

CONVEX HULLS OF MULTIDIMENSIONAL RANDOM WALKS

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ABSTRACT. Consider the convex hull of the first n steps of a random walk S_k in \mathbb{R}^d . We show that for planar symmetric random walks, the probability p_n that the hull does not include the origin does not depend on the distribution of increments of S_k . This extends the well known result by Sparre Andersen that the probability that a one-dimensional symmetric walk stays positive is distribution-free.

We used the developed approach to obtain general results on geometric properties of convex hulls of random walks in any dimension. In particular, we give formulas for expected number of faces, volume, surface area, and other intrinsic volumes, including a multidimensional generalization of the Spitzer-Widom formula (1961): It holds that

$$\mathbb{E}V_1(\text{conv}(0, S_1, \dots, S_n)) = \sum_{i=1}^n \frac{\mathbb{E}|S_i|}{i},$$

where V_1 denotes the first intrinsic volume, which up to a factor equals the mean width. Our method also works for convex hulls of random walk bridges and more general, for partial sums of exchangeable random vectors.

As an application to geometry, these results imply the formula by Gao and Vitale [*Discrete Comput. Geom.* 26 (2001)] for intrinsic volumes of special path-simplexes (canonic orthoschemes). The present paper was partly motivated by a direct connection between the yet unknown *spherical* intrinsic volumes of these simplexes and the probabilities p_n .

Key words: convex hull, random walk, distribution-free probability, random polytope, intrinsic volume, spherical intrinsic volume, average number of faces, average surface area, persistence probability, orthoscheme, path-simplex, Wiener spiral.

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1. INTRODUCTION

Let $S_n = X_1 + \dots + X_n$ be a random walk in \mathbb{R}^d . This paper was motivated by the following question: What is the probability that the convex hull of the first n steps of the walk does not include the origin? This is a natural generalization to higher dimensions of the classical problem to find the probability that a one-dimensional random walk stays positive (or negative) by the time n . We gave a partial answer to the posed question and moreover, extended the developed approach obtaining general results on geometric characteristics of the convex hull such as expected number of faces, volume, surface area, and other intrinsic volumes.

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We were inspired by the famous result of Sparre Andersen [22, Theorem 2] that the probability that a one-dimensional random walk stays positive does not depend on the distribution of its increments if the later is continuous and symmetric; moreover,

$$\mathbb{P}(S_1 > 0, \dots, S_n > 0) = \frac{(2n-1)!!}{(2n)!!}. \quad (1)$$

The other well-known fact on symmetric distributions, which is due to Wendel [30], concerns the convex hull of i.i.d. random vectors X_1, \dots, X_n in \mathbb{R}^d : if

$$\mathbb{P}(X_1 \in h) = 0 \quad \text{for any hyperplane } h \subset \mathbb{R}^d \text{ through the origin} \quad (\text{H}_0)$$

and the distribution of X_1 is centrally symmetric, i.e.

$$X_1 \stackrel{d}{=} -X_1, \quad (\text{S})$$

then

$$\mathbb{P}(0 \notin \text{conv}(X_1, \dots, X_n)) = \frac{1}{2^{n-1}} \sum_{k=0}^{d-1} \binom{n-1}{k}. \quad (2)$$

Wendel assumed (H₀) to ensure that with probability one, X_1, \dots, X_n are in *general position*, that is any d of these vectors are linearly independent. We will need a stronger assumption

$$\mathbb{P}(X_1 \in h) = 0 \quad \text{for any affine hyperplane } h \subset \mathbb{R}^d, \quad (\text{H})$$

which in particular guarantees that any one-dimensional projection of X_1 has a continuous distribution.

There is a similarity between the results of Sparre Andersen and Wendel that stems from the use of combinatorial arguments in their proofs. This was our motivation to prove a two-dimensional version of (1) presented below. It is complemented by a distribution-free expression for the angle Ω_n at which $\text{conv}(S_1, \dots, S_n)$ is observed from 0; note that $\Omega_n \leq \pi$ only if 0 is not in the interior of the convex hull, otherwise $\Omega_n = 2\pi$.

Theorem 1. *Let $d = 2$, and assume that (H) and (S) hold. Then*

$$\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n)) = \sum_{k=1}^n \frac{(2n-2k-1)!!}{k(2n-2k)!!}, \quad (3)$$

and moreover,

$$\mathbb{E}(\pi - \Omega_n)^+ = \frac{2\pi(2n-1)!!}{(2n)!!}. \quad (4)$$

Let us discuss some corollaries. It is not hard to derive from (3) (see Sec. 7 below) that

$$\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n)) \sim \frac{\log n}{\sqrt{\pi n}}, \quad d = 2; \quad (5)$$

here and everywhere below the asymptotic is as $n \rightarrow \infty$. Remarkably, this probability is of a higher order of asymptotic than its one-dimensional counterpart (1), where

$$\frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n-3/2)}{\Gamma(1/2)\Gamma(n-1)} \sim \frac{1}{\sqrt{\pi n}}. \quad (6)$$

The discrepancy between the opening angles of the conic hull of $\text{conv}(S_1, \dots, S_n)$ and of any half-plane that contains the hull satisfies

$$\mathbb{E}(\pi - \Omega_n \mid 0 \notin \text{conv}(S_1, \dots, S_n)) \sim \frac{2\pi}{\log n}, \quad d = 2.$$

Finally, since for symmetric random walks it holds that

$$\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n)) = \mathbb{P}(-S_n \notin \text{conv}(S_1 - S_n, \dots, 0)) = \mathbb{P}(S_n \notin \text{conv}(0, S_1, \dots, S_{n-1})),$$

the expected number of times a symmetric planar walk updates its convex hull satisfies

$$\sum_{k=1}^n \mathbb{P}(S_k \notin \text{conv}(0, S_1, \dots, S_{k-1})) \sim \frac{\sqrt{n} \log n}{2\sqrt{\pi}}, \quad d = 2.$$

Unfortunately, our methods do not allow to generalize (3) to higher dimensions. Based on numerical simulations for dimensions $d = 3$ and 4 , we made the following conjecture.

Conjecture. *Let $d \geq 3$, and assume that (H) and (S) hold. Then $\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n))$ does not depend on the distribution of X_1 for any $n \geq 1$.*

The argument below supports this conjecture at least for dimension three. In any dimension, the opening solid angle Ω_n of the convex hull observed from 0 satisfies¹

$$\mathbb{E}\left(\frac{1}{2} - \frac{\Omega_n}{|\mathbb{S}^{d-1}|}\right)^+ = \frac{1}{2}\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n) | U^\perp), \quad (7)$$

where U is a random vector that is uniformly distributed over the unit sphere \mathbb{S}^{d-1} and independent with the random walk S_n , and $\cdot | u^\perp$ denotes the orthogonal projection onto the hyperplane u^\perp through the origin that is orthogonal to a direction $u \in \mathbb{S}^{d-1}$. For any direction u , $\tilde{S}_n := S_n | u^\perp$ is a $(d-1)$ -dimensional random walk which satisfies assumptions of Theorem 1, hence (7) combined with (3) imply that

$$\mathbb{E}(2\pi - \Omega_n)^+ = 2\pi \sum_{k=1}^n \frac{(2n - 2k - 1)!!}{k(2n - 2k)!!}, \quad d = 3,$$

which again is a distribution-free formula. Moreover, (4) immediately follows by (1) and (7).

