Optimal Bounds for Convergence of Expected Spectral Distributions to the Semi-Circular Law

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Abstract

Let $\mathbf{X} = (X_{jk})_{j,k=1}^n$ denote a Hermitian random matrix with entries X_{jk}, which are independent for $1 \le j \le k \le n$. We consider the rate of convergence of the empirical spectral distribution function of the matrix $\mathbf{X}$ to the semi-circular law assuming that $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$ and that

$$\sup_{n\ge1}\sup_{1\le j,k\le n}\mathbf{E}|X_{jk}|^4 =: \mu_4 < \infty,$$

and

$$\sup_{1\le j,k\le n}|X_{jk}| \le D_0 n^{\frac14}.$$

By means of a recursion argument it is shown that the Kolmogorov distance between the expected spectral distribution of the Wigner matrix $\mathbf{W} = \frac{1}{\sqrt{n}}\mathbf{X}$ and the semicircular law is of order $O(n^{-1})$.

1 Introduction

Consider a family $\mathbf{X} = \{X_{jk}\}$, $1 \le j \le k \le n$, of independent real random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$, for any $n \ge 1$. Assume that $X_{jk} = X_{kj}$, for $1 \le k < j \le n$, and introduce the symmetric

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matrices

$$\mathbf{W} = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}.$$

The matrix $\mathbf{W}$ has a random spectrum $\{\lambda_1, \dots, \lambda_n\}$ and an associated spectral distribution function $\mathcal{F}_n(x) = \frac{1}{n}\,\mathrm{card}\,\{j \le n : \lambda_j \le x\}$, $x \in \mathbb{R}$. Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions $F_n(x) = \mathbf{E}\,\mathcal{F}_n(x)$. Let $G(x)$ denote the semi-circular distribution function with density $g(x) = G'(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{I}_{[-2,2]}(x)$, where $\mathbb{I}_{[a,b]}(x)$ denotes the indicator–function of the interval $[a, b]$. The rate of convergence to the semi-circular law has been studied by several authors. We proved in [13] that the Kolmogorov distance between $\mathcal{F}_n(x)$ and the distribution function $G(x)$, $\Delta_n^* := \sup_x |\mathcal{F}_n(x) - G(x)|$ is of order $O_P(n^{-\frac{1}{2}})$ (i.e. $n^{\frac{1}{2}}\Delta_n^*$ is bounded in probability). Bai et al.[1], [2] and Girko [8] showed that $\Delta_n := \sup_x |F_n(x) - G(x)| = O(n^{-\frac{1}{2}})$. Bobkov, Götze and Tikhomirov [4] proved that Δ_n and $\mathbf{E}\Delta_n^*$ have order $O(n^{-\frac{2}{3}})$ assuming a Poincaré inequality for the distribution of the matrix elements. For the Gaussian Unitary Ensemble respectively for the Gaussian Orthogonal Ensemble, see [12] respectively [21], it has been shown that $\Delta_n = O(n^{-1})$. Denote by $\gamma_{n1} \le \ldots \le \gamma_{nn}$, the quantiles of G, i.e. $G(\gamma_{nj}) = \frac{j}{n}$, and introduce the notation $\mathrm{llog}_n := \log\log n$. Erdös et al. [6], [7] showed, for matrices with elements X_{jk} which have a uniformly sub exponential decay, i.e.

$$\Pr\{|X_{jk}| > t\} \le A\exp\{-t^{\varkappa}\}, \tag{1.1}$$

for some $\varkappa > 0$, $A > 0$ and for any $t \ge 1$, the following result

$$\Pr\Big\{\, \exists\, j : |\lambda_j - \gamma_{nj}| \ge (\log n)^{C\,\mathrm{llog}_n}\Big[\min(j, N-j+1)\Big]^{-\frac{1}{3}} n^{-\frac{2}{3}}\Big\} \le C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\}, \tag{1.2}$$

for n large enough. It is straightforward to check that this bound implies that

$$\Pr\Big\{\sup_x |\mathcal{F}_n(x) - G(x)| \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}\Big\} \ge 1 - C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\}. \tag{1.3}$$

From the last inequality it is follows that $\mathbf{E}\Delta_n^* \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}$. Similar results were obtained in [20, Theorem 32], assuming additionally that the distributions of the entries of matrices have vanishing third moment.

In this paper we derive the optimal bound for the rate of convergence of the expected spectral distribution to the semi–circular law. Using arguments similar to those used in [17] we provide a self-contained proof based on recursion methods developed in the papers of Götze and Tikhomirov [13], [9] and [22]. It follows from the results of Gustavsson [14] that the best possible bound in the Gaussian case for the rate of convergence in probability is $O(n^{-1}\sqrt{\log n})$. The best possible bound for Δ_n is of order $O(n^{-1})$. For Gaussian matrices such bounds were obtained in [12] and [21]. Our setup includes the case that the distributions of $X_{jk} = X_{jk}^{(n)}$ may depend on n. In the following we shall investigate the rate of convergence of expected spectral distribution function $F_n(x) = \mathbf{E}\mathcal{F}_n(x)$ to the semi-circular distribution function by estimating the quantity Δ_n. The main result of this paper is the following

Theorem 1.1. *Let* $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. *Assume that*

$$\sup_{n\geq 1}\sup_{1\leq j,k\leq n}\mathbf{E}|X_{jk}|^4 =: \mu_4 < \infty. \tag{1.4}$$

Assume as well that there exists a constant D_0 *such that for all* $n \geq 1$

$$\sup_{1\leq j,k\leq n}|X_{jk}| \leq D_0 n^{\frac{1}{4}}. \tag{1.5}$$

Then, there exists a positive constant $C = C(D_0, \mu_4)$ *depending on* D_0 *and* μ_4 *only such that*

$$\Delta_n = \sup_x |F_n(x) - G(x)| \leq Cn^{-1}. \tag{1.6}$$

Corollary 1.1. *Let* $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. *Assume that*

$$\sup_{n\geq 1}\sup_{1\leq j,k\leq n}\mathbf{E}|X_{jk}|^8 =: \mu_8 < \infty. \tag{1.7}$$

Then, there exists a positive constant $C = C(\mu_8)$ *depending on* μ_8 *only such that*

$$\Delta_n \leq Cn^{-1}. \tag{1.8}$$

Remark 1.2. *Note that the bound* (1.6) *in Theorem 1.1 and the bound* (1.8) *in the Corollary 1.1 are not improvable and coincide with the corresponding bounds in the Gaussian case.*

We state here as well the results for the Stieltjes transform of the expected spectral distribution of the matrix $\mathbf{W}$. Let $\mathbf{R}$ denote the resolvent matrix of the matrix $\mathbf{W}$,

$$\mathbf{R} := \mathbf{R}(z) = (\mathbf{W} - z\mathbf{I})^{-1}.$$

Here and in what follows $\mathbf{I}$ denotes the unit matrix of corresponding dimension. For any distribution function $F(x)$ we define the Stieltjes transform $s_F(z)$, for $z = u + iv$ with $v > 0$, via formula

$$s_F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x).$$

Denote by $m_n(z)$ the Stieltjes transform of the distribution function $\mathcal{F}_n(x)$. It is a well-known fact that

$$m_n(z) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_j - z} = \frac{1}{n} \operatorname{Tr} \mathbf{R}.$$

By $s(z)$ we denote the Stieltjes transform of the semi-circular law,

$$s(z) = \frac{-z + \sqrt{z^2 - 4}}{2}.$$

The Stieltjes transform of the semi-circular distribution satisfies the equation

$$s^2(z) + zs(z) + 1 = 0 \tag{1.9}$$

(see, for example, [13, equality (4.20)]).

