ON ONE GENERALIZATION OF THE ELLIPTIC LAW FOR RANDOM MATRICES

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Abstract. We consider the products of $m \ge 2$ independent large real random matrices with independent vectors $(X_{jk}^{(q)}, X_{kj}^{(q)})$ of entries. The entries $X_{jk}^{(q)}, X_{kj}^{(q)}$ are correlated with $\rho = \mathbb{E}\, X_{jk}^{(q)} X_{kj}^{(q)}$. The limit distribution of the empirical spectral distribution of the eigenvalues of such products doesn't depend on ρ and equals to the distribution of mth power of the random variable uniformly distributed on the unit disc.

1. Introduction

Let $m \ge 1$ be a fixed integer and $\mathbf{X}^{(q)} = n^{-1/2}\{X_{jk}^{(q)}\}_{j,k=1}^{n}, q = 1, ..., m$, be independent random matrices with real entries. We suppose that the random variables $X_{j,k}^{(q)}$, $1 \le j, k \le n, q = 1, ..., m$, are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfy the following conditions **(C0)**:

a) random vectors $(X_{jk}^{(q)}, X_{kj}^{(q)})$ are mutually independent for $1 \le j < k \le n$;

b) for any $1 \le j \le k \le n$

$$\mathbb{E}\, X_{jk}^{(q)} = 0 \text{ and } \mathbb{E}(X_{jk}^{(q)})^2 = 1;$$

c) for any $1 \le j < k \le n$

$$\mathbb{E}(X_{jk}^{(q)} X_{kj}^{(q)}) = \rho, |\rho| \le 1;$$

d) diagonal entries and off-diagonal entries are independent.

We say that the random variables $X_{j,k}^{(q)}$, $1 \le j, k \le n, q = 1, ..., m$, satisfy the condition (**UI**) if the squares of $X_{jk}^{(q)}$'s are uniformly integrable , i.e.

$$\max_{q,j,k} \mathbb{E}\, |X_{jk}^{(q)}|^2 \mathbb{I}\{|X_{jk}^{(q)}| > M\} \to 0 \quad \text{as} \quad M \to \infty. \tag{1.1}$$

Here and in what follows $\mathbb{I}\{B\}$ denotes the indicator of the event B.

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The random variables $X_{jk}^{(q)}$ may depend on n, but for simplicity we shall not make this explicit in our notations. Denote by $\lambda_1, ..., \lambda_n$ the eigenvalues of the matrix $\mathbf{W} := \prod_{q=1}^m \mathbf{X}^{(q)}$ and define the empirical spectral measure of the eigenvalues by

$$\mu_n(B) = \frac{1}{n}\#\{1 \le i \le n : \lambda_i \in B\}, \quad B \in \mathcal{B}(\mathbb{C}),$$

where $\mathcal{B}(\mathbb{C})$ is a Borel σ-algebra of $\mathbb{C}$.

We say that the sequence of random probability measures $m_n(\cdot)$ converges weakly in probability to the probability measure $m(\cdot)$ if for all continues and bounded functions $f : \mathbb{C} \to \mathbb{C}$ and all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{C}} f(x) m_n(dz) - \int_{\mathbb{C}} f(x) m(dz)\right| > \varepsilon\right) = 0.$$

We denote a weak convergence by the symbol $\xrightarrow{weak}$.

A fundamental problem in the theory of random matrices is to determine the limiting distribution of μ_n as the size of the random matrix tends to infinity. The following theorem gives the solution of this problem for the matrices which satisfy the conditions (**C0**) and (**UI**).

Theorem 1.1. *Let $m \ge 2$ and $\mathbf{X}^{(q)} = n^{-1/2}\{X_{jk}^{(q)}\}_{j,k=1}^n, q = 1, ..., m$, be independent random matrices such that the random variables $X_{jk}^{(q)}, j, k = 1, ..., n, q = 1, ..., m$, satisfy the conditions* (**C0**) *and* (**UI**). *Assume that $|\rho| < 1$. Then $\mu_n \xrightarrow{weak} \mu$ in probability, and μ has the density g:*

$$g(x, y) = \frac{1}{\pi m (x^2 + y^2)^{\frac{m-1}{m}}} \mathbb{I}\{x^2 + y^2 \le 1\},$$

which doesn't depend on ρ.

Remark. Theorem 1.1 was announced in the talk of F. Götze "Spectral Distribution of Random Matrices and Free Probability", Advanced School and Workshop on Random Matrices and Growth Models, Trieste, Italy. Recently O'Rourke, Renfrew, Soshnikov and Vu, see [15] , proved the result of Theorem 1.1 under additional assumptions on the moments of $X_{jk}^{(q)}$.

Remark. Girko [6] showed that for $m = 1$ under the additional assumptions that the distribution of r.v.'s $X_{jk}^{(1)}$ has a density the limit measure μ has a density of uniform distribution on the ellipse $\mathcal{E} = \{(x, y) : \frac{x^2}{(1-\rho)^2} + \frac{y^2}{(1+\rho)^2} \le 1\}$. This result is called "elliptic law". For Gaussian matrices the elliptic law was proved in [18]. The elliptic law without assumption on the density of distribution of entries X_{jk} was proved by Naumov in [13]. Nguyen and O'Rourke in [14] and Götze, Naumov, Tikhomirov in [7] extended the elliptic law on the case when $X_{jk}^{(1)}$'s have only finite second moment and non-identical distribution.

Remark. For $m = 1$ and $\rho = 0$ we have the circular law, i.e. the limit distribution μ is uniform distribution on the unit disc. The circular law was first proved by Ginibre in [4] for matrices with independent standard complex Gaussian entries. Girko in [5] have considered the general case under assumption that the distributions of entries have bounded densities and the fourth moments of entries are finite. Z. Bai (see [1]) rely on the fruitful Girko's ideas gave a correct proof of the circular law under the same assumptions. Götze and Tikhomirov in [10] have proved the circular law without assumption on the density of entries, but assuming the sub-Gaussian distributions of r.v.'s $X_{jk}^{(1)}$. Later Pan and Zhou in [17] proved the circular law assuming that $\mathbb{E}\,|X_{jk}^{(1)}|^4 < \infty$. Götze and Tikhomirov in [8] proved the circular law assuming the logarithmic second moments ($\mathbb{E}\,|X_{jk}^{(1)}|^2 |\log|X_{jk}^{(1)}||^{\alpha} < \infty$ with some α sufficiently large). And finally Tao and Vu in [19] proved the Circular law for i.i.d. case under the assumption on the second moments only.

Remark. In the case $\rho = 0$ and $X_{jk}^{(q)}$ and $X_{kj}^{(q)}$ are independent for $1 \le j < k \le n$, Theorem 1.1 was proved by Götze and Tikhomirov in [8]. See also the result of O'Rourke and Soshnikov [16].

1.1. **Proof of the elliptic law.** In the following we shall give the proof of Theorem 1.1. We shall use the logarithmic potential approach first suggested for the proof of the circular law by Götze and Tikhomirov in [10]. This approach was developed in many papers (see, for instance [8], [9] and [2]). We define the logarithmic potential of the empirical spectral measure of the matrix $\mathbf{W}$ by the formula

$$U_n(z) = -\int_{\mathbb{C}} \ln|w - z| \mu_n(dw)$$

and will prove that

$$\lim_{n\to\infty} U_n(z) = U(z) := -\int_{\mathbb{C}} \ln|w - z| \mu(dw).$$

Let us denote by $s_1 \ge s_2 \ge ... \ge s_n$ the singular values of $\mathbf{W} - z\mathbf{I}$ and introduce the empirical spectral measure $\nu_n(\cdot, z)$ of squares of singular values. We can rewrite the logarithmic potential of μ_n via the logarithmic moments of measure ν_n by

$$U_{\mu_n}(z) = -\int_{\mathbb{C}} \ln|z - w| \mu_n(dw) = -\frac{1}{n} \ln|\det(\mathbf{W} - z\mathbf{I})|$$
$$= -\frac{1}{2n} \ln\det(\mathbf{W} - z\mathbf{I})^* (\mathbf{W} - z\mathbf{I}) = -\frac{1}{2}\int_0^{\infty} \ln x \nu_n(dx).$$

This allows us to consider the Hermitian matrices $(\mathbf{W} - z\mathbf{I})^*(\mathbf{W} - z\mathbf{I})$ instead of $\mathbf{W}$. To prove Theorem 1.1 we need the following lemma.

Lemma 1.2. *Suppose that for a.a. $z \in \mathbb{C}$ there exists a probability measure ν_z on $[0, \infty)$ such that*
a) $\nu_n \xrightarrow{weak} \nu_z$ as $n \to \infty$ in probability
b) $\ln$ *is uniformly integrable in probability with respect to $\{\nu_n\}_{n\ge1}$.*

Then there exists a probability measure μ such that
a) $\mu_n \xrightarrow{weak} \mu$ *as* $n \to \infty$ *in probability*
b) for a.a. $z \in \mathbb{C}$

$$U_\mu(z) = -\int_0^\infty \ln x \nu_z(dx).$$

Proof. See [2][Lemma 4.3] for the proof. □

Proof of Theorem 1.1. From Lemma 1.2 it follows that to prove Theorem 1.1 it is enough to check conditions a) and b) and show that ν_z determines the logarithmic potential of the measure μ. In Theorem 2.1 we find the limit distribution of singular values of the shifted matrix $\mathbf{W}(z) = \mathbf{W} - z\mathbf{I}$ (Section 2). The solution of this problem is divided into several steps. We make symmetrization of one-sided distribution functions. Then we reduce the problem to the case of truncated random variables. Next we show that the limit of empirical distribution of singular values of product of matrices with truncated random variables is the same as one of the product of matrices with Gaussian entries. Finally, we show that the limit of expected distributions of singular values of matrices with Gaussian entries exists and its Stieltjes transform $s(z)$ satisfies the following system of equations

$$\begin{aligned}
&1 + ws(\alpha, z) + (-1)^{m+1} w^m s(\alpha, z)^{m+1} = 0, \\
&(w - \alpha)^2 + (w - \alpha) - 4|z|^2 s(\alpha, z) = 0.
\end{aligned}$$

From the paper [9] we know that the measure with the Stieltjes transform $s(z)$ which satisfies this system of equations determines the logarithmic potential of the measure μ.

In Section 3, Lemma 3.9 we show that $\ln(\cdot)$ is uniformly integrable in probability with respect to $\{\nu_n\}_{n \ge 1}$. □

By C (with an index or without it) we shall denote generic absolute constants, whereas $C(\,\cdot\,,\,\cdot\,)$ will denote positive constants depending on arguments. For any matrix $\mathbf{A}$ we shall denote by $\|\mathbf{A}\|_2$ the Frobenius norm of matrix $\mathbf{A}$ ($\|\mathbf{A}\|_2^2 = \operatorname{Tr} \mathbf{A}\mathbf{A}^*$) and by $\|\mathbf{A}\|$ we shall denote the operator norm of matrix $\mathbf{A}$ ($\|\mathbf{A}\| = \sup_{\mathbf{x}:\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|$). Here and in the what follows $\mathbf{A}^*$ denotes the adjoined (transposed and complex conjugate) matrix $\mathbf{A}$

2. The limit distribution for singular values distribution of shifted matrices

In this Section we prove that there exists the limit distribution for the empirical spectral distribution of the matrices $\mathbf{W} - z\mathbf{I}$. Let $s_1 \ge \ldots \ge s_n$ denote the singular values of the matrix $\mathbf{W} - z\mathbf{I}$. By $\mathcal{G}_n(x, z)$ we denote the empirical spectral distribution function of the matrix $(\mathbf{W} - z\mathbf{I})(\mathbf{W} - z\mathbf{I})^*$ (the distribution function of the uniform distribution on the squared singular values of the matrix $\mathbf{W} - z\mathbf{I}$). This distribution function corresponds to the measure $\nu_n(\cdot, z)$ introduced in the previous section. Let $G_n(x, z) := \mathbb{E}\,\mathcal{G}_n(x, z)$.

We say the entries $X^{(q)}_{j,k}$, $1 \le j, k \le n, q = 1, ..., m$, of the matrices $\mathbf{X}^{(q)}$ satisfy Lindeberg's condition ($\mathbf{L}$) if

$$\text{for all } \tau > 0 \quad L_n(\tau) := \max_{q=1,\dots,n} \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}\, X_{ij}^2 \mathbb{I}(|X_{ij}| \ge \tau\sqrt{n}) \to 0 \text{ as } n \to \infty.$$

It easy to see that ($\mathbf{UI}$) $\Rightarrow$ ($\mathbf{L}$)

We prove the following Theorem

Theorem 2.1. *Let $X^{(q)}_{jk}$'s satisfy the conditions* ($\mathbf{C0}$) *and* ($\mathbf{UI}$)*. Then there exists a distribution function $G(x, z)$ such that:*
1) $G_n(x, z) \to G(x, z)$ as $n \to \infty$;
2) the Stieltjes transform $s(\alpha, z)$ of the distribution function $G(x, z)$, defined by the equality $s(\alpha, z) := \int \frac{1}{x-\alpha} dG(x, z)$, satisfies the following system of equations:

$$1 + ws(\alpha, z) + (-1)^{m+1} w^m s(\alpha, z)^{m+1} = 0$$
$$(w - \alpha)^2 + (w - \alpha) - 4|z|^2 s(\alpha, z) = 0,$$

where $\mathrm{Im}(w - \alpha) > 0$ *for* $\mathrm{Im}\,\alpha > 0$.

Remark. It is well-known that the distribution function with Stieltjes transform satisfying the system exists and unique. Moreover, this distribution is finitely supported and has a density. (See, for instance [9]). In particular, if $G_n(x, z)$ convergence to $G(x, z)$ then this convergence is uniformly in $x \in \mathbb{R}$, i.e.

$$\lim_{n\to\infty} \Delta_n(z) = \sup_x |G_n(x, z) - G(x, z)| \to 0.$$

Remark. By Lemma 4.4 one may show that $\mathcal{G}_n(x, z)$ weakly converges in probability to $G(x, z)$.

2.1. **The proof of Theorem 2.1.** As we noted before we divide the proof into several steps.

2.1.1. *Symmetrization.* We will use the following "symmetrization" of one-sided distributions. Let ξ^2 be a positive random variable with the distribution function $F(x)$. Define $\widetilde{\xi} := \varepsilon\xi$ where ε denotes a Rademacher random variable with $\mathbb{P}\{\varepsilon = \pm 1\} = 1/2$ which is independent of ξ. Let $\widetilde{F}(x)$ denote the distribution function of $\widetilde{\xi}$. It satisfies the equation

$$\widetilde{F}(x) = 1/2(1 + \mathrm{sgn}\{x\}\, F(x^2)), \tag{2.1}$$

Lemma 2.2. *For any one-sided distribution function $F(x)$ and $G(x)$ we have*

$$\sup_{x\ge 0} |F(x) - G(x)| = 2 \sup_x |\widetilde{F}(x) - \widetilde{G}(x)|,$$

where $\widetilde{F}(x)$ ($\widetilde{G}(x)$) denotes the symmetrization of $F(x)$ ($G(x)$ respectively) according to (2.1).

Proof. By (2.1), we have for any $x \geq 0$

$$F(x) = 2\widetilde{F}(\sqrt{x}) - 1$$
$$G(x) = 2\widetilde{G}(\sqrt{x}) - 1.$$

This implies

$$\sup_{x\geq 0} |F(x) - G(x)| = 2 \sup_{x\geq 0} |\widetilde{F}(\sqrt{x}) - \widetilde{G}(\sqrt{x})| = 2 \sup_{x} |\widetilde{F}(x) - \widetilde{G}(x)|.$$

Thus Lemma is proved. □

We apply this Lemma to the distribution of the squared singular values of the matrix $\mathbf{W} - z\mathbf{I}$. Introduce the following matrices

$$\mathbf{V} = \begin{pmatrix} \mathbf{W} & \mathbf{O} \\ \mathbf{O} & \mathbf{W}^* \end{pmatrix}, \quad \mathbf{J}(z) = \begin{pmatrix} \mathbf{O} & z\mathbf{I} \\ z\mathbf{I} & \mathbf{O} \end{pmatrix}, \quad \mathbf{J} = \mathbf{J}(1), \quad \text{and} \quad \mathbf{V}(z) = \mathbf{V}\mathbf{J} - \mathbf{J}(z),$$
$$\mathbf{R} := (\mathbf{V}(z) - \alpha\mathbf{I})^{-1},$$

where $\mathbf{I}$ denotes the unit matrix of the corresponding order and $\alpha = u+iv \in \mathbb{C}^+$ $(v > 0)$. Note that $\mathbf{V}(z)$ is a Hermitian matrix. The eigenvalues of the matrix $\mathbf{V}(z)$ are $-s_1, \ldots, -s_n, s_n, \ldots, s_1$. Note that the symmetrization of the distribution function $\mathcal{G}_n(x, z)$ is a function $\widetilde{\mathcal{G}}_n(x, z)$ which is the empirical distribution function of the eigenvalues of the matrix $\mathbf{V}(z)$. According to Lemma 2.2, we get

$$\Delta_n(z) := \sup_x |\mathcal{G}_n(x, z) - G(x, z)| = 2 \sup_x |\widetilde{\mathcal{G}}_n(x, z) - \widetilde{G}(x, z)| =: 2\widetilde{\Delta}_n(z).$$

Up to now we shall proof that $\lim_{n\to\infty} \widetilde{\Delta}_n(z) = 0$. In what follows we shall consider symmetrizing distribution function only. We shall omit symbol "$\widetilde{\cdot}$" in the corresponding notation.

2.1.2. *Truncation.* We shall now modify the random matrices $\mathbf{X}^{(q)}$, $q = 1, \ldots, m$, by truncation of its entries. Let $\{\tau_n\}$ is a sequence such that

$$\lim_{n\to\infty} L_n(\tau_n) = 0$$

and

$$\lim_{n\to\infty} \tau_n\sqrt{n} = \infty.$$

It is well-known that such sequence there exists since $\lim_{n\to\infty} L_n(\tau) = 0$ for any $\tau > 0$ and $L_n(\tau)$ is non-decreasing function of τ.