Our initial interest in the probabilities $\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n))$ emerged from two different areas. As mentioned above, we were interested in a multidimensional version of the one-dimensional *persistence problem* of finding the probability that a stochastic process (a random walk in our case) stays above a certain level. Such problems has drawn a lot of attention over the past few years from both mathematical and theoretical physics communities, see the survey papers by Aurzada and Simon [2] and Bray et al. [5]. On the

¹Indeed, consider any set $A \subset \mathbb{R}^d$. If $0 \notin \text{Int}(\text{conv}(A))$, then by the definition of the solid angle,

$$\Omega(\text{conv}(A)) := \left| \left\{ \frac{x}{|x|}, x \in \text{conv}(A) \right\} \right| = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \mathbb{1}(0 \in \text{conv}(A) | u^\perp) \sigma(du),$$

hence $\frac{1}{2} - \frac{\Omega(\text{conv}(A))}{|\mathbb{S}^{d-1}|} = \frac{1}{2}\mathbb{P}(0 \notin \text{conv}(A) | U^\perp)$. If $0 \in \text{Int}(\text{conv}(A))$, the l.h.s. of the last equation is negative (minus one half) and the r.h.s. equals zero, and thus $(\frac{1}{2} - \frac{\Omega(\text{conv}(A))}{|\mathbb{S}^{d-1}|})^+ = \frac{1}{2}\mathbb{P}(0 \notin \text{conv}(A) | U^\perp)$ holds true for any A .

other side, there is a direct connection to geometry: for random walks with Gaussian increments, $\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n))$ equals the doubled d -th *spherical intrinsic volume* of a certain path-simplex, *the canonic orthoscheme* in \mathbb{R}^n . Spherical intrinsic volume are spheric analogues of Euclidean intrinsic volumes. The open problem of finding the spheric intrinsic volumes of the canonic path simplex was our second motivation to study the probabilities $\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n))$. The details are explained below in Section 5.

We used the approach developed for our initial problem to obtain general results on geometric properties of the convex hulls of multidimensional random walks. For details, we refer the reader directly to Sections 3 and 4.

Average geometric characteristics of convex hulls of planar random walks, for example, expected number of faces and average perimeter, were studied in many papers starting with Spitzer and Widom [23] and followed by few other works that include Baxter [4], Snyder and Steele [19], Wade and Xu [29]. It seems that higher-dimensional versions were first considered by Barndorff-Nielsen and Baxter [3] whose work was overlooked by most of the followers, including ourselves. To the best of our knowledge, the probabilities $\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n))$ were not considered until the very recent works by Eldan [6] and Tikhomirov and Youssef [25], who obtain asymptotic estimates as dimension $d \rightarrow \infty$ for few special types of random walks.

The paper by Abramson et al. [1] gives an overview and the latest account of the very fine description available for the structure of the largest convex minorants of one-dimensional random walks. There are many related papers that consider random walks as the initial step in their studies of convex hulls of continuous time Lévy processes, and of course there is a huge number of works on convex hulls of Brownian motions. These topics are beyond the scope of the current paper; a lot of references can be found in Pitman and Uribe Bravo [16]. There is a survey of results on random convex hulls by Majumdar et al. [14].

This paper is organised as follows. In the next section we present our main tool, a somewhat technical Proposition 1. It immediately implies Theorem 1 and also allows us to study other geometric properties of the convex hull. This is done in Section 3, where we present Theorem 3, a general result on geometric properties of the convex hull of a random walk in any dimension. The theorem also holds true for random walk bridges and more general, for partial sums of exchangeable random vectors. Many applications are given in Section 4. In Section 5 we consider the special case of random walks with Gaussian increments and explain connections of our problem with geometry. The proofs are contained in the last two sections. In Section 6 we present some combinatorial results, which are used in Section 7 to prove Proposition 1 and Theorem 3.

2. THE MAIN TOOL

Denote by

$$C_n := \text{conv}(S_0, S_1, \dots, S_n)$$

the convex hull of the first n steps *including the origin* $S_0 := 0$. To avoid trivialities, we will always assume that $n \geq d$. With probability one C_n is a convex polytope with the boundary

of the form

$$\partial C_n = \bigcup_{f \in \mathcal{F}_n} f, \quad (8)$$

where \mathcal{F}_n is the set of all $(d-1)$ -dimensional faces of C_n . Almost surely, each face f is a $(d-1)$ -dimensional simplex of the form

$$f = \text{conv}(S_{i_1(f)}, \dots, S_{i_d(f)}) \quad (9)$$

for some indices $0 \leq i_1(f) < \dots < i_d(f) \leq n$. It is instructive to think that f is obtained by shifting the simplex with vertices $0, S_{i_2(f)} - S_{i_1(f)}, \dots, S_{i_d(f)} - S_{i_1(f)}$ by $S_{i_1(f)}$. We say that the ordered $(d-1)$ -tuple $(i_2(f) - i_1(f), \dots, i_d(f) - i_1(f))$ is the *temporal structure* of the face and the ordered d -tuple $(i_1(f), \dots, i_d(f))$ is the *pinned temporal structure*.

Let us express the probability that \mathcal{F}_n contains a face of a given temporal structure. In order to stress combinatorial nature of our result, we prove it in a more general setting considering partial sums $S_k = X_1 + \dots + X_k, 1 \leq k \leq n$, of n -exchangeable increments X_1, \dots, X_n . Recalling the definition, this means that for any permutation σ of length n , $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ has the same distribution as (X_1, \dots, X_n) . We will assume that

$$\mathbb{P}(S_1, \dots, S_n \text{ are in general position}) = 1 \quad (\text{G})$$

to ensure that the faces of C_n still are simplexes with probability one. In other words, any d vectors of S_1, \dots, S_n are linearly independent. This is true, for example, when X_1, \dots, X_n have a joint density.

If the increments are independent, then S_n is a random walk and (G) holds if (H) is satisfied. Our main example of partial sums with dependent exchangeable increments is a *random walk bridge* of length $n+1$, that is a random walk conditioned to return to the origin at time $n+1$. In this special case there exists an explicit distribution-free formula for the probability that \mathcal{F}_n contains a face of a given pinned temporal structure. However, our main interest is in the other special case that allows such a formula, namely S_n is a random walk with symmetrically distributed increments.

Proposition 1. *For any $d \geq 1$, let $0 \leq i_1 < \dots < i_d \leq n$ be any indices.*

(1) *If S_k are partial sums of n -exchangeable random vectors in \mathbb{R}^d and (G) holds, then*

$$\sum_{i=0}^{n-i_d+i_1} \mathbb{P}(\text{conv}(S_i, S_{i+i_2-i_1}, \dots, S_{i+i_d-i_1}) \in \mathcal{F}_n) = \frac{2}{(i_2 - i_1) \cdot \dots \cdot (i_d - i_{d-1})}. \quad (10)$$

Moreover, if S_k is a random walk bridge of length $n+1$, then

$$\mathbb{P}(\text{conv}(S_{i_1}, \dots, S_{i_d}) \in \mathcal{F}_n) = \frac{2}{(i_2 - i_1) \cdot \dots \cdot (i_d - i_{d-1})(n - i_d + i_1 + 1)}. \quad (11)$$

(2) If S_k is a random walk² in \mathbb{R}^d and (H) and (S) hold, then

$$\mathbb{P}(\text{conv}(S_{i_1}, \dots, S_{i_d}) \in \mathcal{F}_n) = \frac{(2i_1 - 1)!! (2n - 2i_d - 1)!!}{(2i_1)!! (2n - 2i_d)!!} \prod_{k=1}^{d-1} \frac{2}{i_{k+1} - i_k}. \quad (12)$$

