CONVEX HULLS OF RANDOM WALKS, HYPERPLANE ARRANGEMENTS, AND WEYL CHAMBERS

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Abstract. We give an explicit formula for the probability that the convex hull of an n-step random walk in $\mathbb{R}^d$ with centrally symmetric density of increments does not contain the origin. This is a distribution-free result that extends the one-dimensional formula of Sparre Andersen (1949): any symmetric continuously distributed random walk stays positive with probability $\frac{(2n-1)!!}{2^n n!}$.

This probabilistic problem is shown to be equivalent to any of the following geometric ones: 1) Find the number of Weyl chambers of type B_n intersected by a generic linear subspace in $\mathbb{R}^n$ of codimension d; 2) Find the conic intrinsic volumes of a Weyl chamber of type B_n. We solve the first geometric problem using the theory of hyperplane arrangements. A corollary of our method is a new simple proof of the general formula by Klivans and Swartz (2011) relating the coefficients of the characteristic polynomial of a linear hyperplane arrangement to the conic intrinsic volumes of the chambers constituting its complement.

Results analogous to the B_n case are obtained for the Weyl chambers of type A_{n-1} (yielding the probability that the convex hull of a generic random walk bridge contains the origin), type D_n (with a less natural probabilistic interpretation), and direct products of Weyl chambers (yielding the probability that the joint convex hull of several random walks or bridges contains the origin). The formulas again do not depend on the distribution of increments. The simplest case of products of the form $B_1 \times \ldots \times B_1$ recovers the Wendel formula (1962) for the probability that the convex hull of an i.i.d. multidimensional sample from a centrally symmetric distribution does not contain the origin.

We give an asymptotic analysis of our results showing that for a symmetric random walk or a random walk bridge in $\mathbb{R}^d$ with $d \to \infty$, the transition from non-absorption to absorption of the origin by the convex hull occurs approximately after $n \approx e^{2d}$ or $n \approx e^d$ steps, respectively.

1. Introduction

1.1. **The absorption problem for random walks.** Let $S_k = \xi_1 + \dots + \xi_k$ be a random walk in $\mathbb{R}^d, d \geq 1$, with independent identically distributed increments $\xi_1, \dots, \xi_n$. We study the probability that the convex hull of the first n steps of the walk does not contain the origin. In other words, the trajectory $S_1, \dots, S_n$ belongs

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to some open linear half-space with 0 at its boundary. This question is a natural generalization to higher dimensions of the problem to find the probability that a one-dimensional random walk does not change its sign by the time n. We will refer to $\mathbb{P}[0 \in \mathrm{Conv}(S_1, S_2, \dots, S_n)]$ as to the *absorption probability* and to the question of its computation as to the *absorption problem.*

The probabilities that a one-dimensional random walk stays positive (or negative) were fully understood by the mid-1950's. There were no results on the absorption problem for random walks in higher dimensions until the very recent papers of Eldan [**?**], Tikhomirov and Youssef [**?** **?**], Vysotsky and Zaporozhets [**?**]. This is despite of the fact that the convex hulls of multidimensional random walks, Brownian motions, and other Lévy processes are very popular objects of studies; we refer to [**?**] for references to general surveys on this broad subject.

Our first result is as follows.

Theorem 1.1. *Let $S_1, \dots, S_n$ be a random walk in $\mathbb{R}^d$ with i.i.d. increments that have absolutely continuous centrally symmetric (i.e., $S_1 \stackrel{d}{=} -S_1$) distribution. Then*

(1)
$$\mathbb{P}[0 \notin \mathrm{Conv}(S_1, S_2, \dots, S_n)] = \frac{2}{2^n n!}\big(B(n, d-1) + B(n, d-3) + \ldots + B(n, d \bmod 2)\big),$$

where $B(n, k)$ are the coefficients of the polynomial

$$(t+1)(t+3)\ldots(t+2n-1) = \sum_{k=0}^{n} B(n,k) t^k.$$

It is remarkable that in any dimension the absorption probabilities are distribution-free being independent of the density $f(x) = f(-x)$ of increments of the walk. This fact was conjectured in Vysotsky and Zaporozhets [**?**], who found an explicit formula for the absorption probabilities in the planar case $d = 2$. Specifying to $d = 1$ and noticing that $B(n, 0) = (2n-1)!!$, we recover the famous distribution-free result of Sparre Andersen on the probability that a continuous symmetric random walk stays positive:

$$\mathbb{P}[S_1 > 0, \dots, S_n > 0] = \frac{(2n-1)!!}{2^n n!} = \frac{1}{2^{2n}}\binom{2n}{n}. \tag{2}$$

Moreover, we will show that the absorption probability for *any* (not necessarily absolutely continuous) symmetric random walk is lower bounded by the right-hand side in (1), see Proposition 2.5 below. With one more exception, all the other results of this paper are based on the absolute continuity assumption for the distribution of increments, which we follow through the end of Section 1.

1.2. **Equivalent geometric problems.** Our proof rests on a newly established direct connection between this probabilistic problem and an equivalent geometric problem concerning Weyl chambers. We solve the geometric problem using the theory of hyperplane arrangements and find the absorption probabilities explicitly. This method is totally different from the one used by [**?**], and it is not even clear how to check directly that the formulas for the absorption probabilities match for $d = 2$.

Let us state the equivalent geometric problem. A *Weyl chamber* of type B_n is any of the $2^n n!$ cones in $\mathbb{R}^n$ of the form

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 < \varepsilon_1 x_{\sigma(1)} < \varepsilon_2 x_{\sigma(2)} < \ldots < \varepsilon_n x_{\sigma(n)}\},$$

where $\sigma(1), \dots, \sigma(n)$ is a permutation of $1, \dots, n$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$. Equivalently, the Weyl chambers are the regions of $\mathbb{R}^n$ formed by the arrangement of n^2 hyperplanes $x_i = 0$ and $x_i \pm x_j = 0, i \neq j$, which constitute the hyperfaces of the chambers.

It turns out that under the assumptions of Theorem 1.1, we have

$$\mathbb{P}[0 \in \operatorname{Conv}(S_1, S_2, \dots, S_n)] = \frac{N_{n,d}}{2^n n!}, \tag{3}$$

where $N_{n,d}$ is the *constant* number of Weyl chambers intersected by a generic non-random subspace of $\mathbb{R}^n$ of codimension d. The exact meaning of "generic" will be explained in Section 3, where we compute the value of $N_{n,d}$ using the formulas of Whitney and Zaslavsky from the theory of *hyperplane arrangements*; see Theorem 3.4 below.

There is another connection between the absorption problem and a problem in spheric convex geometry: we show that for centrally symmetric random walks with density,

$$\mathbb{P}[0 \notin \operatorname{Conv}(S_1, S_2, \dots, S_n)] = \frac{2}{2^n n!}\big(\upsilon_{d-1}(W_n) + \upsilon_{d-3}(W_n) + \dots + \upsilon_{d \bmod 2}(W_n)\big),$$

where W_n is a Weyl chamber of type B_n and υ_k are the *conic intrinsic volumes*. Hence finding the later quantities for Weyl chambers solves the probabilistic problem under the given assumptions. Conic intrinsic volumes are the analogues in spheric geometry for intrinsic volumes of convex sets in Euclidean geometry. Intrinsic volumes include such fundamental geometric characteristics of a set as its volume, surface area, and mean width.

The described connections between the geometric problems via the convex hulls problem gave us insight to obtain a new simple proof of the general formula by Klivans and Swartz [**?**] which relates the coefficients of the characteristic polynomial of a linear hyperplane arrangement to the conic intrinsic volumes of the chambers constituting its complement, see Theorem 4.1 below. We used this result to find the conic intrinsic volumes $\upsilon_k(W_n)$ of a Weyl chamber of type B_n, see Theorem 4.2. The particular value $\upsilon_1(W_n)$, which corresponds to the planar absorption probability, was found in [**?**] (the conic hull of the orthoscheme path-simplex considered there is exactly the standard Weyl chamber).

The explicit expression in Theorem 1.1 easily yields the asymptotic decay of the absorption probabilities for a fixed dimension d as the number of steps n tends to infinity, see Theorem 5.1. It also allows an asymptotic analysis as the dimension increases and $n = n(d)$ grows accordingly to make the absorption happen with a non-trivial probability which follows a central limit theorem, see Theorem 5.2. In particular, our result shows that the phase transition from non-absorption to absorption occurs as the number of steps reaches $n \approx e^{2d}$. In Theorem 5.4 we give the sharp asymptotic for the absorption and non-absorption probabilities in the respective large deviations regions $n = e^{2d/c}$ and $n = e^{2dc}$ for any $c > 1$. This refines much less precise bounds for the large deviations regions by Eldan [**?**] and by Tikhomirov and Youssef [**? ?**], see the discussion in Sec. 5.2. We also obtained a version of this result for simple random walks, which of course do not satisfy the absolute continuity assumption. In this case a one-sided bound for the large deviation regions is given by Theorem 5.5.

1.3. **Extensions to the other types of increments.** The Coxeter group B_n is the symmetry group of the regular cube $[-1,1]^n$. The $2^n n!$ elements of this group act on $\mathbb{R}^n$ by permuting the coordinates in arbitrary way and multiplying any number of coordinates by -1. This is a finite reflection group generated by reflections along the edges (one-dimensional faces) of any Weyl chamber of type B_n. Any Weyl chamber W_n of type B_n is a fundamental region for the action of B_n, which means that the sets $gW_n, g \in B_n$, are disjoint and their closures constitute the entire $\mathbb{R}^n$. We refer to [**?**] for an introduction to finite reflection groups.

Our method actually applies for convex hulls of not only symmetric random walks but also of any sequence of partial sums $S_k = \xi_1 + \dots + \xi_k, k = 1, \dots, n$, with the joint distribution of their increments $\xi_1, \dots, \xi_n \in \mathbb{R}^d$ being invariant under the action of B_n, that is

$$(\xi_1, \dots, \xi_n) \stackrel{d}{=} (\varepsilon_1 \xi_{\sigma(1)}, \dots, \varepsilon_n \xi_{\sigma(n)})$$

for any permutation $\sigma(1), \dots, \sigma(n)$ of $1, \dots, n$ and any $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$. In this case we say that the tuple $(\xi_1, \dots, \xi_n)$ is *symmetrically exchangeable*. There are many important examples of such tuples with non-i.i.d. entries: say, for $d = 1$, so is the tuple of coordinates of any rotational invariant non-Gaussian random vector in $\mathbb{R}^n$. The exact statement of the extended version of Theorem 1.1 is given below in Theorem 2.1 of Section 2.

It turns out that our approach can be extended to treat the absorption problem for partial sums with the distribution of increments invariant under the action of some other finite groups. These are the reflection groups of the type A_{n-1} and D_n, and the direct product of reflection groups. Let us explain.

The Coxeter group A_{n-1} is the symmetry group of the regular simplex, i.e. the convex hull of the standard basis vectors in $\mathbb{R}^n$. The $n!$ elements of this group act on $\mathbb{R}^n$ by permuting the coordinates. The tuple $(\xi_1, \dots, \xi_n)$ of random vectors in $\mathbb{R}^d$ is called *exchangeable* if its distribution is invariant under the action of A_{n-1}, that is

$$(\xi_1, \dots, \xi_n) \stackrel{d}{=} (\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})$$

for any permutation $\sigma(1), \dots, \sigma(n)$ of $1, \dots, n$. We will use the standard notation $\mathrm{Sym}(n)$ for the symmetric group on $\{1, \dots, n\}$ of such permutations.

The action of A_{n-1} leaves the hyperplane $x_1 + \ldots + x_n = 0$ invariant. Restricting the action of A_{n-1} to this hyperplane allows to apply our method to the convex hulls $\mathrm{Conv}(S_1, \dots, S_{n-1})$ of partial sums with exchangeable increments that satisfy the condition $\xi_1 + \dots + \xi_n = 0$ a.s. This covers the absorption problem for random walk bridges under the appropriate absolute continuity assumption. The respective analogue of (3) rests on counting the number of Weyl chambers of type A_{n-1} intersected by a generic subspace of the hyperplane $x_1 + \ldots + x_n = 0$ of codimension d. The full treatment is given by Theorem 2.3 of the next section.

The Coxeter group D_n is the subgroup of B_n of index two whose action on $\mathbb{R}^n$ changes signs of an even number of coordinates. The corresponding Theorem 2.6 on the distributions invariant under the action of D_n, applies to the convex hulls of symmetric random walks which are allowed to choose the sign of the last jump, see the discussion in Section 2.

With the exception of Theorem 5.5, which covers only simple random walks, all the results discussed above in Section 1.2 are proved for convex hulls of partial sums with their increments invariant under the action of the three reflection groups

A_{n-1}, B_n, and D_n. The corresponding geometric statements concern the Weyl chambers of the respective types.

Finally, we consider the increments invariant under the action of direct products of reflection groups. We restrict ourselves to direct products of groups of type B. The corresponding Theorem 2.8 solves the absorption problem for the joint convex hull of several symmetric random walks with possibly different number of steps (under the appropriate absolute continuity assumption).

In the particular case that all the random walks have the same distribution of increments and each walk has only one step, Theorem 2.8 implies the well known result of Wendel [**?**]: if $\xi_1, \dots, \xi_n$ are i.i.d. random vectors in $\mathbb{R}^d$ with an absolutely continuous symmetric distribution, then

$$\mathbb{P}(0 \notin \operatorname{Conv}(\xi_1, \dots, \xi_n)) = \frac{1}{2^{n-1}} \sum_{k=0}^{d-1} \binom{n-1}{k}. \tag{4}$$

Thus our approach brings together the classical distribution-free results (2) and (4) by Sparre Andersen and Wendel on symmetrically distributed random variables.