Introduce the random variables $X_{jk}^{(q,c)} = X_{jk}^{(q)}\mathbb{I}(|X_{jk}^{(q)}| \leq c\tau_n\sqrt{n})$ and $\overline{X}_{jk}^{(q,c)} = X_{jk}^{(q,c)} - \mathbb{E}\, X_{jk}^{(q,c)}$. Introduce the matrices $\mathbf{X}^{(q,c)} = \frac{1}{\sqrt{n}}\{X_{jk}^{(q,c)}\}_{j,k=1}^n$ and $\overline{\mathbf{X}}^{(q,c)} = \frac{1}{\sqrt{n}}\{\overline{X}_{jk}^{(q,c)}\}_{j,k=1}^n$. We define the corresponding matrices $\mathbf{W}^{(c)}, \overline{\mathbf{W}}^{(c)}, \mathbf{V}^{(c)}, \overline{\mathbf{V}}^{(c)}$ and $\mathbf{R}^{(c)}, \overline{\mathbf{R}}^{(c)}$ replacing $\mathbf{X}^{(q)}$ in the notation of $\mathbf{V}, \mathbf{W}$ and $\mathbf{R}$ by $\mathbf{X}^{(q,c)}, \overline{\mathbf{X}}^{(q,c)}$.

Denote by $s_1^{(c)} \geq \ldots \geq s_n^{(c)}$ and $\overline{s}_1^{(c)} \geq \ldots \geq \overline{s}_n^{(c)}$ – the singular values of the random matrices $\mathbf{W}^{(c)} - z\mathbf{I}$ and $\overline{\mathbf{W}}^{(c)} - z\mathbf{I}$ respectively. We define the empirical distribution functions of the matrices $\mathbf{V}^{(c)}(z)$ and $\overline{\mathbf{V}}^{(c)}(z)$ by

$$\mathcal{G}_n^{(c)}(x,z) = \frac{1}{2n}\sum_{k=1}^n \mathbb{I}(s_k^{(c)} \leq x) + \frac{1}{2n}\sum_{k=1}^n \mathbb{I}(-s_k^{(c)} \leq x)$$
$$\overline{\mathcal{G}}_n^{(c)}(x,z) = \frac{1}{2n}\sum_{k=1}^n \mathbb{I}(\overline{s}_k^{(c)} \leq x) + \frac{1}{2n}\sum_{k=1}^n \mathbb{I}(-\overline{s}_k^{(c)} \leq x)$$

Let $s_n(\alpha,z)$, $s_n^{(c)}(\alpha,z)$ and $\overline{s}_n^{(c)}(\alpha,z)$ denote the Stieltjes transforms of the distribution functions $G_n(x,z)$, $G_n^{(c)}(x) := \mathbb{E}\,\mathcal{G}_n^{(c)}(x,z)$ and $\overline{G}_n^{(c)}(x,z) = \mathbb{E}\,\overline{\mathcal{G}}_n^{(c)}(x,z)$ respectively.

Lemma 2.3. *Under the assumptions of Theorem 1.1 the following holds: for any $\delta > 0$*

$$\lim_{n\to\infty} |s_n(z,\alpha) - \overline{s}_n^{(c)}(\alpha,z)| = 0$$

uniformly in $\alpha = u + iv$ with $v \geq \delta$.

Proof. We compare the Stieltjes transforms $s_n(\alpha,z)$, $s_n^{(c)}(\alpha,z)$ and $\overline{s}_n^{(c)}(\alpha,z)$ sequentially. First we note that

$$s_n(\alpha,z) = \frac{1}{2n}\mathbb{E}\operatorname{Tr}\mathbf{R}, \qquad \text{and} \qquad s_n^{(c)}(\alpha,z) = \frac{1}{2n}\mathbb{E}\operatorname{Tr}\mathbf{R}^{(c)}. \tag{2.2}$$

Applying the resolvent equality

$$(\mathbf{A}+\mathbf{B}-\alpha\mathbf{I})^{-1} = (\mathbf{A}-\alpha\mathbf{I})^{-1} - (\mathbf{A}-\alpha\mathbf{I})^{-1}\mathbf{B}(\mathbf{A}+\mathbf{B}-\alpha\mathbf{I})^{-1},$$

we get

$$|s_n(\alpha,z) - s_n^{(c)}(\alpha,z)| \leq \frac{1}{2n}\mathbb{E}\,|\operatorname{Tr}\mathbf{R}^{(c)}(\mathbf{V}-\mathbf{V}^{(c)})\mathbf{J}\mathbf{R}|. \tag{2.3}$$

Let

$$\mathbf{H}^{(\nu)} = \begin{pmatrix} \mathbf{X}^{(\nu)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-\nu+1)*} \end{pmatrix} \quad \text{and} \quad \mathbf{H}^{(\nu,c)} = \begin{pmatrix} \mathbf{X}^{(\nu,c)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-\nu+1,c)*} \end{pmatrix}$$

Introduce the matrices

$$\mathbf{V}_{a,b} = \prod_{q=a}^{b}\mathbf{H}^{(q)}, \quad \mathbf{V}_{a,b}^{(c)} = \prod_{q=a}^{b}\mathbf{H}^{(q,c)}, \tag{2.4}$$

($\mathbf{V}_{a,b} = \mathbf{I}$ if $a > b$). We have

$$\mathbf{V} - \mathbf{V}^{(c)} = \sum_{q=1}^{m}\mathbf{V}_{1,q-1}^{(c)}(\mathbf{H}^{(q)} - \mathbf{H}^{(q,c)})\mathbf{V}_{q+1,m}. \tag{2.5}$$

Inequalities $\max\{\|\mathbf{R}\|, \|\mathbf{R}^{(c)}\|\} \le v^{-1}$, $\|\operatorname{Tr}\mathbf{AB}| \le \|\mathbf{A}\|_2\,\|\mathbf{B}\|_2$, inequality (2.3), and the representations (2.5) together imply

$$|s_n(\alpha,z) - s_n^{(\alpha,c)}(z)| \le$$

$$\frac{C}{\sqrt{n}} \sum_{q=1}^{m-1} \mathbb{E}^{\frac12}\,\|(\mathbf{H}^{(q)} - \mathbf{H}^{(q,c)})\|_2^2 \frac{1}{\sqrt{n}}\,\mathbb{E}^{\frac12}\,\|\mathbf{V}^{(c)}_{q+1,m}\mathbf{R}\mathbf{R}^{(c)}\mathbf{V}_{1,q-1}\|_2^2. \tag{2.6}$$

We use here that $\operatorname{Tr}\mathbf{AB} = \operatorname{Tr}\mathbf{BA}$ as well. Applying well-known inequalities for matrix norms $\|\mathbf{AB}\|_2 \le \|\mathbf{A}\|\,\|\mathbf{B}\|_2$ and relation $\|\mathbf{AB}\|_2 = \|\mathbf{BA}\|_2$ together, we get

$$\mathbb{E}\,\|\mathbf{V}_{q+1,m}\mathbf{R}\mathbf{R}^{(c)}\mathbf{V}^{(c)}_{1,q-1}\|_2^2 \le \frac{C}{v^4}\,\mathbb{E}\,\|\mathbf{V}^{(c)}_{1,q-1}\mathbf{V}_{q+1,m}\|_2^2$$

In view of Lemma 4.2, we obtain

$$\mathbb{E}\,\|\mathbf{V}_{q+1,m}\mathbf{R}\mathbf{R}^{(c)}\mathbf{V}^{(c)}_{1,q-1}\|_2^2 \le \frac{Cn}{v^4}. \tag{2.7}$$

Direct calculations show that, for any $q = 1, \ldots, m$,

$$\frac{1}{n}\,\mathbb{E}\,\|\mathbf{X}^{(q)} - \mathbf{X}^{(q,c)}\|_2^2 \le \frac{C}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbb{E}\,|X^{(q)}_{jk}|^2 I_{\{|X^{(q)}_{jk}| \ge c\tau_n\sqrt{n}\}} \le CL_n(\tau_n).$$

This inequality implies that

$$\max_{1\le q\le m}\mathbb{E}\,\|\mathbf{H}^{(q)} - \mathbf{H}^{(q,c)}\|_2 \le CL_n(\tau_n). \tag{2.8}$$

Inequalities (2.6), (2.7) and (2.8) together imply

$$|s_n(\alpha,z) - s_n^{(c)}(\alpha,z)| \le \frac{C\sqrt{L_n(\tau_n)}}{v^2}.$$

Furthermore, we compare the Stieltjes transforms $s_n^{(c)}(\alpha,z)$ and $\overline{s}_n^{(c)}(\alpha,z)$. By definition of $X^{(c)}_{jk}$, we have

$$|\,\mathbb{E}\,X^{(q,c)}_{jk}| = |\,\mathbb{E}\,X^{(q)}_{jk}\mathbb{I}\{|X_{jk}| \ge c\tau_n\sqrt{n}\}| \le \frac{1}{c\tau_n\sqrt{n}}\,\mathbb{E}\,|X^{(q)}_{jk}|^2 I\{|X_{jk}| \ge c\tau_n\sqrt{n}\}.$$

This implies that

$$\|\,\mathbb{E}\,\mathbf{X}^{(q,c)}\|_2^2 \le \frac{C}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}|\,\mathbb{E}\,X^{(q,c)}_{jk}|^2 \le \frac{CL_n(\tau_n)}{c\tau_n^2}. \tag{2.9}$$

Note that $\overline{\mathbf{H}}^{(q,c)} = \mathbf{H}^{(q,c)} - \mathbb{E}\,\mathbf{H}^{(q,c)}$. Similar to the inequality (2.6) we get

$$|s_n^{(c)}(\alpha,z) - \overline{s}_n^{(c)}(\alpha,z)| \le \sum_{q=1}^{m}\frac{1}{\sqrt{n}}\|\,\mathbb{E}\,\mathbf{H}^{(q,c)}\|_2\frac{1}{\sqrt{n}}\,\mathbb{E}^{\frac12}\,\|\widehat{\mathbf{V}}^{(c)}_{q+1,m}\mathbf{R}^{(c)}\widehat{\mathbf{R}}^{(c)}\widehat{\mathbf{V}}^{(c)}_{1,q-1}\|_2^2.$$

Analogously to inequality (2.7), we get

$$\mathbb{E}\,\|\widehat{\mathbf{V}}^{(c)}_{q+1,m}\mathbf{R}^{(c)}\widehat{\mathbf{R}}^{(c)}\widehat{\mathbf{V}}^{(c)}_{1,q-1}\|_2^2 \le \frac{Cn}{v^4}. \tag{2.10}$$

By the inequality (2.9),

$$\|\,\mathbb{E}\,\mathbf{X}^{(q,c)}\|_2 \le \frac{C\sqrt{L_n(\tau_n)}}{c\tau_n}.$$

This implies that

$$\max_{1\le q\le m}\|\,\mathbb{E}\,\mathbf{H}^{q,c)}\|_2 \le 2\max_{1\le q\le m}\|\,\mathbb{E}\,\mathbf{X}^{(q,c)}\|_2 \le \frac{C\sqrt{L_n(\tau_n)}}{c\tau_n}. \tag{2.11}$$

The inequalities (2.10) and (2.11) together imply that

$$|s_n^{(c)}(\alpha,z) - \ s_n^{(c)}(\alpha,z)| \le \frac{C\sqrt{L_n(\tau_n)}}{\sqrt{n}\tau_n v^2}. \tag{2.12}$$

□

According to Lemma 2.3 the matrices $\mathbf{W}$ and $\overline{\mathbf{W}}^{(c)}$ have the same limit distribution. In the what follows we shall assume without loss of generality that for any $n\ge 1$ and $q=1,\dots,m$ and $j,k=1,\dots,n$,

$$\mathbb{E}\,X_{jk}^{(q)} = 0 \quad \text{and} \quad |X_{jk}^{(q)}| \le c\tau_n\sqrt{n} \tag{2.13}$$

with $\tau_n\to 0$ such that

$$L_n(\tau_n)\to 0 \qquad \text{and} \qquad \tau_n\sqrt{n}\to\infty \qquad \text{as} \qquad n\to\infty.$$

We also have that

$$\frac{1}{n^2}\sum_{j,k=1}^{n}|\,\mathbb{E}(X_{jk}^{(q)})^2 - 1| \le CL_n(\tau_n), \tag{2.14}$$

$$\frac{1}{n^2}\sum_{j,k=1}^{n}|\,\mathbb{E}\,X_{jk}^{(q)}X_{kj}^{(q)} - \rho| \le CL_n(\tau_n). \tag{2.15}$$

2.1.3. *The universality of the limit distribution of singular values of shifted matrices.* In this Section we show that the limit distribution of singular values of product of random matrices satisfying assumptions of Theorem 2.1 doesn't depend on the distribution of matrix entries. Let $\mathbf{Y}^{(1)},\dots,\mathbf{Y}^{(m)}$ be $n\times n$ independent random matrices with independent Gaussian entries $n^{-1/2}Y_{jk}^{(q)}$ such that

$$\mathbb{E}\,Y_{jk}^{(q)} = 0, \qquad \mathbb{E}(Y_{jk}^{(q)})^2 = 1, \qquad \text{for any} \quad q=1,\dots,m,\ j,k=1\dots,n;$$
$$\mathbb{E}\,Y_{jk}^{(q)}Y_{kj}^{(q)} = \rho \qquad \text{for any} \quad q=1,\dots,m, 1\le j<k\le n.$$

Vectors $(Y_{jk}^{(q)}, Y_{kj}^{(q)})$ and r.v.'s $Y_{ll}^{(q)}$ for $q=1,\dots,m$, $1\le j<k\le n$ and $l=1,\dots,n$, are mutually independent. For any $\varphi\in[0,\frac{\pi}{2}]$ and any $\nu=1,\dots,m$, introduce the matrices

$$\mathbf{Z}^{(\nu)}(\varphi) = \mathbf{X}^{(\nu)}\cos\varphi + \mathbf{Y}^{(\nu)}\sin\varphi$$

where

$$[\mathbf{Z}^{(q)}(\varphi)]_{jk} = \frac{1}{\sqrt{n}}Z_{jk}^{(q)} = \frac{1}{\sqrt{n}}(X_{jk}^{(q)}\cos\varphi + Y_{jk}^{(q}\sin\varphi).$$

We define the matrices $\mathbf{W}(\varphi)$, $\mathbf{H}^{(q)}(\varphi)$, $\mathbf{V}(\varphi)$, $\widehat{\mathbf{V}}(\varphi)$, $\mathbf{R}(\varphi)$ by

$$\mathbf{W}(\varphi) = \prod_{\nu=1}^{m} \mathbf{Z}^{(\nu)}(\varphi), \quad \mathbf{H}^{(\nu)}(\varphi) = \begin{bmatrix} \mathbf{Z}^{(\nu)}(\varphi)) & \mathbf{O} \\ \mathbf{O} & \mathbf{Z}^{(m-\nu+1)}(\varphi) \end{bmatrix}$$

$$\mathbf{V}(\varphi) = \prod_{\nu=1}^{m} \mathbf{H}^{(\nu)}(\varphi), \quad \widehat{\mathbf{V}}(\varphi) = \mathbf{V}(\varphi)\mathbf{J}, \quad \mathbf{R}(\varphi) = (\widehat{\mathbf{V}}(\varphi) - \mathbf{J}(z) - \alpha\mathbf{I})^{-1}.$$

Recall that $\mathbf{I}$ (with sub-index or without it) denotes the unit matrix of corresponding order, $\mathbf{J}(z) = \begin{bmatrix} \mathbf{O} & z\mathbf{I} \\ \overline{z}\mathbf{I} & \mathbf{O} \end{bmatrix}$ and $\mathbf{O}$ denotes the matrix with zero-entries.

In these notation the matrices $\mathbf{W}(0)$, $\mathbf{H}^{(\nu)}(0)$, $\mathbf{V}(0)$, $\widehat{\mathbf{V}}(0)$, $\mathbf{R}(0)$ are generated by the matrices $\mathbf{X}^{(\nu)}$, $\nu = 1, \dots, m$, and $\mathbf{W}(\frac{\pi}{2})$, $\mathbf{H}^{(\nu)}(\frac{\pi}{2})$, $\mathbf{V}(\frac{\pi}{2})$, $\widehat{\mathbf{V}}(\frac{\pi}{2})$, $\mathbf{R}(\frac{\pi}{2})$ are generated by $\mathbf{Y}^{(\nu)}$, $\nu = 1, \dots, m$. Let $s_n(\alpha, z, \varphi)$ denote the Stieltjes transform of symmetrized expected distribution function of singular values of the matrix $\mathbf{W}(\varphi) - z\mathbf{I}$. Then $s_n(\alpha, z, 0) = s_n(\alpha, z)$ denote the Stieltjes transform of distribution function $G_n(x, z)$ and $s_n(\alpha, z, \frac{\pi}{2})$ denote the Stieltjes transform of symmetrized expected distribution function of singular values of the matrix $\mathbf{W}(\frac{\pi}{2}) - z\mathbf{I}$ generated by $\mathbf{Y}^{(q)}$, $q = 1, \dots, m$. We prove the following Lemma.

Lemma 2.4. *Under the assumptions of Theorem 1.1 the following holds: for any* $\delta > 0$

$$|s_n(\alpha, z, \frac{\pi}{2}) - s_n(\alpha, z, 0)| \to 0 \quad as \quad n \to \infty$$

uniformly in $\alpha = u + iv$ *with* $v \geq \delta$.