The first application of this result concerns the expected number of faces in \mathcal{F}'_n , the (possibly empty) set of faces of C_n that contain the origin as a vertex. By the equations

$$\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n)) = \mathbb{P}(0 \in \partial C_n) \quad (13)$$

and (12), it is immediate that for symmetric random walks,

$$\mathbb{E}|\mathcal{F}'_n| = 2 \sum_{1 \leq i_2 < \dots < i_d \leq n} \frac{(2n - 2i_d - 1)!!}{i_2 (2n - 2i_d)!!} \prod_{k=2}^{d-1} \frac{1}{i_{k+1} - i_k}. \quad (14)$$

This implies the first part of Theorem 1 since for $d = 2$, it holds that

$$|\mathcal{F}'_n| = \begin{cases} 2, & 0 \notin \text{conv}(S_1, \dots, S_n), \\ 0, & 0 \in \text{conv}(S_1, \dots, S_n). \end{cases}$$

By the same reasoning, we obtain a version of Theorem 1 for random walk bridges:

Theorem 2. *Let S_k be any random walk in \mathbb{R}^2 that satisfies (H). Then*

$$\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n) | S_{n+1} = 0) = \sum_{k=1}^n \frac{1}{k(n-k+1)}.$$

We stress that for this result no assumptions other than (H) are required.

For the asymptotic, it follows that (see Sec. 7) for a random walk under (S) and (H),

$$\mathbb{E}_{RW} |\mathcal{F}'_n| \sim \frac{2(\log n)^{d-1}}{\sqrt{\pi n}}, \quad n \rightarrow \infty, \quad (15)$$

while for a random walk bridge of length $n + 1$,

$$\mathbb{E}_{Br} |\mathcal{F}'_n| \sim \frac{2d(\log n)^{d-1}}{n}, \quad n \rightarrow \infty. \quad (16)$$

3. GEOMETRIC PROPERTIES OF THE CONVEX HULL

For a further application of Proposition 1, we sum in (10) over all possible indices to obtain that the expected number of faces in the convex hull satisfies

$$\mathbb{E}|\mathcal{F}_n| = 2 \sum_{\substack{j_1 + \dots + j_{d-1} \leq n \\ j_1, \dots, j_{d-1} \geq 1}} \frac{1}{j_1 \cdot \dots \cdot j_{d-1}}. \quad (17)$$

² In actual fact, (12) is true for the partial sums of n -exchangeable random vectors X_1, \dots, X_n if (G) holds and all the 2^n n -tuples $(\pm X_1, \dots, \pm X_n)$ have the same distribution. For example, if $d = 1$ so are the coordinates of a random vector X in \mathbb{R}^n with any rotationally invariant distribution. In this case X_1, \dots, X_n are not i.i.d. unless X is a multiple of a standard Gaussian vector.

Comparing (17) and (11), we see that

$$\mathbb{E}^{(d)}|\mathcal{F}_n| = \sum_{k \leq n} \mathbb{E}_{B_r}^{(d-1)}|\mathcal{F}'_{k-1}|,$$

where the upper indices show the dimension, hence by (16),

$$\mathbb{E}|\mathcal{F}_n| \sim 2(\log n)^{d-1}. \quad (18)$$

We stress that these formulas are valid under (G) only, and (S) is not required.

For $d = 2$, (17) was proved by Baxter [4]. We first generalized his argument to higher dimensions but then found a more direct and intuitive proof for Part 1 of Proposition 1 presented below in Section 7. Later we discovered that such generalization was already done by Barndorff-Nielsen and Baxter [3] who extended the proof of [4].

We followed the steps of Baxter [4] and Snyder and Steele [19] (both papers considered only the planar case) to obtain the following generalization of (17). Let $g : \mathbb{R}^{d \times (d-1)} \rightarrow \mathbb{R}$ be any Borel function. As we noted above, with probability one C_n is a convex polytope with faces of the form (9), hence we can represent any geometric property of a face f in terms of

$$g(S_{i_2(f)} - S_{i_1(f)}, \dots, S_{i_d(f)} - S_{i_{d-1}(f)})$$

for some symmetric g . This quantity has the same expectation for all faces with the same temporal structure, and a conditional version of (10) (see (31) below) readily yields the following result.

Theorem 3. *Let S_k be partial sums of n -exchangeable random vectors in \mathbb{R}^d , $d \geq 1$. If (G) holds, then for*

$$G_n := \sum_{f \in \mathcal{F}_n} g(S_{i_2(f)} - S_{i_1(f)}, \dots, S_{i_d(f)} - S_{i_{d-1}(f)}),$$

we have that

$$\mathbb{E}G_n = 2 \sum_{1 \leq i_1 < \dots < i_{d-1} \leq n} \frac{\mathbb{E}g(S_{i_1}, S_{i_2} - S_{i_1}, \dots, S_{i_{d-1}} - S_{i_{d-2}})}{i_1(i_2 - i_1) \cdot \dots \cdot (i_{d-1} - i_{d-2})}.$$

Notice that if S_n is a random walk that satisfies (H), then the $d - 1$ arguments of g in the definition of G_n are independent. In that case $\mathbb{E}G_n$ can be written as

$$\mathbb{E}G_n = 2 \sum_{\substack{j_1 + \dots + j_{d-1} \leq n \\ j_1, \dots, j_{d-1} \geq 1}} \frac{\mathbb{E}g\left(S_{j_1}^{(1)}, S_{j_2}^{(2)}, \dots, S_{j_{d-1}}^{(d-1)}\right)}{j_1 \cdot \dots \cdot j_{d-1}}, \quad (19)$$

where $S_n^{(1)}, \dots, S_n^{(d-1)}$ are independent copies of the random walk S_n .

We proved Theorem 3 being unaware of the work of Barndorff-Nielsen and Baxter [3] who gave no general statement of this type but did a similar consideration and obtained many of the results discussed in the next section on applications of Theorem 3. Our proof uses both combinatorial and probabilistic reasoning and in our opinion, is more transparent than the one of [3].

The later proof rests on the smart combinatorial argument proposed by Baxter [4]. It is based on the simple fact that all the $n!$ possible permutations of the increments X_1, \dots, X_n

do not change the distribution of the partial sums. Similarly, Wendel's proof of (2) uses that all the 2^n possible n -tuples $(\pm X_1, \dots, \pm X_n)$ have the same distribution. If this holds true, his argument works for *any* random vectors X_1, \dots, X_n in general position, hence the assumption that X_i are i.i.d. is actually superfluous. Both proofs of Baxter and Wendel rely on the corresponding properties of *deterministic* sequences. The original proof of the Sparre Andersen result does not allow such a nice description as it combines a simple combinatorial argument with some clever counting resting on the additivity of probability. The well-known proof of this result given by Feller [8, Sec. XII.6] offers much more clear combinatorial approach but heavily uses the independence of increments.

4. APPLICATIONS OF THEOREM 3

Let us derive some corollaries of Theorem 3. In this section we always assume that S_n is a random walk that satisfies (H), and impose no other conditions. As in (19), let $S_n^{(1)}, \dots, S_n^{(d)}$ be independent copies of the walk S_n .

First of all, by considering $g(x_1, \dots, x_{d-1}) \equiv 1$ in (19), we obtain (17), a formula for the expected number of faces of the convex hull of S_n . Less trivial applications are as follows.