Introduce for $z = u + iv$ and a positive constant $A_0 > 0$

$$v_0 := A_0 n^{-1}, \text{ and } \gamma := \gamma(z) := |2 - |u||. \tag{1.10}$$

For any $0 < \varepsilon < \frac{1}{2}$, and $A_0 > 0$, define a region $\mathbb{G} = \mathbb{G}(A_0, n\varepsilon) \subset \mathbb{C}_+$, by

$$\mathbb{G} := \{z = u + iv \in \mathbb{C}_+ : -2 + \varepsilon \le u \le 2 - \varepsilon,\, v \ge v_0/\sqrt{\gamma(z)}\}. \tag{1.11}$$

Let $a > 0$ be a positive number such that

$$\frac{1}{\pi} \int_{|u| \le a} \frac{1}{u^2 + 1} du = \frac{3}{4}. \tag{1.12}$$

We prove the following result.

Theorem 1.3. *Let* $\frac{1}{2} > \varepsilon > 0$ *be a sequence of positive numbers in*(1.11) *such that*

$$\varepsilon^{\frac{3}{2}} = 2v_0 a. \tag{1.13}$$

Assuming the conditions of Theorem 1.1, there exists a positive constant $C = C(D_0, A_0, \mu_4)$ *depending on* D, A_0 *and* μ_4 *only, such that, for* $z \in \mathbb{G}$

$$|\mathbf{E}m_n(z) - s(z)| \le \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2-4|^{\frac{1}{4}}}.$$

1.1 Sketch of the Proof

1. We start with an estimate of the Kolmogorov-distance to the Wigner distribution via an integral over the difference of the corresponding Stieltjes transforms along a contour in the upper half-plane using a smoothing inequality (2.1) and Cauchy's formula developed by the authors in [11] The resulting bound (2.3) involves an integral over a segment at a fixed distance, say $V = 4$, from the real axis and a segment $u + iA_0 n^{-1}(2-|u|)^{-\frac{1}{2}}$, $|u| \le 2$ at a distance of order n^{-1} but avoiding to come close to the endpoints ± 2 of the support. These segments are part of the boundary of an n-dependent region $\mathbb{G}$ where bounds of Stieltjes transforms are needed. Since the Stieltjes-transform and the diagonal elements $R_{jj}(z)$ of the resolvent of the Wigner-matrix $\mathbf{W}$ are uniformly bounded on the segment with $\operatorname{Im} z = V$ by $1/V$ (see Section 3.1) proving a bound of order $O(n^{-1})$ for the latter segment near the x-axis is the essential problem.

2. In order to investigate this crucial part of the error we start with the 2nd resolvent or self-consistency equation for the expected Stieltjes transform resp. the quantities $R_{jj}(z)$ of $\mathbf{W}$ (see (3.2) below) based on the difference of the resolvent of $\mathbf{W}^{(j)}$ (jth row and column removed) and $\mathbf{W}$. For the equivalent representation for the difference of Stieltjes transforms (see (3.3)) we have to show an error bound of order $O((nv)^{-\frac{3}{2}}|z^2-4|^{-\frac{1}{4}})$ for $z \in \mathbb{G}$. To prove this bound we use a recursive version of this representations as in (5.32). Obviously bounds for $\mathbf{E}|R_{jj}|^p$ for $z = u + iv$ close to the real line are needed for the sufficiently large p, ($p = O(\log n)$), which follow once the error terms ε_j are small in the region $\mathbb{G}$. But proving that $\mathbf{E}|\varepsilon_j|^p$ is small requires in turn again bounds of $\mathbf{E}|R_{jj}|^{2p}$.
An approach suggested recently in [17] turns out to be very fruitful in dealing with this recursion problem. Assuming that $\mathbf{E}|R_{jj}|^{2p} \le C_0^{2p}$ for some $z = u + iv$, we can show that $\mathbf{E}|R_{jj}|^p \le C_0^p$ with $z = u + iv/s_0$ with some fixed scale factor $s_0 > 1$. This allows us to prove by induction a bound of type $\mathbf{E}|R_{jj}|^q \le C_0^q$ for some fixed q (independent of n) and $z = u + iv$ with

$v \geq Cn^{-1}$ starting with $\mathbf{E}|R_{jj}|^p \leq C_0^p$ for $p = s_0^q$ and $z = u + iv$ for fixed $v = 4$, say. The latter assumption can be easily verified.
Note that one of the errors, that is ε_{j2}, in (3.2) is a quadratic form in independent random variables. Thus, in case that $\mathbf{W}$ has entries with exponential or even sub-Gaussian tails, inequalities for quadratic forms of independent random variables, like [11], Lemma 3.8, or [17], Proposition A.1 could be applied.
Assuming eight moments in Corollary 1.1 or four moments and a truncation condition in Theorem 1.1 only, we can't use these strong tail estimates for quadratic forms anymore. Our solution is an recursive application of Burkholder's inequality for the pth moment resulting in a bound involving moments of order $p/2$ of another quadratic form in independent variables in each step. This is the crucial part of the moment recursion for R_{jj} described above. Details of this procedure are described in Sections 5.1 and 5.2.
3. In Section 6 we prove a bound for the error $\Lambda_n := \mathbf{E}m_n(z) - s(z)$ of the form $(nv)^{-\frac{3}{4}} + (nv)^{-\frac{3}{2}}|z^2 - 4|^{-\frac{1}{4}}$ which suffices to prove the rate $O(n^{-1})$ in Theorem 1.1. Here we use a series of martingale-type decompositions to evaluate the *expectation* $\mathbf{E}m_n(z)$ combined with the bound $\mathbf{E}|\Lambda_n|^2 \leq C(nv)^{-2}$ of Lemma 7.24 in the Appendix which is again based on a recursive inequality for $\mathbf{E}|\Lambda_n|^2$ in (7.71). A direct application of this bound to estimate the error terms ε_{j4} would result in a less precise bound of order $O(n^{-1}\log n)$ in Theorem 1.1. Bounds of such type will be shown for the Kolmogorov distance of the *random* spectral distribution to Wigner's law in a separate paper. For the expectation we provide sharper bounds in Section 6.2 involving $m_n'(z)$.
4. The necessary auxiliary bounds for all these steps are collected in the Appendix.

2 Bounds for the Kolmogorov Distance of Spectral Distributions via Stieltjes Transforms

To bound the error Δ_n we shall use an approach developed in previous work of the authors, see [13].
We modify the bound of the Kolmogorov distance between an arbitrary distribution function and the semi-circular distribution function via their Stieltjes transforms obtained in [13, Lemma 2.1]. For $x \in [-2, 2]$ define $\gamma(x) := 2 - |x|$. Given $\frac{1}{2} > \varepsilon > 0$ introduce the interval $\mathbb{J}_\varepsilon = \{x \in [-2, 2] : \gamma(x) \geq \varepsilon\}$ and $\mathbb{J}'_\varepsilon = \mathbb{J}_{\varepsilon/2}$. For a distribution function F denote by $S_F(z)$ its Stieltjes transform.