Proof. By Newton–Leibnitz formula we have

$$s_n(\alpha, z, \frac{\pi}{2}) - s_n(\alpha, z, 0) = \int_0^{\frac{\pi}{2}} \frac{\partial s_n(\alpha, z, \varphi)}{\partial \varphi} d\varphi.$$

Applying the formula for the derivative of matrix resolvent we get

$$\frac{\partial s_n(\alpha, z, \varphi)}{\partial \varphi} = -\frac{1}{2n} \mathbb{E} \operatorname{Tr} \mathbf{R}(\varphi) \frac{\partial \mathbf{V}(\varphi)}{\partial \varphi} \mathbf{J}\mathbf{R}(\varphi). \tag{2.16}$$

We shall omit in what follows the argument φ in the notations of $\mathbf{R}$ and $\mathbf{V}$ if it doesn't confuse. By the definition of the matrix $\mathbf{V}$ and $\mathbf{V}_{a,b}$ (see (2.4)), we have

$$\frac{\partial \mathbf{V}}{\partial \varphi} = \sum_{q=1}^{m} \mathbf{V}_{1,q-1} \frac{\partial \mathbf{H}^{(q)}}{\partial \varphi} \mathbf{V}_{q+1,m}.$$

Furthermore, by the definition of $\mathbf{H}^{(q)}$, for $q = 1, \dots, m$, we have

$$\frac{\partial \mathbf{H}^{(q)}}{\partial \varphi} = \sum_{j=1}^{n} \sum_{k=1}^{n} \Big(\frac{\partial \mathbf{H}^{(q)}}{\partial Z_{jk}^{(q)}} \frac{dZ_{jk}^{(q)}}{d\varphi} + \frac{\partial \mathbf{H}^{(q)}}{\partial Z_{jk}^{(m-q+1)}} \frac{dZ_{jk}^{(m-q+1)}}{d\varphi} \Big),$$

where we denote by $\mathbf{e}_j = (0, \dots, 0, 1, \dots, 0)^T$ the column vector of the dimension $2n$ with all zero entries except j-th one, which equal to 1, $j = 1, \dots, 2n$. In these notations we have

$$\frac{\partial\,\mathbf{H}^{(q)}}{\partial Z_{jk}^{(q)}} = \frac{1}{\sqrt{n}}\mathbf{e}_j\mathbf{e}_k^T, \quad \frac{\partial\,\mathbf{H}^{(q)}}{\partial Z_{jk}^{(m-q+1)}} = \frac{1}{\sqrt{n}}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T,$$

for $j,k = 1,\dots,n$. By the definition of $Z_{jk}^{(q)}$, we have

$$\frac{dZ_{jk}^{(q)}}{d\varphi} = -X_{jk}^{(q)}\sin\varphi + Y_{jk}^{(q)}\cos\varphi.$$

After a simple calculation we get

$$\frac{\partial\mathbf{V}}{\partial\varphi} = \frac{1}{\sqrt{n}}\sum_{q=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\Bigg(\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}(-X_{jk}^{(q)}\sin\varphi + Y_{jk}^{(q)}\cos\varphi)$$
$$+\mathbf{V}_{1,q-1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{q+1,m}(-X_{jk}^{(m-q+1)}\sin\varphi + Y_{jk}^{(m-q+1)}\cos\varphi)\Bigg).$$

Introduce the following functions

$$u_{jk}^{(q)} = -\operatorname{Tr}\mathbf{R}\mathbf{V}_{1,q-1}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R},$$
$$v_{jk}^{(q)} = \operatorname{Tr}\mathbf{R}\mathbf{V}_{1,q-1}\mathbf{e}_{k+n}{\mathbf{e}_{j+n}}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R},$$

for $q = 1,\dots,m$, and $j,k = 1,\dots,n$. In these notations we have

$$\frac{\partial s_n(z,\varphi)}{\partial\varphi} = \Xi_1 + \Xi_2,$$

where

$$\Xi_1 = \frac{1}{2n\sqrt{n}}\sum_{q=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbb{E}(-X_{jk}^{(q)}\sin\varphi + Y_{jk}^{(q)}\cos\varphi))u_{jk}^{(q)}$$
$$\Xi_2 = \frac{1}{2n\sqrt{n}}\sum_{q=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbb{E}(-X_{jk}^{(m-q+1)}\sin\varphi + Y_{jk}^{(m-q+1)}\cos\varphi))v_{jk}^{(q)}.$$

First we investigate Ξ_1. Let $\xi_{jk}^{(q)} = X_{jk}^{(q)}\cos\varphi + Y_{jk}^{(q)}\sin\varphi$. In what follows we shall consider the functions $u_{jk}^{(q)} = u_{jk}^{(q)}(\xi_{jk}^{(q)}, \xi_{kj}^{(q)})$ as functions of $X_{jk}^{(q)}, X_{kj}^{(q)}, Y_{jk}^{(q)}$

and $Y_{kj}^{(q)}$. Applying Taylor's formula, we may write

$$\begin{aligned} u_{jk}^{(q)}(\xi_{jk}^{(q)},\xi_{kj}^{(q)}) &= u_{jk}^{(q)}(0,0) + \xi_{jk}^{(q)}\frac{\partial u_{jk}^{(q)}}{\partial \xi_{jk}^{(q)}}(0,0) + \xi_{kj}^{(q)}\frac{\partial u_{jk}^{(q)}}{\partial \xi_{kj}^{(q)}}(0,0) \\ &+ \mathbb{E}_\theta(\xi_{jk}^{(q)})^2(1-\theta)\frac{\partial^2 u_{jk}^{(q))}}{\partial {\xi_{jk}^{(q)}}^2}(\theta\xi_{jk}^{(q)},\theta\xi_{kj}^{(q)}) \\ &+ 2\,\mathbb{E}_\theta\,\xi_{kj}^{(q)}\xi_{jk}^{(q)}\frac{\partial^2 u_{jk}^{(q)}}{\partial \xi_{jk}^{(q)}\partial \xi_{kj}^{(q)}}(\theta\xi_{jk}^{(q)},\theta\xi_{kj}^{(q)}) \\ &+ \mathbb{E}_\theta(\xi_{kj}^{(q)})^2(1-\theta)\frac{\partial^2 u_{jk}^{(q))}}{\partial {\xi_{kj}^{(q)}}^2}(\theta\xi_{jk}^{(q)},\theta\xi_{kj}^{(q)}). \end{aligned} \tag{2.17}$$

Here θ are uniformly distributed on $[0,1]$ and is independent of all $X_{jk}^{(q)}$ and $Y_{jk}^{(q)}$, and $\mathbb{E}_\theta$ denotes the expectation with respect to the random variable θ. Furthermore, we introduce the random variables

$$\widehat{\xi}_{jk}^{(q)} = -X_{jk}^{(q)}\sin\varphi + Y_{jk}^{(q)}\cos\varphi.$$

Multiplying (2.17) by $\widehat{\xi}_{jk}^{(q)}$ and taking expectation, we rewrite Ξ_1 as $\Xi_1 = \Xi_{11} + \Xi_{12}$, where

$$\begin{aligned} \Xi_{11} &= \mathbb{E}\,\widehat{\xi}_{jk}^{(q)}\xi_{jk}^{(q)}\,\mathbb{E}\,\frac{\partial u_{jk}^{(q)}}{\partial \xi_{jk}^{(q)}}(0,0) + \mathbb{E}\,\widehat{\xi}_{jk}^{(q)}\xi_{kj}^{(q)}\,\mathbb{E}\,\frac{\partial u_{jk}^{(q)}}{\partial \xi_{kj}^{(q)}}(0,0), \\ \Xi_{12} &= \mathbb{E}\,\widehat{\xi}_{jk}^{(q)}(\xi_{jk}^{(q)})^2(1-\theta)\frac{\partial^2 u_{jk}^{(q))}}{\partial {\xi_{jk}^{(q)}}^2}(\theta\xi_{jk}^{(q)},\theta\xi_{kj}^{(q)}) \\ &+ 2\,\mathbb{E}\,\widehat{\xi}_{jk}^{(q)}\xi_{jk}^{(q)}\xi_{kj}^{(q)}\frac{\partial^2 u_{jk}^{(q)}}{\partial \xi_{jk}^{(q)}\partial \xi_{kj}^{(q)}}(\theta\xi_{jk}^{(q)},\theta\xi_{kj}^{(q)}) \\ &+ \mathbb{E}(\xi_{kj}^{(q)})^2\widehat{\xi}_{jk}^{(q)}(1-\theta)\frac{\partial^2 u_{jk}^{(q))}}{\partial {\xi_{kj}^{(q)}}^2}(\theta\xi_{jk}^{(q)},\theta\xi_{kj}^{(q)}). \end{aligned}$$

It is straightforward to check, that

$$\begin{aligned} \mathbb{E}\,\widehat{\xi}_{jk}^{(q)}\xi_{jk}^{(q)} &= \cos\varphi\sin\varphi\,\mathbb{E}[(Y_{jk}^{(q)})^2 - (X_{jk}^{(q)})^2] \\ \mathbb{E}\,\widehat{\xi}_{jk}^{(q)}\xi_{kj}^{(q)} &= \cos\varphi\sin\varphi\,\mathbb{E}[Y_{jk}^{(q)}Y_{kj}^{(q)} - X_{jk}^{(q)}X_{kj}^{(q)}] \end{aligned}$$

We introduce the following matrices

$$\mathbf{B}_{jk}^{(q)} := \mathbf{V}_{1,q-1}\mathbf{e}_j\mathbf{e}_k{}^T\mathbf{V}_{q+1,m}.$$

In these notations we get $u_{jk}^{(q)} = -\operatorname{Tr}\mathbf{B}_{jk}^{(q)}\mathbf{J}\mathbf{R}^2$. It is easy to check that

$$\frac{\partial u_{jk}^{(q)}}{\partial \xi_{jk}^{(q)}}(\theta_1\xi_{jk}^{(q)}, \theta\xi_{kj}^{(q)}) = -\operatorname{Tr}\frac{\partial \mathbf{B}_{jk}^{(q)}}{\partial \xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}^2 + \operatorname{Tr}\mathbf{B}_{jk}^{(q)}\mathbf{J}\mathbf{R}^2\frac{\partial \mathbf{V}}{\partial \xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}$$
$$+ \operatorname{Tr}\mathbf{B}_{jk}^{(q)}\mathbf{J}\mathbf{R}\frac{\partial \mathbf{V}}{\partial \xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}^2 = I_1 + I_2 + I_3.$$

Furthermore,

$$\frac{\partial \mathbf{B}_{jk}^{(q)}}{\partial \xi_{jk}^{(q)}} = \frac{1}{\sqrt{n}}\mathbf{V}_{1,m-q}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{m-q+2,q-1}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{q+1,m}\mathbb{I}\{m-q \le q-1\}$$
$$+ \frac{1}{\sqrt{n}}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m-q}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{m-q+2,m}\mathbb{I}\{m-q \ge q\},$$

and

$$\frac{\partial \mathbf{V}}{\partial \xi_{jk}^{(q)}} = \frac{1}{\sqrt{n}}\mathbf{V}_{1,q-1}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{q+1,m} + \frac{1}{\sqrt{n}}\mathbf{V}_{1,m-q}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{V}_{m-q+2,m}. \tag{2.18}$$

Note that $[\mathbf{V}_{m-q+2,q-1}]_{j,j+n} = 0$ and $[\mathbf{V}_{q+1,m-q}]_{k+n,k} = 0$. These equalities imply that

$$I_1 = 0.$$

Using (2.18) we get

$$I_2 = I_{21} + I_{22}, \tag{2.19}$$

where

$$I_{21} = \frac{1}{\sqrt{n}}\operatorname{Tr}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R},$$
$$I_{22} = \frac{1}{\sqrt{n}}\operatorname{Tr}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2\mathbf{V}_{1,m-q}\mathbf{e}_{k+n}{\mathbf{e}_{j+n}}^T\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R}$$

We shall bound each term in (2.19). Note that

$$I_{21} = \frac{1}{\sqrt{n}}[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2\mathbf{V}_{1,q-1}]_{kj}[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q-1}]_{kj}.$$

It is straightforward to check that

$$|I_{21}| \le C\,v^{-3}n^{-1/2}\|{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\|_2^2\|\mathbf{V}_{1,q-1}\mathbf{e}_j\|_2^2.$$

Note that the random variables in the r.h.s of the last inequality conditionally independent with respect to $\xi_{jk}^{(q)}$ and $\xi_{kj}^{(q)}$. We may write

$$\mathbb{E}\left\{|I_{21}|\middle|\xi_{jk}^{(q)}, \xi_{kj}^{(q)}\right\} \le$$
$$\frac{C}{v^3n^{1/2}}\mathbb{E}\left\{\|{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\|_2^2\middle|\xi_{jk}^{(q)}, \xi_{kj}^{(q)}\right\}\mathbb{E}\left\{\|\mathbf{V}_{1,q-1}\mathbf{e}_j\|_2^2\middle|\xi_{jk}^{(q)}, \xi_{kj}^{(q)}\right\}.$$

Applying Lemma 4.3, we get

$$\mathbb{E}\left\{|I_{21}|\middle|\xi_{jk}^{(q)}, \xi_{kj}^{(q)}\right\} \le Cn^{-1/2}v^{-3}.$$

Similarly we estimate I_{22} and I_3. It follows from these bounds, (2.14) and (2.15) that

$$|\Xi_{11}| \le Cv^{-3}L_n(\tau_n).$$

We now estimate Ξ_{12}. Without loss of generality we may assume that

$$\max\Big\{|\xi_{jk}^{(q)}|, |\xi_{kj}^{(q)}|, |\widehat{\xi}_{jk}^{(q)}|, |\widehat{\xi}_{kj}^{(q)}|\Big\} \le C\tau_n\sqrt{n}.$$

If we prove that there exists a constant C such that, for any $q = 1, \dots, m$, $1 \le j, k \le n$,

$$\max\Big\{\Big|\,\mathbb{E}\Big\{\frac{\partial^2 u_{jk}^{(q)}}{\partial {\xi_{jk}^{(q)}}^2}(\theta\xi_{jk}^{(q)}, \theta\xi_{kj}^{(q)})\Big|\xi_{jk}^{(q}, \xi_{kj}^{(q)}\Big\}\Big|, \Big|\,\mathbb{E}\{\frac{\partial^2 u_{jk}^{(q)}}{\partial\xi_{jk}^{(q)}\partial\xi_{kj}^{(q)}}(\theta\xi_{jk}^{(q)}, \theta\xi_{kj}^{(q)})\Big|\xi_{jk}^{(q}, \xi_{kj}^{(q)}\Big\}\Big|\Big\}$$

$$\le Cn^{-1}v^{-4}, \tag{2.20}$$

we get

$$|\,\mathbb{E}\,\widehat{\xi}_{jk}^{(q)}u_{jk}^{(q)}(\xi_{jk}^{(q)}, \xi_{kj}^{(q)})| \le \frac{C\tau_n}{\sqrt{n}}.$$

The last bound implies that

$$|\Xi_{12}| \le C\tau_n v^{-4}. \tag{2.21}$$

Furthermore,

$$\begin{aligned}
&\frac{\partial^2 u_{jk}^{(q)}}{\partial {\xi_{jk}^{(q)}}^2}(\theta\xi_{jk}^{(q)}, \theta\xi_{kj}^{(q)}) = -2\operatorname{Tr}\mathbf{B}_{jk}^{(q)}\mathbf{J}\mathbf{R}^2\frac{\partial\mathbf{V}}{\partial\xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}\frac{\partial\mathbf{V}}{\partial\xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}\\
&-2\operatorname{Tr}\mathbf{B}_{jk}^{(q)}\mathbf{J}\mathbf{R}\frac{\partial\mathbf{V}}{\partial\xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}^2\frac{\partial\mathbf{V}}{\partial\xi_{jk}^{(q)}}\mathbf{J}\mathbf{R} - 2\operatorname{Tr}\mathbf{B}_{jk}^{(q)}\mathbf{J}\mathbf{R}\frac{\partial\mathbf{V}}{\partial\xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}\frac{\partial\mathbf{V}}{\partial\xi_{jk}^{(q)}}\mathbf{J}\mathbf{R}^2\\
&= T_1 + T_2 + T_3.
\end{aligned}$$

We bound T_1 now. The estimates for T_2, T_3 may be written down in the similar way. Using (2.18) we get

$$T_1 = T_{11} + \dots + T_{14}, \tag{2.22}$$

where

$$T_{11} = -2\frac{1}{n}\ \operatorname{Tr}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2 \\ \times \mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R},$$

$$T_{12} = -2\frac{1}{n}\ \operatorname{Tr}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2 \\ \times \mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-q}\mathbf{e}_{k+n}{\mathbf{e}_{j+n}}^T\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R},$$

$$T_{13} = -2\frac{1}{n}\ \operatorname{Tr}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2 \\ \times V_{1,m-q}\mathbf{e}_{k+n}{\mathbf{e}_{j+n}}^T\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R},$$

$$T_{14} = -2\frac{1}{n}\ \operatorname{Tr}\mathbf{V}_{1,q-1}\mathbf{e}_j{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2 \\ \times \mathbf{V}_{1,m-q}\mathbf{e}_{k+n}{\mathbf{e}_{j+n}}^T\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-q}\mathbf{e}_{k+n}{\mathbf{e}_{j+n}}^T\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R},$$

We shall bound each term in (2.22). Note that

$$T_{11} = -2\frac{1}{n}[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2\mathbf{V}_{1,q-1}]_{kj}[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q-1}]_{kj}[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q-1}]_{kj}.$$

It is straightforward to check that

$$|T_{31}| \le C\,v^{-4}n^{-1}\|{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\|_2^3\|\mathbf{V}_{1,q-1}\mathbf{e}_j\|_2^3.$$

Note that the random variables in the r.h.s of the last inequality conditionally independent with respect to $\xi_{jk}^{(q)}$ and $\xi_{kj}^{(q)}$. We may write

$$\mathbb{E}\left\{|T_{31}|\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\} \le \\ \frac{C}{v^4n}\,\mathbb{E}\left\{\|{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\|_2^3\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\}\mathbb{E}\left\{\|\mathbf{V}_{1,q-1}\mathbf{e}_j\|_2^3\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\}. \tag{2.23}$$

Applying Lemma 4.3, we get

$$\mathbb{E}\left\{|T_{31}|\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\} \le Cn^{-1}v^{-4}.$$

Furthermore, we represent T_{32} in the form

$$T_{32} = -2\frac{1}{n}[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}^2\mathbf{V}_{1,q-1}]_{k,j}[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-q}]_{k,k+n}[\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q-1}]_{j+n,j}.$$

Similar to (2.23) we get

$$|T_{32}| \le C\,v^{-4}n^{-1}\,\|{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\|_2^2\,\|\mathbf{V}_{1,q-1}\mathbf{e}_j\|_2^2 \\ \times\|\mathbf{V}_{1,m-q}\mathbf{e}_{k+n}\|_2\,\|{\mathbf{e}_{j+n}}^T\mathbf{V}_{m-q+2,m}\|_2.$$