Let us explain the structure of the paper. Section 2 contains explicit solutions of the absorption problem for different types of increments under the most general assumptions. These results are proved in Section 6. In Sections 3 and 4 we provide some basic facts from the theory of hyperplane arrangements and conic convex geometry, and prove our new results on the geometric problems equivalent to the absorption problem. An asymptotic analysis of absorption probabilities in given in Section 5. The paper concludes with a list of open questions.

2. Convex hulls of random walks and bridges

In this section we present explicit solutions of the absorption problems for partial sums with increments jointly invariant under the action of groups of several types.

2.1. **Type A_{n-1}: Random walk bridges.** The Coxeter group A_{n-1} is the symmetric group $\operatorname{Sym}(n)$ which acts on $\mathbb{R}^n$ by permuting the coordinates. The number of elements of this group is $n!$. The action of this group leaves the following hyperplane invariant:

$$L = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \ldots + x_n = 0\},$$

which explains why the subscript $n-1$ rather than n appears in the standard notation A_{n-1}. Note that the group A_{n-1} is the symmetry group of the regular simplex with n vertices (defined as the convex hull of the standard basis in $\mathbb{R}^n$).

Theorem 2.1. *Let $(\xi_1, \dots, \xi_n)$ be an exchangeable tuple of random vectors in $\mathbb{R}^d$, that is for every permutation $\sigma \in \operatorname{Sym}(n)$ it holds*

$$(\xi_1, \dots, \xi_n) \stackrel{d}{=} (\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}). \tag{5}$$

Assume that $\xi_1 + \ldots + \xi_n = 0$ a.s. and that $(\xi_1, \dots, \xi_{n-1})$ has a joint density w.r.t. the Lebesgue measure on $\mathbb{R}^{d(n-1)}$. Denoting the partial sums by $S_k = \xi_1 + \ldots + \xi_k$, we have

$$\mathbb{P}[0 \in \operatorname{Conv}(S_1, \dots, S_{n-1})] = \frac{2}{n!} \left(\left[{n \atop d+2} \right] + \left[{n \atop d+4} \right] + \dots \right), \tag{6}$$

where $\left[{n \atop k}\right]$ *are the Stirling numbers of the first kind defined by the formula*

$$t(t+1)\dots(t+n-1) = \sum_{k=1}^{\infty} \left[{n \atop k}\right] t^k. \tag{7}$$

Remark 2.2. The sum in (6) is finite as per our notation, $\left[{n \atop k}\right] = 0$ for $k \notin \{1, \dots, n\}$. For the same reason, the absorption probability is 0 if $n \le d+1$. Combining (6) with the identity

$$\left[{n \atop 1}\right] + \left[{n \atop 3}\right] + \dots = \left[{n \atop 2}\right] + \left[{n \atop 4}\right] + \dots = \frac{n!}{2} \tag{8}$$

(which can be obtained by taking $t = \pm 1$ in (7)), we obtain the following formula for the non-absorption probability:

$$\mathbb{P}[0 \notin \mathrm{Conv}(S_1, \dots, S_{n-1})] = \frac{2}{n!} \left(\left[{n \atop d}\right] + \left[{n \atop d-2}\right] + \dots \right). \tag{9}$$

Theorem 2.1 applies to random walk bridges. Namely, let $\xi_1, \dots, \xi_n$ be i.i.d. random vectors in $\mathbb{R}^d$ having a Lebesgue density that does not vanish at the origin. Then, the conditional distribution of $\xi_1, \dots, \xi_n$ given that $\xi_1 + \dots + \xi_n = 0$ is well-defined and satisfies the assumptions of Theorem 2.1. In the one-dimensional case $d = 1$ we obtain from (9) the probability that a random walk bridge does not change its sign:

$$\mathbb{P}[S_1, \dots, S_{n-1} > 0 \text{ or } S_1, \dots, S_{n-1} < 0 \,|\, S_n = 0] = \frac{2}{n}, \tag{10}$$

since $\left[{n \atop 1}\right] = (n-1)!$. In fact, Sparre Andersen [**?**] showed that the probability of staying positive and the probability of staying negative are $1/n$ each. Theorem 2.1 can be viewed as a multidimensional generalization of this classical formula.

Theorem 2.1 can also be applied to a bridge of the following type. Let $\eta_1, \dots, \eta_n$ be i.i.d. random vectors in $\mathbb{R}^d$ having a Lebesgue density. Let $M = \eta_1 + \dots + \eta_n$ and define $\xi_i = \eta_i - M/n$. It is easy to check that $\xi_1, \dots, \xi_n$ satisfy the assumptions of Theorem 2.1.

It is remarkable that Theorem 2.1 (as well as similar theorems stated below) is distribution-free, that is the probability in (6) does not depend on the distribution of $\xi_1, \dots, \xi_n$. No moment conditions on the random vectors are imposed. Let us stress that the existence of the density imposed in Theorem 2.1 is essential. For example, for the bridge of a simple random walk on $\mathbb{Z}$ (which makes jumps ± 1 with probability 1/2), it is known that (for even n)

$$\mathbb{P}[S_1, \dots, S_{n-1} > 0 \text{ or } S_1, \dots, S_{n-1} < 0 \,|\, S_n = 0] = \frac{1}{n-1},$$

which is clearly different from (10). In particular, without absolute continuity, the absorption probability becomes distribution dependent.

The number $\frac{1}{n!}\left[{n \atop k}\right]$ turns out to be the k-th conic intrinsic volume of the Weyl chamber of type A_{n-1}, as shown below in Section 4.3. It can be also interpreted as the probability of having k records in n i.i.d. observations from a continuous distribution [**?** , Lecture 13], or as

$$\frac{1}{n!}\left[{n \atop k}\right] = \mathbb{P}[\delta_1 + \dots + \delta_n = k],$$

where $\delta_1, \dots, \delta_n$ are independent random variables with $\delta_i \sim \text{Bernoulli}(\frac{1}{i})$, which may be interpreted as the record indicators.

2.2. **Type B_n: Symmetric random walks.** The Coxeter group B_n is the symmetry group of the regular cube $[-1,1]^n$ (or of its dual, the regular crosspolytope). The elements of this group act on $\mathbb{R}^n$ by permuting the coordinates in arbitrary way and multiplying any number of coordinates by -1. The number of elements of this group is $2^n n!$.

Theorem 2.3. *Let $(\xi_1, \dots, \xi_n)$ be a symmetrically exchangeable tuple of random vectors in $\mathbb{R}^d$, that is for every permutation $\sigma \in \text{Sym}(n)$ and every $\varepsilon_1, \dots, \varepsilon_n \in \{-1,+1\}$ it holds*

$$(\xi_1, \dots, \xi_n) \stackrel{d}{=} (\varepsilon_1 \xi_{\sigma(1)}, \dots, \varepsilon_n \xi_{\sigma(n)}). \tag{11}$$

Assume also that $(\xi_1, \dots, \xi_n)$ has a joint density w.r.t. the Lebesgue measure on $\mathbb{R}^{dn}$. Denoting the partial sums by $S_k = \xi_1 + \dots + \xi_k$, we have

$$\mathbb{P}[0 \in \text{Conv}(S_1, S_2, \dots, S_n)] = \frac{2}{2^n n!}(B(n, d+1) + B(n, d+3) + \dots), \tag{12}$$

where $B(n,k)$ are the coefficients of the polynomial

$$(t+1)(t+3)\dots(t+2n-1) = \sum_{k=0}^{\infty} B(n,k) t^k. \tag{13}$$

Remark 2.4. Again, the absorption probability is 0 if $n \le d$. By taking $t = \pm 1$ in (13) we obtain the identity

$$B(n,1) + B(n,3) + \dots = B(n,2) + B(n,4) + \dots = 2^{n-1} n!. \tag{14}$$

It follows that the non-absorption probability is given by

$$\mathbb{P}[0 \notin \text{Conv}(S_1, S_2, \dots, S_n)] = \frac{2}{2^n n!}(B(n, d-1) + B(n, d-3) + \dots). \tag{15}$$

For example, Theorem 2.3 can be applied when $\xi_1, \dots, \xi_n$ are independent random vectors in $\mathbb{R}^d$ having a common density f that is centrally symmetric w.r.t. the origin, that is $f(t) = f(-t)$ for all $t \in \mathbb{R}^d$. Importantly, the probability in (12) does not depend on f. This proves the conjecture of **?**] for general d. Specifying (15) to $d = 1$ and noting that $B(n,0) = (2n-1)!!$ we obtain the probability that the symmetric random walk stays positive:

$$\mathbb{P}[S_1 > 0, S_2 > 0, \dots, S_n > 0] = \frac{(2n-1)!!}{2^n n!} = \frac{1}{2^{2n}} \binom{2n}{n}.$$

This recovers the other classical result of Sparre Andersen [**?**]. For a simple random walk the formula is different: by the reflection principle,

$$\mathbb{P}[S_1 > 0, S_2 > 0, \dots, S_n > 0] = \frac{1}{2^{n+1}} \binom{n}{\lfloor n/2 \rfloor}.$$

The numbers $B(n,k)$ are called the B-analogs of the (signless) Stirling numbers of the first kind; see entries A028338 (or A039757 for the signed version) in [**?**]. They satisfy the recurrence relation

$$B(n,k) = (2n-1)B(n-1,k) + B(n-1,k-1)$$

and are explicitly given by $B(n,k) = \sum_{i=k}^{n} 2^{n-i}\binom{i}{k}\left[{n \atop i}\right]$. These numbers were studied in detail by **?**]. There is a probabilistic representation of $B(n,k)$ (which follows directly from (13)):

$$\frac{B(n,k)}{2^n n!} = \mathbb{P}[\delta_1 + \ldots + \delta_n = k],$$

where $\delta_1, \ldots, \delta_n$ are independent random variables with $\delta_i \sim \text{Bernoulli}(\frac{1}{2i})$ for $1 \le i \le n$. Geometrically, $B(n,k)/(2^n n!)$ is the k-th conic intrinsic volume of the Weyl chamber of type B_n, see Section 4.3.

As we explained above, the existence of density is essential. In the general case it is still possible to obtain a one-sided bound for the absorption probability.

Proposition 2.5. *Let $(\xi'_1, \ldots, \xi'_n)$ be a symmetrically exchangeable tuple of random vectors in $\mathbb{R}^d$. Denoting the partial sums by $S'_k = \xi'_1 + \ldots + \xi'_k$, we have*

$$\mathbb{P}[0 \in \text{Conv}(S'_1, S'_2, \ldots, S'_n)] \ge \frac{2}{2^n n!}(B(n, d+1) + B(n, d+3) + \ldots).$$

The main example to keep in mind is the simple random walk on $\mathbb{Z}^d$:

$$\mathbb{P}[\xi'_1 = e_i] = \mathbb{P}[\xi'_1 = -e_i] = \frac{1}{2d}, \; i = 1, \ldots, d,$$

where $e_1, \ldots, e_d$ is the standard basis in $\mathbb{R}^d$.

Proof. Define $\xi_i = \xi'_i + \varepsilon\delta_i$, where $\delta_1, \ldots, \delta_n$ are i.i.d. standard normal random vectors in $\mathbb{R}^d$ (independent of $\xi'_1, \ldots, \xi'_n$) and $\varepsilon \ne 0$, and let $S_k = \xi_1 + \ldots + \xi_k$. The convex hull $H_{n,d} := \text{Conv}(S_1, \ldots, S_n)$ is obtained from $H'_{n,d} := \text{Conv}(S'_1, \ldots, S'_n)$ by a random distortion. The tuple $(\xi_1, \ldots, \xi_n)$ satisfies the assumptions of Theorem 2.3, and the claim follows by

$$\mathbb{P}[0 \in H_{n,d}] \le \mathbb{P}[0 \in H'_{n,d}] + \mathbb{P}[0 \in H_{n,d}, 0 \notin H'_{n,d}],$$

where the left-hand side does not depend on ε and the second term on the right-hand side vanishes as $\varepsilon \to 0$ since $H'_{n,d}$ is a closed set.

Note that the difference in probabilities occurs because if 0 is on the boundary of $H'_{n,d}$, then even a small distortion possibly gets $H_{n,d}$ aside of 0. $\square$

In Section 6.3 we will present another proof of Proposition 2.5 which follows our geometric interpretation in terms of intersections of Weyl chambers.

2.3. **Type D_n.** The Coxeter group D_n acts on $\mathbb{R}^n$ by permuting the coordinates in an arbitrary way and by multiplying any even number of coordinates by -1. It is a subgroup of B_n of index 2 and the number of its elements is $2^{n-1}n!$.