Proposition 2.1. *Let $v > 0$ and $a > 0$ and $\frac{1}{2} > \varepsilon > 0$ be positive numbers such that*

$$\frac{1}{\pi}\int_{|u|\le a}\frac{1}{u^2+1}du = \frac{3}{4} =: \beta, \tag{2.1}$$

and

$$2va \le \varepsilon^{\frac{3}{2}}. \tag{2.2}$$

If G denotes the distribution function of the standard semi-circular law, and F is any distribution function, there exist some absolute constants C_1 and C_2 such that

$$\Delta(F,G) := \sup_x |F(x) - G(x)|$$
$$\le 2\sup_{x\in\mathbb{J}'_\varepsilon}\Big|\mathrm{Im}\int_{-\infty}^{x}(S_F(u+i\frac{v}{\sqrt{\gamma}}) - S_G(u+i\frac{v}{\sqrt{\gamma}}))du\Big| + C_1 v + C_2\varepsilon^{\frac{3}{2}}.$$

Remark 2.2. *For any $x \in \mathbb{J}_\varepsilon$ we have $\gamma = \gamma(x) \ge \varepsilon$ and according to condition (2.2), $\frac{av}{\sqrt{\gamma}} \le \frac{\varepsilon}{2}$.*

For a proof of this Proposition see [11, Proposition 2.1] .

Lemma 2.1. *Under the conditions of Proposition 2.1, for any $V > v$ and $0 < v \le \frac{\varepsilon^{3/2}}{2a}$ and $v' = v/\sqrt{\gamma}, \gamma = 2 - |x|$, $x \in \mathbb{J}'_\varepsilon$ as above, the following inequality holds*

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{-\infty}^{x}(\mathrm{Im}\,(S_F(u+iv') - S_G(u+iv'))du\right|$$
$$\le \int_{-\infty}^{\infty}|S_F(u+iV) - S_G(u+iV)|du$$
$$+\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{v'}^{V}(S_F(x+iu) - S_G(x+iu))\,du\right|.$$

Proof. Let $x \in \mathbb{J}'_\varepsilon$ be fixed. Let $\gamma = \gamma(x)$. Put $z = u + iv'$. Since $v' = \frac{v}{\sqrt{\gamma}} \le \frac{\varepsilon}{2a}$, see (2.2), we may assume without loss of generality that $v' \le 4$ for $x \in \mathbb{J}'_\varepsilon$. Since the functions $S_F(z)$ and $S_G(z)$ are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write for $x \in \mathbb{J}'_\varepsilon$

$$\int_{-\infty}^{x}\mathrm{Im}\,(S_F(z) - S_G(z))du = \mathrm{Im}\{\lim_{L\to\infty}\int_{-L}^{x}(S_F(u+iv') - S_G(u+iv'))du\}.$$

By Cauchy's integral formula, we have

$$\begin{aligned}\int_{-L}^{x}(S_F(z)-S_G(z))du &= \int_{-L}^{x}(S_F(u+iV)-S_G(u+iV))du\\ &+\int_{v'}^{V}(S_F(-L+iu)-S_G(-L+iu))du\\ &-\int_{v'}^{V}(S_F(x+iu)-S_G(x+iu))du.\end{aligned}$$

Denote by ξ (resp. η) a random variable with distribution function $F(x)$ (resp. $G(x)$). Then we have

$$|S_F(-L+iu)| = \left|\mathbf{E}\frac{1}{\xi+L-iu}\right| \le {v'}^{-1}\Pr\{|\xi|>L/2\}+\frac{2}{L},$$

for any $v' \le u \le V$. Similarly,

$$|S_G(-L+iu)| \le {v'}^{-1}\Pr\{|\eta|>L/2\}+\frac{2}{L}.$$

These inequalities imply that

$$\left|\int_{v'}^{V}(S_F(-L+iu)-S_G(-L+iu))du\right| \to 0 \quad \text{as} \quad L\to\infty,$$

which completes the proof. □

Combining the results of Proposition 2.1 and Lemma 2.1, we get

Corollary 2.2. *Under the conditions of Proposition 2.1 the following inequality holds*

$$\begin{aligned}\Delta(F,G) \le 2\int_{-\infty}^{\infty}|S_F(u+iV)-S_G(u+iV)|du + C_1 v_0 + C_2\varepsilon^{\frac{3}{2}}\\ +2\sup_{x\in\mathbb{J}'_\varepsilon}\int_{v'}^{V}|S_F(x+iu)-S_G(x+iu)|du, \qquad (2.3)\end{aligned}$$

where $v' = \frac{v_0}{\sqrt{\gamma}}$ *with* $\gamma = 2-|x|$ *and* $C_1, C_2 > 0$ *denote absolute constants.*

3 Proof of Theorem 1.1

Proof. We shall apply Corollary 2.2 to prove the Theorem 1.1. We choose $V = 4$ and v_0 as defined in (1.10) and use the quantity $\varepsilon = (2av_0)^{\frac{2}{3}}$.

3.1 Estimation of the First Integral in (2.2) for $V = 4$

Denote by $\mathbb{T} = \{1, \dots, n\}$. In the following we shall systematically use for any $n \times n$ matrix $\mathbf{W}$ together with its resolvent $\mathbf{R}$, its Stieltjes transform m_n etc. the corresponding quantities $\mathbf{W}^{(\mathbb{A})}$, its resolvent $\mathbf{R}^{(\mathbb{A})}$ and its Stieltjes transform $m_n^{(\mathbb{A})}$ for the corresponding sub matrix with entries $X_{jk}, j, k \notin \mathbb{A}$, $\mathbb{A} \subset \mathbb{T} = \{1, \dots, n\}$. Observe that

$$m_n^{(\mathbb{A})}(z) = \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{A}}} \frac{1}{\lambda^{(\mathbb{A})} - z}.$$

Let $\mathbb{T}_{\mathbb{A}} = \mathbb{T} \setminus \mathbb{A}$. By $\mathfrak{M}^{(\mathbb{J})}$ we denote the σ-algebra generated by X_{lk} with $l, k \in \mathbb{T}_{\mathbb{J}}$. If $\mathbb{A} = \emptyset$ we shall omit the set $\mathbb{A}$ as exponent index.