Applying Hölder's inequality, we get

$$\mathbb{E}\left\{|T_{32}|\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\} \le \frac{C}{v^4n}\,\mathbb{E}^{\frac12}\left\{\|{\mathbf{e}_k}^T\mathbf{V}_{q+1,m}\|_2^4\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\}\mathbb{E}^{\frac12}\left\{\|\mathbf{V}_{1,q-1}\mathbf{e}_j\|_2^4\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\} \\ \times\mathbb{E}^{\frac12}\left\{\|\mathbf{V}_{1,m-q}\mathbf{e}_{k+n}\|_2^2\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\}\mathbb{E}^{\frac12}\left\{\|{\mathbf{e}_{j+n}}^T\mathbf{V}_{m-q+2,m}\|_2^2\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\}.$$

Using Lemma 4.3, we get

$$\mathbb{E}\left\{|T_{12}|\Big|\xi_{jk}^{(q)},\xi_{kj}^{(q)}\right\} \le C\,v^{-4}n^{-1}.$$

Analogously we get the bounds for other terms T_{1l}, for $l = 3, 4$. We have

$$\mathbb{E}\left\{|T_1|\Big|\xi_{jk}^{(q)}, \xi_{kj}^{(q)}\right\} \le C\, v^{-4} n^{-1}.$$

This proves (2.20) and (2.21). Similarly we may estimate the term Ξ_2

$$|\Xi_2| \le C\tau_n v^{-4}.$$

It follows that there exists some $\delta > 0$ such that

$$\lim_{n\to\infty} |s_n(\alpha, z, \frac{\pi}{2}) - s_n(\alpha, z, 0)| = 0,$$

for all $v \ge \delta$. The last inequality proves the Lemma 2.4. □

2.1.4. *The Limit Distribution of Singular Values of* $\mathbf{V}(z)$ *in the Gaussian case.* In this Section we find the limit distribution of singular values of shifted products of Gaussian random matrices. Recall that

$$\mathbf{H}^{(\nu)} = \begin{pmatrix} \mathbf{Y}^{(\nu)} & \mathbf{O} \\ \mathbf{O} & \mathbf{Y}^{(\mathbf{m}-\nu+\mathbf{1})*} \end{pmatrix}, \quad \mathbf{J}(z) := \begin{pmatrix} \mathbf{O} & z\,\mathbf{I} \\ \overline{z}\,\mathbf{I} & \mathbf{O} \end{pmatrix}, \text{ and } \mathbf{J} := \mathbf{J}(1).$$

For any $1 \le a, b \le m$, put

$$\mathbf{V}_{[a,b]} = \begin{cases} \prod_{k=a}^{b} \mathbf{H}^{(k)}, & \text{for } a \le b, \\ \mathbf{I} & \text{otherwise,} \end{cases}$$

and

$$\mathbf{V}(z) := \mathbf{V}\mathbf{J} - \mathbf{J}(z), \quad \mathbf{R} = (\mathbf{V}(z) - \alpha\mathbf{I})^{-1}.$$

It is straightforward to check

$$s_n(\alpha, z) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{R}(\alpha, z)]_{jj}$$

$$= \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{R}(\alpha, z)]_{j+nj+n} = \frac{1}{2n}\sum_{j=1}^{2n} \mathbb{E}[\mathbf{R}(\alpha, z)]_{jj}. \tag{2.24}$$

We introduce the following functions

$$t_n(\alpha, z) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{R}(\alpha, z)]_{j+n,j}, \quad u_n(\alpha, z) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{R}(\alpha, z)]_{j,j+n}.$$

We prove the following statement

Statement 2.5. *Let r.v.'s* $Y_{jk}^{(q)}$, $q = 1, \dots, m$, $j, k = 1, \dots n$ *are Gaussian and satisfy the conditions* (**C0**). *Then the following limit exists*

$$g = g(\alpha, z) = \lim_{n\to\infty} s_n(\alpha, z),$$

and satisfy the system equations

$$\begin{aligned} &1 + wg + (-1)^{m+1} w^{m-1} g^{m+1} = 0, \\ &g(w-\alpha)^2 + (w-\alpha) - g|z|^2 = 0, \end{aligned} \tag{2.25}$$

with a function $w = w(\alpha, z)$ *such that* $\operatorname{Im}(w - \alpha) > 0$.

Corollary 2.6. *Under the assumptions of Theorem 2.1 for any $z \in \mathbb{C}$ there exists a distribution function $G(x,z)$ such that $\lim_{n\to\infty} G_n(x,z) = G(x,z)$ and $g = g(\alpha, z) = \int_{-\infty}^{\infty} \frac{1}{x-\alpha} dG(x,z)$ satisfy the system of equations* (2.25) *and*

$$\Delta_n(z) := \sup_x |G_n(x,z) - G(x,z)| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.26}$$

Remark. Note that the second equation of (2.57) implies

$$\operatorname{Im} g = -\operatorname{Im}\Big\{\frac{w-\alpha}{(w-\alpha)^2 - |z|^2}\Big\} = \frac{\operatorname{Im}\{w-\alpha\}(|w-\alpha|^2 + |z|^2)}{|(w-\alpha)^2 - |z|^2|^2}.$$

This equality implies that $\operatorname{Im}(w-\alpha) > 0$.

Proof. Statement 2.5. In what follows we shall denote by $\varepsilon_n(\alpha, z)$ a generic error function such that $|\varepsilon_n(\alpha,z)| \le \frac{C\tau_n^q}{v^r}$ for some positive constants C, q, r. By the resolvent equality, we may write

$$\begin{aligned} 1 + \alpha s_n(\alpha, z) &= \frac{1}{2n} \mathbb{E} \operatorname{Tr} \mathbf{V}(z)\mathbf{R}(\alpha, z) \\ &= \frac{1}{2n} \mathbb{E} \operatorname{Tr} \mathbf{VJR}(\alpha, z) - \frac{1}{2} z t_n(\alpha, z) - \frac{1}{2}\overline{z} u_n(\alpha, z). \end{aligned} \tag{2.27}$$

In the following we shall write $\mathbf{R}$ instead of $\mathbf{R}(\alpha, z)$. Introduce the notation

$$\mathcal{A} := \frac{1}{2n} \mathbb{E} \operatorname{Tr} \mathbf{VJR}$$

and represent $\mathcal{A}$ as follows

$$\mathcal{A} = \frac{1}{2}\mathcal{A}_1 + \frac{1}{2}\mathcal{A}_2,$$

where

$$\mathcal{A}_1 = \frac{1}{n}\sum_{j=1}^n \mathbb{E}[\mathbf{VJR}]_{jj}, \quad \mathcal{A}_2 = \frac{1}{n}\sum_{j=1}^n \mathbb{E}[\mathbf{VJR}]_{j+n,j+n}.$$

By definition of the matrix $\mathbf{V}$ and the matrix $\mathbf{H}^{(1)}$, we have

$$\mathcal{A}_1 = \frac{1}{n\sqrt{n}} \sum_{j,k=1}^n \mathbb{E}\, Y_{jk}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{kj}.$$

In the Gaussian case we may represent the random variables $Y_{jk}^{(q)}$ and $Y_{kj}^{(q)}$ in the form

$$\begin{aligned} Y_{jk}^{(q)} &= a\xi_{jk}^{(q)} + b\eta_{jk}^{(q)}, \\ Y_{kj}^{(q)} &= a\xi_{jk}^{(q)} - b\eta_{jk}^{(q)}, \end{aligned} \tag{2.28}$$

where $a = \sqrt{\frac{1+\rho}{2}}$, $b = \sqrt{\frac{1-\rho}{2}}$ and $\xi_{jk}^{(q)}$, $\eta_{jk}^{(q)}$ are mutually independent standard Gaussian r.v.'s. We shall use the well-known equality for the standard Gaussian r.v. ξ and any smooth function f

$$\mathbb{E}\, \xi f(\xi) = \mathbb{E}\, f'(\xi). \tag{2.29}$$

First we represent $\mathcal{A}_1$ in the form

$$\mathcal{A}_1 = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{13},$$

where

$$\mathcal{A}_{11} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \mathbb{E}\, Y_{jk}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{kj},$$

$$\mathcal{A}_{12} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbb{E}\, Y_{jj}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{jj},$$

$$\mathcal{A}_{13} = \frac{1}{n\sqrt{n}} \sum_{j=2}^{n} \sum_{k=1}^{j-1} \mathbb{E}\, Y_{jk}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{kj}.$$

First we note that

$$|\mathcal{A}_{12}| \le \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbb{E}^{\frac{1}{2}} |\mathbf{e}_j^T \mathbf{V}_{2,m}\mathbf{JR}\mathbf{e}_j|^2 \le \frac{1}{\sqrt{n}} \Big(\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\, |\mathbf{e}_j^T \mathbf{V}_{2,m}\mathbf{JR}\mathbf{e}_j|^2 \Big)^{\frac{1}{2}}$$
$$\le \frac{1}{v\sqrt{n}} \Big(\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\, \|\mathbf{e}_j^T \mathbf{V}_{2,m}\|^2 \Big)^{\frac{1}{2}} \le \frac{C}{v\sqrt{n}}.$$

We use here the inequalities $\|\mathbf{JR}\mathbf{e}_j\| \le \|\mathbf{JR}\| \le v^{-1}$ and $|\mathbf{e}_j^T \mathbf{V}_{2,m}\mathbf{JR}\mathbf{e}_j| \le \||\mathbf{e}_j^T \mathbf{V}_{2,m}\|\|\mathbf{JR}\mathbf{e}_j\|$. We may write now

$$\mathcal{A}_{12} = \varepsilon_n(\alpha, z). \tag{2.30}$$

Furthermore, we consider $\mathcal{A}_{11}$ and $\mathcal{A}_{13}$. Using (2.28), we get

$$\mathcal{A}_{11} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (a\, \mathbb{E}\, \xi_{jk}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{kj} + b\, \mathbb{E}\, \eta_{jk}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{kj}).$$

Applying (2.29), we get

$$\mathcal{A}_{11} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(a\, \mathbb{E} \left[\frac{\partial \mathbf{V}_{2,m}\mathbf{JR}}{\partial \xi_{jk}^{(1)}} \right]_{kj} + b\, \mathbb{E} \left[\frac{\partial \mathbf{V}_{2,m}\mathbf{JR}}{\partial \eta_{jk}^{(1)}} \right]_{kj} \right).$$

A simple calculation shows that

$$\mathcal{A}_{13} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (a\, \mathbb{E}\, \xi_{jk}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{kj} - b\, \mathbb{E}\, \eta_{jk}^{(1)} [\mathbf{V}_{2,m}\mathbf{JR}]_{jk}).$$

By the equality (2.29), we have

$$\mathcal{A}_{13} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \Big(a\, \mathbb{E} \Big[\frac{\partial \mathbf{V}_{2,m}\mathbf{JR}}{\partial \xi_{jk}^{(1)}} \Big]_{jk} - b\, \mathbb{E} \Big[\frac{\partial \mathbf{V}_{2,m}\mathbf{JR}}{\partial \eta_{jk}^{(1)}} \Big]_{jk} \Big).$$

Note that for $1 \le j < k \le n$

$$\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial \xi_{jk}^{(1)}} = a\left(\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial Y_{jk}^{(1)}} + \frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial Y_{kj}^{(1)}}\right),$$

$$\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial \eta_{jk}^{(1)}} = b\left(\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial Y_{jk}^{(1)}} - \frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial Y_{kj}^{(1)}}\right). \tag{2.31}$$

Computing the matrix derivatives

$$\begin{aligned}
\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial Y_{jk}^{(1)}} &= \frac{1}{\sqrt{n}}\mathbf{V}_{2,m-1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{J}\mathbf{R} \\
&- \frac{1}{\sqrt{n}}\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{2,m}\mathbf{J}\mathbf{R} - \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T\mathbf{J}\mathbf{R}, \\
\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial Y_{kj}^{(1)}} &= \frac{1}{\sqrt{n}}\mathbf{V}_{2,m-1}\mathbf{e}_{j+n}\mathbf{e}_{k+n}^T\mathbf{J}\mathbf{R} \\
&- \frac{1}{\sqrt{n}}\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{e}_k\mathbf{e}_j^T\mathbf{V}_{2,m}\mathbf{J}\mathbf{R} - \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}\mathbf{e}_{j+n}\mathbf{e}_{k+n}^T\mathbf{J}\mathbf{R}.
\end{aligned} \tag{2.32}$$

Combining the equalities (2.31) and (2.32), we get

$$\begin{aligned}
\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial \xi_{jk}^{(1)}} =& a\frac{1}{\sqrt{n}}(\mathbf{V}_{2,m-1}(\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T + \mathbf{e}_{j+n}\mathbf{e}_{k+n}^T)\mathbf{J}\mathbf{R} \\
&- \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}(\mathbf{e}_j\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_j^T)\mathbf{V}_{2,m}\mathbf{J}\mathbf{R} \\
&- \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}(\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T + \mathbf{e}_{j+n}\mathbf{e}_{k+n}^T)\mathbf{J}\mathbf{R}) \\
\frac{\partial \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}}{\partial \eta_{jk}^{(1)}} =& - b\frac{1}{\sqrt{n}}(\mathbf{V}_{2,m-1}(\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T - \mathbf{e}_{j+n}\mathbf{e}_{k+n}^T)\mathbf{J}\mathbf{R} \\
&- \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}(\mathbf{e}_j\mathbf{e}_k^T - \mathbf{e}_k\mathbf{e}_j^T)\mathbf{V}_{2,m}\mathbf{J}\mathbf{R} \\
&- \mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}(\mathbf{e}_{k+n}\mathbf{e}_{j+n}^T - \mathbf{e}_{j+n}\mathbf{e}_{k+n}^T)\mathbf{J}\mathbf{R}).
\end{aligned}$$

Using the previous steps we may write

$$\mathcal{A}_{11} + \mathcal{A}_{13} = \mathcal{A}_{111} + \ldots + \mathcal{A}_{114}, \tag{2.33}$$

where

$$\mathcal{A}_{111} = -\frac{2\rho}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jj}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{kk},$$

$$\mathcal{A}_{112} = -\frac{1}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \mathbb{E}\Big([\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jk}^2 + [\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{kj}^2\Big),$$

$$\mathcal{A}_{113} = -\frac{1}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \Big(\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}]_{kk+n}[\mathbf{J}\mathbf{R}]_{j+n,j} + \mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}]_{j,j+n}[\mathbf{J}\mathbf{R}]_{k+n,k}\Big),$$

$$\mathcal{A}_{114} = -\frac{\rho}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \Big(\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}]_{k,j+n}[\mathbf{J}\mathbf{R}]_{k+n,j} + \mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}]_{j,k+n}[\mathbf{J}\mathbf{R}]_{j+n,k}\Big).$$

We use here $[V_{2,m-1}]_{k,k+n} = [V_{2,m-1}]_{j,k+n} = [V_{2,m-1}]_{k,j+n} = [V_{2,m-1}]_{j,j+n} = 0$ and $a^2 + b^2 = 1$, $a^2 - b^2 = \rho$. We prove the following lemma.

Lemma 2.7. *Suppose the conditions of Theorem 2.1 hold, we have*

$$\max\{|\mathcal{A}_{112}|, |\mathcal{A}_{114}|\} \le \frac{C}{nv^2}.$$

Proof. It is straightforward to check that

$$|\mathcal{A}_{112}| \le \frac{1}{n^2}\,\mathbb{E}\,\|\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\|_2^2,$$

$$|\mathcal{A}_{114}| \le \frac{1}{n^2}\,\mathbb{E}^{\frac12}\,\|\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{2,m-1}\|_2^2\,\mathbb{E}^{\frac12}\,\|\mathbf{J}\mathbf{R}\|_2^2.$$

Using well-known properties of Frobenius norm for matrices, $\|\mathbf{A}\mathbf{B}\|_2 = \|\mathbf{B}\mathbf{A}\|_2$ and $\|\mathbf{A}\mathbf{B}\|_2 \le \|\mathbf{A}\|\|\mathbf{B}\|_2$, we get

$$|\mathcal{A}_{112}| \le \frac{1}{n^2v^2}\,\mathbb{E}\,\|\mathbf{V}_{2,m}\|_2^2,$$

$$|\mathcal{A}_{114}| \le \frac{1}{n^2}\,\mathbb{E}^{\frac12}\,\|\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}\|_2^2\,\mathbb{E}^{\frac12}\,\|\mathbf{J}\mathbf{R}\|_2^2.$$

Furthermore, we note

$$\begin{aligned}
\mathbb{E}\,\|\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}\|_2^2 &= \mathbb{E}\,\|{\mathbf{H}^{(1)}}^{-1}\mathbf{V}_{1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}\|_2^2 \\
&= \mathbb{E}\,\|{\mathbf{H}^{(1)}}^{-1}(\mathbf{I} + \alpha\mathbf{R} + \mathbf{J}(z)\mathbf{R})\mathbf{V}_{1,m-1}\|_2^2 \\
&\le \mathbb{E}\left(\|\mathbf{V}_{2,m-1} + {\mathbf{H}^{(1)}}^{-1}(\alpha\mathbf{I} + \mathbf{J}(z))\mathbf{R}\mathbf{V}_{1,m-1}\|_2^2\right) \\
&\le 2\Big(\mathbb{E}\,\|\mathbf{V}_{2,m-1}\|_2^2 + \frac{(|\alpha| + |z|)^2}{v^2}\,\mathbb{E}\,\|\mathbf{V}_{1,m-1}{\mathbf{H}^{(1)}}^{-1}\|_2^2\Big) \\
&\le 2\Big(1 + \frac{(|\alpha| + |z|)^2}{v^2}\Big)\,\mathbb{E}\,\|\mathbf{V}_{2,m-1}\|_2^2.
\end{aligned}$$

Applying Lemma 4.3, we conclude the proof. □

By Lemma 2.7, we may write

$$\mathcal{A}_1 = \mathcal{A}_{111} + \mathcal{A}_{113} + \varepsilon_n(\alpha, z).$$