Theorem 2.6. *Let $\xi_1, \ldots, \xi_n$ be random vectors in $\mathbb{R}^d$ such that for every permutation $\sigma \in \text{Sym}(n)$ and every $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, +1\}$ with $\varepsilon_1 \ldots \varepsilon_n = +1$, we have*

$$(\xi_1, \ldots, \xi_n) \stackrel{d}{=} (\varepsilon_1 \xi_{\sigma(1)}, \ldots, \varepsilon_n \xi_{\sigma(n)}). \tag{16}$$

Assume that $(\xi_1, \ldots, \xi_n)$ has a joint density w.r.t. the Lebesgue measure on $\mathbb{R}^{dn}$. Denoting $S_k = \xi_1 + \ldots + \xi_k$, $1 \le k \le n$, and $S^*_n = \xi_1 + \ldots + \xi_{n-1} - \xi_n$, we have

$$\mathbb{P}[0 \in \text{Conv}(S_1, \ldots, S_{n-1}, S_n, S^*_n)] = \frac{2}{2^{n-1}n!}(D(n, d+1) + D(n, d+3) + \ldots), \tag{17}$$

where $D(n,k)$ are the coefficients of the polynomial

$$(t+1)(t+3)\dots(t+2n-3)(t+n-1) = \sum_{k=0}^{\infty} D(n,k)t^k. \tag{18}$$

Remark 2.7. The non-absorption probability is given by

$$\mathbb{P}[0 \notin \operatorname{Conv}(S_1,\dots,S_{n-1},S_n,S_n^*)] = \frac{2}{2^{n-1}n!}(D(n,d-1)+D(n,d-3)+\dots). \tag{19}$$

For example, Theorem 2.6 can be applied when $\xi_1,\dots,\xi_n$ are i.i.d. random vectors having a density that is centrally symmetric w.r.t. the origin. It is easy to show (see (49) below) that

$$\operatorname{Conv}(S_1,\dots,S_{n-1},S_n,S_n^*) = \operatorname{Conv}(S_1,\dots,S_{n-1},S_n) \cup \operatorname{Conv}(S_1,\dots,S_{n-1},S_n^*),$$

hence the probabilistic problem corresponding to the symmetry group D_n concerns the convex hull of a symmetric random walk that is allowed to choose the sign of its last jump.

The numbers

$$D(n,k) = (n-1)B(n-1,k) + B(n-1,k-1)$$

are called the D-analogs of the (signless) Stirling numbers of the first kind; see entry A039762 in [**?**] for the signed version. The kth conic intrinsic volume of the Weyl chamber of type D_n will be shown to be $D(n,k)/(2^{n-1}n!)$ (see Section 4.3) and we have

$$\frac{D(n,k)}{2^{n-1}n!} = \mathbb{P}[\delta_1+\dots+\delta_n = k],$$

where $\delta_1,\dots,\delta_n$ are independent random variables with $\delta_i \sim \text{Bernoulli}(\frac{1}{2i})$ for $1 \le i \le n-1$ and $\delta_n \sim \text{Bernoulli}(\frac{1}{n})$.

2.4. **Direct products of reflection groups.** So far we considered probabilistic problems related to irreducible reflection groups. It is known that a general reflection group can be represented as a direct sum of irreducible ones. In this section we state formulas for the probability that a *joint* convex hull of *several* random walks contains the origin. The corresponding symmetry groups are direct products of finite reflection groups. To be specific, we restrict ourselves to direct products of the form $B_{n_1}\times\dots\times B_{n_r}$, which contain only groups of type B. Here $r \ge 1$ corresponds to the number of random walks and n_i, where $1 \le i \le r$, stands for the number of their steps.

It is straightforward to extend our results to products of the form $A_{n_1}\times\dots A_{n_r}$, which corresponds to joint convex hulls of several random bridges, and even to mixed direct products containing groups of all 3 types A, B, D. We omit such extension because it requires complicated notation.

Theorem 2.8. *Let $\xi_1^{(1)},\dots,\xi_{n_1}^{(1)},\dots,\xi_1^{(r)},\dots,\xi_{n_r}^{(r)}$ be random vectors in $\mathbb{R}^d$ such that for every permutations $\sigma^{(1)} \in \operatorname{Sym}(n_1),\dots,\sigma^{(r)} \in \operatorname{Sym}(n_r)$ and every signs $\varepsilon_1^{(1)},\dots,\varepsilon_{n_1}^{(1)},\dots,\varepsilon_1^{(r)},\dots,\varepsilon_{n_r}^{(r)} \in \{-1,+1\}$, we have*

$$(\xi_1^{(1)},\dots,\xi_{n_1}^{(1)},\dots,\xi_1^{(r)},\dots,\xi_{n_r}^{(r)}) \stackrel{d}{=} (\varepsilon_1^{(1)}\xi_{\sigma_1(1)}^{(1)},\dots,\varepsilon_{n_1}^{(1)}\xi_{\sigma_1(n_1)}^{(1)},\dots,\varepsilon_1^{(r)}\xi_{\sigma_r(1)}^{(r)},\dots,\varepsilon_{n_r}^{(r)}\xi_{\sigma_r(n_r)}^{(r)}). \tag{20}$$

Assume that $(\xi_1^{(1)},\dots,\xi_{n_1}^{(1)},\dots,\xi_1^{(r)},\dots,\xi_{n_r}^{(r)})$ *has a joint density on* $\mathbb{R}^{(n_1+\dots+n_r)d}$. *Denoting the partial sums* $S_k^{(i)}=\xi_1^{(i)}+\dots+\xi_k^{(i)}$, $1\le i\le r$, $1\le k\le n_i$, *we have*

(21)
$$\mathbb{P}[0\in \mathrm{Conv}(S_1^{(1)},\dots,S_{n_1}^{(1)},\dots,S_1^{(r)},\dots,S_{n_r}^{(r)})]=\frac{2(P(d+1)+P(d+3)+\dots)}{2^{n_1}n_1!\dots 2^{n_r}n_r!},$$

where the $P(k)$*'s (which also depend on* $r,n_1,\dots,n_r$*) are the coefficients of the polynomial*

$$\prod_{i=1}^{r}((t+1)(t+3)\dots(t+2n_i-1))=\sum_{k=0}^{\infty}P(k)t^k. \tag{22}$$

Remark 2.9. Condition (20) is satisfied if $\xi_1^{(1)},\dots,\xi_{n_1}^{(1)},\dots,\xi_1^{(r)},\dots,\xi_{n_r}^{(r)}$ are independent random vectors such that the density of $\xi_k^{(i)}$ depends only on i and is centrally symmetric w.r.t. the origin in $\mathbb{R}^d$. In this particular case, Theorem 2.8 allows to compute the probability that the origin is absorbed by the joint convex hull of r independent random walks with i.i.d. steps, where different walks are allowed to have different distributions of increments.

Example 2.10 (Type B_1^r: the Wendel formula)**.** Let us consider the particular case $n_1=\dots=n_r=1$ that all random walks have unit length. This corresponds to the direct product of r groups $\mathbb{Z}/2\mathbb{Z}$, where each factor acts on $\mathbb{R}$ by multiplication by ±1.

The random vectors $\xi^{(1)}:=\xi_1^{(1)},\dots,\xi^{(r)}:=\xi_1^{(r)}$ with values in $\mathbb{R}^d$ satisfy

$$(\xi^{(1)},\dots,\xi^{(r)})\overset{d}{=}(\pm\xi_1^{(1)},\dots,\pm\xi_1^{(r)}) \tag{23}$$

for all 2^r choices of the signs. Then (22), which defines $P(k)$'s, takes the form $(t+1)^r=\sum_{k=0}^{r}P(k)t^k$ so that $P(k)=\binom{r}{k}$. Theorem 2.8 asserts that

$$\mathbb{P}[0\notin \mathrm{Conv}(\xi^{(1)},\dots,\xi^{(r)})]=\frac{1}{2^{r-1}}\left(\binom{r}{d-1}+\binom{r}{d-3}+\dots+\binom{r}{d \bmod 2}\right).$$

Using the recursive property of the Pascal triangle, we obtain

$$\mathbb{P}[0\notin \mathrm{Conv}(\xi^{(1)},\dots,\xi^{(r)})]=\frac{1}{2^{r-1}}\sum_{k=0}^{d-1}\binom{r-1}{k}.$$

This formula is due to **?**], who assumed that $\xi^{(1)},\dots,\xi^{(r)}$ are i.i.d., which is stronger than (23). Wendel's proof is essentially based on the Schläfli's formula (38) presented below, see [**?** , Section 8.2.1].

The same result can be obtained if one considers the symmetry group A_1^r since its action on $\mathbb{R}^{2r}$ is isomorphic to the action of B_1^r on $\mathbb{R}^r$.

3. Hyperplane arrangements

3.1. **The main formula for the number of regions.** A *linear hyperplane arrangement* (or simply "*arrangement*") $\mathcal{A}$ is a finite set of distinct hyperplanes in $\mathbb{R}^n$ that pass through the origin. The literature on hyperplane arrangements [**?**], [**?**] considers the more general concept of *affine* hyperplane arrangements (the hyperplanes are not required to pass through the origin) but in the present work we study only the linear case.

The *rank* of an arrangement $\mathcal{A}$, denoted by $\operatorname{rank}(\mathcal{A})$, is the dimension of the space spanned by the normals to the hyperplanes in $\mathcal{A}$. Equivalently, the rank is the codimension of the intersection of all hyperplanes in the arrangement:

$$\operatorname{rank}(\mathcal{A}) = n - \dim\left(\bigcap_{H\in\mathcal{A}} H\right).$$

The *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ of the arrangement $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B}\subset\mathcal{A}} (-1)^{\#\mathcal{B}} t^{n-\operatorname{rank}(\mathcal{B})}, \tag{24}$$

where # denotes the number of elements and $\operatorname{rank}(\varnothing) = 0$ under convention that the intersection over the empty set of hyperplanes is $\mathbb{R}^n$. The original definition of the characteristic polynomial is much more complicated and uses the notions of the intersection poset of $\mathcal{A}$ and the Möbius function on it; see [**?** , Section 1.3]. For our goals we need only the above equivalent definition. The equivalence was proved by Whitney; see, e.g., [**?** , Lemma 2.3.8] or [**?** , Theorem 2.4].

Denote by $\mathcal{R}(\mathcal{A})$ the set of open connected components ("regions" or "chambers") of the complement $\mathbb{R}^n \setminus \cup_{H\in\mathcal{A}} H$ of the hyperplanes. The following fundamental result due to Zaslavsky [**?**] (see also [**?** , Theorem 2.5]) expresses the number of regions of the arrangement $\mathcal{A}$ in terms of its characteristic polynomial:

$$\#\mathcal{R}(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1). \tag{25}$$

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ and let L_{n-d} be a linear subspace in $\mathbb{R}^n$ of codimension $d \le n-1$. We say that L_{n-d} is in *general position* with respect to $\mathcal{A}$ if for any subset $\mathcal{B}\subset\mathcal{A}$

$$\dim\left(\bigcap_{H\in\mathcal{B}} (H\cap L_{n-d})\right) = \begin{cases} n-d-\operatorname{rank}(\mathcal{B}), & \text{if } \operatorname{rank}(\mathcal{B}) \le n-d,\\ 0, & \text{if } \operatorname{rank}(\mathcal{B}) \ge n-d.\end{cases} \tag{26}$$

Our aim is to find a formula for the number of regions in $\mathcal{R}(\mathcal{A})$ which are intersected by L_{n-d}. This number can be expressed as the number of regions in the induced arrangement $\mathcal{A}|L_{n-d}$ that is the arrangement in $L_{n-d} \cong \mathbb{R}^{n-d}$ defined by

$$\mathcal{A}|L_{n-d} = \{H\cap L_{n-d} \colon H\in\mathcal{A}\}.$$

In this definition we assume that the linear subspace L_{n-d} is in general position w.r.t. $\mathcal{A}$. This is needed to verify that every $H \cap L_{n-d}$ indeed has codimension 1 in L_{n-d} and that all these hyperplanes are indeed distinct. We obviously have that

$$\#\{R\in\mathcal{R}(\mathcal{A})\colon R\cap L_{n-d} \ne \varnothing\} = \#\mathcal{R}(\mathcal{A}|L_{n-d}).$$

Lemma 3.1. *Let $\mathcal{A}$ be a linear hyperplane arrangement in $\mathbb{R}^n$ and let L_{n-d} be a linear subspace in $\mathbb{R}^n$ of codimension d that is in general position w.r.t. $\mathcal{A}$. Let*

$$\chi_{\mathcal{A}}(t) = \sum_{k=0}^{n} (-1)^{n-k} a_k t^k \tag{27}$$

be the characteristic polynomial of $\mathcal{A}$. Then the characteristic polynomial of $\mathcal{A}$ restricted to L_{n-d} is given by

$$\chi_{\mathcal{A}|L_{n-d}}(t) = \sum_{k=0}^{d} (-1)^{n-k} a_k + \sum_{k=d+1}^{n} (-1)^{n-k} a_k t^{k-d}.$$

Remark 3.2. It is easy to see that $a_n = 1$, $a_{n-1} = \#\mathcal{A}$. Moreover, the sequence $a_0, \dots, a_n$ is strictly positive [**?** , Corollary 3.5] and unimodal [**?** , p. 17].