Corollary 1 (Expected surface area). *We have that*

$$\mathbb{E} \text{Vol}_{d-1}(\partial C_n) = \frac{2}{(d-1)!} \sum_{\substack{j_1 + \dots + j_{d-1} \leq n \\ j_1, \dots, j_{d-1} \geq 1}} \frac{\mathbb{E} \sqrt{\left| \det \left(\left\langle S_{j_m}^{(m)}, S_{j_\ell}^{(\ell)} \right\rangle \right)_{m, \ell=1}^{d-1} \right|}}{j_1 \cdot \dots \cdot j_{d-1}}.$$

For $d = 2$ this gives a formula by Spitzer and Widom [23] on the average perimeter:

$$\mathbb{E} \text{Vol}_1(\partial C_n) = 2 \sum_{j=1}^n \frac{\mathbb{E}|S_j|}{j}. \quad (20)$$

A higher-dimensional version of this result was first obtained by Barndorff-Nielsen and Baxter [3].

Proof. Applying (19) with

$$g(x_1, \dots, x_{d-1}) = \text{Vol}_{d-1}(\text{conv}(0, x_1, \dots, x_{d-1})) = \frac{1}{(d-1)!} \sqrt{\left| \det \left(\langle x_m, x_l \rangle \right)_{m, l=1}^{d-1} \right|},$$

we get that

$$\begin{aligned} \mathbb{E} \text{Vol}_{d-1}(\partial C_n) &= 2 \sum_{\substack{j_1 + \dots + j_{d-1} \leq n \\ j_1, \dots, j_{d-1} \geq 1}} \frac{\mathbb{E} \text{Vol}_{d-1} \left(\text{conv}(0, S_{j_1}^{(1)}, \dots, S_{j_{d-1}}^{(d-1)}) \right)}{j_1 \cdot \dots \cdot j_{d-1}} \\ &= \frac{2}{(d-1)!} \sum_{\substack{j_1 + \dots + j_{d-1} \leq n \\ j_1, \dots, j_{d-1} \geq 1}} \frac{\mathbb{E} \sqrt{\left| \det \left(\left\langle S_{j_m}^{(m)}, S_{j_\ell}^{(\ell)} \right\rangle \right)_{m, \ell=1}^{d-1} \right|}}{j_1 \cdot \dots \cdot j_{d-1}}. \end{aligned}$$

□

Corollary 2 (Expected volume). *We have that*

$$\mathbb{E} \text{Vol}_d(C_n) = \frac{1}{d!} \sum_{\substack{j_1 + \dots + j_d \leq n \\ j_1, \dots, j_d \geq 1}} \frac{\mathbb{E} \left| \det \left[S_{j_1}^{(1)}, \dots, S_{j_d}^{(d)} \right] \right|}{j_1 \cdot \dots \cdot j_d}. \quad (21)$$

A version of this result was first obtained by Barndorff-Nielsen and Baxter [3].

Proof. Denote by $' : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ the projection to the first d coordinates. Let \tilde{S}_n be any $(d+1)$ -dimensional random walk such that $\tilde{S}'_n = S_n$ and its last coordinate is distributed continuously and independently of S_n . The convex hull \tilde{C}_n of \tilde{S}_n satisfies $(\partial\tilde{C}_n)' = C_n$, and the pre-image under $'$ of any point from $\text{Int}(C_n)$ consists of exactly two points. By applying (19) to \tilde{S}_n with

$$g(x_1, \dots, x_d) = \text{Vol}_d(\text{conv}(0, x'_1, \dots, x'_d)) = \frac{1}{d!} \left| \det[x'_1, \dots, x'_d] \right|$$

and using (8), we get that

$$\begin{aligned} \mathbb{E} \text{Vol}_d(C_n) &= \sum_{\substack{j_1 + \dots + j_d \leq n \\ j_1, \dots, j_d \geq 1}} \frac{\mathbb{E} \text{Vol}_d \left(\text{conv}(0, S_{j_1}^{(1)}, \dots, S_{j_d}^{(d)}) \right)}{j_1 \cdot \dots \cdot j_d} \\ &= \frac{1}{d!} \sum_{\substack{j_1 + \dots + j_d \leq n \\ j_1, \dots, j_d \geq 1}} \frac{\mathbb{E} \left| \det \left[S_{j_1}^{(1)}, \dots, S_{j_d}^{(d)} \right] \right|}{j_1 \cdot \dots \cdot j_d}. \end{aligned} \quad (22)$$

□

Let us present an approach that unifies the examples we considered above. Volume and surface area are the special cases of the so called *intrinsic volumes* V_0, \dots, V_d , which naturally arise as the coefficients in the *Steiner formula*: for any convex set $K \subset \mathbb{R}^d$,

$$\text{Vol}_d(K + rB_d) = \sum_{k=0}^d \kappa_{d-k} V_k(K) r^{d-k}, \quad r \geq 0,$$

where B_d denotes a d -dimensional unit ball and $\kappa_k := \pi^{k/2} / \Gamma(\frac{k}{2} + 1)$ is the volume of B_k . In particular, $V_0(K) = 1$, $V_d(K) = \text{Vol}_d(K)$, $2V_{d-1}(K) = \text{Vol}_{d-1}(\partial K)$, and $V_1(K)$ equals the mean width of K , up to a constant factor. The latter relation readily follows from the other definition of intrinsic volumes, which sometimes is called the *Crofton formula*:

$$V_k(K) := \binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \int_{\mathcal{L}_k^d} \text{Vol}_k(K|L) d\mu_k(L), \quad (23)$$

where \mathcal{L}_k^d is the Grassmanian of all k -dimensional linear subspaces of \mathbb{R}^d equipped with the Haar probability measure μ_k , and $K|L$ is the orthogonal projection of K onto L . Roughly speaking, the k -th intrinsic volume of K is, up to a constant factor, the mean k -dimensional

volume of the projection of K onto a uniformly chosen random k -dimensional linear subspace in \mathbb{R}^d . The normalization constant $\binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}}$ is chosen so that the intrinsic volumes of K do not depend on whether we consider K as a subset of \mathbb{R}^d or embed it into any higher-dimensional Euclidean space. For an extensive account on integral geometry we refer the reader to the books Santaló [17] and Schneider and Weil [18].

Corollary 3 (Expected intrinsic volumes). *We have that*

$$\mathbb{E}V_k(C_n) = \frac{1}{k!} \sum_{\substack{j_1+\dots+j_k \leq n \\ j_1, \dots, j_k \geq 1}} \frac{\mathbb{E} \sqrt{\left| \det \left(\left\langle S_{j_m}^{(m)}, S_{j_\ell}^{(\ell)} \right\rangle \right)_{m, \ell=1}^k \right|}}{j_1 \cdots j_k}, \quad k = 1, \dots, d.$$

In particular, the Spitzer-Widom formula (20) naturally extends to any dimension:

$$\mathbb{E}V_1(C_n) = \sum_{j=1}^n \frac{\mathbb{E}|S_j|}{j}.$$

Proof. For any $L \in \mathcal{L}_k^d$, $\tilde{S}_n := S_n|L$ is a k -dimensional random walk satisfying (H) and its convex hull is $\tilde{C}_n = C_n|L$. Hence by (22) it holds that

$$\mathbb{E} \text{Vol}_k(C_n|L) = \sum_{\substack{j_1+\dots+j_k \leq n \\ j_1, \dots, j_k \geq 1}} \frac{\mathbb{E} \text{Vol}_k(\text{conv}(0, S_{j_1}^{(1)}, \dots, S_{j_k}^{(k)})|L)}{j_1 \cdots j_k}.$$