Lemma 2.8. *Under the assumptions of Theorem 2.1 we have*

$$|\mathcal{A}_{111}| \le \frac{C}{nv^2}(1 + v^{-2}).$$

Proof. A simple calculation shows that

$$I := \frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jj}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{kk} = \frac{1}{n^2}\sum_{1\le j\ne k\le n} \mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jj}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{kk}$$

By Lemma 2.7

$$\frac{1}{n^2}\sum_{j=1}^{n}\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jj}^2 \le \frac{C}{nv^2}.$$

We may write

$$I = \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jj}\right)^2 + \varepsilon_n(\alpha, z).$$

Applying Lemma 4.5, we obtain

$$\left|I - \left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{j,j}\right)^2\right| \le \frac{C}{nv^2}(1 + v^{-2}). \tag{2.34}$$

Note that $\mathbf{H}^{(q)}$, $q = 1, \dots, m$ have a symmetric joint distribution of entries, i.e. $\mathbf{H}^{(q)}$ has the same joint distribution of entries as $-\mathbf{H}^{(q)}$, for any $q = 1, \dots, m$. It follows immediately that

$$\mathbb{E}\,\mathrm{Tr}\,\mathbf{V}_{2,m}\mathbf{J}\mathbf{R} = 0. \tag{2.35}$$

To prove (2.35) we may replace the matrices $\mathbf{H}^{(1)}$ and $\mathbf{H}^{2)}$ in the definition of $\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}$ by $-\mathbf{H}^{(1)}$ and $-\mathbf{H}^{(2)}$. The resolvent matrix $\mathbf{R}$ still the same, since $\prod_{q=1}^{m}\mathbf{H}^{(q)} = (-\mathbf{H}^{(1)})(-\mathbf{H}^{(2)})\prod_{q=3}^{m}\mathbf{H}^{(q)}$ and we get

$$\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jj} = -\,\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}]_{jj} = 0.$$

The inequality (2.34)) and equality (2.35) together imply the result of Lemma. Thus Lemma 2.8 is proved. □

Finally, we prove that

$$\begin{aligned}\mathcal{A}_1 = &-\frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}]_{k,k+n}[\mathbf{J}\mathbf{R}]_{j+n,j}\\ &-\frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\mathbb{E}[\mathbf{V}_{2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-1}]_{j,j+n}[\mathbf{J}\mathbf{R}]_{k+n,k} + \varepsilon_n(\alpha, z).\end{aligned}$$

This equality we may rewrite as follows

$$\mathcal{A}_1 = -\frac{1}{n^2}\,\mathbb{E}\sum_{1\le j\ne k\le n}([\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{k,k+n}[\mathbf{JR}]_{j+n,j} + \mathbb{E}[\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j,j+n}[\mathbf{JR}]_{k+n,k}) + \varepsilon_n(\alpha, z). \tag{2.36}$$

It is straightforward to check that

$$\frac{1}{n^2}|\,\mathbb{E}\sum_{j=1}^{n}[\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j,j+n}[\mathbf{JR}]_{j+n,j}| \le \frac{C}{n^{\frac{3}{2}}v}\,\mathbb{E}^{\frac{1}{2}}\,\|\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}\|_2^2 \le \frac{C}{nv^2}. \tag{2.37}$$

Relations (2.36) and (2.37) together imply

$$\mathcal{A}_1 = -\frac{1}{n^2}\,\mathbb{E}\sum_{j=1}^{n}\sum_{k=1}^{n}([\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{k,k+n}[\mathbf{JR}]_{j+n,j} + \mathbb{E}[\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j,j+n}[\mathbf{JR}]_{k+n,k}) + \varepsilon_n(\alpha, z).$$

By Lemma 4.4, Lemma 4.5 and $\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{JR}]_{j,j+n} = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{JR}]_{j+n,j} = s_n(\alpha, z)$, we get

$$\mathcal{A}_1 = -s_n(\alpha, z)\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}([\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j,j+n} + [\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j+n,j}) + \varepsilon_n(\alpha, z). \tag{2.38}$$

Consider now the quantity $\mathcal{A}_2$. Similar to (2.38), we obtain

$$\mathcal{A}_2 = -s_n(\alpha, z)(\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}([\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j+n,j} + [\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j,j+n}) + \varepsilon_n(\alpha, z).$$

Introduce the notation, for $\nu = 2, \ldots, m$

$$f_q = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{V}_{q,m}\mathbf{JRV}_{1,m-q+1}]_{j,j+n} \tag{2.39}$$

and

$$f_{m+1} = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{JR}]_{j,j+n} = s_n(\alpha, z).$$

We rewrite the equality (2.38) using these notations

$$\mathcal{A}_1 = -f_2 s_n(\alpha, z) + \varepsilon_n(\alpha, z). \tag{2.40}$$

We shall investigate the asymptotic of f_q, for $q = 2, \ldots, m$. By definition of the matrices $\mathbf{V}_{q,m}$ and $\mathbf{H}^{(q)}$, we have

$$f_q = \frac{1}{n\sqrt{n}}\sum_{k,j=1}^{n}\mathbb{E}\,Y_{jk}^{(q)}[\mathbf{V}_{q+1,m}\mathbf{JRV}_{1,m-q+1}]_{k,j+n}.$$

We represent f_q in the form

$$f_q = f_{q1} + f_{q2} + f_{q3}, \tag{2.41}$$

where

$$f_{q1} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \mathbb{E}\, Y_{jk}^{(q)} [\mathbf{V}_{q+1,m}\mathbf{JRV}_{1,m-q+1}]_{k,j+n},$$

$$f_{q2} = \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbb{E}\, Y_{jj}^{(q)} [\mathbf{V}_{q+1,m}\mathbf{JRV}_{1,m-q+1}]_{j,j+n},$$

$$f_{q3} = \frac{1}{n\sqrt{n}} \sum_{j=2}^{n} \sum_{k=1}^{j-1} \mathbb{E}\, Y_{jk}^{(q)} [\mathbf{V}_{q+1,m}\mathbf{JRV}_{1,m-q+1}]_{k,j+n}.$$

Similarly to the previous steps we get

$$\begin{aligned} f_q &= \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[\mathbf{V}_{q+1,m}\mathbf{JRV}_{1,m-q}]_{k,k+n} \\ &- \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[\mathbf{V}_{q+1,m}\mathbf{JRV}_{1,m-q}]_{k,k+n} \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{V}_{m-q+2,m}\mathbf{JRV}_{1,m-q+1}]_{j+n,j+n} \\ &= f_{\nu+1}(1 - \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{V}_{m-q+2,m}\mathbf{JRV}_{1,m-q+1}]_{j+n,j+n}) + \varepsilon_n(\alpha, z). \end{aligned}$$

Note that

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{V}_{m-\nu+2,m}\mathbf{JRV}_{1,m-\nu+1}]_{j+n,j+n} = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{V}_{1,m}\mathbf{JR}]_{j+n,j+n}.$$

Furthermore,

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{V}_{1,m}\mathbf{JR}]_{j+n,j+n} = 1 + \alpha s_n(\alpha, z) + \overline{z} u_n(\alpha, z). \tag{2.42}$$

Relations (2.39)–(2.42) together imply

$$f_q = f_{q+1}(-\alpha s_n(\alpha, z) - \overline{z} u_n(\alpha, z)) + \varepsilon_n(\alpha, z).$$

By induction we get

$$f_2 = (-1)^{m-1}(\alpha s_n(\alpha, z) + \overline{z} u_n(\alpha, z))^{m-1} s_n(\alpha, z) + \varepsilon_n(\alpha, z). \tag{2.43}$$

Relations (2.40) and (2.43) together imply

$$\mathcal{A}_1 = (-1)^{m}(\alpha s_n(\alpha, z) + \overline{z} u_n(\alpha, z))^{m-1} s_n^2(\alpha, z) + \varepsilon_n(\alpha, z). \tag{2.44}$$

Introduce now the notations

$$h_q = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[\mathbf{V}_{q,m}\mathbf{JRV}_{1,m-q+1}]_{j+n,j},$$

for $q = 2, \dots, m$, and

$$h_{m+1} = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{JR}]_{j+n,j} = s_n(\alpha, z).$$

Similar to (2.43) we get that

$$h_2 = (-1)^{m-1}(\alpha s_n(\alpha,z) + z t_n(\alpha,z))^{m-1} s_n(\alpha,z) + \varepsilon_n(\alpha,z). \tag{2.45}$$

and

$$\mathcal{A}_2 = (-1)^{m}(\alpha s_n(\alpha,z) + z t_n(\alpha,z))^{m-1} s_n^2(\alpha,z) + \varepsilon_n(\alpha,z). \tag{2.46}$$

Consider now the function $t_n(\alpha, z)$ which we may represent as follows

$$\alpha t_n(\alpha,z) = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{V}(z)\mathbf{R}]_{j+n,j}.$$

By definition of the matrix $\mathbf{H}^{(1)}$, we may write

$$\alpha t_n(\alpha,z) = \frac{1}{n}\sum_{j,k=1}^{n}\mathbb{E}\, Y_{jk}^{(m)}[\mathbf{V}_{2,m}\mathbf{JR}]_{j+n,k} - \overline{z}\ s_n(\alpha,z). \tag{2.47}$$

The first term in the r.h.s. of (2.47) we represent in the form

$$\mathcal{B}_1 := \frac{1}{n}\sum_{j,k=1}^{n}\mathbb{E}\, Y_{jk}^{(m)}[\mathbf{V}_{2,m}\mathbf{JR}]_{j+n,k} = \mathcal{B}_{11} + \mathcal{B}_{12} + \mathcal{B}_{13},$$

where

$$\mathcal{B}_{11} = \frac{1}{n}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\mathbb{E}\, Y_{jk}^{(m)}[\mathbf{V}_{2,m}\mathbf{JR}]_{j+n,k},$$

$$\mathcal{B}_{11} = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\, Y_{jk}^{(m)}[\mathbf{V}_{2,m}\mathbf{JR}]_{j+n,k},$$

$$\mathcal{B}_{13} = \frac{1}{n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}\mathbb{E}\, Y_{jk}^{(m)}[\mathbf{V}_{2,m}\mathbf{JR}]_{j+n,k},$$

Previous relations together imply

$$\begin{aligned}\alpha t_n(\alpha,z) &= -\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[\mathbf{V}_{2,m}\mathbf{JRV}_{1,m-1}]_{j+n,j}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[\mathbf{R}]_{k+n,k} \\ &- \overline{z}s_n(\alpha,z) + \varepsilon_n(\alpha,z) \\ &= h_2\ t_n(\alpha,z) - \overline{z}\ s_n(\alpha,z) + \varepsilon_n(\alpha,z).\end{aligned}$$

Applying the equality (2.45), we obtain

$$\begin{aligned}\alpha t_n(\alpha,z) = (-1)^{m}(\alpha s_n(\alpha,z) + z t_n(\alpha,z))^{m-1} s_n(\alpha,z) t_n(\alpha,z) \\ - \overline{z}\ s_n(\alpha,z) + \varepsilon_n(\alpha,z).\end{aligned} \tag{2.48}$$

Analogously we obtain

$$\alpha u_n(\alpha,z) = (-1)^m(\alpha s_n(\alpha,z)+\overline{z}u_n(\alpha,z))^{m-1}s_n(\alpha,z)u_n(\alpha,z) \\ - z\, s_n(\alpha,z)+\varepsilon_n(\alpha,z). \tag{2.49}$$

Since $|\alpha|\ge v$, we may rewrite these equation as follows

$$\begin{aligned} t_n(\alpha,z) &= (-1)^m(\alpha s_n(\alpha,z)+zt_n(\alpha,z))^{m-1}\alpha^{-1}s_n(\alpha,z)t_n(\alpha,z) \\ &\qquad - \overline{z}\, s_n(\alpha,z)\alpha^{-1}+\varepsilon_n(\alpha,z) \\ u_n(\alpha,z) &= (-1)^m(\alpha s_n(\alpha,z)+\overline{z}u_n(\alpha,z))^{m-1}\alpha^{-1}s_n(\alpha,z)u_n(\alpha,z) \\ &\qquad - z\, s_n(\alpha,z)\alpha^{-1}+\varepsilon_n(\alpha,z). \end{aligned}$$

The rest of the proof is the same as in the proof of Theorem 3.1, [9], p. 11-13. For the readers convenience we repeat it here. We note that, for some numerical constant $C>0$,

$$\begin{aligned} |\alpha s_n(\alpha,z)| &\le 1+\Big|\frac{1}{2n}\mathbb{E}\operatorname{Tr}\mathbf{RV}\Big| \le 1+v^{-1}\frac{C}{n}(\mathbb{E}^{\frac12}\|\mathbf{W}\|_2+n|z|) \\ &\le C(1+\frac{|z|}{v}), \end{aligned} \tag{2.50}$$

and

$$\max\{|\overline{z}t_n(\alpha,z)|,|zu_n(\alpha,z)|\}\le\frac{|z|}{v}. \tag{2.51}$$

Introduce notation

$$\begin{aligned} P &:= P(\alpha,z)=\alpha s_n(\alpha,z)+\overline{z}u_n(\alpha,z) \\ Q &:= Q(\alpha,z)=\alpha s_n(\alpha,z)+zt_n(\alpha,z). \end{aligned}$$

Multiplying (2.48) by z and (2.49) by $\overline{z}$ and subtracting the second one from the first equation, we obtain

$$\begin{aligned} zt_n(\alpha,z)-\overline{z}u_n(\alpha,z) &= (zt_n(\alpha,z)-\overline{z}u_n(\alpha,z)) \\ &\quad\times s_n(\alpha,z)zt_n(\alpha,z)\alpha^{-1}(P^{m-2}+QP^{m-3}+\cdots+Q^{m-2}) \\ &\quad+Q^{m-1}s_n(\alpha,z)\alpha^{-1}(zt_n(\alpha,z)-\overline{z}u_n(\alpha,z))+\varepsilon(\alpha,z). \end{aligned} \tag{2.52}$$

Using inequalities (2.50), (2.51) and $|s_n(\alpha,z)|\le v^{-1}$, we get

$$\begin{aligned} |s_n(\alpha,z)zt_n(\alpha,z)\alpha^{-1}(P^{m-2}+QP^{m-3}+\cdots+Q^{m-2})| &\le \frac{C^{m-1}m(1+\frac{|z|}{v})^{m-2}}{v^3}, \\ |Q^{m-1}s_n(\alpha,z)\alpha^{-1}| &\le \frac{C^{m-1}(1+\frac{|z|}{v})^{m-2}}{v^3}. \end{aligned} \tag{2.53}$$

From relations (2.52) and (2.53) we may conclude that there exists $V_0 = V_0(m,z)$ depending on m and z such that for all $v\ge V_0$

$$zt_n(\alpha,z)=\overline{z}u_n(\alpha,z)+\varepsilon_n(\alpha,z). \tag{2.54}$$

The last relation implies that

$$\mathcal{A}_1=\mathcal{A}_2+\varepsilon_n(\alpha,z). \tag{2.55}$$

Relations (2.27), 2.44), (2.46), (2.54, and (2.55) together imply

$$1+\alpha s_n(\alpha,z)=(-1)^m(\alpha s_n(\alpha,z)+z\ t_n(\alpha,z))^{m-1}s_n^2(\alpha,z) \\ -z\ t_n(\alpha,z)+\varepsilon_n(\alpha,z). \tag{2.56}$$

Introduce the notations

$$g_n:=s_n(\alpha,z),\quad w_n:=\alpha+\frac{z\ t_n(\alpha,z)}{g_n}.$$

Using these notations we may rewrite the equations (2.56) and (2.54) as follows

$$1+w_ng_n=(-1)^mg_n^{m+1}w_n^{m-1}+\varepsilon_n(\alpha,z) \\ (w_n-\alpha)+(w_n-\alpha)^2g_n-g_n|z|^2=\varepsilon_n(\alpha,z). \tag{2.57}$$

Let $n,n'\to\infty$. Consider the difference $g_n-g_{n'}$. From the first inequality it follows that

$$|g_n-g_{n'}|\le\frac{|\varepsilon_{n,n'}(\alpha,z)|+|w_n-w_{n'}||g_n+(-1)^{m+1}g_{n'}^{m+1}(w_n^{m-2}+\dots+w_{n'}^{m-2})|}{|w_n+(-1)^{m+1}y_{n'}^{m+1}(w_n+(-1)^{m+1}w_n^{m-1}(g_n^m+\dots+g_{n'}^m)|}$$

Note that $\max\{|g_n|,\ |g_{n'}|\}\le\frac{1}{v}$ and $\max\{|w_n|,\ |w_{n'}|\}\le C+v$ for some positive constant $C=C(m)$ depending of m. We may choose a sufficiently large V_0' such that for any $v\ge V_0'$ we obtain

$$|g_n-g_{n'}|\le\frac{|\varepsilon_{n,n'}(\alpha,z)|}{v}+\frac{C}{v}|w_n-w_{n'}|. \tag{2.58}$$

Furthermore, the second equation in (2.57) implies that

$$(w_n-w_{n'})(1+g_n(w_n+w_{n'}-2\alpha)) \\ =(g_n-g_{n'})((w_n-\alpha)^2-|z|^2)+\varepsilon_{n,n'}(\alpha,z).$$

It is straightforward to check that $\max\{|w_n-\alpha|,|w_{n'}-\alpha|\}\le(1+|\varepsilon_n(\alpha,z)|)|z|$. This implies that there exists V_1 such that for any $v\ge V_1$

$$|w_n-w_{n'}|\le|\varepsilon_{n,n'}(\alpha,z)|+4|z|^2|g_n-g_n'|. \tag{2.59}$$

Inequalities (2.58) and (2.59) together imply that there exists a constant $V_0=\max\{V_0',V_1\}$ such that for any $v\ge V_0$

$$|g_n-g_n'|\le|\varepsilon_{n,n'}(\alpha,z)|,$$

where $\varepsilon_{n,n'}(\alpha,z)\to0$ as $n,n'\to\infty$ uniformly with respect to $v\ge V_0$ and $|u|\le C$ ($\alpha=u+iv$).