Proof. It follows from (26) that for any subset $\mathcal{B} \subset \mathcal{A}$

$$\operatorname{rank}(\mathcal{B}|L_{n-d}) = \begin{cases} \operatorname{rank}(\mathcal{B}), & \text{if } \operatorname{rank}(\mathcal{B}) \le n-d, \\ n-d, & \text{if } \operatorname{rank}(\mathcal{B}) \ge n-d, \end{cases} \tag{28}$$

and $\#(\mathcal{B}|L_{n-d}) = \#\mathcal{B}$. Using (24) (in dimension $n-d$) and then (28) we obtain

$$\begin{aligned} \chi_{\mathcal{A}|L_{n-d}}(t) &= \sum_{\mathcal{B}\subset\mathcal{A}} (-1)^{\#\mathcal{B}} t^{n-d-\operatorname{rank}(\mathcal{B}|L_{n-d})} \\ &= \sum_{k=0}^{d} \sum_{\substack{\mathcal{B}\subset\mathcal{A} \\ \operatorname{rank}(\mathcal{B})=n-k}} (-1)^{\#\mathcal{B}} + \sum_{k=d+1}^{n} \sum_{\substack{\mathcal{B}\subset\mathcal{A} \\ \operatorname{rank}(\mathcal{B})=n-k}} (-1)^{\#\mathcal{B}} t^{n-d-\operatorname{rank}(\mathcal{B})}. \end{aligned}$$

After noting that by (24),

$$\sum_{\substack{\mathcal{B}\subset A \\ \operatorname{rank}(\mathcal{B})=n-k}} (-1)^{\#\mathcal{B}} = (-1)^{n-k} a_k,$$

we obtain the required formula. □

Now we are ready to state the main result of this section.

Theorem 3.3. *Let L_{n-d} be linear subspace in $\mathbb{R}^n$ of codimension d that is in general position w.r.t. to a linear hyperplane arrangement $\mathcal{A}$. The number of regions in $\mathcal{R}(\mathcal{A})$ intersected by L_{n-d} is given by*

$$\#\{R \in \mathcal{R}(\mathcal{A})\colon R \cap L_{n-d} \ne \varnothing\} = 2(a_{d+1} + a_{d+3} + \dots),$$

where the a_k's are defined by (27) *and we set $a_k = 0$ for $k \notin \{0, \dots, n\}$.*

Proof. By (25) and Lemma 3.1, we have

$$\#\{R \in \mathcal{R}(\mathcal{A})\colon R \cap L_{n-d} \ne \varnothing\} = \begin{cases} \sum_{k=0}^{n} a_k - 2\sum_{k=0}^{s} a_{2k}, & \text{if } d = 2s+1, \\ \sum_{k=0}^{n} a_k - 2\sum_{k=1}^{s} a_{2k-1}, & \text{if } d = 2s, \end{cases}$$

where we used that $\mathcal{A}|L_{n-d}$ is an arrangement in dimension $n-d$. To complete the proof, we need to show that

$$a_0 + a_2 + \dots = a_1 + a_3 + \dots. \tag{29}$$

By the second part of Zaslavsky's theorem [**?** , Theorem 2.5], the number of *bounded* regions in $\mathcal{R}(\mathcal{A})$ is (up to the sign) given by $\chi_{\mathcal{A}}(1) = \sum_{k=0}^{n} (-1)^{n-k} a_k$. Since we are dealing only with linear hyperplane arrangements, there are no bounded regions, whence the statement follows. □

3.2. **Special case: the reflection arrangements.** The above results can be applied to the *reflection arrangements* in $\mathbb{R}^n$ of the types A_{n-1}, B_n, D_n. These arrangements consist of the hyperplanes

$$\begin{aligned} \mathcal{A}(A_{n-1})&\colon \quad x_i = x_j, \quad 1 \le j < j \le n, \\ \mathcal{A}(B_n)&\colon \quad x_i = x_j, \quad x_i = -x_j, \quad x_k = 0, \quad 1 \le i < j \le n, \quad 1 \le k \le n, \\ \mathcal{A}(D_n)&\colon \quad x_i = x_j, \quad x_i = -x_j, \quad 1 \le i < j \le n. \end{aligned}$$

Theorem 3.4. *Let L_{n-d} be a linear subspace in $\mathbb{R}^n$ of codimension d that is in general position w.r.t. to one of the reflection arrangement $\mathcal{A}(A_{n-1}), \mathcal{A}(B_n), \mathcal{A}(D_n)$. Then number of regions in this arrangement intersected by L_{n-d} is given, respectively, by*

$$\mathcal{R}(\mathcal{A}(A_{n-1})|L_{n-d}) = 2\left(\begin{bmatrix} n \\ d+1 \end{bmatrix} + \begin{bmatrix} n \\ d+3 \end{bmatrix} + \dots\right),$$
$$\mathcal{R}(\mathcal{A}(B_n)|L_{n-d}) = 2(B(n, d+1) + B(n, d+3) + \dots),$$
$$\mathcal{R}(\mathcal{A}(D_n)|L_{n-d}) = 2(D(n, d+1) + D(n, d+3) + \dots).$$

Proof. The characteristic polynomials of the reflection arrangements are (see pp. 63–64 and Corollary 2.2 on p. 28 in [**?**])

$$\chi_{\mathcal{A}(A_{n-1})}(t) = t(t-1)\dots(t-(n-1)) = \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k, \tag{30}$$

$$\chi_{\mathcal{A}(B_n)}(t) = (t-1)(t-3)\dots(t-(2n-1)) \sum_{k=0}^{n} (-1)^{n-k} B(n,k) t^k, \tag{31}$$

$$\chi_{\mathcal{A}(D_n)}(t) = (t-1)(t-3)\dots(t-(2n-3))(t-(n-1)) = \sum_{k=0}^{n} (-1)^{n-k} D(n,k) t^k,$$

see (7), (13), (18). We stress that $\mathcal{A}(A_{n-1})$ is an arrangement in $\mathbb{R}^n$, hence its characteristic polynomial has degree n. □

3.3. **Non-general position.** The following lemma compares the number of open and closed chambers intersected by an arbitrary linear subspace with the respective number of chambers for a linear subspace in general position.

Lemma 3.5. *Let $\mathcal{A}$ be a linear arrangement in $\mathbb{R}^n$ and let L_{n-d}, L'_{n-d} be linear subspaces in $\mathbb{R}^n$ of codimension d. If L_{n-d} is in general position w.r.t. $\mathcal{A}$, then*

$$\#\{R \in \mathcal{R}(\mathcal{A})\colon \bar{R} \cap L_{n-d} \neq \varnothing\} = \#\{R \in \mathcal{R}(\mathcal{A})\colon R \cap L_{n-d} \neq \varnothing\} \tag{32}$$

and

$$\#\{R \in \mathcal{R}(\mathcal{A})\colon \bar{R} \cap L'_{n-d} \neq \{0\}\} \geq \#\{R \in \mathcal{R}(\mathcal{A})\colon \bar{R} \cap L_{n-d} \neq \{0\}\},$$
$$\#\{R \in \mathcal{R}(\mathcal{A})\colon R \cap L'_{n-d} \neq \varnothing\} \leq \#\{R \in \mathcal{R}(\mathcal{A})\colon R \cap L_{n-d} \neq \varnothing\}.$$

Proof. Let us prove the first statement. Let $R \in \mathcal{R}(\mathcal{A})$ be any chamber. It is possible to show that if L_{n-d} does not intersect with R but has a non-trivial intersection with $\bar{R}$, then either L_{n-d} is contained in a hyperplane from $\mathcal{A}$ that forms a hyperface of R or L_{n-d} contains a face of R of some positive dimension. Both options contradict the general position assumption.

Now we prove the second statement; the proof of the third one is analogous. Let $\mathbf{Gr}(n-d, n)$ be the Grassmannian of all $(n-d)$-dimensional linear subspaces in $\mathbb{R}^n$ endowed with the standard metric. The set of subspaces which are in general position w.r.t. $\mathcal{A}$ is dense in $\mathbf{Gr}(n-d, n)$. For any chamber $R \in \mathcal{R}(\mathcal{A})$, the set

$$\{M \in \mathbf{Gr}(n-d, n)\colon \bar{R} \cap M = \{0\}\}$$

is open in $\mathbf{Gr}(n-d, n)$. Therefore, there exists a neighborhood U of L'_{n-d} such that for all linear subspaces $M \in U$ we have

$$\{R \in \mathcal{R}(\mathcal{A})\colon \bar{R} \cap M = \{0\}\} \supset \{R \in \mathcal{R}(\mathcal{A})\colon \bar{R} \cap L'_{n-d} = \{0\}\}.$$

We finish the proof by taking $M \in U$ that is in general position w.r.t. $\mathcal{A}$ and noting that by (32) and Theorem 3.3, it holds

$$\#\{R \in \mathcal{R}(\mathcal{A})\colon \bar{R} \cap M = \{0\}\} = \#\{R \in \mathcal{R}(\mathcal{A})\colon \bar{R} \cap L_{n-d} = \{0\}\}.$$

□

4. Connection with conic intrinsic volumes

4.1. **Definition of conic intrinsic volumes.** We call a set $C \subset \mathbb{R}^n$ a *convex cone* if for any $x, y \in C$ and $a, b > 0$ it holds $ax + by \in C$. In the 1940's a spherical counterpart of the Steiner formula was developed in [**? ? ?**]. In its modern form (see [**?** , Section 6.5], [**?** , Section IV], and [**? ? ?**]), this formula expresses the size of angular expansions of a closed convex cone C in $\mathbb{R}^n$:

$$\mathbb{P}(\operatorname{dist}^2(\theta, C) \le \lambda) = \sum_{k=0}^{n} \beta_{k,n}(\lambda)\upsilon_k(C), \tag{33}$$

where θ is a random variable uniformly distributed on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\beta_{k,n}(\cdot)$ is the distribution function of a Beta distribution with parameters $(n-k)/2$ and $n/2$. Since the functions $\beta_{1,n}, \dots, \beta_{n,n}$ are linearly independent, the formula defines the coefficients $\upsilon_k(C)$ uniquely. The quantities $\upsilon_0(C), \dots, \upsilon_n(C)$ are called the *conic intrinsic volumes* of the cone C. The normalization is chosen so that these quantities do not depend on the dimension of the enclosing space. Note that the kth conic intrinsic volume $\upsilon_k(C)$ equals the $(k-1)$th spherical intrinsic volume of $C \cap \mathbb{S}^{n-1}$ considered in [**? ?**].

The kth conic intrinsic volume $\upsilon_k(C)$ corresponds to the $(k-1)$th spherical intrinsic volume in [**? ?**].

Following the notation of [**?**], for each $k \in \{0, \dots, n\}$, define the kth *half-tail functional* by

$$h_k(C) = \upsilon_k(C) + \upsilon_{k+2}(C) + \dots, \tag{34}$$

where we set $\upsilon_k(C) = 0$ for $k \notin \{0, \dots, n\}$.

The conic intrinsic volumes satisfy a version of the Gauss–Bonnet theorem (see, e.g., [**?** , Theorem 6.5.5] or [**?** , p. 28]):

$$h_0(C) = \upsilon_0(C) + \upsilon_2(C) + \dots = \frac{1}{2}, \quad h_1(C) = \upsilon_1(C) + \upsilon_3(C) + \dots = \frac{1}{2}. \tag{35}$$

The conic analogue of the Crofton formula (see, e.g., [**?** , pp. 261–262] or [**?** , Equation 5.10]) is the following relation: for every $d \in \{0, \dots, n-1\}$,

$$h_{d+1}(C) = \frac{1}{2}\mathbb{P}[C \cap W_{n-d} \ne \{0\}], \tag{36}$$

where W_{n-d} is a random $(n-d)$-dimensional linear subspace in $\mathbb{R}^n$ chosen w.r.t. the uniform distribution on the Grassmannian.

4.2. **Characteristic polynomial of linear arrangement and conic intrinsic volumes.** Let $\mathcal{A}$ be a linear hyperplane arrangement in $\mathbb{R}^n$ with characteristic polynomial

$$\chi_{\mathcal{A}}(t) = \sum_{k=0}^{n} (-1)^{n-k} a_k t^k.$$

The next theorem, conjectured by **?**] and proved by **?**], relates the coefficients of the characteristic polynomial to the conic intrinsic volumes of the regions of the arrangement. We will give a completely different (and very short) proof of this theorem.