We shall use the representation

$$R_{jj} = \frac{1}{-z + \frac{1}{\sqrt{n}} X_{jj} - \frac{1}{n} \sum_{k,l \in \mathbb{T}_j} X_{jk} X_{jl} R_{kl}^{(j)}}, \tag{3.1}$$

(see, for example, [13, equality (4.6)]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)} \varepsilon_j R_{jj}, \tag{3.2}$$

where $\varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4}$ with

$$\varepsilon_{j1} := \frac{1}{\sqrt{n}} X_{jj}, \quad \varepsilon_{j2} := -\frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jk} X_{jl} R_{kl}^{(j)},$$
$$\varepsilon_{j3} := -\frac{1}{n} \sum_{k \in \mathbb{T}_j} (X_{jk}^2 - 1) R_{kk}^{(j)}, \quad \varepsilon_{j4} := \frac{1}{n} (\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(j)}).$$

Let

$$\Lambda_n := \Lambda_n(z) := m_n(z) - s(z) = \frac{1}{n} \operatorname{Tr} \mathbf{R} - s(z).$$

Summing equality (3.2) in $j = 1, \dots, n$ and solving with respect Λ_n, we get

$$\Lambda_n = m_n(z) - s(z) = \frac{T_n}{z + m_n(z) + s(z)}, \tag{3.3}$$

where

$$T_n = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j R_{jj}.$$

Obvious bounds like $|z+s(z)|\ge 1$, $|\lambda_j-z|^{-1}\le v^{-1}$, $\max\{|R^{(\mathbb{J})}_{jj}(z)|,\,|m^{(\mathbb{J})}_n(z)|\}\le v^{-1}$, imply that for $V=4$ and for any $\mathbb{J}\subset\mathbb{T}$,

$$|m^{(\mathbb{J})}_n(z)|\le\frac14\le\frac12|z+s(z)|,$$
$$|s(z)-m^{(\mathbb{J})}_n(z)|\le\frac12\le\frac12|z+s(z)|,\ \text{a.s.}$$

and therefore,

$$|z+m^{(\mathbb{J})}_n(z)+s(z)|\ge\frac12|z+s(z)|,\quad |z+m^{(\mathbb{J})}_n(z)|\ge\frac12|s(z)+z|. \tag{3.4}$$

Using equality (3.3), we may write

$$\begin{aligned}\mathbf{E}\Lambda_n&=\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_jR_{jj}}{z+m_n(z)+s(z)}\\&=\sum_{\nu=1}^4\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_{j\nu}s(z)}{z+m_n(z)+s(z)}+\sum_{\nu=1}^3\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_{j\nu}(R_{jj}-s(z))}{z+m_n(z)+s(z)}.\end{aligned}$$

We use that $\mathbf{E}\{\varepsilon_{j\nu}\Big|\mathfrak{M}^{(j)}\}=0$, for $\nu=1,2,3$ and obtain

$$\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_{j\nu}s(z)}{z+m_n(z)+s(z)}=-\mathbf{E}\frac{\varepsilon_{j\nu}\varepsilon_{j4}s(z)}{(z+m^{(j)}_n(z)+s(z))(z+m_n(z)+s(z))}.$$

Thus, according to inequalities (3.4), Lemmas 7.8, 7.9, 7.10, 7.12 in the Appendix and equation (1.9), we obtain

$$|\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_{j\nu}s(z)}{z+m_n(z)+s(z)}|\le 4|s(z)|^3\frac1n\sum_{j=1}^n\mathbf{E}|\varepsilon_{j4}\varepsilon_{j\nu}|\le\frac{C|s(z)|^2}{n^{\frac32}},$$

where C depends on μ_4 only. For $\nu=4$, Lemma 7.12 in the Appendix, inequality (3.4) and relation (1.9) yield

$$\frac1n\sum_{j=1}^n\frac{|s(z)||\varepsilon_{j4}|}{|z+m_n(z)+s(z)|}\le\frac Cn|s(z)|^2 \tag{3.5}$$

with some absolute constant C. Furthermore, applying the Cauchy – Schwartz inequality and inequality (3.4) and relation (1.9), we get

$$\Big|\sum_{\nu=1}^3\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_{j\nu}(R_{jj}-s(z))}{z+m_n(z)+s(z)}\Big|\le C|s(z)|\sum_{\nu=1}^3\frac1n\sum_{j=1}^n\mathbf{E}^{\frac12}|\varepsilon_{j\nu}|^2\mathbf{E}^{\frac12}|R_{jj}-s(z)|^2. \tag{3.6}$$

We may rewrite the representation (3.2) using $\Lambda_n = m_n(z) - s(z)$ and (1.9) as (compare (3.1))

$$R_{jj} = s(z) - s(z)\varepsilon_j R_{jj} - s(z)\Lambda_n R_{jj}. \tag{3.7}$$

Applying representations (3.3) and (3.7) together with (3.4) and $|R_{jj}| \le \frac{1}{4}$, we obtain

$$\mathbf{E}|R_{jj}(z) - s(z)|^2 \le C|s(z)|^2 \frac{1}{n}\sum_{l=1}^{n} \mathbf{E}|\varepsilon_l|^2. \tag{3.8}$$

Combining inequalities (3.6) and (3.8), we get

$$\Big|\sum_{\nu=1}^{4}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\varepsilon_{j\nu}(R_{jj}-s(z))}{z+m_n(z)+s(z)}\Big| \le C|s(z)|^2\sum_{\nu=1}^{4}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}|\varepsilon_{j\nu}|^2. \tag{3.9}$$

Applying now Lemmas 7.8, 7.9, 7.10 and 7.12, we get

$$\left|\sum_{\nu=1}^{3}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\varepsilon_{j\nu}(R_{jj}-s(z))}{z+m_n(z)+s(z)}\right| \le \frac{C|s(z)|^2}{n}. \tag{3.10}$$

Inequality (3.5) and (3.10) together imply

$$|\mathbf{E}\Lambda_n| \le \frac{C}{n}|s(z)|^2. \tag{3.11}$$

Consider now the integral

$$Int(V) = \int_{-\infty}^{\infty} |\mathbf{E}m_n(u+iV) - s(u+iV)|du$$

for $V = 4$. Using inequality (3.11), we have

$$|Int(V)| \le \frac{C}{n}\int_{-\infty}^{\infty}|s(u+Vi)|^2du.$$

Finally, we note that

$$\int_{-\infty}^{\infty}|s(z)|^2dx \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x-u)^2+V^2}du dF_n(x) \le \frac{\pi}{V}. \tag{3.12}$$

Therefore,

$$\int_{-\infty}^{\infty}|\mathbf{E}m_n(u+iV) - s(u+iV)|du \le \frac{C}{n}. \tag{3.13}$$

3.2 Estimation of the Second Integral in (2.3)

To finish the proof of Theorem 1.1 we need to bound the second integral in (2.3) for $z \in \mathbb{G}$ and $v_0 = A_0 n^{-1}$, where $\varepsilon = (2av_0)^{\frac{2}{3}}$ is defined in such a way that condition (2.2) holds. We shall use the results of Theorem 1.3. According to these results we have, for $z \in \mathbb{G}$,

$$|\mathbf{E}m_n(z) - s(z)| \le \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2-4|^{\frac{1}{4}}}. \tag{3.14}$$

We have

$$\int_{v_0/\sqrt{\gamma}}^{V} |\mathbf{E}(m_n(x+iv) - s(x+iv))|dv \le \frac{1}{n}\int_{\frac{v_0}{\sqrt{\gamma}}}^{V} \frac{dv}{v^{\frac{3}{4}}} + \frac{1}{n\sqrt{n}\gamma^{\frac{1}{4}}}\int_{\frac{v_0}{\sqrt{\gamma}}}^{V} \frac{dv}{v^{\frac{3}{2}}}.$$

After integrating we get

$$\int_{v_0/\sqrt{\gamma}}^{V} |\mathbf{E}(m_n(x+iv) - s(x+iv))|dv \le \frac{C}{n} + \frac{C\gamma^{\frac{1}{4}}}{n\sqrt{n}\gamma^{\frac{1}{4}}v_0^{\frac{1}{2}}} \le \frac{C}{n}. \tag{3.15}$$

Inequalities (3.13) and (3.15) complete the proof of Theorem 1.1. Thus Theorem 1.1 is proved. □

4 The proof of Corollary 1.1

To prove the Corollary 1.1 we consider truncated random variables $\widehat{X}_{jl}$ defined by

$$\widehat{X}_{jl} := X_{jl}\mathbb{I}\{|X_{jl}| \le cn^{\frac{1}{4}}\}. \tag{4.1}$$

Let $\widehat{\mathcal{F}}_n(x)$ denote the empirical spectral distribution function of the matrix $\widehat{\mathbf{W}} = \frac{1}{\sqrt{n}}(\widehat{X}_{jl})$.