Since $g_n,g_{n'}$ are locally bounded analytic functions in the upper half-plane we may conclude by Montel's Theorem (see, for instance, [3], p. 153, Theorem 2.9) that there exists an analytic function g_0 in the upper half-plane such that $\lim g_n=g_0$. Since g_n are Nevanlinna functions, (that is analytic functions mapping the upper half-plane into itself) g_0 will be a Nevanlinna function too and there exists some distribution function $G(a,z)$ such that

$$g_0=\int_{-\infty}^{\infty}\frac{1}{a-\alpha}dG(a,z)$$

and

$$\Delta_n(z) := \sup_a |G_n(a,z) - G(a,z)| \to 0 \quad \text{as} \quad n \to \infty.$$

The function g_0 satisfies the equations (2.25). Thus Proposition 2.5 is proved. □

The Lemma 2.4 and Proposition 2.5 together conclude the proof of Theorem 2.1. Thus Theorem 2.1 is proved.

3. The minimal singular value of matrix $\mathbf{V}(z)$

We shall use the following theorem which was proved in [7].

Theorem 3.1. *Assume that X_{jk}, $1 \le j,k \le n$, satisfy the conditions* **(C0)** *and* **(UI)**. *Let $\mathbf{X} = \{X_{jk}\}$ denote a $n \times n$ random matrix with the entries X_{jk} and let $\mathbf{M}_n$ denote a non-random matrix with $\|\mathbf{M}_n\| \le Kn^Q =: K_n$ for some $K > 0$ and $Q \ge 0$. Then there exist constants $C, A, B > 0$ depending on K, Q and ρ such that*

$$\mathbb{P}(s_n \le n^{-B}) \le Cn^{-A}, \tag{3.1}$$

Lemma 3.2. *Under the conditions of Theorem 1.1 there exists a constant C such that for any $k \le n(1 - C\Delta_n^{\frac{1}{m+1}}(z))$,*

$$\mathbb{P}\{s_k \le \Delta_n(z)\} \le C\Delta_n^{\frac{1}{m+1}}(z).$$

Proof. We may write, for any $k = 1, \dots, n$,

$$\mathbb{P}\{s_k \le \Delta_n(z)\} \le \mathbb{P}\{\overline{\mathcal{G}}_n(s_k, z) \le \overline{\mathcal{G}}_n(\Delta_n(z), z)\} \le \mathbb{P}\{\frac{n-k}{n} \le \overline{\mathcal{G}}_n(\Delta_n(z), z)\}.$$

Applying Chebyshev's inequality, we obtain

$$\mathbb{P}\{s_k \le \Delta_n(z)\} \le \frac{n\,\mathbb{E}\,\overline{\mathcal{G}}_n(\Delta_n(z), z)}{n-k} \le \frac{n(\overline{G}(\Delta_n(z), z) + 2\Delta_n(z)}{n-k}.$$

It is straightforward to check that from the system of equations (2.25) it follows

$$\overline{G}(\Delta_n(z), z) \le C\Delta_n^{\frac{2}{m+1}}(z).$$

The last inequality concludes the proof of Lemma 3.2. □

Lemma 3.3. *Let $n_1 := [n - n\delta_n] + 1$ and $n_2 := [n - n^\gamma]$ for any sequence $\delta_n \to 0$, and some $0 < \gamma < 1$. Under the conditions of Theorem 1.1 we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{n_1 \le j \le n_2} \ln s_j(\mathbf{X}^{(q)}) = 0, \quad \textit{for} \quad q = 1, \dots, m-1,$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{n_1 \le j \le n_2} \ln s_j(\mathbf{X}^{(m)} + \mathbf{M}_n) = 0,$$

where $||\mathbf{M}_n|| \le n^Q$ for some $Q > 0$.

Proof. The claim follows from the bound

$$s_j(\mathbf{X}^{(\nu)} + \mathbf{M}_n) \ge c\frac{n-j}{n}, \quad 1 \le j \le n - n^{\gamma}. \tag{3.2}$$

To prove this we need the following simple Lemma.

Lemma 3.4. *Let* $\lim_{n\to\infty} \delta_n = 0$ *and let* q_j*, for* $n_1 \le j \le n_2$ *with* $0 < \gamma < 1$ *denote numbers satisfying the inequalities*

$$n^Q \ge q_j \ge c\frac{n-j}{n}$$

for some constant $Q > 0$*. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{n_1 \le j \le n_2} \ln q_j = 0.$$

Proof. Note that

$$0 \le \frac{1}{n} \sum_{n_1 \le j \le n_2:\ q_j \ge 1} \ln q_j \le Q n^{-(1-\gamma)} \ln n \to 0, \text{ as } n \to \infty.$$

Without loss of generality we may assume that $0 < q_j \le 1$. By the conditions of Lemma 3.4, we have

$$0 \ge \frac{1}{n} \sum_{n_1 \le j \le n_2} \ln q_j \ge \frac{1}{n} \sum_{n_1 \le j \le n_2} \ln\{\frac{n-j}{n}\} = A.$$

After summation and using Stirling's formula, we get

$$|A| \le \frac{1}{n} \ln\left\{\frac{n_1!}{n_2! n^{n_2 - n_1}}\right\} \le \delta_n |\ln \delta_n| + (1-\gamma) n^{\gamma - 1} \ln n \to 0 \text{ as } n \to \infty. \tag{3.3}$$

This proves Lemma 3.4. □

We continue the proof of Lemma 3.3. It remains to prove the inequality (3.2). Similar result for matrices with independent entries was proved by Tao and Vu in [19] (see inequality (8.4) in [19]). It represents the crucial result in their proof of the circular law assuming the second moment only. For completeness we give here a simple modification of their proof for the case of random matrices with correlated entries. We start from the following

Statement 3.5. *Let* $1 \le d \le n - n^{\gamma}$ *with* $\frac{8}{15} < \gamma < 1$*. and* $0 < c < 1$*, and* $\mathbb{H}$ *be a (deterministic)* d*-dimensional subspace of* $\mathbb{C}^n$*. Let* X_j *be independent random variables with* $\mathbb{E}\, X_j = 0$ *and* $\mathbb{E}\, |X_j|^2 = 1$*, squares of which are uniformly integrable , i.e.*

$$\max_j \mathbb{E}\, |X_j|^2 \mathbb{I}\{|X_j| > M\} \to 0 \quad \text{as} \quad M \to \infty. \tag{3.4}$$

Let $\mathbf{x}^T = (X_1, \ldots, X_n) + (m_1, \ldots, m_N)$ *where* $\mathbf{m}^T = (m_1 + \ldots, m_n)$ *is non-random vector. Then*

$$\mathbb{P}\{\mathrm{dist}(\mathbf{x} + \mathbf{m}, \mathbb{H}) \le c\sqrt{n-d}\} = O(\exp\{-n^{\frac{\gamma}{8}}\}), \tag{3.5}$$

where $\mathrm{dist}(X, \mathbb{H})$ *denotes the Euclidean distance between a vector* X *and a subspace* $\mathbb{H}$ *in* $\mathbb{C}^n$*.*

Proof. It was proved by Tao and Vu in [19] (see Proposition 5.1). Here we sketch their proof. As shown in [19] we may reduce the problem to the case that $\mathbb{E}\,X = 0$. For this it is enough to consider vectors $\mathbf{x}'$ and $\mathbf{v}$ such that $\mathbf{x} = \mathbf{x}' + \mathbf{v}$ and $\mathbb{E}\,\mathbf{x}' = 0$. Instead of the subspace $\mathbb{H}$ we may consider subspace $\mathbb{H}' = \mathrm{span}(\mathbb{H}, \mathbf{v})$ and note that

$$\mathrm{dist}(\mathbf{x}, \mathbb{H}) \geq \mathrm{dist}(\mathbf{x}', \mathbb{H}'). \tag{3.6}$$

The claim follows now from a corresponding result for random vectors with mean zero. In what follows we assume that $\mathbb{E}\,\mathbf{x} = 0$. We reduce the problem to vectors with bounded coordinates. Let $\xi_j = \mathbb{I}\{|X_j| \geq n^{\frac{1-\gamma}{2}}\}$, where X_j denotes the j-th coordinate of a vector $\mathbf{x}$. Note that $p_n := \mathbb{E}\,\xi_j \leq n^{-(1-\gamma)}$. Applying Chebyshev's inequality, we get, for any $h > 0$

$$\mathbb{P}\{\sum_{j=1}^{n} \xi_j \geq 2n^{\gamma}\} \leq \exp\{-hn^{\gamma}\} \exp\{np_n(\mathrm{e}^h - 1 - h)\}.$$

Choosing $h = \frac{1}{4}$, we obtain

$$\mathbb{P}\{\sum_{j=1}^{n} \xi_j \geq 2n^{\gamma}\} \leq \exp\{-\frac{n^{\gamma}}{8}\}. \tag{3.7}$$

Let $J \subset \{1, \dots, n\}$ and $E_J := \{\prod_{j \in J}(1 - \xi_j) \prod_{j \notin J} \xi_j = 1\}$. Inequality (3.7) implies

$$\mathbb{P}\{\bigcup_{J:|J| \geq n - 2n^{\gamma}} E_J\} \geq 1 - \exp\{-\frac{n^{\gamma}}{8}\}.$$

Let J with $|J| \geq n - 2n^{\gamma}$ be fixed. Without loss of generality we may assume that $J = 1, \dots, n'$ with some $n - 2n^{\gamma} \leq n' \leq n$. It is now sufficient to prove that

$$\Pr\{\mathrm{dist}(\mathbf{x}, \mathbb{H}) \leq c\sqrt{n - d} | E_J\} = O(\exp\{-\frac{n^{\gamma}}{8}\}). \tag{3.8}$$

Let π denote the orthogonal projection $\pi : \mathbb{C}^n \to \mathbb{C}^{n'}$. We note that

$$\mathrm{dist}(\mathbf{x}, \mathbb{H}) \geq \mathrm{dist}(\pi(\mathbf{x}), \pi(\mathbb{H})). \tag{3.9}$$

Let $\widetilde{X}$ be a random variable X conditioned on the event $|X| \leq n^{1-\gamma}$ and let $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \dots, \widetilde{x}_n)$. The relation (3.8) will follow now from

$$\mathbb{P}\{\mathrm{dist}(\widetilde{\mathbf{x}'}, \mathbb{H}') \leq c\sqrt{n - d}\,\big||x_j| \leq n^{1-\gamma}, j \notin J\} = O(\exp\{-\frac{n^{\gamma}}{8}\}),$$

where $\mathbb{H}' = \pi(\mathbb{H})$ and $\widetilde{\mathbf{x}'} = \pi(\widetilde{\mathbf{x}})$. We may represent the vector $\widetilde{\mathbf{x}}$as $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}'} + \mathbf{v}$, where $\mathbf{v} = \mathbb{E}\,\widetilde{\mathbf{x}}$ and $\mathbb{E}\,\widetilde{\mathbf{x}'} = 0$. We reduce the claim to the bound

$$\mathbb{P}\{\mathrm{dist}(\widetilde{\mathbf{x}'}, \mathbb{H}'') \leq c\sqrt{n - d}\,\big||x_j| \leq n^{1-\gamma}, j \notin J\} = O(\exp\{-\frac{n^{\gamma}}{8}\}), \tag{3.10}$$

where $\mathbb{H}'' = \mathrm{span}(v, \mathbb{H}')$. In what follows we shall omit the symbol $'$ in the notations. To prove (3.10) we shall apply the following result of Maurey. Let

$\mathbb{X}$ denote a normed space and f denote a convex function on $\mathbb{X}$. Define the functional Q as follows

$$Qf(x) := \inf_{y\in\mathbb{X}}[f(y) + \frac{\|x-y\|^2}{4}].$$

Definition 3.6. *We say that a measure μ satisfies the convex property (τ) if for any convex function f on $\mathbb{X}$*

$$\int_{\mathbb{X}} \exp\{Qf\} d\mu \int_{\mathbb{X}} \exp\{-f\} d\mu \le 1.$$

We reformulate the following result of Maurey (see [12], Theorem 3). Following Maurey we shall say that ν has diameter ≤ 1 as a short way to express that ν is supported by a set of diameter ≤ 1.

Theorem 3.7. *Let $(\mathbb{X}_i)$ be a family of normed spaces; for each i, let ν_i be a probability measure with diameter ≤ 1 on $\mathbb{X}_i$. If ν is the product of a family (ν_i), then ν satisfies the convex property (τ).*

As corollary of Theorem 3.7 we get

Corollary 3.8. *Let ν_i be a probability measure with diameter ≤ 1 on $\mathbb{X}$, $i = 1, \dots, n$. Let g denote a convex 1-Lipshitz function on $\mathbb{X}^n$. Let $M(g)$ denote a median of g. If ν is the product of the family (ν_i), then*

$$\nu\{|g - M(g)| \ge h\} \le 4\exp\{-\frac{h^2}{4}\}.$$

Applying Corollary 3.8 to ν_i, being the distribution of $\widetilde{x}_i$, we get

$$\mathbb{P}\left\{|\mathrm{dist}(\widetilde{\mathbf{x}}, \mathbb{H}) - M(\mathrm{dist}(\widetilde{\mathbf{x}}, \mathbb{H}))| \ge rn^{\frac{1-\gamma}{2}}\right\} \le 4\exp\{-r^2/16\}. \tag{3.11}$$

The last inequality implies that there exists a constant $C > 0$ such that

$$|\,\mathbb{E}\,\mathrm{dist}(\widetilde{\mathbf{x}}, \mathbb{H}) - M(\mathrm{dist}(\widetilde{\mathbf{x}}, \mathbb{H}))| \le Cn^{\frac{1-\gamma}{2}}, \tag{3.12}$$

and

$$\mathbb{E}\,\mathrm{dist}(\widetilde{\mathbf{x}}, \mathbb{H}) \ge \sqrt{\mathbb{E}(\mathrm{dist}(\widetilde{\mathbf{x}}, \mathbb{H}))^2} - Cn^{\frac{1-\gamma}{2}}. \tag{3.13}$$

By Lemma 5.3 in [19]

$$\mathbb{E}(\mathrm{dist}(\widetilde{\mathbf{x}}, \mathbb{H}))^2 = (1 - o(1))(n - d). \tag{3.14}$$

Since $n - d \ge n^\gamma$ the inequalities (3.12), (3.13) and (3.14) together imply (3.5).

Now we prove (3.2). We repeat the proof of Tao and Vu [19], inequality (8.4). Fix j. Let $\mathbf{A}_n = \mathbf{X}^{(\nu)} - z\mathbf{M}_n$ and let $\mathbf{A}'_n$ denote a matrix formed by the first $n' = n - k$ rows of $\sqrt{n}\mathbf{A}_n$ with $k = j/2$. Let σ_l (σ'_l), $1 \le l \le n - k$, be the singular values of $\mathbf{A}_n$ $(\mathbf{A}'_n)$ (in decreasing order). By the interlacing property and re-normalizing we get

$$\sigma_{n-j} \ge \frac{1}{\sqrt{n}}\sigma'_{n-j}.$$

By Lemma A.4 in [19]

$$T := {\sigma'_1}^{-2} + \cdots + {\sigma'_{n-k}}^{-2} = \mathrm{dist}_1^{-2} + \cdots + \mathrm{dist}_{n-k}^{-2}, \tag{3.15}$$

with

$$\mathrm{dist}_j = \mathrm{dist}(\mathbf{x}_j, \mathbb{H}_j),$$

where $\mathbf{x}_j$ is the j-th row of matrix $\mathbf{A}'_n$ and $\mathbb{H}_j$ denotes hyperplane generated by the $n'-1$ rows $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{n'}$. Let π_j denote the projector onto $\mathbb{R}_j^{n-1}$ in $\mathbb{R}^n$ defined by $\pi_j(\mathbf{x}) = (X_1, \ldots, X_{j-1}, 0.X_{j+1}, \ldots, X_n)$. Then we have

$$\mathrm{dist}(\mathbf{x}_j, \mathbb{H}_j) \geq \mathrm{dist}(\pi_j(\mathbf{x}), \pi_j(\mathbb{H}_j)).$$

Note that vector $\pi_j(\mathbf{x})$ and subspace $\pi_j(\mathbb{H}_j)$ are independent and vector $\pi_j(\mathbf{x})$ has independent coordinates. From (3.15)

$$T \geq (j-k){\sigma'}_{n-j}^{-2} = \frac{j}{2}{\sigma'}_{n-j}^{-2} \geq \frac{j}{2n}\sigma_{n-j}^{-2}.$$

Applying Proposition 3.5, we get that with probability $1 - \exp\{-n^\gamma\}$

$$T \leq \frac{n}{j}.$$

Combining the last inequalities, we get (3.2). Thus Proposition 3.5 is proved. □

This finishes the proof of Lemma. □

Lemma 3.9. *Assume the assumptions of Theorem 1.1 hold, then* $\ln(\cdot)$ *is uniformly integrable in probability with respect to* $\{\nu_n\}_{n\geq 1}$.