Theorem 4.1. *For every linear hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$,*

$$a_k = \sum_{R\in\mathcal{R}(\mathcal{A})} \upsilon_k(R), \quad k=0,\dots,n.$$

Proof. Let W_{n-d+1} be a random $(n-d+1)$-dimensional linear subspace in $\mathbb{R}^n$ distributed according to the uniform measure on the Grassmannian, where $d \in \{1,\dots,n\}$. With probability one, W_{n-d+1} is in general position w.r.t. $\mathcal{A}$. Thus, by Theorem 3.3 we have

$$\#\{R\in\mathcal{R}(\mathcal{A})\colon R\cap W_{n-d+1}\neq\varnothing\} = 2(a_d+a_{d+2}+\dots) \quad \text{a.s.}$$

On the other hand, for any $R\in\mathcal{R}(\mathcal{A})$ it follows from (36) that

$$h_d(R) = \frac{1}{2}\mathbb{E}[\mathbb{1}_{R\cap W_{n-d+1}\neq\varnothing}].$$

Summing up over all $R\in\mathcal{R}(\mathcal{A})$ and combining the formulas together, we obtain that for all $d\in\{1,\dots,n\}$,

$$\sum_{R\in\mathcal{R}(\mathcal{A})} h_d(R) = \frac{1}{2}\mathbb{E}[\#\{R\in\mathcal{R}(\mathcal{A})\colon R\cap W_{n-d+1}\neq\varnothing\}] = a_d+a_{d+2}+\dots.$$

By (29), (35), and the equation above for $d=1$, it also holds

$$\sum_{R\in\mathcal{R}(\mathcal{A})} h_0(R) = a_0+a_2+\dots.$$

Using the formula $\upsilon_k(R)=h_k(R)-h_{k+2}(R)$ (which follows from (34)) we obtain that for all $k\in\{0,\dots,n\}$,

$$\sum_{R\in\mathcal{R}(\mathcal{A})}\upsilon_k(R) = \sum_{R\in\mathcal{R}(\mathcal{A})} h_k(R) - \sum_{R\in\mathcal{R}(\mathcal{A})} h_{k+2}(R) = a_k.$$

This completes the proof. □

4.3. **Conic intrinsic volumes of the Weyl chambers.** The *Weyl chambers* of type A_{n-1}, B_n, D_n are the following convex cones in $\mathbb{R}^n$:

$$\begin{aligned}
\mathcal{C}(A_{n-1}) &:= \{(x_1,\dots,x_n)\in\mathbb{R}^n\colon x_1<x_2<\dots<x_n\},\\
\mathcal{C}(B_n) &:= \{(x_1,\dots,x_n)\in\mathbb{R}^n\colon 0<x_1<x_2<\dots<x_n\},\\
\mathcal{C}(D_n) &:= \{(x_1,\dots,x_n)\in\mathbb{R}^n\colon 0<|x_1|<x_2<\dots<x_n\}.
\end{aligned}$$

Each Weyl chamber $C=\mathcal{C}(G)$ is a *fundamental domain* for the corresponding reflection group $G=A_{n-1},B_n$ or D_n. This means that all the cones of the form $gC, g\in G$, which are also called Weyl chambers without any risk of confusion, are disjoint and that $\cup_{g\in G} g\bar{C}=\mathbb{R}^n$ holds true.

Theorem 4.2. *The conic intrinsic volumes of the Weyl chambers of types A_{n-1}, B_n, D_n are given by*

$$\upsilon_k(\mathcal{C}(A_{n-1})) = \frac{1}{n!}\begin{bmatrix} n\\ k\end{bmatrix}, \quad \upsilon_k(\mathcal{C}(B_n)) = \frac{B(n,k)}{2^n n!}, \quad \upsilon_k(\mathcal{C}(D_n)) = \frac{D(n,k)}{2^{n-1}n!},$$

for $k=0,1,\dots,n$, where $\begin{bmatrix} n\\ k\end{bmatrix}$, $B(n,k)$, $D(n,k)$ are as in (7), (13), (18), respectively.

Proof. To be specific, consider the $\mathcal{A}_{n-1}$ case. The coefficients of the characteristic polynomial of the corresponding hyperplane arrangement are $a_k = \left[{n \atop k}\right]$; see the proof of Theorem 3.4. The regions in $\mathcal{R}(\mathcal{A}_{n-1})$ are the $n!$ isometric Weyl chambers of the type A_{n-1}. By Theorem 4.1 we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix} = n! \upsilon_k(\mathcal{C}(A_{n-1})),$$

which proves the required formula. The B_n and D_n cases are analogous. □

4.4. **Random arrangements of hyperplanes in general position.** Let $\mathcal{A}$ be a linear arrangement in $\mathbb{R}^n$ consisting of $m \geq n$ hyperplanes in general position (in our terminology, this means that $\mathbb{R}^n$ is in general position w.r.t. $\mathcal{A}$, see (26)). By (24) we have that

$$\chi_{\mathcal{A}}(t) = \sum_{k=0}^{n-1} (-1)^k \binom{m}{k} t^{n-k} + \sum_{k=n}^{m} (-1)^k \binom{m}{k}. \tag{37}$$

Applying (25) and using that the alternating binomial coefficients add up to zero, we get

$$\mathcal{R}(\mathcal{A}) = \sum_{k=0}^{n-1} \binom{m}{k} + \sum_{k=n}^{m} (-1)^{k+n} \binom{m}{k} = \sum_{k=0}^{n-1} \binom{m}{k} - \sum_{k=0}^{n-1} (-1)^{k+n} \binom{m}{k},$$

hence by the recursive property of the Pascal triangle,

$$\mathcal{R}(\mathcal{A}) = 2 \sum_{k=0}^{n-1} \binom{m-1}{k} =: C(m, n). \tag{38}$$

This well-known formula, proved by Schläfli [**?** , pp.209–212] for a general dimension, goes back to Steiner [**?**] for $n = 3$; see also [**?** , Lemma 8.2.1] for a simple inductive proof and references. We already saw this formula in Example 2.10.

Let $X_1, \ldots, X_m$ be i.i.d. random vectors on the unit sphere $\mathbb{S}^{n-1}$ such that their common distribution is centrally symmetric and assigns no mass to any $(n-2)$-dimensional great subsphere. The hyperplanes $X_1^\perp, \ldots, X_m^\perp$, which are in general position a.s., divide $\mathbb{R}^n$ into $C(m, n)$ random cones. We choose one of these cones uniformly at random to obtain the *random Schläfli cone* C_m in $\mathbb{R}^n$ introduced by Hug and Schneider in [**?**].

The next result of [**?**] (given there with a slightly different notation) calculates the expected intrinsic volumes of a random Schläfli cone. This theorem easily follows from Theorem 4.1.

Theorem 4.3. *For any random Schläfli cone C_m in $\mathbb{R}^n$, it holds*

$$\mathbb{E}\, \upsilon_k(C_m) = \frac{1}{C(m, n)} \binom{m}{n-k}, \quad k = 0, \ldots, n.$$

Proof. Let $\mathcal{A}$ be the arrangement consisting of the hyperplanes $X_1^\perp, \ldots, X_m^\perp$. We have

$$\mathbb{E}\, \upsilon_k(C_m) = \frac{1}{C(m, n)} \mathbb{E} \sum_{R \in \mathcal{R}(\mathcal{A})} \nu_k(R).$$

At the other hand, it follows from Theorem 4.1 and (37) that

$$\sum_{R\in\mathcal{R}(\mathcal{A})} \upsilon_k(R) = \binom{m}{n-k} \quad \text{a.s.,}$$

which concludes the proof. $\square$

Remark 4.4. As readily seen from the proof, this result holds true for any *deterministic* random vectors $X_1, \dots, X_m$ that are in general position. The essential randomness here is in the uniform measure on $C(m,n)$ elements of $\mathcal{R}(\mathcal{A})$.

5. Asymptotic results

In this section we use the exact expressions of Section 2 to study the asymptotic behavior of the probability that the convex hull of a symmetric random walk or of a random walk bridge absorbs the origin. The cases A_{n-1}, B_n and D_n are very similar. We work in the setting of Theorems 2.1, 2.3, 2.6, respectively, and recall that these require absolute continuity assumptions and the symmetry of increments in the cases B_n and D_n. Let $H_{n,d} \subset \mathbb{R}^d$ denote the convex hull considered in any of these theorems.

5.1. **Constant dimension.** The following theorem gives the asymptotics of the non-absorption probability in the case that the dimension d is fixed and the number of steps n tends to infinity. In the case $d = 2$ this result was obtained by **?**]. We write $x_n \sim y_n$ (as $n \to \infty$) if $\lim_{n\to\infty} x_n/y_n = 1$.

Theorem 5.1. *If the dimension $d \geq 2$ is fixed, then under assumptions of any of Theorems 2.1, 2.3, 2.6,*

$$\mathbb{P}[0 \notin H_{n,d}] \underset{n\to\infty}{\sim} \begin{cases} \frac{2(\log n)^{d-1}}{(d-1)!n}, & \text{in the } A_{n-1} \text{ case,} \\ \frac{(\log n)^{d-1}}{2^{d-2}(d-1)!\sqrt{\pi n}}, & \text{in the } B_n \text{ and } D_n \text{ cases.} \end{cases}$$

Proof. In the A_{n-1} case, we can use the well-known asymptotics of the Stirling numbers [**?**]: for a fixed $k \in \mathbb{N}$,

$$\frac{1}{(n-1)!} \begin{bmatrix} n \\ k \end{bmatrix} \underset{n\to\infty}{\sim} \frac{(\log n)^{k-1}}{(k-1)!}.$$

Substituting this formula into the statement of Theorem 2.1 (see also (9)) and noting that the term $\left[{n \atop d}\right]$ dominates all other terms, we obtain

$$\mathbb{P}[0 \notin H_{n,d}] = \frac{2}{n!} \left(\begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n \\ d-2 \end{bmatrix} + \dots \right) \underset{n\to\infty}{\sim} \frac{2}{n!} \begin{bmatrix} n \\ d \end{bmatrix} \underset{n\to\infty}{\sim} \frac{2(\log n)^{d-1}}{(d-1)!n}.$$

In the B_n case, the definition of $B(n,k)$ given in (13) yields

$$\begin{aligned} B(n,k) &= (2n-1)!! \sum_{1\leq i_1<\dots<i_k\leq n} \frac{1}{(2i_1-1)\dots(2i_k-1)} \\ &\underset{n\to\infty}{\sim} \frac{(2n-1)!!}{k!} \left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1}\right)^k \\ &\underset{n\to\infty}{\sim} \frac{(2n-1)!!}{2^k k!} (\log n)^k, \end{aligned}$$

where, in order to pass from the first line to the second one, one shows (by an omitted standard argument) that the contribution of all terms where at least two indices i_m and i_l are equal is $o((\log n)^k)$. Substituting the above asymptotic for $B(n,k)$ into the statement of Theorem 2.3 (see also (15)), and noting that the term $B(n,d-1)$ dominates all other terms, we obtain

$$\mathbb{P}[0 \notin H_{n,d}] = \frac{2}{2^n n!}(B(n,d-1)+B(n,d-3)+\dots) \underset{n\to\infty}{\sim} \frac{2B(n,d-1)}{2^n n!}.$$

Using the asymptotics of $B(n,d-1)$ and the Stirling formula, we obtain

$$\mathbb{P}[0 \notin H_{n,d}] \underset{n\to\infty}{\sim} 2\frac{(2n-1)!!}{2^n n!}\frac{(\log n)^{d-1}}{2^{d-1}(d-1)!} \underset{n\to\infty}{\sim} \frac{(\log n)^{d-1}}{2^{d-2}(d-1)!\sqrt{\pi n}}.$$

The computation for the D_n case is similar to the one for the B_n case and yields the same result. □

5.2. **Central limit theorem and large deviations.** Consider the convex hull $H_{n,d}$ of a symmetric random walk (or a random walk bridge) of length n in a *high* dimension d. It is clear that if n is sufficiently small, then the absorption probability should be close to 0, whereas for sufficiently large n the absorption probability should be close to 1. Hence, at some value of n (which is a function of d) there should be a phase transition from absorption to non-absorption. The question where this phase transition occurs was studied by **?**] and **? ?**]. **?**] proved that for some constants $0 < c_1 < c_2$, non-absorption (respectively, absorption) occurs with high probability provided that $n < \mathrm{e}^{c_1 d/\log d}$ (respectively, $n > \mathrm{e}^{c_2 d \log d}$). **? ?**] removed the $\log d$ factor by replacing the bounds above by $n < \mathrm{e}^{c_1 d}$ and $n > \mathrm{e}^{c_2 d}$, respectively.

In this section we provide a precise description of the location of this phase transition. It will be convenient for us to make $d = d(n)$ a function of n rather than considering n as a function of d. **?**] and **? ?**] studied the following models: a Brownian motion sampled at the points of a Poisson point process (in [**?**] and [**?**]), the simple random walk (in [**?**] and [**?**]), and a Brownian motion sampled at times $1,\dots,n$ (in [**?**]). We work under the assumptions of any of Theorems 2.1, 2.3, 2.6. In particular, our results are valid for a Brownian motion sampled at times $1,\dots,n$ and can be easily adapted to a Brownian motion sampled at the times of a Poisson process since the jumps in this model are exchangeable. However, we should stress that the simple random walk on $\mathbb{Z}^d$ is not covered by our results. This model will be studied separately in Section 5.3.

The next theorem shows that the absorption probabilities $\mathbb{P}[0 \in H_{n,d(n)}]$ exhibit a phase transition at $d(n) \approx \frac{1}{2}\log n$ for symmetric random walks and $d(n) \approx \log n$ for random walk bridges.

Theorem 5.2. *Let the dimension $d = d(n)$ be such that for some $a \in \mathbb{R}$,*

$$d(n) = u\log n + a\sqrt{u \log n} + o(\sqrt{\log n}),$$

as $n \to \infty$, where $u = 1$ in the A_{n-1} case and $u = \frac{1}{2}$ in the B_n and D_n cases. Then under the assumptions of any of Theorems 2.1, 2.3, 2.6,

$$\lim_{n\to\infty} \mathbb{P}[0 \notin H_{n,d(n)}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} \mathrm{e}^{-t^2/2}dt. \tag{39}$$

Proof. Consider the case A_{n-1} first. By Theorem 2.1 (see (9)), we have

$$\mathbb{P}[0 \notin H_{n,d(n)}] = \frac{2}{n!}\left(\begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n \\ d-2 \end{bmatrix} + \dots\right). \tag{40}$$

The classical result of Goncharov (see, e.g., [**?** , Sec. IX.5]) states that the Stirling numbers of the first type satisfy a central limit theorem (CLT) of the form

$$\frac{1}{n!}\sum_{k=1}^{d(n)}\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} \mathrm{e}^{-t^2/2}dt. \tag{41}$$

On the other hand, the Stirling numbers of the first kind are unimodal (in k) being the (signed) coefficients of a hyperplane arrangement, see Remark 3.2 and (30). Combining this fact with (41) and the unimodality of the standard normal density, we see that the mode m_n of the sequence $\left[{n \atop k}\right]$ has the asymptotic

$$m_n = u\log n + o(\sqrt{\log n}). \tag{42}$$

Hence if $a < 0$, then $\left[{n \atop k}\right]$ are monotonously increasing for $k < d(n)$, and thus

$$\frac{1}{n!}\sum_{k=1}^{d(n)}\begin{bmatrix} n \\ k \end{bmatrix} \leq \mathbb{P}[0 \notin H_{n,d(n)}] \leq \frac{1}{n!}\sum_{k=1}^{d(n)+1}\begin{bmatrix} n \\ k \end{bmatrix}. \tag{43}$$

By the Goncharov CLT (41), this implies the required (39) for $a < 0$. The proof for $a > 0$ follows analogously by considering the complement probabilities $\mathbb{P}[0 \in H_{n,d(n)}]$ and using the monotonicity of $\left[{n \atop k}\right]$ for $k > d(n)$. Finally, the case $a = 0$ is easily treated using the continuity of the standard normal density. This concludes the proof of Theorem 5.2 in the case A_{n-1}.