After integrating this equation over \mathcal{L}_k^d with respect to μ_k , normalizing according to the definition of the intrinsic volume, and applying the Fubini theorem to both sides we get that

$$\mathbb{E}V_k(C_n) = \sum_{\substack{j_1+\dots+j_k \leq n \\ j_1, \dots, j_k \geq 1}} \frac{\mathbb{E}V_k(\text{conv}(0, S_{j_1}^{(1)}, \dots, S_{j_k}^{(k)}))}{j_1 \cdots j_k}.$$

The linear dimension of $K = \text{conv}(0, S_{j_1}^{(1)}, \dots, S_{j_k}^{(k)})$, which is a convex hull of $k+1$ points, equals k , hence $V_k(K) = \text{Vol}_k(K)$ and the claim follows. \square

5. APPLICATIONS TO GEOMETRY VIA THE GAUSSIAN CASE

In this section we always assume that X_1, \dots, X_n are independent standard Gaussian vectors in \mathbb{R}^d . Consider the Gaussian random $d \times n$ matrix A with the columns X_1, \dots, X_n . Its rows Y_1, \dots, Y_d are standard Gaussian vectors in \mathbb{R}^n . It is known that the linear span of Y_1, \dots, Y_d (which are in general position with probability one) is a random d -dimensional linear subspace of \mathbb{R}^n uniformly distributed on the Grassmanian \mathcal{L}_d^n with respect to the Haar probability measure. Using this fact and the Crofton formula (23), it can be shown that for any convex body $K \subset \mathbb{R}^n$

$$V_d(K) = \frac{(2\pi)^{d/2}}{d! \kappa_d} \mathbb{E} \text{Vol}_d(\text{conv}(\{Ax : x \in K\})).$$

This equation is a finite-dimensional version of a general result due to Sudakov [24] (for $d = 1$) and Tsirelson [26, 27, 28] (for general d) on Gaussian measures in infinite-dimensional spaces.

Consider the simplex $T_n \subset \mathbb{R}^n$ with vertices

$$(0, 0, \dots, 0), (1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, \dots, 1),$$

which we call *Schläfli canonical orthoscheme*. Such simplexes are also called *path-simplexes*.

Now

$$\text{conv}(\{Ax : x \in T_n\}) = C_n,$$

which implies that

$$V_d(T_n) = \frac{(2\pi)^{d/2}}{d! \kappa_d} \mathbb{E} \text{Vol}_d(C_n). \quad (24)$$

Combining this with (21), we obtain that

$$V_d(T_n) = \frac{(2\pi)^{d/2}}{(d!)^2 \kappa_d} \sum_{\substack{j_1 + \dots + j_d \leq n \\ j_1, \dots, j_d \geq 1}} \frac{\mathbb{E} \left| \det \left[S_{j_1}^{(1)}, \dots, S_{j_d}^{(d)} \right] \right|}{j_1 \cdot \dots \cdot j_d},$$

where $S_1^{(1)}, \dots, S_d^{(d)}$ are independent standard Gaussian random walks in \mathbb{R}^d . Let M be a $d \times d$ matrix with independent standard normal entries. Then $\mathbb{E} |\det M| = \mathbb{E} \sqrt{\det(MM^\top)}$, where MM^\top is a Wishart matrix whose determinant has a well known distribution and moments. Hence (see for example Kabluchko and Zaporozhets [12])

$$\mathbb{E} \left| \det \left[S_{j_1}^{(1)}, \dots, S_{j_d}^{(d)} \right] \right| = \frac{d! \kappa_d}{(2\pi)^{d/2}} \sqrt{j_1 \cdot \dots \cdot j_d},$$

which implies that

$$V_d(T_n) = \frac{1}{d!} \sum_{\substack{j_1 + \dots + j_d \leq n \\ j_1, \dots, j_d \geq 1}} \frac{1}{\sqrt{j_1 \cdot \dots \cdot j_d}}. \quad (25)$$

This result was first obtained by Gao and Vitale [10] who considered a direct geometric approach using a formula for intrinsic volumes of a convex polytope. The simplex $T_n/\sqrt{n} \in \mathbb{R}^n$ is a finite-dimensional approximation of the convex hull T of a *Wiener spiral*³ in a Hilbert space as introduced by Kolmogorov in 1940; [10] calls T the Brownian motion body. Note that T is isometric to the subset of non-increasing functions in $L_2[0, 1]$ that take values in $[0, 1]$. Gao and Vitale [10] used (25) to prove that

$$V_d(T) = \frac{\kappa_d}{d!}.$$

Due to Tsirelson [26, 27, 28], the normalized d -th intrinsic volume of T is equal to expected volume of the convex hull of a d -dimensional Brownian motion, see Kabluchko and Zaporozhets [13] for details. The latter quantity was calculated by Eldan [7] using direct methods.

³This is the deterministic curve $\{W(t), t \in [0, 1]\}$, where W is a standard Wiener process, in the Hilbert space of square-integrable centred random variables.

Let us consider the unit sphere \mathbb{S}^n in \mathbb{R}^{n+1} . By saying that $K \subset \mathbb{S}^n$ is convex we mean that the conic hull of K in \mathbb{R}^{n+1} is convex and line-free. Following Santaló (see [17, Section IV.4]), we can use a spherical counterpart of the Crofton formula for a convex body K in \mathbb{S}^n to define

$$U_k(K) := \frac{1}{2} \int_{S_{n-k}^n} \mathbb{1}_{\{K \cap s \neq \emptyset\}} d\nu_{n-k}(s),$$

where S_k^n denotes the space of k -dimensional great subspheres of \mathbb{S}^n equipped with the rotationally invariant probability measure ν_k . The functionals U_k can be considered as *spherical* counterparts of the Euclidean *intrinsic volumes* V_k . However, there are other possible definition of spherical intrinsic volumes. For basic facts from spherical integral geometry we refer the reader to Gao et al. [9], McCoy and Tropp [15], and Schneider and Weil [18, Sec. 6.5].

Similarly to (24), it can be shown (see Götze et al. [11] for details) that

$$U_d(\tilde{T}_n) = \frac{1}{2} \mathbb{P}(0 \in C_n),$$

where \tilde{T}_n denotes the intersection of the conic hull of T_n with \mathbb{S}^{n-1} . To eliminate any misunderstanding, in the abstract and Section 1 we meant exactly $U_d(\tilde{T}_n)$ by the spherical intrinsic volumes of the canonic orthoscheme T_n . It follows from (1) and Theorem 1 that

$$U_1(\tilde{T}_n) = \frac{1}{2} - \frac{(2n-1)!!}{(2n)!!}, \quad U_2(\tilde{T}_n) = \frac{1}{2} - \sum_{k=1}^n \frac{(2n-2k-1)!!}{2k(2n-2k)!!},$$

but the values of the other spherical intrinsic volumes of \tilde{T}_n remain unknown since we do not know how to calculate $\mathbb{P}(0 \in C_n)$ for $d \geq 3$ even in the Gaussian case.

6. SOME COMBINATORIAL ARGUMENT

For any $x_1, \dots, x_n \in \mathbb{R}^d$, denote by

$$s_0 := 0, \quad s_k := x_1 + \dots + x_k, \quad k = 1, \dots, n,$$

the sequence of partial sums. For any permutation $\sigma = (\sigma(1), \dots, \sigma(n))$, denote

$$s_0(\sigma) := 0, \quad s_k(\sigma) := x_{\sigma(1)} + \dots + x_{\sigma(k)}, \quad k = 1, \dots, n.$$

We first proved a simple combinatorial statement, which generalizes two-dimensional Lemma 1 from Baxter [4] to higher dimensions. Later we found this result in the paper by Barndorff-Nielsen and Baxter [3]. For the reader's convenience we present the proof here.