Proof of Lemma 3.9. It is enough to check that

$$\lim_{t\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{P}\left(\int_0^\infty |\ln x| \nu_n(dx) > t\right) = 0 \tag{3.16}$$

Let $k_0 = [n(1 - C\Delta_n^{\frac{1}{m+1}}(z))]$. We introduce the event

$$\Omega_0 := \Omega_{0,n} := \{\omega \in \Omega : s_n(\mathbf{X}^{(q)}) \geq n^{-b}, q = 1, ..., m-1,$$
$$s_n(\mathbf{X}^{(m)} + \mathbf{M}_n) \geq n^{-b}, s_{k_0} \geq \Delta_n(z)\}.$$

for some $b > 0$ which will be chosen later and $\mathbf{M}_n = -z(\prod_{i=1}^{m-1} \mathbf{X}^{(q)})^{-1}$. Note that the matrices $\mathbf{X}^{(m)}$ and $\mathbf{M}_n$ are independent and it follows from Theorem 3.1 that $\|\mathbf{M}_n\|_2 \leq n^Q$ for some $Q > 0$ with probability close to one. From Theorem 3.1 and Lemma 3.3 we conclude that $\overline{\lim}_{n\to\infty} \mathbb{P}(\Omega_0^c) = 0$. It follows that it is enough to prove that

$$\lim_{t\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{P}\left(\int_0^\infty |\ln x| \nu_n(dx) > t, \Omega_0\right) = 0$$

We may split the integral $\int_0^\infty |\ln x|\nu_n(dx)$ into three terms

$$T_1 := -\int_0^{\Delta_n} \ln x \nu_n(dx, z),$$
$$T_2 := \int_{\Delta_n}^{\Delta_n^{-1}} |\ln x|\nu_n(dx, z),$$
$$T_3 := \int_{\Delta_n^{-1}}^\infty \ln x \nu_n(dx, z).$$

Denote by $n' := k_0 + 1$ and $n'' := [n - n^{1-\gamma}]$. We consider the term T_1 which we may rewrite as

$$T_1 = -\frac{1}{n}\sum_{i=n'+1}^n \ln s_i.$$

We shall use the following well-known fact. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices and let $s_1(\mathbf{A}) \geq \cdots \geq s_n(\mathbf{A})$ resp. ($s_1(\mathbf{B}) \geq \cdots \geq s_n(\mathbf{B})$ and $s_1(\mathbf{AB}) \geq \cdots \geq s_n(\mathbf{AB})$) denote the singular value of a matrix $\mathbf{A}$ (and the matrices $\mathbf{B}$ and $\mathbf{AB}$ respectively). Then we have

$$\prod_{j=k}^n s_j(\mathbf{AB}) \geq \prod_{j=k}^n s_j(\mathbf{A}) s_j(\mathbf{B}), \tag{3.17}$$

and

$$\prod_{j=1}^n s_j(\mathbf{AB}) = \prod_{j=1}^n s_j(\mathbf{A}) s_j(\mathbf{B}),$$

for any $1 \leq k \leq n$ (see, for instance [11], p.171, Theorem 3.3.4). From (3.17) it follows that

$$T_1 \leq -\frac{1}{n}\sum_{q=1}^{m-1}\sum_{i=n'+1}^n \ln s_i(\mathbf{X}^{(q)}) - \frac{1}{n}\sum_{i=n'+1}^n \ln s_i(\mathbf{X}^{(m)} + \mathbf{M}_n) =$$
$$-\frac{1}{n}\sum_{q=1}^{m-1}\sum_{i=n'+1}^{n''} \ln s_i(\mathbf{X}^{(q)}) - \frac{1}{n}\sum_{i=n'+1}^{n''} \ln s_i(\mathbf{X}^{(m)} + \mathbf{M}_n)$$
$$-\frac{1}{n}\sum_{q=1}^{m-1}\sum_{i=n''+1}^{n} \ln s_i(\mathbf{X}^{(q)}) - \frac{1}{n}\sum_{i=n''+1}^{n} \ln s_i(\mathbf{X}^{(m)} + \mathbf{M}_n)$$

From Lemma 3.3, inequality (3.3) and definition of Ω_0 it follows that

$$T_1 \leq Cn^{\gamma-1}\ln n + \Delta_n|\ln \delta_n| \to 0 \text{ as } n \to \infty$$

For the term T_3 we may write the bound

$$T_3 \leq \Delta_n|\ln \Delta_n| \int_0^\infty x^2 \nu_n(dx, z) \to 0 \text{ as } n \to \infty,$$

where we have used the fact that $x^{-2}\ln x$ is a decreasing function for $x \geq \sqrt{e}$. It remains to estimate T_2. Integrating by parts and using (2.26) we write

$$\mathbb{E}\, T_2 \leq C\Delta_n|\ln \Delta_n| + \int_{\Delta_n}^{\Delta_n^{-1}} |\ln x| dG(x, z) < \infty$$

Using Markov's inequality we finish the proof of Lemma. □

4. Appendix

Lemma 4.1. *Under the conditions of Theorem 1.1 we have, for any* $j,k = 1,\ldots,n,$ *and for any* $1 \le \alpha \le \beta \le m$,

$$\mathbb{E}[\mathbf{V}_{\alpha,\beta}]_{jk} = 0$$

Proof. For $\alpha = \beta$ the claim is easy. Let $\alpha < \beta$. Direct calculations show that

$$\mathbb{E}[\mathbf{V}_{\alpha,\beta}]_{jk} = \frac{1}{n^{\frac{\beta-\alpha}{2}}}\sum_{j_1=1}^{p_\alpha}\sum_{j_2=1}^{p_{\alpha+1}}\cdots\sum_{j_{\beta-\alpha}=1}^{p_{\beta-1}}\mathbb{E}\,X^{(\alpha)}_{j,j_1}X^{(\alpha+1)}_{j_1,j_2}\cdots X^{(\beta)}_{j_{\beta-\alpha},k} = 0$$

Thus the Lemma is proved. □

In all Lemmas below we shall assume that

$$\mathbb{E}\,X^{(\nu)}_{jk} = 0, \quad \mathbb{E}\,|X^{(\nu)}_{jk}|^2 = 1, \quad |X^{(\nu)}_{jk}| \le c\tau_n\sqrt{n} \quad \text{a. s.} \tag{4.1}$$

Lemma 4.2. *Under the conditions of Theorem 1.1 assuming (4.1), we have, for any* $1 \le \alpha \le \beta \le m$,

$$\mathbb{E}\,\|\mathbf{V}_{\alpha,\beta}\|_2^2 \le Cn$$

Proof. We shall consider the case $\alpha < \beta$ only. The other cases are obvious. Direct calculation shows that

$$\mathbb{E}\,\|\mathbf{V}_{\alpha,\beta}\|_2^2 \le \frac{C}{n^{\beta-\alpha+1}}\sum_{j=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_{\beta-\alpha}=1}^{n}\sum_{k=1}^{n}\mathbb{E}[X^{(\alpha)}_{j,j_1}X^{(\alpha+1)}_{j_1,j_2}\cdots X^{(\beta)}_{j_{\beta-\alpha},k}]^2$$

By independents of random variables, we get

$$\mathbb{E}\,\|\mathbf{V}_{\alpha,\beta}\|_2^2 \le Cn$$

Thus the Lemma is proved. □

Lemma 4.3. *Under the condition of Theorem 1.1 and assumption (4.1) we have, for any* $j,k = 1,\ldots,n$, *and* $r \ge 1$,

$$\mathbb{E}\,\|\mathbf{V}_{\alpha,\beta}\mathbf{e}_k\|_2^{2r} \le C_r, \quad \mathbb{E}\,\|\mathbf{V}_{\alpha,\beta}\mathbf{e}_{j+n}\|_2^{2r} \le C_r \tag{4.2}$$

and

$$\mathbb{E}\,\|{\mathbf{e}_j}^T\mathbf{V}_{\alpha,\beta}\|_2^{2r} \le C_r, \quad \mathbb{E}\,\|{\mathbf{e}_{k+n}}^T\mathbf{V}_{\alpha,\beta}\|_2^{2r} \le C_r, \tag{4.3}$$

with some positive constant C_r *depending on* r. *Moreover, for any* $q = 1,\ldots,m$ *and any* $l,s = 1,\ldots,n$,

$$\mathbb{E}\left\{\|{\mathbf{e}_j}^T\mathbf{V}_{\alpha,\beta}\|_2^{2r}\Big|X^{(q)}_{ls}, X^{(q)}_{sl}\right\} \le C_r. \tag{4.4}$$

and

$$\mathbb{E}\left\{\|\mathbf{V}_{\alpha,\beta}\mathbf{e}^{(\beta)}_{j+n}\|_2^{2r}\Big|X^{(\nu)}_{lq}, X^{(q)}_{sl}\right\} \le C_r. \tag{4.5}$$

Proof. By definition of the matrices $\mathbf{V}_{\alpha,\beta}$, we may write

$$\|\mathbf{e}_j^T \mathbf{V}_{\alpha,\beta}\|_2^2 = \frac{1}{n^{\beta-\alpha}} \sum_{l=1}^n \left| \sum_{j_\alpha=1}^n \cdots \sum_{j_{\beta-1}=1}^n X_{jj_\alpha}^{(\alpha)} \cdots X_{j_{\beta-1}l}^{(\beta)} \right|^2$$

Using this representation, we get

$$\mathbb{E}\,\|\mathbf{e}_j^T \mathbf{V}_{\alpha,\beta}\|_2^{2r} = \frac{1}{n^{(\beta-\alpha)r}}$$

$$\times \sum_{l_1=1}^n \cdots \sum_{l_r=1}^n \mathbb{E} \prod_{q=1}^r \left(\sum_{j_\alpha=1}^n \cdots \sum_{j_{\beta-1}=1}^n \sum_{\widehat{j}_\alpha=1}^n \cdots \sum_{\widehat{j}_{\beta-1}=1}^n A^{(l_q)}_{(j_\alpha,\ldots,j_{\beta-1},\widehat{j}_\alpha,\ldots,\widehat{j}_{\beta-1})} \right) \tag{4.6}$$

where

$$A^{(l_q)}_{(j_\alpha,\ldots,j_{\beta-1},\widehat{j}_1,\ldots,\widehat{j}_{\beta-1})} :=$$

$$X_{jj_\alpha}^{(\alpha)} \overline{X}_{j\widehat{j}_\alpha}^{(\alpha)} X_{j_\alpha j_{\alpha+1}}^{(\alpha+1)} X_{\widehat{j}_\alpha \widehat{j}_{\alpha+1}}^{(\alpha+1)} \cdots X_{j_{\beta-2}j_{\beta-1}}^{(\beta-1)} X_{\widehat{j}_{\beta-2}\widehat{j}_{\beta-1}}^{(\beta-1)} X_{j_{\beta-1}l_q}^{(\beta)} X_{\widehat{j}_{\beta-1}l_q}^{(\beta)}. \tag{4.7}$$

Rewriting the product on the r.h.s of (4.6), we get

$$\mathbb{E}\,\|\mathbf{e}_j^T \mathbf{V}_{\alpha,\beta}\|_2^{2r} = \frac{1}{n^{(\beta-\alpha)r}} \sum\nolimits^{**} \mathbb{E} \prod_{q=1}^r A^{(l_q)}_{(j_\alpha^{(q)},\ldots,j_{\beta-1}^{(q)},\widehat{j}_1^{(\nu)},\ldots,\widehat{j}_{\beta-1}^{(q)})}, \tag{4.8}$$

where $\sum^{**}$ is taken over all set of indices $j_\alpha^{(q)},\ldots,j_{\beta-1}^{(q)}, l_q$ and $\widehat{j}_\alpha^{(\nu)},\ldots,\widehat{j}_{\beta-1}^{(q)}$ where $j_k^{(q)}, \widehat{j}_k^{(q)} = 1,\ldots,p_k$, $k = \alpha,\ldots,\beta-1$, $l_q = 1,\ldots,n$ and $q = 1,\ldots,r$. Note that the summands in the right hand side of (4.7) is equal 0 if there is at least one term in the product (4.7) which appears only one time. This implies that the summands in the right hand side of (4.8) is not equal zero only if the union of all sets of indices in r.h.s of (4.7) consist from at least r different terms and each term appears at least twice.

Introduce the random variables, for $q = \alpha+1,\ldots,\beta-1$,

$$\zeta^{(q)}_{j_{q-1}^{(1)},\ldots,j_{q-1}^{(r)},j_q^{(1)},\ldots,j_q^{(r)},\widehat{j}_{q-1}^{(1)},\ldots,\widehat{j}_{q-1}^{(r)},\widehat{j}_q^{(1)},\ldots,\widehat{j}_q^{(r)}} :=$$

$$X^{(q)}_{j_{q-1}^{(1)},j_q^{(1)}} \cdots X^{(q)}_{j_{q-1}^{(r)},j_q^{(r)}} X^{(q)}_{\widehat{j}_{q-1}^{(1)},\widehat{j}_q^{(1)}}, \cdots X^{(q)}_{\widehat{j}_{q-1}^{(r)},\widehat{j}_q^{(r)}},$$

and

$$\zeta^{(\alpha)}_{j_1^{(1)},\ldots,j_1^{(r)},\widehat{j}_1^{(1)},\ldots,\widehat{j}_1^{(r)}} := X^{(\alpha)}_{jj_1^{(\alpha)}} \cdots X^{(\alpha)}_{j_a^{(r)} j_{a+1}^{(r)}} X^{(\alpha)}_{j\widehat{j}_a^{(1)}} \cdots X^{(\alpha)}_{\widehat{j}_a^{(r)},\widehat{j}_{a+1}^{(r)}}$$

$$\zeta^{(\beta)}_{j_{\beta-1}^{(1)},\ldots,j_{\beta-1}^{(r)},\widehat{j}_{\beta-1}^{(1)},\ldots,\widehat{j}_{\beta-1}^{(r)},l_q} := X^{(\beta)}_{j_{\beta-1}^{(1)} j_\beta^{(1)}} \cdots X^{(\beta)}_{j_{\beta-1}^{(r)} l_q} X^{(\beta)}_{\widehat{j}_{\beta-1}^{(1)},l_q}, \cdots X^{(\beta)}_{\widehat{j}_{\beta-1}^{(r)},l_q}.$$

Assume that the set of indices $j_\alpha^{(1)},\ldots,j_\alpha^{(r)},\widehat{j}_\alpha^{(1)},\ldots,\widehat{j}_\alpha^{(r)}$ contains t_α different indexes, say $i_1^{(\alpha)},\ldots,i_{t_\alpha}^{(\alpha)}$ with multiplicities $k_1^{(\alpha)},\ldots,k_{t_\alpha}^{(\alpha)}$ respectively, $k_1^{(\alpha)} + \ldots + k_{t_\alpha}^{(\alpha)} = 2r$. Note that $\min\{k_1^{(\alpha)},\ldots,k_{t_\alpha}^{(\alpha)}\} \geq 2$. Otherwise,

$|\mathbb{E}\,\zeta^{(\alpha)}_{j_a^{(1)},\dots,j_a^{(r)},\widehat{j}_\alpha^{(1)},\dots,\widehat{j}_\alpha^{(r)}}| = 0$. By assumption (4.1), we have

$$|\mathbb{E}\,\zeta^{(\alpha)}_{j_\alpha^{(1)},\dots,j_\alpha^{(r)},\widehat{j}_\alpha^{(1)},\dots,\widehat{j}_\alpha^{(r)}}| \le C(\tau_n\sqrt{n})^{2r-2t_\alpha} \tag{4.9}$$

Similar bounds we get for $|\mathbb{E}\,\zeta^{(\beta)}_{j_{\beta-1}^{(1)},\dots,j_1^{(r)},\widehat{j}_{\beta-1}^{(1)},\dots,\widehat{j}_{\beta-1}^{(r)},l_q}|$. Assume that the set of indexes $\{j_{\beta-1}^{(1)},\dots,j_{\beta-1}^{(r)},\ \widehat{j}_{\beta-1}^{(1)},\dots,\widehat{j}_{\beta-1}^{(r)}\}$ contains $t_{\beta-1}$ different indices, say, $i_1^{(\beta-1)},\dots,i_{t_{\beta-1}}^{(\alpha)}$ with multiplicities
$k_1^{(\beta-1)},\dots,k_{t_{\beta-1}}^{(\alpha)}$ respectively, $k_1^{(\beta-1)}+\dots+k_{t_{\beta-1}}^{(\alpha)} = 2r$. Then

$$|\mathbb{E}\,\zeta^{(\beta)}_{j_{\beta-1}^{(1)},\dots,j_1^{(r)},\widehat{j}_{\beta-1}^{(1)},\dots,\widehat{j}_{\beta-1}^{(r)},l_q}| \le C(\tau_n\sqrt{n})^{2r-2t_{\beta-1}} \tag{4.10}$$

Furthermore, assume that for $\alpha+1\le q\le \beta-2$ there are t_q different pairs of indices, say, $(i_\alpha,i'_\alpha),\dots(i_{t_\beta},i'_{t_\beta})$ in the set
$\{j_\alpha^{(1)},\dots,j_\alpha^{(r)},\widehat{j}_\alpha^{(1)},\dots,\widehat{j}_\alpha^{(r)},\dots,j_{\beta-1}^{(1)},\dots,j_{\beta-1}^{(r)},\widehat{j}_{\beta-1}^{(1)},\dots,\widehat{j}_{\beta-1}^{(r)},l_1,l_r\}$ with multiplicities
$k_1^{(q)},\dots,k_{t_q}^{(q)}$. Note that

$$k_1^{(q)}+\dots+k_{t_q}^{(q)} = 2r$$

and

$$|\mathbb{E}\,\zeta^{(q)}_{j_{q-1}^{(1)},\dots,j_{q-1}^{(r)},j_q^{(1)},\dots,j_q^{(r)},\widehat{j}_{q-1}^{(1)},\dots,\widehat{j}_{q-1}^{(r)},\widehat{j}_q^{(1)},\dots,\widehat{j}_q^{(r)}}| \le C(\tau_n\sqrt{n})^{2r-2t_q}. \tag{4.11}$$

Inequalities (4.9)-(4.11) together yield

$$|\mathbb{E}\prod_{q=1}^{r} A^{(l_q)}_{(j_\alpha^{(q)},\dots,j_{\beta-1}^{(q)},\widehat{j}_1^{(q)},\dots,\widehat{j}_{\beta-1}^{(q)})}| \le C(\tau_n\sqrt{n})^{2r(\beta-\alpha)-2(t_1+\dots+t_{\beta-\alpha})}. \tag{4.12}$$

It is straightforward to check that the number $\mathcal{N}(t_\alpha,\dots,t_\beta)$ of sequences of indices
$\{j_\alpha^{(1)},\dots,j_\alpha^{(r)},\widehat{j}_\alpha^{(1)},\dots,\widehat{j}_\alpha^{(r)},\dots,j_{\beta-1}^{(1)},\dots,j_{\beta-1}^{(r)},\widehat{j}_{\beta-1}^{(1)},\dots,\widehat{j}_{\beta-1}^{(r)},l_1,\dots,l_r\}$ with $t_\alpha,\dots,t_\beta$ of different pairs satisfies the inequality

$$\mathcal{N}(t_\alpha,\dots,t_\beta) \le Cn^{t_\alpha+\dots+t_\beta}, \tag{4.13}$$

with $1\le t_i\le r,\quad i=\alpha,\dots,\beta$. Note that in the case $t_\alpha=\dots=t_b=r$ the inequalities (4.9)– (4.11) imply

$$\mathbb{E}\,\zeta^{(q)}_{j_{q-1}^{(1)},\dots,j_{q-1}^{(r)},j_\nu^{(1)},\dots,j_q^{(r)},\widehat{j}_{q-1}^{(1)},\dots,\widehat{j}_{q-1}^{(r)},\widehat{j}_\nu^{(1)},\dots,\widehat{j}_q^{(r)}} \le C \tag{4.14}$$