We now turn to the case B_n. We will use the powerful theory of mod-Poisson convergence developed in [**? ?**]. Once established, the mod-Poisson convergence yields many limit theorems besides the CLT.

Let X_n be an integer-valued random variable with the distribution

$$\mathbb{P}[X_n = k] = \frac{1}{2^n n!}B(n,k), \quad k = 0,\dots,n. \tag{44}$$

Note that the probabilities indeed sum up to one by (13).

We claim that X_n satisfies a central limit theorem of the form

$$\frac{X_n - \frac{1}{2}\log n}{\sqrt{\frac{1}{2}\log n}} \xrightarrow[n\to\infty]{d} \mathrm{N}(0,1), \tag{45}$$

where $\mathrm{N}(0,1)$ is the standard normal law. This is analogous to (41).

Denote by $(x)_n = x(x+1)\dots(x+n-1)$ the rising factorial. By the definition of $B(n,k)$ given in (13), the moment generating function of X_n is

$$\mathbb{E}[\mathrm{e}^{zX_n}] = \frac{1}{2^{2n}n!}\cdot\frac{(\mathrm{e}^z)_{2n}}{(\frac{1}{2}\mathrm{e}^z)_n}.$$

Recall that $(x)_n \sim n^x\Gamma(n)/\Gamma(x)$ as $n\to\infty$. This holds locally uniformly in $x \in \mathbb{C}$ as follows from the Euler formula for $\Gamma(x)$ in the form of the infinite product, which converges uniformly on compact sets excluding the poles of the Gamma function. Using this asymptotic and the Stirling formula, we obtain

$$\lim_{n\to\infty}\frac{\mathbb{E}[\mathrm{e}^{zX_n}]}{\mathrm{e}^{(\frac{1}{2}\log n)(\mathrm{e}^z-1)}} = \frac{2^{\mathrm{e}^z}\Gamma(\frac{1}{2}\mathrm{e}^z)}{2\sqrt{\pi}\Gamma(\mathrm{e}^z)}. \tag{46}$$

As explained above, this convergence is locally uniform in $\mathbb{C}$.

The denominator $\mathrm{e}^{\frac{1}{2}\log n(\mathrm{e}^z-1)}$ on the left-hand side is the moment generating function of a Poisson distribution with parameter $\frac{1}{2}\log n$. Hence (46) states that

X_n converges in the mod-Poisson sense. This implies the CLT (45) by the general theory of mod-Poisson convergence, see [**?** , Proposition 2.4, Part (2)].

The rest of the proof is completely analogous to the case A_{n-1}. The probability mass function of X_n is unimodal by Remark 3.2 and (31), and its mode satisfies (42) by the established CLT (45). By Theorem 2.3 (see (15)) combined with the definition of X_n given in (44) and the unimodality of $B(n, k)$, we see that if $a < 0$, then

$$\mathbb{P}[X_n \le d(n) - 1] \le \mathbb{P}[0 \notin H_{n,d(n)}] \le \mathbb{P}[X_n \le d(n)]. \tag{47}$$

This is analogous to (43) and proves the required (39) in the B_n case for $a < 0$. The case $a \ge 0$ is covered as above.

The D_n case is completely analogous to the B_n case and yields the same asymptotic. □

Corollary 5.3. *It holds*

$$\lim_{n\to\infty} \mathbb{P}[0 \in H_{n,d(n)}] = \begin{cases} 0, & \text{if } \limsup_{n\to\infty} \frac{d(n)}{u \log n} < 1, \\ 1, & \text{if } \liminf_{n\to\infty} \frac{d(n)}{u \log n} > 1, \\ 1/2, & \text{if } d(n) \sim u \log n. \end{cases}$$

Proof. The third case follows by taking $a = 0$ in Theorem 5.2. The other two cases also follow from Theorem 5.2 since the probability $\mathbb{P}[0 \notin H_{n,d}]$ is decreasing in d if we restrict d to be even or odd; see Theorems 2.1, 2.3, 2.6. □

In the next theorem we give the exact asymptotic for the absorption probabilities.

Theorem 5.4. *Suppose that for some $x > 0$, $d(n) \sim ux \log n$ as $n \to \infty$, where $u = 1$ in the A_{n-1} case and $u = \frac{1}{2}$ in the B_n and D_n cases. Then under the assumptions of any of Theorems 2.1, 2.3, 2.6,*

$$\mathbb{P}[0 \in H_{n,d(n)}] \sim \frac{n^{-u(x\log x - x + 1)}}{\sqrt{2\pi x u \log n}} \frac{Lx}{x-1}, \quad \text{if } x < 1,$$
$$\mathbb{P}[0 \notin H_{n,d(n)}] \sim \frac{n^{-u(x\log x - x + 1)}}{\sqrt{2\pi x u \log n}} \frac{Lx}{x-1}, \quad \text{if } x > 1,$$

as $n \to \infty$, where $L = \frac{1}{\Gamma(x)}$ in the A_{n-1} case and $L = \frac{2^x \Gamma(x/2)}{2\sqrt{\pi}\Gamma(x)}$ in the B_n and D_n cases.

Note that the function $x \log x - x + 1$ is the large deviations function of a standard Poisson distribution.

Proof. For the A_{n-1} case, the second statement follows by (43) and Example 3.8 in [**?**]. For the B_n and D_n cases, this holds by (46), (47), and the general theory of mod-Poisson convergence [**?** , Theorem 3.3] using the fact that the limit function on the right-hand side of (46) is entire analytic and non-vanishing for real z.

The first statement of the theorem is proved analogously. □

5.3. **Large deviations for simple random walks.** The results of the previous section require existence of density of increments and do not cover simple random walks and bridges on $\mathbb{Z}^d$ (considered by **?**] and **?**]). Without the absolute continuity assumption, we can prove only a lower bound for the absorption probability. The following result states that convex hull of a simple random walk on $\mathbb{Z}^d$ absorbs the origin with a high probability provided that $n > e^{(2+\varepsilon)d}$. It strengthens the results of [? ?], which give similar estimates under the assumptions $n > \mathrm{e}^{c_2 d \log d}$ and $n > \mathrm{e}^{c_2 d}$, respectively, with c_2 sufficiently large. Our theorem shows that one can take $c_2 = 2 + \varepsilon$ with any $\varepsilon > 0$.

Theorem 5.5. *For every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) \in (0,1)$ and $C > 0$ such that a simple random walk S_k on $\mathbb{Z}^d$ of length $n > \mathrm{e}^{(2+\varepsilon)d}$ satisfies*

$$\mathbb{P}[0 \in \mathrm{Conv}(S_1, \dots, S_n)] \geq 1 - Cn^{-\delta}.$$

Proof. For a random walk with a density, by Theorem 5.4 we have $\mathbb{P}[0 \notin H_{n,d}] \leq Cn^{-\delta}$ for some $\delta \in (0,1)$ since $d < \frac{1}{2+\varepsilon} \log n$ and $x \log x - x + 1 < 1$ for $x \in (0,1)$. Then the claim follows by Proposition 2.5 and Theorem 2.3. □

Remark 5.6. Results similar to Proposition 2.5 and Theorem 5.5 can be obtained for a simple random walk bridge. The condition on n in Theorem 5.5 should be changed to $n > \mathrm{e}^{(1+\varepsilon)d}$.

It is natural to assume that Theorem 5.5 is sharp in the following sense:

Conjecture 5.7. For every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) \in (0,1)$ and $C > 0$ such that a simple random walk S_k on $\mathbb{Z}^d$ of length $n < \mathrm{e}^{(2-\varepsilon)d}$ satisfies

$$\mathbb{P}[0 \notin \mathrm{Conv}(S_1, \dots, S_n)] \geq 1 - Cn^{-\delta}.$$

6. Proofs: Random convex hulls and Weyl chambers

In this section we prove our main probabilistic results Theorems 2.1, 2.3, 2.6, and 2.8 on exact absorption probabilities under existence of densities of increments. Here we also prove Proposition 2.5, which estimates absorption probabilities for general random walks. The proofs are based on the same general approach but particular details are rather different. For the reason of notation, we present together the proofs of Theorems 2.1, 2.3, and 2.6, and prove the other two results separately.

6.1. Reflection groups: Proofs of Theorems 2.1, 2.3, and 2.6.

We identify the elements of the Coxeter groups A_{n-1}, B_n, and D_n with orthogonal transformations $g : \mathbb{R}^n \to \mathbb{R}^n$. The Weyl chambers $\mathcal{C}(A_{n-1}), \mathcal{C}(B_n)$, and $\mathcal{C}(D_n)$ are respective fundamental domains for the actions of these groups. The action of A_{n-1} leaves the hyperplane L given by the equation $x_1 + \ldots + x_n = 0$ invariant, hence $\mathcal{C}(A_{n-1}) \cap L$ is a fundamental domain for the action of A_{n-1} restricted on L.

Let $\xi_1, \dots, \xi_n$ be random vectors with values in $\mathbb{R}^d$ (written as columns), and let A be the random $d \times n$-matrix with columns $\xi_1, \dots, \xi_n$. We regard A as a random linear operator $A : \mathbb{R}^n \to \mathbb{R}^d$. The kernel $\mathrm{Ker}\, A$ of this operator is a random subspace of $\mathbb{R}^n$. Recall that $H_{n,d}$ is the common notation for the convex hulls of the partial sums of ξ_i considered in Theorems 2.1, 2.3, 2.6.

Lemma 6.1. *Let G be any of the Coxeter groups A_{n-1}, B_n, D_n, and let C denote one of the respective domains $\mathcal{C}(A_{n-1}) \cap L, \mathcal{C}(B_n)$, $\mathcal{C}(D_n)$. Under the respective invariance assumptions* (5) *and $S_n = 0$ a.s.,* (11), (16), *for every $g \in G$,*

$$\mathbb{P}[0 \in H_{n,d}] = \mathbb{P}[\mathrm{Ker}\, A \cap (g\bar{C}) \neq \{0\}].$$

Proof. We are interested in the probability of the event

$$E := \{\mathrm{Ker}\, A \cap (g\bar{C}) \neq \{0\}\} = \{\mathrm{Ker}(Ag) \cap \bar{C} \neq \{0\}\}.$$

Denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{R}^n$, and recall that

$$S_1 = \xi_1, \;\; S_2 = \xi_1 + \xi_2, \;\; \dots, \;\; S_n = \xi_1 + \dots + \xi_n.$$

Type A_{n-1}. The elements of A_{n-1} are the orthogonal transformations of the form $g_\sigma : \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma \in \mathrm{Sym}(n)$ is a permutation on n elements, and

$$g_\sigma(e_k) = e_{\sigma(k)}, \quad k = 1, \dots, n.$$

It is easy to check that the columns of the matrix Ag are $\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}$. Hence,

$$E = \{\exists x \in \bar{C} \backslash \{0\} \colon \xi_{\sigma(1)} x_1 + \dots + \xi_{\sigma(n)} x_n = 0\}.$$

There is a bijective correspondence between $x = (x_1, \dots, x_n) \in \bar{C}\backslash\{0\}$ and $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}_{\geq 0} \backslash \{0\}$ given by

$$y_1 = x_2 - x_1, \;\; \dots, \;\; y_{n-1} = x_n - x_{n-1}$$

or, equivalently,

$$x_1 = y_0, \;\; x_2 = y_0 + y_1, \;\; \dots, \;\; x_n = y_0 + \dots + y_{n-1},$$

where $y_0 \in \mathbb{R}$ is chosen to fulfil the condition $x_1 + \dots + x_n = 0$. So the event E occurs if and only if for some $y_0, \dots, y_{n-1}$ with the restrictions above,

$$\xi_{\sigma(1)} y_0 + \xi_{\sigma(2)}(y_0 + y_1) + \dots + \xi_{\sigma(n)}(y_0 + \dots + y_{n-1}) = 0.$$

Rearranging the terms, we can write this as

$$y_0(\xi_{\sigma(1)} + \dots + \xi_{\sigma(n)}) + y_1(\xi_{\sigma(2)} + \dots + \xi_{\sigma(n)}) + \dots + y_{n-1}\xi_{\sigma(n)} = 0.$$

