\left(0, \frac{\pi - 3}{\pi}\right),$$

which proves the claim. $\square$

Remark 6.1. Self-normalized or studentized sums of the form

$$R_n := \frac{\zeta_1 + \ldots + \zeta_n}{\sqrt{\zeta_1^2 + \ldots + \zeta_n^2}} \quad \text{or} \quad T_n := \frac{\zeta_1 + \ldots + \zeta_n}{\sqrt{(\zeta_1 - \bar{\zeta}_n)^2 + \ldots + (\zeta_n - \bar{\zeta}_n)^2}},$$

where $\zeta_1, \zeta_2, \ldots$ are i.i.d. random variables and $\bar{\zeta}_n = \frac{1}{n}(\zeta_1 + \ldots + \zeta_n)$, have been extensively studied in the literature; see, e.g., [10], with main emphasis on the central case $\mathbb{E}[\zeta_i] = 0$. The non-central case $\mathbb{E}[\zeta_i] \neq 0$ has been analyzed by Bentkus et al. [5] (who studied T_n) and by Omey and Van Gulck [18] (who studied $1/R_n^2$ and related quantities). After some calculations, our central limit theorem for W_{Q_n} could be deduced from [18, Theorem 3.1(v)], but since these authors studied $1/R_n^2$ instead of R_n it was easier to provide a direct proof.

Proof of Theorem 3.7. We prove (26). Using representation (41), we obtain

$$(49) \qquad W_{S_{n-1}} \overset{d}{=} \frac{\max_{1 \le i \le n} \eta_i - \min_{1 \le i \le n} \eta_i}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}}.$$

It is known from extreme-value theory that the range of the standard normal sample satisfies

$$(50) \qquad Z_n := u_n \left(\max_{1 \le i \le n} \eta_i - \min_{1 \le i \le n} \eta_i - 2u_n\right) \xrightarrow[n \to \infty]{d} G_1 + G_2,$$

where $u_n \sim \sqrt{2 \log n}$ is as in (16). In fact, this follows from the asymptotic independence [17, Theorem 1.8.3 on p. 28] of $\max_{1 \le i \le n} \eta_i$ and $-\min_{1 \le i \le n} \eta_i$ which both have limiting Gumbel distribution as in (14). Define Y_n as in (43) and observe that Y_n has limiting standard normal distribution by the central limit theorem. We have

$$\frac{\max_{1 \le i \le n} \eta_i - \min_{1 \le i \le n} \eta_i}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} = \frac{2u_n + \frac{Z_n}{u_n}}{\sqrt{n + \sqrt{2n}Y_n}} = \frac{2u_n}{\sqrt{n}} \frac{1 + \frac{Z_n}{2u_n^2}}{\sqrt{1 + \sqrt{2/n}Y_n}}.$$

Noting that both Z_n and Y_n are $O_P(1)$ and expanding into a Taylor series, we obtain

$$\frac{\max_{1 \le i \le n} \eta_i - \min_{1 \le i \le n} \eta_i}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} = \frac{2u_n}{\sqrt{n}} \left(1 + \frac{Z_n}{2u_n^2} + O_P\left(\frac{1}{u_n^4}\right)\right),$$

where we have used that $u_n \sim \sqrt{2 \log n}$ and hence, the term with Y_n is negligible. It follows from (50) that

$$u_n\sqrt{n} \left(\frac{\max_{1 \le i \le n} \eta_i - \min_{1 \le i \le n} \eta_i}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}} - \frac{2u_n}{\sqrt{n}}\right) \xrightarrow[n \to \infty]{d} G_1 + G_2,$$

which, in view of (49), implies (26).

The proof of (27) is analogous but instead of (42) it uses the representation

$$(51) \qquad W_{C_n} \overset{d}{=} \frac{2 \max_{1 \le i \le n} |\eta_i|}{\sqrt{\eta_1^2 + \ldots + \eta_n^2}}.$$

together with the limit relation

$$(52) \qquad Z'_n := u_n \left(\max_{1 \le i \le n} |\eta_i| - u_{2n} \right) \xrightarrow[n \to \infty]{d} G_1$$

following from the asymptotic independence of the maximum and the minimum. $\square$

References

[1] R. Adler and J. Taylor. *Random fields and geometry.* Springer, 2009.

[2] F. Affentranger and R. Schneider. Random projections of regular simplices. *Discrete Comput. Geom.*, 7(3):219–226, 1992.

[3] A. Balakrishnan. Research problem no. 9: Geometry. *Bull. Amer. Math. Soc.*, 69:737–738, 1963.

[4] A. Balakrishnan. Signal selection for space communication channels. In *Advances in Communication Systems*, pages 1–31. Academic Press, 1965.

[5] V. Bentkus, B.-Y. Jing, Q.-M. Shao, and W. Zhou. Limiting distributions of the non-central t-statistic and their applications to the power of t-tests under non-normality. *Bernoulli*, 13(2):346–364, 2007.

[6] A. Bulinski and A. Shashkin. *Limit theorems for associated random fields and related systems.* World Scientific, 2007.

[7] S. Chatterjee. An error bound in the Sudakov–Fernique inequality. *Preprint*, 2005. Available at http://arxiv.org/abs/math/0510424.

[8] S. Chatterjee. *Superconcentration and related topics.* Springer, 2014.

[9] T. M. Cover and eds. B. Gopinath. Open problems in communication and computation. Papers of a Workshop on Problems in Communication and Computation, Morristown, New Jersey, summers of 1984 and 1985 and Palo Alto, California, summer 1986). New York etc.: Springer–Verlag, 1987.

[10] V. H. de la Peña, T. L. Lai, and Q.-M. Shao. *Self-normalized processes. Limit theory and statistical applications.* Berlin: Springer, 2009.

[11] S. R. Finch. Mean width of a regular cross-polytope. *Preprint*, 2011. Available at http://arxiv.org/abs/1112.0499.

[12] S. R. Finch. Mean width of a regular simplex. *Preprint*, 2011. Available at http://arxiv.org/abs/1111.4976.

[13] S. R. Finch. Width distributions for convex regular polyhedra. *Preprint*, 2011. Available at http://arxiv.org/abs/1110.0671.

[14] E. Gilbert. A comparison of signalling alphabets. *Bell Syst. Tech. J.*, 31(3): 504–522, 1952.

[15] P. Gritzmann and V. Klee. On the complexity of some basic problems in computational convexity II: Volume and mixed volumes. Bisztriczky, T. (ed.) et al., Polytopes: abstract, convex and computational. Proceedings of the NATO Advanced Study Institute. Kluwer Academic Publishers. NATO ASI Ser., Ser. C, Math. Phys. Sci. 440, 373–466, 1994.

[16] K. Joag-Dev and F. Proschan. Negative association of random variables with applications. *Ann. Stat.*, 11:286–295, 1983.

[17] M. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and related properties of random sequences and processes.* Springer Series in Statistics. Springer-Verlag, New York–Berlin, 1983.

[18] E. Omey and S. Van Gulck. Domains of attraction of the real random vector (X, X^2) and applications. *Publ. Inst. Math., Nouv. Sér.*, 86:41–53, 2009.

[19] J. Pickands. Moment convergence of sample extremes. *Ann. Math. Stat.*, 39: 881–889, 1968.

[20] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[21] V. Sudakov. Geometric problems in the theory of infinite-dimensional probability distributions. *Trudy Mat. Inst. Steklov*, 141(4):3–191, 1976. English translation: AMS.

[22] C. Weber. *Elements of detection and signal design.* McGraw–Hill, 1968.

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