Lemma 4.1. *Assuming the conditions of Theorem 1.1 there exists a constant $C > 0$ depending on μ_8 only such that*

$$\mathbf{E}\{\sup_x |\mathcal{F}_n(z) - \widehat{\mathcal{F}}_n(x)|\} \le \frac{C}{n}.$$

Proof. We shall use the rank inequality of Bai. See [3], Theorem A.43, p. 503. According this inequality

$$\mathbf{E}\{\sup_x |\mathcal{F}_n(x) - \widehat{\mathcal{F}}_n(x)|\} \le \frac{1}{n}\mathbf{E}\{\text{rank}(\mathbf{X} - \widehat{\mathbf{X}})\}.$$

Observing that the rank of a matrix is not larger then numbers of its non-zero entries, we may write

$$\mathbf{E}\{\sup_x |\mathcal{F}_n(x) - \widehat{\mathcal{F}}_n(x)|\} \le \frac{1}{n}\sum_{j,k=1}^n \mathbf{E}\mathbb{I}\{|X_{jk}| \ge Cn^{\frac14}\} \le \frac{1}{n^3}\sum_{j,k=1}^n \mathbf{E}|X_{jk}|^8 \le \frac{C\mu_8}{n}.$$

Thus, the Lemma is proved. □

Note that in the bound of the first integral in (2.3) we used the condition (1.4) only. We shall compare the Stieltjes transform of the matrix $\widehat{\mathbf{W}}$ and the matrix obtained from $\widehat{\mathbf{W}}$ by centralizing and normalizing its entries. Introduce $\widetilde{X}_{jk} = \widehat{X}_{jk} - \mathbf{E}\widehat{X}_{jk}$ and $\widetilde{\mathbf{W}} = \frac{1}{\sqrt{n}}(\widetilde{X}_{jk})_{j,k=1}^n$. We normalize the r.v.'s $\widetilde{X}_{jk}$. Let $\sigma_{jk}^2 = \mathbf{E}|\widetilde{X}_{jk}|^2$. We define the r.v.'s $\breve{X}_{jk} = \sigma_{jk}^{-1}\widetilde{X}_{jk}$. Finally, let $\breve{m}_n(z)$ (resp. $\widehat{m}_n(z)$, $\widetilde{m}_n(z)$) denote Stieltjes transform of empirical spectral distribution function of the matrix $\breve{\mathbf{W}} = \frac{1}{\sqrt{n}}(\breve{X}_{jk})_{j,k=1}^n$ (resp. $\widehat{\mathbf{W}}$, $\widetilde{\mathbf{W}}$).

Remark 4.1. *Note that*

$$|\breve{X}_{jl}| \le D_1 n^{\frac14}, \quad \mathbf{E}\breve{X}_{jl} = 0 \textit{ and } \mathbf{E}\breve{X}_{jk}^2 = 1, \tag{4.2}$$

for some absolute constant D_1. That means that the matrix $\breve{\mathbf{W}}$ satisfies the conditions of Theorem 1.3.

Lemma 4.2. *There exists some absolute constant C depending on μ_8 such that*

$$\mathbf{E}|\widetilde{m}_n(z) - \breve{m}_n(z)| \le \frac{C}{n^{\frac32}v^{\frac32}}.$$

Proof. Note that

$$\breve{m}_n(z) = \frac1n \mathrm{Tr}\,(\breve{\mathbf{W}} - z\mathbf{I})^{-1} =: \frac1n \mathrm{Tr}\,\breve{\mathbf{R}},\ \widetilde{m}_n(z) = \frac1n \mathrm{Tr}\,(\widetilde{\mathbf{W}} - z\mathbf{I})^{-1} =: \frac1n \mathrm{Tr}\,\widetilde{\mathbf{R}}.$$

Therefore,

$$\widetilde{m}_n(z) - \breve{m}_n(z) = \frac1n \mathrm{Tr}\,(\widetilde{\mathbf{R}} - \breve{\mathbf{R}}) = \frac1n \mathrm{Tr}\,(\widetilde{\mathbf{W}} - \breve{\mathbf{W}})\widetilde{\mathbf{R}}\widehat{\mathbf{R}}. \tag{4.3}$$

Using the simple inequalities $|\mathrm{Tr}\,\mathbf{AB}| \le \|\mathbf{A}\|_2\|\mathbf{B}\|_2$ and $\|\mathbf{AB}\|_2 \le \|\mathbf{A}\|\|\mathbf{B}\|_2$, we get

$$\mathbf{E}|\widetilde{m}_n(z) - \breve{m}_n(z)| \le n^{-1}\mathbf{E}^{\frac12}\|\widetilde{\mathbf{R}}\|^2\|\breve{\mathbf{R}}\|_2^2\mathbf{E}^{\frac12}\|\widetilde{\mathbf{W}} - \breve{\mathbf{W}}\|_2^2. \tag{4.4}$$

Furthermore, we note that,

$$\widetilde{\mathbf{W}} - \breve{\mathbf{W}} = \frac{1}{\sqrt{n}}((1-\sigma_{jk})\breve{X}_{jk}), \tag{4.5}$$

and

$$\|\widetilde{\mathbf{W}} - \breve{\mathbf{W}}\|_2 \le \max_{1\le j,k\le n}\{1-\sigma_{jk}\}\|\breve{\mathbf{W}}\|_2.$$

Since

$$0 < 1-\sigma_{jk} \le 1-\sigma_{jk}^2 \le Cn^{-\frac{3}{2}}\mu_8,$$

therefore

$$\mathbf{E}\|\widetilde{\mathbf{W}} - \breve{\mathbf{W}}\|_2^2 \le C\mu_8^2 n^{-2}. \tag{4.6}$$

Applying Lemma 7.6 inequality (7.11) in the Appendix , we get

$$\mathbf{E}^{\frac{1}{2}}\|\widetilde{\mathbf{R}}\|^2\|\breve{\mathbf{R}}\|_2^2 \le v^{-1}\mathbf{E}\|\breve{\mathbf{R}}\|_2^2 \le v^{-1}\Big(\sum_{j,k}\mathbf{E}|\breve{R}_{jk}|^2\Big)^{\frac{1}{2}} \le v^{-\frac{3}{2}}\sqrt{n}(\operatorname{Im}\breve{m}_n(z))^{\frac{1}{2}}. \tag{4.7}$$

Usin now inequalities (4.6) and (4.7), we obtain

$$\mathbf{E}|\widetilde{m}_n(z) - \breve{m}_n(z)| \le Cn^{-\frac{3}{2}}v^{-\frac{3}{2}}\Big(\frac{1}{n}\sum_{j=1}^n\mathbf{E}|\breve{\mathbf{R}}_{jj}|\Big)^{\frac{1}{2}}.$$