Lemma 1. *Let $x_0, x_1, \dots, x_n \in \mathbb{R}^d$, and let H be a closed half-space such that*

$$x_0, x_0 + s_n \in \partial H \quad \text{and} \quad x_0 + s_j - s_i \notin \partial H, \quad 0 \leq i < j \leq n-1.$$

There exists exactly one cyclic permutation $\sigma = (k+1, \dots, n, 1, \dots, k)$ such that

$$x_0, x_0 + s_1(\sigma), \dots, x_0 + s_n(\sigma) \in H.$$

Proof. By the assumption, there exists exactly one point $x_0 + s_k$ among $\{x_0 + s_i\}_{i=0}^{n-1} \cap (\text{Int}(H))^c$ that is at the maximum distance (possibly zero) from ∂H . Then $\sigma := (k+1, \dots, n, 1, \dots, k)$ is a required permutation, and it is unique by the uniqueness of k . \square

Our next goal is to obtain stochastic versions of this result. For any permutation $\sigma = (\sigma(1), \dots, \sigma(n))$, introduce the partial sums

$$S_0(\sigma) := 0, S_k(\sigma) := X_{\sigma(1)} + \dots + X_{\sigma(k)}, \quad k = 1, \dots, n.$$

For any points $x_1, \dots, x_d \in \mathbb{R}^d$, define

$$H_{\pm}(x_1, \dots, x_d) := \{z \in \mathbb{R}^d : \pm \det[x_2 - x_1, \dots, x_d - x_1, z - x_1] \geq 0\}.$$

If there is a unique hyperplane through these points, then this definition gives a rule to distinguish between the two half-spaces H_+ and H_- lying to the different sides of the hyperplane. If such a hyperplane is not unique, then $H_{\pm} = \mathbb{R}^d$.

Lemma 2. *Let $S_k = X_1 + \dots + X_k$, $1 \leq k \leq n$, be partial sums of exchangeable random vectors $X_1, \dots, X_n \in \mathbb{R}^d$ that satisfy (G). For any indices $1 \leq i_1 < \dots < i_{d-2} \leq n-1$,*

$$\mathbb{P}(S_1, \dots, S_n \in H_{\pm}(0, S_{i_1}, \dots, S_{i_{d-2}}, S_n)) = \frac{1}{i_1(i_2 - i_1) \cdot \dots \cdot (n - i_{d-2})}$$

and moreover,

$$\mathbb{E}(S_1, \dots, S_n \in H_{\pm}(0, S_{i_1}, \dots, S_{i_{d-2}}, S_n) | S_{i_1}, \dots, S_{i_{d-2}}, S_n) = \frac{1}{i_1(i_2 - i_1) \cdot \dots \cdot (n - i_{d-2})}.$$

This statement is a little generalization of the well-known fact that the chance that the trajectory of any continuously distributed one-dimensional random walk S_n lies above the line joining $(0, 0)$ with (n, S_n) equals $1/n$, see Feller [8, Sec. XII.9]. The fact follows from Lemma 2 if we consider the two-dimensional walk $\tilde{S}_n := (n, S_n)$ with deterministic first component.

Proof. With probability one, there exists a unique half-plane through $0, S_{i_1}, \dots, S_{i_{d-2}}, S_n$. Supposing otherwise, we add any other point S_k and arrive to a contradiction with (G). Hence almost surely,

$$H_{\pm}(S) := H_{\pm}(0, S_{i_1}, \dots, S_{i_{d-2}}, S_n)$$

are half-spaces. Put $i_0 := 0, i_{d-1} := n$ and denote by \mathcal{S} the set of $(i_1 - i_0) \cdot \dots \cdot (i_{d-1} - i_{d-2})$ permutations of length n that are products of $d-1$ cyclic permutations of the form

$$(k_j + 1, \dots, i_j, i_{j-1} + 1, \dots, k_j), \quad (26)$$

where $i_{j-1} + 1 \leq k_j \leq i_j$. Note that any $\sigma \in \mathcal{S}$ does not change H_{\pm} , i.e. $H_{\pm}(S) = H_{\pm}(S(\sigma))$ since $S_k = S_k(\sigma)$ for $k \in \{i_1, \dots, i_{d-2}, n\}$, and the sequences of partial sums S and $S(\sigma)$ have the same distribution by the exchangeability of the increments.

Note that $S_{i_j}, X_{i_j+1}, \dots, X_{i_{j+1}}$ and the half-space $H_{\pm}(0, S_{i_1}, \dots, S_{i_{d-2}}, S_n)$ satisfy the assumption of Lemma 1 with probability one for any $0 \leq j \leq d-2$. Indeed, if for some $i_j \leq m < \ell < i_{j+1}$ with a positive probability it holds that $S_{i_j} + S_{\ell} - S_m \in \partial H_{\pm}(S)$, then among the partial sums $S_k(\sigma)$ with

$$\sigma = (1, \dots, i_j, m+1, \dots, \ell, i_j+1, i_j+m, \ell+1, \dots, n)$$

there are d points $S_{i_1}(\sigma), \dots, S_{i_{d-1}}(\sigma), S_{i_j+\ell-m}(\sigma)$ that belong to the hyperplane ∂H_{\pm} through 0 , which contradicts (G) by the exchangeability of increments.

By Lemma 1, there exists a unique random permutation $\sigma_{\pm} = \sigma_{\pm}(S) \in \mathcal{S}$ such that $S_1(\sigma_{\pm}), \dots, S_n(\sigma_{\pm}) \in H_{\pm}(S)$. Hence the sum in

$$\mathbb{P}(S_1, \dots, S_n \in H_{\pm}(S_{i_0}, S_{i_1}, \dots, S_{i_{d-1}})) = \frac{1}{|\mathcal{S}|} \sum_{\sigma \in \mathcal{S}} \mathbb{P}(S_1(\sigma), \dots, S_n(\sigma) \in H_{\pm}(S)) \quad (27)$$

equals one a.s. and similarly, for any non-negative Borel function $g : \mathbb{R}^{d \times (d-1)} \rightarrow \mathbb{R}$,

$$\begin{aligned} & \mathbb{E} \left[g(S_{i_1}, \dots, S_{i_{d-2}}, S_n) \mathbb{1}(S_1, \dots, S_n \in H_{\pm}(S_{i_0}, S_{i_1}, \dots, S_{i_{d-1}})) \right] \\ &= \frac{1}{|\mathcal{S}|} \mathbb{E} \left[g(S_{i_1}, \dots, S_{i_{d-2}}, S_n) \sum_{\sigma \in \mathcal{S}} \mathbb{1}(S_1(\sigma), \dots, S_n(\sigma) \in H_{\pm}(S)) \right]. \end{aligned}$$

Thus the claims follow. \square

Recall that a random walk bridge of length n is a random walk conditioned to return to the origin at time n , which is understood as the well-defined limit of conditioning the last step of the walk to be in the ball of a vanishing radius centered at the origin.