The inequalities (4.13), (4.12), (4.14), and the representation (4.6) together conclude the proof of inequalities (4.2) and (4.3). To prove the inequalities (4.4), (4.5) note that in the case $q\notin[\alpha,\beta]$ and $m-q\notin[\alpha,\beta]$ we have

$$\mathbb{E}\left\{\|\mathbf{e}_j{}^T\mathbf{V}_{\alpha,\beta}\|_2^{2r}\Big|X_{ls}^{(q)},X_{ls}^{(q)}\right\} = \mathbb{E}\,\|\mathbf{e}_j{}^T\mathbf{V}_{\alpha,\beta}\|_2^{2r}$$
$$\mathbb{E}\left\{\|\mathbf{V}_{\alpha,\beta}\mathbf{e}_{j+n}\|_2^{2r}\Big|X_{ls}^{(q)},X_{sl}^{(q)}\right\} = \mathbb{E}\,\|\mathbf{V}_{\alpha,\beta}\mathbf{e}_{j+n}\|_2^{2r}.$$

Thus in the case $q \notin [\alpha, \beta]$ and $m - q \notin [\alpha, \beta]$ the inequalities (4.4) and (4.5) are proved. Consider now the case $q \in [\alpha, \beta]$ and $m - q \notin [\alpha, \beta]$. In this case we may write

$$\mathbf{V}_{\alpha,\beta} = \mathbf{V}_{\alpha,q-1}(\mathbf{H}^{(q,l,s)} + X_{ls}^{(q)}\mathbf{e}_l\mathbf{e}_q{}^T + X_{sl}^{(q)}\mathbf{e}_s\mathbf{e}_l{}^T)\mathbf{V}_{q+1,\beta}, \tag{4.15}$$

where the matrix $\mathbf{H}^{(q,l,s)}$ is obtained from the matrix $\mathbf{H}^{(q)}$ by replacement the entries $X_{ls}^{(q)}$ and $X_{sl}^{(q)}$ by zero. Note that the matrix $\mathbf{H}^{(q,l,s)}$ and random variables $X_{ls}^{(q)}$ and $X_{ls}^{(q)}$ are independent. Let $\mathbf{V}_{\alpha,\beta}^{(q,l,s)} = \mathbf{V}_{\alpha,q-1}\mathbf{H}^{(q,l,s)}\mathbf{V}_{q+1,\beta}$. We may rewrite (4.15) in the form

(4.16)

$$\mathbf{V}_{\alpha,\beta} = \mathbf{V}_{\alpha,\beta}^{(\nu,l,q)} + \frac{1}{\sqrt{n}}X_{ls}^{(q)}\mathbf{V}_{\alpha,\nu-1}\mathbf{e}_l\mathbf{e}_s{}^T\mathbf{V}_{q+1,\beta} + \frac{1}{\sqrt{n}}X_{sl}^{(q)}\mathbf{V}_{\alpha,q-1}\mathbf{e}_s\mathbf{e}_l{}^T\mathbf{V}_{q+1,\beta}$$

From the independence of $\mathbf{V}_{\alpha,q-1}$, $\mathbf{V}_{q+1,\beta}$, $X_{ls}^{(q)}$, $X_{sl}^{(q)}$ and $|X_{ls}^{(q)}|/\sqrt{n} \le \tau_n$, the equality (4.16) it follows that

$$\begin{aligned}\mathbb{E}\left\{\|\mathbf{V}_{\alpha,\beta}\mathbf{e}_j^{(q)}\|_2^{2r}\Big|\xi_{ls}^{(q)}, \xi_{ls}^{(q)}\right\} \le & 2^r\Big(\mathbb{E}\,\|\mathbf{V}_{\alpha,\beta}^{(q,l,s)}\mathbf{e}_j\|_2^{2r} \\ & + \tau_n\,\mathbb{E}\,\|\mathbf{V}_{\alpha,\nu-1}\mathbf{e}_l\|_2^{2r}\,\mathbb{E}\,\|\mathbf{e}_q{}^T\mathbf{V}_{q+1,\beta}\mathbf{e}_j\|_2^{2r}\Big).\end{aligned}$$

The last inequality concludes the proof of inequality (4.4) in the case $q \in [\alpha, \beta]$ and $m - q \notin [\alpha, \beta]$. The proof of inequality (4.5) is similar. The proof of both inequalities (4.4) and (4.5) in the cases $q \notin [\alpha, \beta]$ and $m - q \in [\alpha, \beta]$ and $q \in [\alpha, \beta]$ and $m - q \in [\alpha, \beta]$ is analogously. Thus Lemma 4.3 is proved. ☐

Lemma 4.4. *Under conditions of Theorem 1.1 assuming (4.1), we have*

$$\mathbb{E}\left|\frac{1}{n}\left(\operatorname{Tr}\mathbf{R} - \mathbb{E}\operatorname{Tr}\mathbf{R}\right)\right|^2 \le \frac{C}{nv^2}.$$

Proof. We define the following matrices

$$\mathbf{H}^{(q,j)} = \mathbf{H}^{(q)} - \mathbf{e}_j\mathbf{e}_j^T\mathbf{H}^{(q)} - \mathbf{H}^{(q)}\mathbf{e}_j\mathbf{e}_j^T,$$

and

$$\widetilde{\mathbf{H}}^{(m-q+1,j)} = \mathbf{H}^{(m-q+1)} - \mathbf{H}^{(m-q+1)}\mathbf{e}_{j+n}\mathbf{e}_{j+n}^T - \mathbf{e}_{j+n}\mathbf{e}_{j+n}^T\mathbf{H}^{(m-q+1)},$$

for $q = 1, \ldots, m$ and $j = 1, \ldots, n$. For simplicity we shall assume that $q \le m - q + 1$. Define

$$\mathbf{V}^{(q,j)} = \prod_{\beta=1}^{q-1}\mathbf{H}^{(\beta)}\,\mathbf{H}^{(q,j)}\prod_{\beta=q+1}^{m-q}\mathbf{H}^{(\beta)}\widetilde{\mathbf{H}}^{(m-q+1,j)}\prod_{\beta=m-q+2}^{m}\mathbf{H}^{(\beta)}.$$

Let $\mathbf{V}^{(q,j)}(z) = \mathbf{V}^{(q,j)}\mathbf{J} - \mathbf{J}(z)$. We shall use the following inequality. For any Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$ with spectral distribution function $F_A(x)$ and $F_B(x)$ respectively, we have

$$|\operatorname{Tr}(\mathbf{A} - \alpha\mathbf{I})^{-1} - \operatorname{Tr}(\mathbf{B} - \alpha\mathbf{I})^{-1}| \le \frac{\operatorname{rank}(\mathbf{A} - \mathbf{B})}{v}, \tag{4.17}$$

where $\alpha = u + iv$. It is straightforward to show that

$$\operatorname{rank}(\mathbf{V}(z) - \mathbf{V}^{(q,j)}(z)) = \operatorname{rank}(\mathbf{V}\mathbf{J} - \mathbf{V}^{(q,j)}\mathbf{J}) \le 4m. \tag{4.18}$$

The inequalities (4.17) and (4.18) together imply

$$\left| \frac{1}{2n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(q,j)}) \right| \le \frac{C}{nv}.$$

After this remark we may apply a standard martingale expansion procedure. We introduce σ-algebras $\mathcal{F}_{q,j} = \sigma\{X_{lk}^{(q)},\, j < l, k \le n; X_{ps}^{(\beta)},\, \beta = q+1, \dots m,\, p, s = 1, \dots, n,\}$ and use the representation

$$\operatorname{Tr}\mathbf{R} - \mathbb{E}\operatorname{Tr}\mathbf{R} = \sum_{q=1}^{m}\sum_{j=1}^{n}(\mathbb{E}_{q,j-1}\operatorname{Tr}\mathbf{R} - \mathbb{E}_{q,j}\operatorname{Tr}\mathbf{R}),$$

where $\mathbb{E}_{q,j}$ denotes conditional expectation given the σ-algebra $\mathcal{F}_{q,j}$. Note that $\mathcal{F}_{q,n} = \mathcal{F}_{q+1,0}$ and $\mathbb{E}_{q,j-1}\operatorname{Tr}\mathbf{R}^{(q,j)} = \mathbb{E}_{q,j}\operatorname{Tr}\mathbf{R}^{(q,j)}$. □

Lemma 4.5. *Under the conditions of Theorem 1.1 we have, for* $1 \le a \le m$,

$$\mathbb{E}\left|\frac{1}{n}\left(\sum_{k=1}^{n}[\mathbf{V}_{a+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a}]_{k,k+n} - \mathbb{E}\sum_{j=1}^{n}[\mathbf{V}_{a+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a}]_{kk+n}\right)\right|^2 \le \frac{C}{nv^4}.$$

and, for $1 \le a \le m-1$,

$$\mathbb{E}\left|\frac{1}{n}\left(\sum_{k=1}^{n}[\mathbf{V}_{m-a+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{k,k} - \mathbb{E}\sum_{j=1}^{n}[\mathbf{V}_{m-a+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{kk}\right)\right|^2 \le \frac{C}{nv^4}.$$

Proof. We prove the first inequality only. The proof of the other one is similar. Let $\mathbf{H}^{(q,j)}$ and $\widetilde{\mathbf{H}}^{(m-q+1,j)}$ be the matrices defined in the previous Lemma, for $q = 1, \dots, m$ and for $j = 1, \dots, n$. We introduce as well the matrices $\mathbf{X}^{(q,j)} = \mathbf{X}^{(q)} - \mathbf{e}_j\mathbf{e}_j^T\mathbf{X}^{(q)} - \mathbf{X}^{(q)}\mathbf{e}_j\mathbf{e}_j^T$. Note that the matrix $\mathbf{X}^{(q,j)}$ is obtained from the matrix $\mathbf{X}^{(q)}$ by replacing its j-th row and jth column by a row and column of zeros. Similar to the proof of the previous Lemma we introduce the matrices $\mathbf{V}_{c,d}^{(q,j)}$ by replacing in the definition of $\mathbf{V}_{c,d}$ the matrix $\mathbf{H}^{(q)}$ by $\mathbf{H}^{(q,j)}$ and the matrix $\mathbf{H}^{(m-q+1)}$ by $\widetilde{\mathbf{H}}^{(m-q+1,j)}$. For instance, if $c \le m-q+1 \le d$ we get

$$\mathbf{V}_{c,d}^{(q,j)} = \prod_{\beta=c}^{q-1}\mathbf{H}^{(\beta)}\,\mathbf{H}^{(q,j)}\prod_{\beta=q+1}^{m-q}\mathbf{H}^{(\beta)}\widetilde{\mathbf{H}}^{(m-q+1,j)}\prod_{\beta=m-q+1}^{d}\mathbf{H}^{(\beta)}.$$

Let $\mathbf{V}^{(q,j)} := \mathbf{V}_{1,m}^{(q,j)}$ and $\mathbf{R}^{(j)} := (\mathbf{V}^{(q,j)}(z) - \alpha\mathbf{I})^{-1}$. Introduce the following quantities, for $q = 1 \dots, m$ and $j = 1, \dots, n$,

$$\Xi_{q,j} := \sum_{k=1}^{n}[\mathbf{V}_{a+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{kk+n} - \sum_{k=1}^{n}[\mathbf{V}_{a+1,m}^{(q,j)}\mathbf{J}\mathbf{R}^{(q,j)}\mathbf{V}_{1,m-a+1}^{(q,j)}]_{kk+n}$$

We represent them in the following form

$$\Xi_{q,j} := \Xi^{(1)}_{q,j} + \Xi^{(2)}_{q,j} + \Xi^{(3)}_{q,j},$$

where

$$\Xi^{(1)}_{q,j} = \sum_{k=1}^{n}[(\mathbf{V}_{a+1,m} - \mathbf{V}^{(q,j)}_{a+1,m})\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}]_{k,k+n},$$
$$\Xi^{(2)}_{q,j} = \sum_{k=1}^{n}[\mathbf{V}^{(q,j)}_{a+1,m}\mathbf{J}(\mathbf{R} - \mathbf{R}^{(q,j)})\mathbf{J}\mathbf{V}_{1,m-a+1}]_{k,k+n},$$
$$\Xi^{(3)}_{q,j} = \sum_{k=1}^{n}[\mathbf{V}^{(j)}_{a+1,m}\mathbf{J}\mathbf{R}^{(q,j)}(\ V_{1,m-a+1} - \mathbf{V}^{(q,j)}_{1,m-a+1})]_{k,k+n}.$$

Note that

$$\mathbf{V}_{a+1,m} - \mathbf{V}^{(q,j)}_{a+1,m} = \mathbf{V}_{a+1,q-1}(\mathbf{H}^{(q)} - \mathbf{H}^{(q,j)})\mathbf{V}_{q+1,m} + \mathbf{V}_{a+1,q-1}\mathbf{H}^{(q,j)}\mathbf{V}_{q+1,m-\nu}(\widetilde{\mathbf{H}}_{m-q+1} - \widetilde{\mathbf{H}}^{q,j}_{m-q+1})\mathbf{V}_{m-q+2,m}.$$

By definition of the matrices $\mathbf{H}^{q,j}$ and $\widetilde{\mathbf{H}}^{m-q+1,j}$, we have

$$\sum_{k=1}^{n}[(\mathbf{V}_{a+1,m} - \mathbf{V}^{(q,j)}_{a+1,m})\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-q+1}]_{k,k+n} = [\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,q}]_{j,j} + [\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,m-a+1}]_{j+n,j+n},$$

where

$$\widetilde{\mathbf{J}} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

This equality implies that

$$|\Xi^{(1)}_{q,j}| \le |[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,q}]_{j,j+n}| + |[\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,m-q+1}]_{j+n,j+n}|.$$

Using the obvious inequality $\sum_{j=1}^{n} a_{jj}^2 \le \|\mathbf{A}\|_2^2$ for any matrix $\mathbf{A} = (a_{jk})$, $j,k = 1,\dots,n$, we get

$$T_1 := \sum_{j=1}^{n}\mathbb{E}\,|\Xi^{(1)}_{j}|^2 \le \mathbb{E}\,\|\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,q}\|_2^2 + \mathbb{E}\,\|\mathbf{V}_{m-q+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\widetilde{\mathbf{J}}\mathbf{V}_{a+1,m-q+1}\|_2^2.$$

By Lemma 4.2, we get

$$T_1 \le \frac{C}{v^2}\,\mathbb{E}\,\|\mathbf{V}_{a+1,m}\mathbf{V}_{1,m-a+1}\|_2^2 \le \frac{Cn}{v^2} \tag{4.19}$$

Consider now the term

$$T_2 = \sum_{j=1}^{n}\mathbb{E}\,|\Xi^{(2)}_{q,j}|^2.$$

Using that $\mathbf{R}-\mathbf{R}^{(j)}=-\mathbf{R}^{(j)}(\mathbf{V}(z)-\mathbf{V}^{(q,j)}(z))\mathbf{R}$, we get

$$|\Xi_{q,j}^{(2)}|\le|\sum_{k=1}^{n}[\mathbf{V}_{a,m}^{(q,j)}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q-1}\mathbf{e}_j\mathbf{e}_j^T\mathbf{V}_{q,m}\mathbf{R}\mathbf{V}_{1,b}]_{k,k+n}|$$
$$\le[\mathbf{J}\mathbf{H}^{(\alpha+1)}\mathbf{V}_{\alpha+2,m-\alpha}\mathbf{H}^{(m-\alpha+1,j)}\mathbf{V}_{m-\alpha+2,m}\mathbf{R}\mathbf{V}_{1,m-\alpha}\mathbf{V}_{\alpha+1,m}^{(j)}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\alpha}]_{jj}.$$

This implies that

$$T_2\le C\,\mathbb{E}\,\|[\mathbf{V}_{q+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,b}\mathbf{V}_{a,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,q}\|_2^2.$$

It is straightforward to check that

$$T_2\le\frac{C}{v^4}\,\mathbb{E}\,\|\mathbf{V}_{1,\alpha}\mathbf{J}\mathbf{H}^{(\alpha+1)}\mathbf{V}_{\alpha+2,m-\alpha}\mathbf{H}^{(m-\alpha+1,j)}\mathbf{V}_{m-\alpha+2,m}\|_2^2=\mathbb{E}\,\|\mathbf{Q}\|_2^2 \tag{4.20}$$

The matrix on the right hand side of equation (4.20) may be represented in the following form

$$Q=\prod_{q=1}^{m}\mathbf{H}^{(q)^{\varkappa_q}},$$

where $\varkappa_q=0$ or $\varkappa_q=1$ or $\varkappa_q=2$. Since $X_{ss}^{(q)}=0$, for $\varkappa=1$ or $\varkappa=2$, we have

$$\mathbb{E}\,|\mathbf{H}^{(q)}{}_{kl}^{\varkappa}|^2\le\frac{C}{n}.$$

This implies that

$$T_2\le Cn. \tag{4.21}$$

Similar we prove that

$$T_3:=\sum_{j=1}^{n}\mathbb{E}\,|\Xi_{q,j}^{(3)}|^2\le Cn. \tag{4.22}$$

The inequalities (4.19), (4.21) and (4.22) together imply

$$\sum_{j=1}^{n}\mathbb{E}\,|\Xi_{q,j}|^2\le Cn$$

Applying now a martingale expansion with respect to the σ-algebras $\mathcal{F}_j$ generated by the random variables $X_{kl}^{(\alpha+1)}$ with $1\le k\le j$, $1\le l\le n$ and all other random variables $X_{sl}^{(q)}$ except $q=\alpha+1$, we get

$$\mathbb{E}\left|\frac{1}{n}\left(\sum_{k=1}^{n}[\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha}]_{kk+n}-\mathbb{E}\sum_{j=1}^{n}[\mathbf{V}_{\alpha+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\alpha}]_{kk+n}\right)\right|^2\le\frac{C}{nv^4}.$$

Thus the Lemma is proved. □

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