According Remark 4.1, we may apply Corollary 5.14 in Section 5 with $q=1$ to prove the claim. Thus, Lemma 4.2 is proved. ☐

Lemma 4.3. *For some absolute constant $C>0$ we have*

$$\mathbf{E}|\widetilde{m}_n(z) - \widehat{m}_n(z)| \le \frac{C\mu_8}{n^{\frac{3}{2}}v^{\frac{3}{2}}}.$$

Proof. Similar to (4.3), we write

$$\widetilde{m}_n(z) - \widehat{m}_n(z) = \frac{1}{n}\operatorname{Tr}(\widetilde{\mathbf{R}} - \widehat{\mathbf{R}}) = \frac{1}{n}\operatorname{Tr}(\widetilde{\mathbf{W}} - \widehat{\mathbf{W}})\widetilde{\mathbf{R}}\widehat{\mathbf{R}}.$$

This yields

$$\mathbf{E}|\widetilde{m}_n(z) - \widehat{m}_n(z)| \le n^{-1}\mathbf{E}\|\widehat{\mathbf{R}}\|\|\widetilde{\mathbf{R}}\|_2\|\mathbf{E}\widehat{\mathbf{W}}\|_2. \tag{4.8}$$

Furthermore, we note that, by definition (4.1) and condition (1.7), we have

$$|\mathbf{E}\widehat{X}_{jk}| \le Cn^{-\frac{7}{4}}\mu_8. \tag{4.9}$$

Applying Lemma 7.6, inequality (7.11), in the Appendix and inequality (4.9), we obtain using $\|\widehat{\mathbf{R}}\| \le v^{-1}$,

$$\mathbf{E}|\widetilde{m}_n(z) - \widehat{m}_n(z)| \le n^{-\frac{7}{4}} v^{-\frac{3}{2}} \mathbf{E}^{\frac{1}{2}} |\widetilde{m}_n(z)|.$$

By Lemma 4.2,

$$\mathbf{E}|\widetilde{m}_n(z)| \le \mathbf{E}|\breve{m}_n(z)| + C,$$

for some constant C depending on μ_8 and A_0. According to Corollary 5.14 in Section 5.2 with $q = 1$

$$\mathbf{E}|\breve{m}_n(z)| \le \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}|\breve{R}_{jj}| \le C,$$

with a constant C depending on μ_4, D_0. Using these inequalities, we get

$$\mathbf{E}|\widetilde{m}_n(z) - \widehat{m}_n(z)| \le \frac{C\mu_8}{n^{\frac{7}{4}} v^{\frac{3}{2}}} \le \frac{C\mu_8}{n^{\frac{3}{2}} v^{\frac{3}{2}}}.$$

Thus Lemma 4.3 is proved. □

Corollary 4.4. *Assuming the conditions of Corollary 1.1, we have for $z \in \mathbb{G}$,*

$$|\mathbf{E}\widehat{m}_n(z) - s(z)| \le \frac{C}{(nv)^{\frac{3}{2}}} + \frac{C}{n^2 v^2 \sqrt{\gamma}}.$$

Proof. The proof immediately follows from the inequality

$$|\mathbf{E}\widehat{m}_n(z) - s(z)| \le |\mathbf{E}(\widehat{m}_n(z) - \breve{m}_n(z))| + |\mathbf{E}\breve{m}_n(z) - s(z)|,$$

Lemmas 4.2 and 4.3 and Theorem 1.3. □

The proof of Corollary 1.1 follows now from Lemma 4.1, Corollary 2.2, inequality (3.13) and inequality

$$\sup_{x \in \mathbb{J}_\varepsilon} \int_{v_0/\sqrt{\gamma}}^{V} |\mathbf{E}\widehat{m}_n(x + iv) - s(x + iv)| dv \le \frac{C}{n}.$$

5 Resolvent Matrices and Quadratic Forms

The crucial problem in the proof of Theorem 1.3 is the following bound for any $z \in \mathbb{G}$

$$\mathbf{E}|R_{jj}|^p \le C^p,$$

for $j = 1, \ldots, n$ and some absolute constant $C > 0$. To prove this bound we use an approach similar to the proof of Lemma 3.4 in [17]. In oder to arrive at our goal we need additional bounds of quadratic forms of type

$$\mathbf{E}\Big|\frac{1}{n}\sum_{l\ne k} X_{jl}X_{jk}R_{kl}^{(j)}\Big|^p \le \Big(\frac{Cp}{\sqrt{nv}}\Big)^p.$$

To prove this bound we recurrently use Rosenthal's and Burkholder's inequalities.

5.1 The Key Lemma

In this Section we provide auxiliary lemmas needed for the proof of Theorem 1.1.

For any $\mathbb{J} \subset \mathbb{T}$ introduce $\mathbb{T}_{\mathbb{J}} = \mathbb{T}\setminus\mathbb{J}$. We introduce the quantity, for some $\mathbb{J} \subset \mathbb{T}$,

$$B_p^{(\mathbb{J})} := \Big[\frac{1}{n}\sum_{q\in\mathbb{T}_{\mathbb{J}}}\Big(\sum_{r\in\mathbb{T}_{\mathbb{J}}}|R_{qr}^{(\mathbb{J})}|^2\Big)^p\Big].$$

By Lemma 7.6, inequality (7.12) in the Appendix, we have

$$\mathbf{E}B_p^{(\mathbb{J})} \le v^{-p}\frac{1}{n}\sum_{q\in\mathbb{T}_{\mathbb{J}}}\mathbf{E}|R_{qq}^{(\mathbb{J})}|^p. \tag{5.1}$$

Furthermore, introduce the quantities

$$\begin{aligned}
Q_\nu^{(\mathbb{J},k)} &= \sum_{l\in\mathbb{T}_{\mathbb{J},k}}\Big|\sum_{r\in\mathbb{T}_{\mathbb{J},k}\cap\{1,\ldots,l-1\}} X_{kr}a_{lr}^{(\mathbb{J},k,\nu)}\Big|^2,\\
Q_{\nu1}^{(\mathbb{J},k)} &= \sum_{r\in\mathbb{T}_k} a_{rr}^{(\mathbb{J},k,\nu+1)},\\
Q_{\nu2}^{(\mathbb{J},k)} &= \sum_{r\in\mathbb{T}_k}(X_{kr}^2-1)a_{rr}^{(\mathbb{J},k,\nu+1)},\\
Q_{\nu3}^{(\mathbb{J},k)} &= \sum_{r\ne q\in\mathbb{T}_k} X_{kr}X_{kq}a_{qr}^{(\mathbb{J},k,\nu+1)},
\end{aligned} \tag{5.2}$$

where, $a_{qr}^{(\mathbb{J},k,0)}$ are defined recursively via

$$a_{qr}^{(\mathbb{J},k,0)} = \frac{1}{\sqrt{n}} R_{qr}^{(\mathbb{J},k)},$$
$$a_{qr}^{(\mathbb{J},k,\nu+1)} = \sum_{l \in \{\max\{q,r\}+1,\dots,n\} \cap \mathbb{T}_{\mathbb{J},k}} a_{rl}^{(\mathbb{J},k,\nu)} \overline{a}_{lq}^{(\mathbb{J},k,\nu)}, \text{ for } \nu = 0, \dots, L. \tag{5.3}$$