Using the assumption $\xi_1 + \dots + \xi_n = 0$ a.s., y_0 disappears and we can transform the above as

$$y_1 \xi_{\sigma(1)} + y_2(\xi_{\sigma(1)} + \xi_{\sigma(2)}) + \dots + y_{n-1}(\xi_{\sigma(1)} + \dots + \xi_{\sigma(n-1)}) = 0.$$

The exchangeability assumption (5) on the distribution of $(\xi_1, \dots, \xi_n)$ implies that

$$(\xi_{\sigma(1)}, \xi_{\sigma(1)} + \xi_{\sigma(2)}, \dots, \xi_{\sigma(1)} + \dots + \xi_{\sigma(n-1)}) \stackrel{d}{=} (S_1, S_2, \dots, S_{n-1}).$$

Hence we obtain the required relation

$$\begin{aligned}\mathbb{P}[E] &= \mathbb{P}[\exists (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}_{\geq 0} \backslash \{0\} \colon y_1 S_1 + y_2 S_2 + \dots + y_{n-1} S_{n-1} = 0] \\ &= \mathbb{P}[0 \in \mathrm{Conv}(S_1, S_2, \dots, S_{n-1})].\end{aligned}$$

Type B_n. The elements of B_n are the orthogonal transformations of the form $g_{\sigma,\varepsilon} : \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma \in \mathrm{Sym}(n)$ is a permutation on n elements, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$, and

$$g_{\sigma,\varepsilon}(e_k) = \varepsilon_k e_{\sigma(k)}, \quad k = 1, \dots, n.$$

Note that the columns of the matrix Ag are $\varepsilon_1 \xi_{\sigma(1)}, \dots, \varepsilon_n \xi_{\sigma(n)}$, as one can see by computing $(Ag)e_1, \dots (Ag)e_n$. So we can write the event E in the form

$$E = \{\exists x \in \bar{C}\backslash\{0\} \colon \varepsilon_1 \xi_{\sigma(1)} x_1 + \dots + \varepsilon_n \xi_{\sigma(n)} x_n = 0\}. \tag{48}$$

There is a bijection between $x = (x_1, \dots, x_n) \in \bar{C}$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n_{\geq 0} \backslash \{0\}$ given by

$$x_1 = y_1, \;\; x_2 = y_1 + y_2, \;\; \dots, \;\; x_n = y_1 + \dots + y_n.$$

Hence we can write the condition for the event E as

$$\varepsilon_1 \xi_{\sigma(1)} y_1 + \varepsilon_2 \xi_{\sigma(2)}(y_1 + y_2) + \dots + \varepsilon_n \xi_{\sigma(n)}(y_1 + \dots + y_n) = 0,$$

or equivalently,

$$y_1(\varepsilon_1\xi_{\sigma(1)} + \ldots + \varepsilon_n\xi_{\sigma(n)}) + y_2(\varepsilon_2\xi_{\sigma(2)} + \ldots + \varepsilon_n\xi_{\sigma(n)}) + \ldots + y_n\varepsilon_n\xi_{\sigma(n)} = 0.$$

The symmetric exchangeability assumption (11) on the distribution of $(\xi_1, \ldots, \xi_n)$ implies that

$$(\varepsilon_1\xi_{\sigma(1)} + \ldots + \varepsilon_n\xi_{\sigma(n)}, \varepsilon_2\xi_{\sigma(2)} + \ldots + \varepsilon_n\xi_{\sigma(n)}, \ldots, \varepsilon_n\xi_{\sigma(n)}) \stackrel{d}{=} (S_n, S_{n-1}, \ldots, S_1),$$

hence we obtain the required

$$\begin{aligned}\mathbb{P}[E] &= \mathbb{P}[\exists y \in \mathbb{R}^n_{\geq 0}\backslash\{0\} \colon y_1S_n + y_2S_{n-1} + \ldots + y_nS_1 = 0] \\ &= \mathbb{P}[0 \in \operatorname{Conv}(S_1, S_2, \ldots, S_n)],\end{aligned}$$

Type D_n. This case is very similar to the B_n case as the elements of $D_n \subset B_n$ are the orthogonal transformations $g_{\sigma,\varepsilon}$ such that $\varepsilon_1 \ldots \varepsilon_n = 1$. There is a bijective correspondence between $x = (x_1, \ldots, x_n) \in \bar{C}\backslash\{0\}$ and $y = (y_1, \ldots, y_n) \in (\mathbb{R} \times \mathbb{R}^{n-1}_{\geq 0})\backslash\{0\}$ given by

$$x_1 = y_1, \;\; x_2 = |y_1| + y_2, \;\; \ldots, \;\; x_n = |y_1| + y_2 + \ldots + y_n.$$

So we can write the condition defining the event E in (48) in the form

$$\varepsilon_1\xi_{\sigma(1)}y_1 + \varepsilon_2\xi_{\sigma(2)}(|y_1| + y_2) + \ldots + \varepsilon_n\xi_{\sigma(n)}(|y_1| + y_2 + \ldots + y_n) = 0.$$

Rearranging the terms, we obtain the equivalent representation

$$\begin{aligned}&|y_1|((\operatorname{sgn} y_1)\varepsilon_1\xi_{\sigma(1)} + \varepsilon_2\xi_{\sigma(2)} + \ldots + \varepsilon_n\xi_{\sigma(n)}) \\ &\qquad\qquad + y_2(\varepsilon_2\xi_{\sigma(2)} + \ldots + \varepsilon_n\xi_{\sigma(n)}) + \ldots + y_n\varepsilon_n\xi_{\sigma(n)} = 0.\end{aligned}$$

Recall that $S_n^* = \xi_1 + \ldots + \xi_{n-1} - \xi_n$. The invariance assumption (16) on the distribution of $(\xi_1, \ldots, \xi_n)$ implies that

$$\begin{aligned}&(\varepsilon_1\xi_{\sigma(1)} + \ldots + \varepsilon_n\xi_{\sigma(n)}, -\varepsilon_1\xi_{\sigma(1)} + \ldots + \varepsilon_n\xi_{\sigma(n)}, \\ &\qquad\qquad \varepsilon_2\xi_{\sigma(2)} + \ldots + \varepsilon_n\xi_{\sigma(n)}, \ldots, \varepsilon_n\xi_{\sigma(n)}) \stackrel{d}{=} (S_n, S_n^*, S_{n-1}, \ldots, S_1).\end{aligned}$$

So we obtain that

$$\begin{aligned}\mathbb{P}[E] = \mathbb{P}[\exists y \in (\mathbb{R} \times \mathbb{R}^{n-1}_{\geq 0})\backslash\{0\} \colon & y_1S_n + y_2S_{n-1} + \ldots + y_nS_1 = 0 \\ & \text{or } y_1S_n^* + y_2S_{n-1} + \ldots + y_nS_1 = 0],\end{aligned}$$

hence

$$\mathbb{P}[E] = \mathbb{P}[0 \in \operatorname{Conv}(S_1, \ldots, S_{n-1}, S_n) \text{ or } 0 \in \operatorname{Conv}(S_1, \ldots, S_{n-1}, S_n^*)].$$

To complete the proof of Lemma 6.1, we need to argue that

$$\operatorname{Conv}(S_1, \ldots, S_{n-1}, S_n) \cup \operatorname{Conv}(S_1, \ldots, S_{n-1}, S_n^*) = \operatorname{Conv}(S_1, \ldots, S_n, S_n^*). \tag{49}$$

The left-hand side is a subset of the right-hand side by definition of the convex hull. To see the converse inclusion, consider any convex combination

$$x = \alpha_1S_1 + \ldots + \alpha_{n-2}S_{n-2} + \alpha_{n-1}S_{n-1} + \alpha_nS_n + \alpha_n^*S_n^*.$$

Assuming that $\alpha_n \geq \alpha_n^*$ and using the identity $S_n^* = 2S_{n-1} - S_n$, we obtain

$$x = \alpha_1S_1 + \ldots + \alpha_{n-2}S_{n-2} + (\alpha_{n-1} + 2\alpha_n^*)S_{n-1} + (\alpha_n - \alpha_n^*)S_n,$$

which represents x as a convex combination of $S_1, \ldots, S_{n-1}, S_n$. Similarly, if $\alpha_n \leq \alpha_n^*$, we obtain representation of x as a convex combination of $S_1, \ldots, S_{n-1}, S_n^*$. □

We are now ready to conclude the proofs of Theorems 2.1, 2.3, and 2.6. Applying Lemma 6.1 to all $g \in G$ and taking the arithmetic mean, we obtain

$$\mathbb{P}[0 \in H_{n,d}] = \frac{1}{\#G} \sum_{g \in G} \mathbb{P}[\operatorname{Ker} A \cap (g\bar{C}) \neq \{0\}] = \frac{\mathbb{E}N}{\#G}, \tag{50}$$

where the random variable

$$N := \sum_{g \in G} \mathbb{1}_{\{\operatorname{Ker} A \cap (g\bar{C}) \neq \{0\}\}} \tag{51}$$

counts the number of chambers of the form $g\bar{C}$, $g \in G$, intersected by $\operatorname{Ker} A$.

In the A_{n-1} case, N equals the number of closed Weyl chambers of type A_{n-1} (non-trivially) intersected by $\operatorname{Ker} A \cap L$, as readily seen from the equation $\operatorname{Ker} A \cap (g\bar{C}) = (\operatorname{Ker} \cap L) \cap g(\overline{\mathcal{C}(A_{n-1})})$. In the B_n and D_n cases, N equals the number of closed Weyl chambers of respective type that have a non-trivial intersection with $\operatorname{Ker} A$.

We claim that with probability one,
1) under the assumptions of Theorem 2.1, $\operatorname{Ker} A \cap L$ has codimension $\min(d+1, n)$ in $\mathbb{R}^n$ and is in general position w.r.t. the arrangement $\mathcal{A}(A_{n-1})$;
2) under the assumptions of either Theorems 2.3 or 2.6, $\operatorname{Ker} A$ has codimension $\min(d, n)$ in $\mathbb{R}^n$ and is in general position w.r.t. $\mathcal{A}(B_n)$ and $\mathcal{A}(D_n)$.

Our claim implies that by Lemma 3.5, the value of N does not change a.s. if we replace $\bar{C}$ by C in the definition of N. Hence N is a.s. a constant of the value given by Theorem 3.4, and then Theorem 2.1, 2.3, and 2.6 follow.

Thus, it only remains to prove the claim.

Type A_{n-1}. First of all we note that is essential to consider $\operatorname{Ker} A \cap L$ because $\operatorname{Ker} A$ is not in general position w.r.t. $\mathcal{A}(A_{n-1})$. Indeed, the condition $\xi_1 + \ldots + \xi_n = 0$ a.s. implies that $L^\perp \subset \operatorname{Ker} A$ a.s. while $L^\perp$ is the common intersection of all the hyperplanes from $\mathcal{A}(A_{n-1})$.

Let us use $(x_1, \ldots, x_{n-1})$ as coordinates in L. Then $x_n = -(x_1 + \ldots + x_{n-1})$ and the linear subspace $(\operatorname{Ker} A) \cap L$ is given by the equation

$$\xi_1 x_1 + \ldots + \xi_{n-1} x_{n-1} + (\xi_1 + \ldots + \xi_{n-1})(x_1 + \ldots + x_{n-1}) = 0,$$

or, if we recall the notation $S_{n-1} = \xi_1 + \ldots + \xi_{n-1}$, by

$$(S_{n-1} + \xi_1)x_1 + \ldots + (S_{n-1} + \xi_{n-1})x_{n-1} = 0. \tag{52}$$

This is a system of d linear equations in $x_1, \ldots, x_{n-1}$. The subspace of its solutions a.s. has codimension $\min(d, n-1)$ in $\mathbb{R}^{n-1}$ iff the rank of the system is full a.s. The later holds true since the random vectors $S_{n-1} + \xi_1, \ldots, S_{n-1} + \xi_{n-1}$ have a joint Lebesgue density on $\mathbb{R}^{d(n-1)}$. This is because $(\xi_1, \ldots, \xi_{n-1})$ has a Lebesgue density on $\mathbb{R}^{d(n-1)}$ by the assumption of Theorem 2.1, and the transformation $(\xi_1, \ldots, \xi_{n-1}) \mapsto (S_{n-1} + \xi_1, \ldots, S_{n-1} + \xi_{n-1})$ is a non-singular linear map from $\mathbb{R}^{d(n-1)}$ to itself. Thus $(\operatorname{Ker} A) \cap L$ has codimension $\min(d+1, n)$ in $\mathbb{R}^n$ a.s.

Further, for any linear subspace $H \subset \mathbb{R}^n$, the set of all tuples $(S_{n-1}+\xi_1, \ldots, S_{n-1}+\xi_{n-1})$ such that $(\operatorname{Ker} A) \cap L$ is not in general position w.r.t. to $H \cap L$ is a proper algebraic submanifold of $\mathbb{R}^{d(n-1)}$ and, as such, has probability 0. This is because the general position condition is violated iff the system of linear equations obtained from (52) by adding any $n - 1 - \dim(H \cap L)$ linear equations defining $H \cap L$ is not of a full rank. Hence $(\operatorname{Ker} A) \cap L$ is in general position w.r.t. to any arrangement in $\mathbb{R}^n$ and in particular, with $\mathcal{A}(A_{n-1})$.