Lemma 3. *Let S_k be a bridge of length $n+1$ of a random walk in \mathbb{R}^d that satisfies (H). For any indices $0 \leq i_1 < \dots < i_{d-1} \leq n$,*

$$\mathbb{P}(S_1, \dots, S_n \in H_{\pm}(0, S_{i_1}, \dots, S_{i_{d-1}})) = \frac{1}{i_1(i_2 - i_1) \cdot \dots \cdot (n - i_{d-1} + 1)}.$$

This is a multidimensional counterpart of the fact that in dimension one, a bridge of length n of a continuously distributed random walk stays positive with probability $1/n$.

Proof. By our understanding of the conditioning, the bridge satisfies (G) since so does the random walk under (H). By repeating the argument used in the proof of Lemma 2, we see that (27) holds for $i_0 = 0$ and \mathcal{S} defined to be the set of permutations of length $n+1$ that are products of d cyclic permutations of the form (26), where $0 \leq j \leq d-1$ and $i_d = n+1$. \square

7. PROOFS

Proof of Proposition 1. Recall that $0 \leq i_1 < \dots < i_d \leq n$. By (8) and (9),

$$\begin{aligned} \mathbb{P}(\text{conv}(S_{i_1}, \dots, S_{i_d}) \in \mathcal{F}_n) &= \mathbb{P}(0, S_1, \dots, S_n \in H_+(S_{i_1}, \dots, S_{i_d})) \\ &+ \mathbb{P}(0, S_1, \dots, S_n \in H_-(S_{i_1}, \dots, S_{i_d})) \end{aligned} \quad (28)$$

Denote

$$H_{\pm} := H_{\pm}(0, S_{i_2} - S_{i_1}, \dots, S_{i_d} - S_{i_{d-1}}).$$

We first prove (11) of Part 1 of the proposition. If S_k , $1 \leq k \leq n$, is a random walk bridge of length $n+1$, then we make it “complete” by putting $S_{n+1} := 0$ to include the last exchangeable increment of the underlying random walk. Let us transform the trajectory of the complete bridge by moving the part from 1 to i_1 and attaching it to the end of the part from $i_1 + 1$ to $n+1$. Formally, we rewrite the probability $\mathbb{P}(0, S_1, \dots, S_n \in H_{\pm} + S_{i_1})$ in terms of the partial sums of $S_n(\sigma)$ with

$$\sigma = (n - i_1 + 2, n - i_1 + 3, \dots, n + 1, 1, 2, \dots, n - i_1 + 1)$$

and use that $S_{n+1} = 0$. Then

$$\begin{aligned} & \mathbb{P}(0, S_1, \dots, S_n \in H_{\pm}(S_{i_1}, \dots, S_{i_d})) \\ &= \mathbb{P}(0, S_{n-i_1+2}, \dots, S_n, S_1, S_2, \dots, S_{n-i_1+1} \in H_{\pm}(0, S_{i_2-i_1}, \dots, S_{i_d-i_1})), \end{aligned}$$

and (11) follows by (28) and Lemma 3.

For a further consideration, let us use the identity $H_{\pm} + S_{i_1} = H_{\pm} + S_{i_d}$, which holds by $S_{i_d} - S_{i_d-1} \in \partial H_{\pm}$, to split the trajectory of S_n into three parts:

$$\begin{aligned} & \mathbb{P}(0, S_1, \dots, S_n \in H_{\pm}(S_{i_1}, \dots, S_{i_d})) \\ &= \mathbb{P}(0, S_1, \dots, S_{i_1} \in H_{\pm} + S_{i_1}; S_{i_1+1}, \dots, S_{i_d} \in H_{\pm} + S_{i_1}; S_{i_d+1}, \dots, S_n \in H_{\pm} + S_{i_d}). \end{aligned} \quad (29)$$

Now we can easily prove Part 2 of the proposition. If S_n is a random walk, then by conditioning in (29) on $X_{i_1+1}, \dots, X_{i_d}$, which define H_{\pm} , and using independence of the increments, we obtain that

$$\begin{aligned} & \mathbb{P}(0, S_1, \dots, S_n \in H_{\pm}(S_{i_1}, \dots, S_{i_d})) \\ &= \mathbb{P}(-S'_{i_1}, -S'_{i_1-1}, \dots, 0 \in H_{\pm}) \mathbb{P}(S_{i_1+1} - S_{i_1}, \dots, S_{i_d} - S_{i_1} \in H_{\pm}) \mathbb{P}(S'_1, \dots, S'_{n-i_d} \in H_{\pm}) \\ &= \mathbb{P}(S'_1, \dots, S'_{i_1} \in H_{\mp}) \mathbb{P}(S_1, \dots, S_{i_d-i_1} \in H_{\pm}(0, S_{i_2-i_1}, \dots, S_{i_d-i_d-1})) \mathbb{P}(S'_1, \dots, S'_{n-i_d} \in H_{\pm}), \end{aligned}$$

where S'_n is an independent copy of the random walk S_n . Then (12) holds by (28), Lemma 2, and the following simple result.

Lemma 4. *Let S_n be a random walk in \mathbb{R}^d , and let H be a half-space such that $0 \in \partial H$. Assume that (H) and (S) hold. Then*

$$\mathbb{P}(S_1, \dots, S_n \in H) = \frac{(2n-1)!!}{(2n)!!}.$$

Proof. Denote by u the unit vector that is orthogonal to ∂H and belongs to H . The distribution of increments of the one-dimensional random walk $S'_k := \langle S_k, u \rangle$ is continuous and symmetric, hence the result follows by (1) and

$$\mathbb{P}(S_1, \dots, S_n \in H) = \mathbb{P}(S'_1 > 0, \dots, S'_n > 0).$$

□

It only remains to prove (10). Let us rearrange the three parts of the trajectory (from 1 to i_1 , from $i_1 + 1$ to i_d , and from $i_d + 1$ to n , respectively) attaching the beginning of the first part to the end of the second and the beginning of the third part to the end of the first one. Formally, we rewrite the r.h.s of (29) in terms of the partial sums $S_n(\sigma)$ with

$$\sigma = (i_d - i_1 + 1, i_d - i_1 + 2, \dots, i_d, 1, 2, \dots, i_d - i_1, i_d + 1, i_d + 2, \dots, n)$$

and obtain that

$$\begin{aligned} & \mathbb{P}(0, S_1, \dots, S_n \in H_{\pm}(S_{i_1}, \dots, S_{i_d})) \\ &= \mathbb{P}(0, S_{i_d-i_1+1}, \dots, S_{i_d} \in H_{\pm}(\sigma) + S_{i_d}; S_1, \dots, S_{i_d-i_1} \in H_{\pm}(\sigma); S_{i_d+1}, \dots, S_n \in H_{\pm}(\sigma) + S_{i_d}). \end{aligned}$$

On the later event it holds that

$$-S_{i_d} \in H_{\pm}(\sigma) = H_{\pm}(0, S_{i_2-i_1}, \dots, S_{i_d-i_1}),$$

hence $H_{\pm}(\sigma) \subset H_{\pm}(\sigma) + S_{i_d}$, and with probability one S_{i_d} is the most distant point from $H_{\pm}(\sigma)$ (such a point is unique by (G)) unless $i_1 = 0$, when every point is at zero distance:

$$\begin{aligned} \mathbb{P}(0, S_1, \dots, S_n \in H_{\pm}(S_{i_1}, \dots, S_{i_d})) &= \mathbb{P}(S_1, \dots, S_{i_d-i_1} \in H_{\pm}(0, S_{i_2-i_1}, \dots, S_{i_d-i_1}); \\ &\quad \operatorname{argmax}_{0 \leq k \leq n-(i_d-i_1)} \operatorname{dist}(H_{\pm}(0, S_{i_2-i_1}, \dots, S_{i_d-i_1}), S_{i_d-i_1+k}) = i_1), \end{aligned} \quad (30)$$

where argmax denotes *the first* moment when S_k attains the maximum distance from $H_{\pm}(\dots)$; it is introduced only to treat the case that $i_1 = 0$.