Using these notations we have

$$Q_{\nu}^{(\mathbb{J},k)} = Q_{\nu 1}^{(\mathbb{J},k)} + Q_{\nu 2}^{(\mathbb{J},k)} + Q_{\nu 3}^{(\mathbb{J},k)}. \tag{5.4}$$

Lemma 5.1. *Under the conditions of Theorem 1.1 we have*

$$\sum_{r \in \mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,\nu+1)}|^2 \le \Big(\sum_{l,r \in \mathbb{T}_{\mathbb{J},k}}^{n} |a_{lr}^{(\mathbb{J},k,\nu)}|^2 \Big) \Big(\sum_{l \in \mathbb{T}_{\mathbb{J},k}}^{n} |a_{ql}^{(\mathbb{J},k,\nu)}|^2 \Big). \tag{5.5}$$

Moreover,

$$\sum_{q,r \in \mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,\nu+1)}|^2 \le (\sum_{q,r \in \mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,\nu)}|^2)^2. \tag{5.6}$$

Proof. We apply Hölder's inequality and obtain

$$|a_{q,r}^{(\mathbb{J},k,\nu+1)}|^2 \le \sum_{l \in \mathbb{T}_{\mathbb{J},k}} |a_{ql}^{(\mathbb{J},k,\nu)}|^2 \sum_{l \in \mathbb{T}_{\mathbb{J},k}} |a_{lr}^{(\mathbb{J},k,\nu)}|^2.$$

Summing in q and r, (5.5) and (5.6) follow.

□

Corollary 5.2. *Under the conditions of Theorem 1.1 we have*

$$\sum_{q,r \in \mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,\nu)}|^2 \le \left(\left(\operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv} \right) v^{-1} \right)^{2^{\nu}}$$

and

$$\sum_{r \in \mathbb{T}_k} |a_{qr}^{(\mathbb{J},k,\nu)}|^2 \le \left(\left(\operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv} \right) v^{-1} \right)^{2^{\nu}-1} n^{-1} v^{-1} \operatorname{Im} R_{qq}^{(\mathbb{J},k)}.$$

Proof. By definition of $a_{qr}^{(\mathbb{J},k,0)}$, see (5.3), applying (Lemma 7.6 equality (7.11) in the Appendix), we get

$$\sum_{q,r\in\mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,0)}|^2 \le \frac{1}{n}\sum_{q,r\in\mathbb{T}_{\mathbb{J},k}} |R_{qr}^{(\mathbb{J},k)}|^2 \le \left(\operatorname{Im} m_n^{(\mathbb{J})}(z)+\frac{1}{nv}\right)v^{-1}, \tag{5.7}$$

and by definition (5.3),

$$\sum_{r\in\mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,0)}|^2 \le \frac{1}{n}\sum_{r\in\mathbb{T}_{\mathbb{J},k}} |R_{qr}^{(\mathbb{J},k)}|^2. \tag{5.8}$$

The general case follows now by induction in ν, Lemma 5.1, and Lemma 7.6 inequality (7.12) in the Appendix. □

Corollary 5.3. *Under the conditions of Theorem 1.1 we have*

$$a_{rr}^{(\mathbb{J},k,\nu+1)} \le \left(\left(\operatorname{Im} m_n^{(\mathbb{J})}(z)+\frac{1}{nv}\right)v^{-1}\right)^{2^\nu-1} n^{-1}v^{-1}\operatorname{Im} R_{rr}^{(\mathbb{J},k)}. \tag{5.9}$$

Proof. The result immediately follows from the definition of $a_{rr}^{(k,\nu)}$ and Corollary 5.2. □

In what follows we shall use the notations

$$\Psi^{(\mathbb{J})} = \operatorname{Im} m_n^{(\mathbb{J})}(z)+\frac{1}{nv}, \quad (A_{\nu,p}^{(\mathbb{J})})^2 = \mathbf{E}(\Psi^{(\mathbb{J})})^{(2^\nu-1)2p}, \quad T_{\nu,p}^{(\mathbb{J},k)} = \mathbf{E}|Q_\nu^{(\mathbb{J},k)}|^p,$$
$$A_p^{(\mathbb{J})} := 1+\mathbf{E}^{\frac14}|\Psi^{(\mathbb{J})}|^{4p}. \tag{5.10}$$

Let s_0 denote some fixed number (for instance $s_0 = 2^8$). Let A_1 be a constant (to be chosen later) and $0 < v_1 \le 4$ a constant such that $v_0 = A_0 n^{-1} \le v_1$ for all $n \ge 1$.

Lemma 5.4. *Assuming the conditions of Theorem 1.1 and for* $p \le A_1(nv)^{\frac14}$

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le C_0^p, \ \textit{for } v \ge v_1, \ \textit{for all } j = 1,\dots,n, \tag{5.11}$$

we have for $v \ge v_1/s_0$ *and* $p \le A_1(nv)^{\frac14}$*, and* $k \in \mathbb{T}_{\mathbb{J}}$

$$\mathbf{E}(Q_0^{(\mathbb{J},k)})^p \le 6\left(\frac{C_3 p}{\sqrt{2}}\right)^{2p} v^{-p} A_p^{(\mathbb{J})}. \tag{5.12}$$

Proof. Using the representation (5.4) and the triangle inequality, we get

$$\mathbf{E}|Q_{\nu}^{(\mathbb{J},k)}|^p \le 3^p\Big(\mathbf{E}|Q_{\nu1}^{(\mathbb{J},k)}|^p + \mathbf{E}|Q_{\nu2}^{(\mathbb{J},k)}|^p + \mathbf{E}|Q_{\nu3}^{(\mathbb{J},k)}|^p\Big). \tag{5.13}$$

Let $\mathfrak{M}^{(\mathbb{A})}$ denote the σ-algebra generated by r.v.'s $X_{j,l}$ for $j,l \in \mathbb{T}_{\mathbb{A}}$, for any set $\mathbb{A}$. Conditioning on $\mathfrak{M}^{(\mathbb{J},k)}$ ($\mathbb{A} = \mathbb{J} \cup \{k\}$) and applying Rosenthal's inequality (see Lemma 7.1), we get

$$\mathbf{E}|Q_{\nu2}^{(\mathbb{J},k)}|^p \le C_1^p p^p\Big(\mathbf{E}\Big(\sum_{r\in\mathbb{T}_{\mathbb{J},k}} |a_{rr}^{(\mathbb{J},k,\nu+1)}|^2\Big)^{\frac{p}{2}} + \sum_{r\in\mathbb{T}_{\mathbb{J},k}} \mathbf{E}|a_{rr}^{(\mathbb{J},k,\nu+1)}|^p \mathbf{E}|X_{kr}|^{2p}\Big), \tag{5.14}$$

where C_1 denotes the absolute constant in Rosenthal's inequality. By Remark 4.1, we get

$$\mathbf{E}|Q_{\nu2}^{(\mathbb{J},k)}|^p \le C_1^p p^p\Big(\mathbf{E}(\sum_{r\in\mathbb{T}_{\mathbb{J},k}} |a_{rr}^{(\mathbb{J},k,\nu+1)}|^2)^{\frac{p}{2}} + n^{\frac{p}{2}}\frac{1}{n}\sum_{r\in\mathbb{T}_{\mathbb{J},k}} \mathbf{E}|a_{rr}^{(\mathbb{J},k,\nu+1)}|^p\Big). \tag{5.15}$$