Types B_n and D_n. Under the assumptions of either Theorem 2.3 or 2.6, the random vectors $\xi_1,\dots,\xi_n$ have a joint density on $\mathbb{R}^{dn}$. Then the set of $d\times n$-matrices A with a non-full rank $\min(d,n)$ and the set of A such that $\operatorname{Ker} A$ is not in general position w.r.t. any fixed liner subspace $H\subset\mathbb{R}^n$ are proper algebraic submanifolds in $\mathbb{R}^{dn}$ and therefore both have probability zero. Thus with probability one, the linear subspace $\operatorname{Ker} A$ has codimension d in $\mathbb{R}^n$ and is in general position w.r.t. any hyperplane arrangement and in particular, with $\mathcal{A}(B_n)$ and $\mathcal{A}(D_n)$.

6.2. **Direct products of reflection groups: Proof of Theorem 2.8.** The group $G:=B_{n_1}\times\dots\times B_{n_r}$ acts in a natural way on $\mathbb{R}^{n_1}\times\dots\times\mathbb{R}^{n_r}\cong\mathbb{R}^n$, where $n=n_1+\dots+n_r$. The elements of G can be represented as tuples of the form

$$g=(g_{\sigma^{(1)},\varepsilon^{(1)}},\dots,g_{\sigma^{(r)},\varepsilon^{(r)}}),$$

where $\sigma^{(i)}\in\operatorname{Sym}(n_i)$ are permutations, $\varepsilon^{(i)}:=(\varepsilon^{(i)}_1,\dots,\varepsilon^{(i)}_{n_i})\in\{-1,+1\}^{n_i}$, and $g_{\sigma^{(i)},\varepsilon^{(i)}}$ are the orthogonal transformations of $\mathbb{R}^{n_i}$ defined by

$$g_{\sigma^{(i)},\varepsilon^{(i)}}(e^{(i)}_k)=\varepsilon^{(i)}_k e^{(i)}_{\sigma^{(i)}(k)},\quad k=1,\dots,n_i,$$

with $e^{(i)}_1,\dots,e^{(i)}_{n_i}$ being the standard basis of $\mathbb{R}^{n_i}$. The total number of elements in the group G is $2^{n_1}n_1!\dots2^{n_r}n_r!$. Consider the convex cones

$$C^{(i)}:=\{(x^{(i)}_1,\dots,x^{(i)}_{n_i})\in\mathbb{R}^{n_i}\colon 0<x^{(i)}_1<\dots<x^{(i)}_{n_i}\}\subset\mathbb{R}^{n_i}$$

and their direct product

$$C:=C^{(1)}\times\dots\times C^{(r)}\subset\mathbb{R}^{n_1}\times\dots\times\mathbb{R}^{n_r}\equiv\mathbb{R}^n,$$

which is a fundamental domain for the action of G on $\mathbb{R}^n$.

Let A be the $d\times n$ matrix with the columns $\xi^{(1)}_1,\dots,\xi^{(1)}_{n_1},\dots,\xi^{(r)}_1,\dots,\xi^{(r)}_{n_r}$. Recall that $S^{(i)}_k=\xi^{(i)}_1+\dots+\xi^{(i)}_k$, $1\le i\le r$, $1\le k\le n_i$, are the partial sums. The following result is completely analogous to Lemma 6.1.

Lemma 6.2. *Under the assumptions of Theorem 2.8, for every $g\in B_{n_1}\times\dots\times B_{n_r}$,*

$$\mathbb{P}[0\in\operatorname{Conv}(S^{(1)}_1,\dots,S^{(1)}_{n_1},\dots,S^{(r)}_1,\dots,S^{(r)}_{n_r})]=\mathbb{P}[\operatorname{Ker}A\cap(g\bar C)\neq\{0\}].$$

Proof. We are interested in the probability of the event

$$E:=\{\operatorname{Ker}A\cap(g\bar C)\neq\{0\}\}=\{\operatorname{Ker}(Ag)\cap\bar C\neq\{0\}\}.$$

The columns of the matrix Ag are

$$\varepsilon^{(1)}_1\xi^{(1)}_{\sigma^{(1)}(1)},\dots,\varepsilon^{(1)}_{n_1}\xi^{(1)}_{\sigma^{(1)}(n_1)},\dots,\varepsilon^{(r)}_1\xi^{(r)}_{\sigma^{(r)}(1)},\dots,\varepsilon^{(r)}_{n_r}\xi^{(r)}_{\sigma^{(r)}(n_r)},$$

as one can easily check by computing the action of Ag on the standard basis of $\mathbb{R}^n$. So we can write the event E in the form

$$E=\Big\{\exists(x^{(1)},\dots,x^{(r)})\in(\bar C^{(1)}\times\dots\times\bar C^{(r)})\backslash\{0\}\colon \sum_{i=1}^r\Big(\varepsilon^{(i)}_1\xi_{\sigma^{(i)}(1)}x^{(i)}_1+\dots+\varepsilon^{(i)}_n\xi_{\sigma^{(i)}(n_i)}x^{(i)}_{n_i}\Big)=0\Big\}. \tag{53}$$

There are bijective correspondences between $x^{(i)}=(x^{(i)}_1,\dots,x^{(i)}_{n_i})\in\bar C^{(i)}$ and $y^{(i)}=(y^{(i)}_1,\dots,y^{(i)}_{n_i})\in\mathbb{R}^{n_i}_{\ge0}$ given by

$$x^{(i)}_1=y^{(i)}_1,\ \ x^{(i)}_2=y^{(i)}_1+y^{(i)}_2,\ \ \dots,\ \ x^{(i)}_{n_i}=y^{(i)}_1+\dots+y^{(i)}_{n_i}.$$

Applying to each term in (53) the same transformations as in the proof of Lemma 6.1 for the B_n case, we write the event E as follows:

$$E = \Big\{\exists(y^{(1)},\dots,y^{(r)}) \in (\mathbb{R}^{n_1}_{\geq 0}\times\dots\times\mathbb{R}^{n_r}_{\geq 0})\backslash\{0\}\colon \\ \sum_{i=1}^{r}\sum_{k=1}^{n_i} y_k^{(i)}\left(\varepsilon_k^{(i)}\xi_{\sigma^{(i)}(k)}+\dots+\varepsilon_{n_i}^{(i)}\xi_{\sigma^{(i)}(n_i)}\right)=0\Big\}.$$

Since the invariance assumption (20) implies the distributional equality

$$\left\{\varepsilon_k^{(i)}\xi_{\sigma^{(i)}(k)}+\dots+\varepsilon_{n_i}^{(i)}\xi_{\sigma^{(i)}(n_i)}\right\}_{\substack{i=1,\dots,r\\k=1,\dots,n_i}} \stackrel{d}{=} \left\{S^{(i)}_{n_i-k+1}\right\}_{\substack{i=1,\dots,r\\k=1,\dots,n_i}},$$

we obtain that

$$\mathbb{P}[E] = \mathbb{P}\Big[\exists(y_1^{(1)},\dots,y_{n_1}^{(1)},\dots,y_1^{(r)},\dots,y_{n_r}^{(r)})\in\mathbb{R}^n_{\geq 0}\backslash\{0\}\colon \\ \sum_{i=1}^{r}\left(y_1^{(i)}S_{n_i}^{(i)}+y_2^{(i)}S_{n_i-1}^{(i)}+\dots+y_{n_i}^{(i)}S_1^{(i)}\right)=0\Big].$$

The term on the right-hand side is the probability that the joint convex hull of $S_k^{(i)}$, $1\leq k\leq n_i$, $1\leq i\leq r$, contains 0. This proves the lemma. □

We complete the proof of Theorem 2.8 proceeding exactly as in the previous section for the B_n case. Equations (50) and (51) remain valid with $G = B_{n_1}\times\dots\times B_{n_r}$ and $H_{n,d} = \mathrm{Conv}(S_1^{(1)},\dots,S_{n_1}^{(1)},\dots,S_1^{(r)},\dots,S_{n_r}^{(r)})$. Since the random vectors $\xi_1^{(1)},\dots,\xi_{n_1}^{(1)},\dots,\ \xi_1^{(r)},\dots,\xi_{n_r}^{(r)}$ have a joint density on $\mathbb{R}^{nd}$, with probability one, the linear subspace $\mathrm{Ker}\,A$ has codimension d in $\mathbb{R}^n$ and is in general position w.r.t. any hyperplane arrangement.

It is clear that the arrangement $\mathcal{A}$ in $\mathbb{R}^n$ that has regions $\mathcal{R}(\mathcal{A}) = \{gC\}_{g\in G}$ is the product arrangement

$$\mathcal{A}(B_{n_1})\times\dots\times\mathcal{A}(B_{n_r}) := \\ \{H_{i_1}\times\mathbb{R}^{n-n_1}\}_{H\in\mathcal{A}(B_{n_1})}\cup\{\mathbb{R}^{n_1}\times H_{i_2}\times\mathbb{R}^{n-i_1-i_2}\}_{H\in\mathcal{A}(B_{n_2})}\cup\dots\cup\{\mathbb{R}^{n-n_r}\times H\}_{H\in\mathcal{A}(B_{n_1})}.$$

The characteristic polynomial of such arrangement is the product of the individual characteristic polynomials (Lemma 2.50 on p. 43 in [**?**]), hence

$$(-1)^n\chi_{\mathcal{A}(B_{n_1})\times\dots\times\mathcal{A}(B_{n_1})}(-t) = \prod_{i=1}^{r}((t+1)(t+3)\dots(t+2n_i-1)) = \sum_{k=0}^{n}P(k)t^k.$$

By Theorem 3.3, we have

$$N = 2(P(d+1)+P(d+3)+\dots) \quad \text{a.s.},$$

Combining this equation with (50) completes the proof of Theorem 2.8.

6.3. **Random walks with no density: Proof of Proposition 2.5.** Recall that $(\xi_1',\dots,\xi_n')$ is a symmetrically exchangeable tuple of random vectors in $\mathbb{R}^d$ and $H_{n,d}' = \mathrm{Conv}(S_1',\dots,S_n')$ is the convex hull of the partial sums $S_k' = \xi_1'+\dots+\xi_k'$. Let A' be the $d\times n$-matrix with columns $\xi_1',\dots,\xi_n'$. By Lemma 6.1,

$$\mathbb{P}[0\in H_{n,d}'] = \frac{\mathbb{E}N'}{2^n n!},$$

where N' is the numbers of closed Weyl chambers of type B_n (non-trivially) intersected by $\operatorname{Ker} A'$:

$$N' = \sum_{g \in B_n} \mathbb{1}_{\{\operatorname{Ker} A' \cap (g\bar{C}) \neq \{0\}\}}.$$

We imposed no absolute continuity assumption on $(\xi_1', \ldots, \xi_n')$ and we cannot claim that $\operatorname{Ker} A'$ is in general position w.r.t. the arrangement $\mathcal{A}(B_n)$. In particular, the random variable N' need not be a constant a.s. Moreover, we do not even known the exact codimension of $\operatorname{Ker} A'$, but we can claim that it is at most $\min(d, n)$. Let $F \subset \operatorname{Ker} A'$ be any random linear subspace of $\mathbb{R}^n$ of a.s. codimension d. For example, we may define it as follows. Put $\kappa = \min(d - \operatorname{codim}(\operatorname{Ker} A'), n)$ and take $F = \bigcap_{i=1}^{\kappa} X_i^{\perp} \cap \operatorname{Ker} A'$, where $X_i, i = 1, \ldots, n$, are i.i.d. random vectors that are uniformly distributed on $\mathbb{S}^{n-1}$ and independent with A'.

Clearly,

$$\mathbb{P}[0 \in H_{n,d}'] \geq \frac{\mathbb{E}\tilde{N}}{2^n n!}, \text{ where } \tilde{N} := \sum_{g \in B_n} \mathbb{1}_{\{F \cap (g\bar{C}) \neq \{0\}\}}.$$

By Lemma 3.5, we have $\tilde{N} \geq N$ a.s. for N defined by (51) with $G = B_n$. Applying (50) completes the proof.

7. Open questions

Our results, except those of Section 5.3, do not apply to the simple random walk on the lattice $\mathbb{Z}^d$. The next problem does not seem trivial even for $d = 2$.

Problem 7.1. Let $S_1, \ldots, S_n$ be a simple random walk on $\mathbb{Z}^d$ starting at the origin. Compute the probability that $\operatorname{Conv}(S_1, \ldots, S_n)$ contains the origin. Compute exactly the conditional probability that $\operatorname{Conv}(S_1, \ldots, S_{n-1})$ contains the origin given that $S_n = 0$.

Problem 7.2. Prove analogues of Theorems 5.2 and 5.4 for the simple random walks (and bridges) on $\mathbb{Z}^d$.

The answer to the next question should be non distribution-free (for $x \neq 0$) and seems to be unknown even in the case of standard normal increments.

Problem 7.3. Let $S_1, \ldots, S_n$ be a random walk in $\mathbb{R}^d$ starting at the origin. Compute the probability that $\operatorname{Conv}(S_1, \ldots, S_n)$ contains a given point $x \in \mathbb{R}^d$.

The same question makes sense for a Brownian motion.

Problem 7.4. Let $\{B(t) \colon t \geq 0\}$ be a standard Brownian motion in $\mathbb{R}^d$ starting at the origin. Compute the probability that $\operatorname{Conv}\{B(t) \colon 0 \leq t \leq 1\}$ contains a given point $x \in \mathbb{R}^d$.

One may also ask whether results similar to those obtained in the present paper exist for the exceptional Coxeter groups or for another group actions like affine Coxeter groups and Coxeter groups in the hyperbolic space.

References

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