We fix a tuple $(i_2 - i_1, \dots, i_d - i_1)$ and sum in i_1 from 0 to $n - (i_d - i_1)$. Then (10) follows by (28), (30), and Lemma 2. Proposition 1 is now proved.

Let us briefly comment on the statement in Footnote 2. Note that under these assumptions, (1) holds true in the one-dimensional case, see Sparre Andersen [20] or [21, Theorem 4]; formally, both theorems require slightly stronger assumptions. The later theorem yields (12) when applied for the partial sums of $(n - i_d + i_1)$ -exchangeable one-dimensional increments $X'_k := \pm \det(S_{i_2-i_1}, \dots, S_{i_d-i_1}, X_{i_d-i_1+k})$ (where $1 \leq k \leq n - i_d + i_1$) with $n - i_d + i_1$ substituted for n and the first event in the r.h.s. of (30) substituted for C_n .

Proof of Theorem 3. A straightforward extension of the path-transform argument above allows to prove a little strengthening of (30): for any non-negative Borel function $g : \mathbb{R}^{d \times (d-1)} \rightarrow \mathbb{R}$,

$$\begin{aligned} &\mathbb{E} \left[g(S_{i_2} - S_{i_1}, \dots, S_{i_d} - S_{i_{d-1}}) \mathbb{1}(0, S_1, \dots, S_n \in H_{\pm}(S_{i_1}, \dots, S_{i_d})) \right] \\ &= \mathbb{E} \left[g(S_{i_2-i_1}, S_{i_3-i_1} - S_{i_2-i_1}, \dots, S_{i_d-i_1} - S_{i_{d-1}-i_1}) \mathbb{1}(S_1, \dots, S_{i_d-i_1} \in H_{\pm}(0, S_{i_2-i_1}, \dots, S_{i_d-i_1})) \right. \\ &\quad \left. \times \mathbb{1} \left(\operatorname{argmax}_{0 \leq k \leq n-(i_d-i_1)} \operatorname{dist}(H_{\pm}(0, S_{i_2-i_1}, \dots, S_{i_d-i_1}), S_{i_d-i_1+k}) = i_1 \right) \right]. \end{aligned}$$

For any fixed tuple $(i_2 - i_1, \dots, i_d - i_1) =: (i'_1, \dots, i'_{d-1})$, we sum in $i_1 =: i$ to obtain a conditional version of (10):

$$\begin{aligned} &\sum_{i=0}^{n-i'_{d-1}} \mathbb{E}(\operatorname{conv}(S_i, S_{i+i'_1}, \dots, S_{i+i'_{d-1}}) \in \mathcal{F}_n | S_{i+i'_1} - S_i, S_{i+i'_2} - S_{i+i'_1}, \dots, S_{i+i'_{d-1}} - S_{i+i'_{d-2}}) \\ &= \frac{2}{i'_1(i'_2 - i'_1) \cdots (i'_{d-1} - i'_{d-2})}. \end{aligned} \quad (31)$$

Theorem 3 then follows immediately by summation over all temporal structures (i'_1, \dots, i'_{d-1}) .

Proofs of the asymptotics. 1. We claim that for any sequence a_n such that $a_n \sim (\log n)^a n^{-1/2}$ for some $a \geq 0$ it holds that

$$\sum_{k=1}^n \frac{a_{n-k}}{k} \sim \frac{(\log n)^{a+1}}{\sqrt{n}}. \quad (32)$$

In particular, by (6) this implies (5) if we take $a = 0$.

Let us check that the main contribution to the asymptotic comes from the terms $k = o(n)$. Since the sum of a_n diverges, for any $\varepsilon \in (0, 1)$,

$$\sum_{k=\varepsilon n}^n \frac{a_{n-k}}{k} \leq \frac{1}{\varepsilon n} \sum_{k=\varepsilon n}^n a_{n-k} \sim \frac{1}{\varepsilon n} \sum_{k=1}^{(1-\varepsilon)n} \frac{(\log k)^a}{\sqrt{k}} \leq \frac{2(\log n)^a}{\varepsilon \sqrt{n}}.$$

The latter expression is of a smaller order of asymptotic than

$$\sum_{k=1}^{\varepsilon n-1} \frac{a_{n-k}}{k} \sim \sum_{k=1}^{\varepsilon n-1} \frac{(\log(n-k))^a}{k \sqrt{(n-k)}}$$

since

$$\frac{(\log n)^{a+1}}{\sqrt{n}} \sim \frac{(\log(1-\varepsilon)n)^a}{\sqrt{n}} \sum_{k=1}^{\varepsilon n-1} \frac{1}{k} \leq \sum_{k=1}^{\varepsilon n-1} \frac{(\log(n-k))^a}{k \sqrt{(n-k)}} \leq \frac{(\log n)^a}{\sqrt{(1-\varepsilon)n}} \sum_{k=1}^{\varepsilon n-1} \frac{1}{k} \sim \frac{(\log n)^{a+1}}{\sqrt{(1-\varepsilon)n}}.$$

The later inequalities clearly imply (32).

2. We prove (15) by induction in d . The base $d = 2$ holds by (5). Since

$$\begin{aligned} & \sum_{1 \leq i_2 < \dots < i_d \leq n} \frac{(2n - 2i_d - 1)!!}{i_2(2n - 2i_d)!!} \prod_{k=2}^{d-1} \frac{1}{i_{k+1} - i_k} \\ &= \sum_{i_2=1}^{n-d+2} \frac{1}{i_2} \left[\sum_{1 \leq i'_2 < \dots < i'_{d-1} \leq n-i_2} \frac{(2(n-i_2) - 2i'_{d-1} - 1)!!}{i'_2(2(n-i_2) - 2i'_{d-1})!!} \prod_{k=2}^{d-2} \frac{1}{i'_{k+1} - i'_k} \right], \end{aligned}$$

(14) (or (12)) implies that

$$\mathbb{E}_{RW}^{(d)} |\mathcal{F}'_n| = \sum_{k=1}^n \frac{1}{k} \mathbb{E}_{RW}^{(d-1)} |\mathcal{F}'_{n-k}|, \quad (33)$$

where the upper indices show dimension and by definition, $\mathbb{E}_{RW}^{(d)} |\mathcal{F}'_n| := 0$ for $n \leq d - 1$. It remains to use (32) to obtain (15).

3. Arguing as above and using (11) instead of (12), one can easily show that (33) also holds for a random walk bridge of length $n+1$. For any sequence b_n such that $b_n \sim (\log n)^b n^{-1}$ for some $b \geq 0$ it holds that

$$\sum_{k=1}^n \frac{b_{n-k}}{k} \sim \frac{(b+2)(\log n)^{b+1}}{(b+1)n}, \quad (34)$$

where the difference from (32) is due to the fact that the main contribution to the asymptotic comes from the indices k that are either $k = o(n)$ or $k = n - o(n)$. The asymptotic for the base $d = 2$ is then different, namely

$$\mathbb{E}_{Br}^{(2)}|\mathcal{F}'_n| = \sum_{k=1}^n \frac{2}{k(n-k+1)} \sim \frac{4 \log n}{n},$$

but the rest is analogous and (16) follows easily.

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