Analogously conditioning on $\mathfrak{M}^{(\mathbb{J},k)}$ and applying Burkholder's inequality (see Lemma 7.3), we get

$$\begin{aligned}\mathbf{E}|Q_{\nu3}^{(\mathbb{J},k)}|^p \le C_2^p p^p\Big(\mathbf{E}\Big(\sum_{r\in\mathbb{T}_{\mathbb{J},k}} \Big|\sum_{q\in\mathbb{T}_{\mathbb{J},k}\cap\{1,\ldots,r-1\}} X_{kq}a_{rq}^{(\mathbb{J},k,\nu+1)}\Big|^2\Big)^{\frac{p}{2}}\\ + \sum_{q=1}^{n-1}\mathbf{E}\Big|\sum_{r=1}^{q-1} X_{kr}a_{rq}^{(\mathbb{J},k,\nu+1)}\Big|^p\mathbf{E}|X_{kq}|^p\Big),\end{aligned} \tag{5.16}$$

where C_2 denotes the absolute constant in Burkholder's inequality. Conditioning again on $\mathfrak{M}^{(\mathbb{J},k)}$ and applying Rosenthal's inequality, we obtain

$$\begin{aligned}\mathbf{E}|\sum_{r\in\mathbb{T}_{\mathbb{J},k}} X_{kr}a_{rq}^{(\mathbb{J},k,\nu+1)}|^p \le C_1^p p^p\Big(\mathbf{E}(\sum_{r=1}^{q-1}|a_{rq}^{(\mathbb{J},k,\nu+1)}|^2)^{\frac{p}{2}}\\ + \sum_{r\in\mathbb{T}_{\mathbb{J},k}}\mathbf{E}|a_{rq}^{(\mathbb{J},k,\nu+1)}|^p\mathbf{E}|X_{kr}|^p\Big).\end{aligned} \tag{5.17}$$

Combining inequalities (5.16) and (5.17), we get

$$\begin{aligned}\mathbf{E}|Q_{\nu3}^{(\mathbb{J},k)}|^p \le C_2^p p^p\mathbf{E}|Q_{\nu+1}^{(\mathbb{J},k)}|^{\frac{p}{2}} + C_1^pC_2^pp^{2p}\sum_{q\in\mathbb{T}_{\mathbb{J},k}}\mathbf{E}(\sum_{r\in\mathbb{T}_{\mathbb{J},k}}|a_{rq}^{(\mathbb{J},k,\nu+1)}|^2)^{\frac{p}{2}}\mathbf{E}|X_{kq}|^p\\ + C_1^pC_2^pp^{2p}\sum_{q\in\mathbb{T}_{\mathbb{J},k}}\sum_{r\in\mathbb{T}_{\mathbb{J},k}}\mathbf{E}|a_{rq}^{(\mathbb{J},k,\nu+1)}|^p\mathbf{E}|X_{kq}|^p\mathbf{E}|X_{kr}|^p.\end{aligned} \tag{5.18}$$

Using Remark 4.1, this implies

$$\mathbf{E}|Q_{\nu 3}^{(\mathbb{J},k)}|^p \le C_2^p p^p \mathbf{E}|Q_{\nu+1}^{(\mathbb{J},k)}|^{\frac{p}{2}} + C_1^p C_2^p p^{2p} n^{\frac{p}{4}} \frac{1}{n} \sum_{q\in\mathbb{T}_{\mathbb{J},k}} \mathbf{E}(\sum_{r\in\mathbb{T}_{\mathbb{J},k}} |a_{rq}^{(\mathbb{J},k,\nu+1)}|^2)^{\frac{p}{2}}$$
$$+ C_1^p C_2^p p^{2p} n^{\frac{p}{2}} \frac{1}{n^2} \sum_{q\in\mathbb{T}_{\mathbb{J},k}} \sum_{r\in\mathbb{T}_{\mathbb{J},k}} \mathbf{E}|a_{rq}^{(\mathbb{J},k,\nu+1)}|^p. \quad (5.19)$$

Using the definition (5.2) of $Q_{\nu 1}^{(\mathbb{J},k)}$ and the definition (5.3) of coefficients $a_{rr}^{(\mathbb{J},k,\nu+1)}$, it is straightforward to check that

$$\mathbf{E}|Q_{\nu 1}^{(\mathbb{J},k)}|^p \le \mathbf{E}\Big[\sum_{q,r\in\mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,\nu)}|^2\Big]^p. \quad (5.20)$$

Combining (5.15), (5.19) and (5.20), we get by (5.13)

$$3^{-p}\mathbf{E}|Q_{\nu}^{(\mathbb{J},k)}|^p \le C_2^p p^p \mathbf{E}|Q_{\nu+1}^{(\mathbb{J},k)}|^{\frac{p}{2}} + C_1^p C_2^p p^{2p} n^{\frac{p}{4}} \frac{1}{n} \sum_{q\in\mathbb{T}_{\mathbb{J},k}} \mathbf{E}\Big(\sum_{r\in\mathbb{T}_{\mathbb{J},k}} |a_{rq}^{(\mathbb{J},k,\nu+1)}|^2\Big)^{\frac{p}{2}}$$
$$+ C_1^p C_2^p p^{2p} n^{\frac{p}{2}} \frac{1}{n^2} \sum_{q\in\mathbb{T}_{\mathbb{J},k}} \sum_{r\in\mathbb{T}_{\mathbb{J},k}} \mathbf{E}|a_{rq}^{(\mathbb{J},k,\nu+1)}|^p$$
$$+ C_1^p C_2^p \mathbf{E}[\sum_{q,r\in\mathbb{T}_{\mathbb{J},k}} |a_{qr}^{(\mathbb{J},k,\nu)}|^2]^p$$
$$+ C_1^p C_2^p p^p \Big(\mathbf{E}\Big(\sum_{r\in\mathbb{T}_{\mathbb{J},k}} |a_{rr}^{(\mathbb{J},k,\nu+1)}|^2\Big)^{\frac{p}{2}} + n^{\frac{p}{2}} \frac{1}{n} \sum_{r\in\mathbb{T}_{\mathbb{J},k}} \mathbf{E}|a_{rr}^{(\mathbb{J},k,\nu+1)}|^p\Big). \quad (5.21)$$

Applying now Lemma 5.1 and Corollaries 5.2 and 5.3, we obtain

$$\mathbf{E}|Q_{\nu}^{(\mathbb{J},k)}|^p \le C_3^p p^p \mathbf{E}|Q_{\nu+1}^{(\mathbb{J},k)}|^{\frac{p}{2}} + C_3^p \mathbf{E}(\operatorname{Im} m_n(z) + \frac{1}{nv})^{(2^\nu-1)p} v^{-(2^\nu-1)p}$$
$$+ C_3^p p^{2p} v^{-2^\nu p} n^{-\frac{p}{4}} \mathbf{E}(\operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv})^{(2^\nu-1)\frac{2}{p}} \Big(\frac{1}{n} \sum_{q\in\mathbb{T}_{\mathbb{J},k}} |R_{qq}^{(\mathbb{J},k)}|^{\frac{p}{2}}\Big)$$
$$+ C_3^p p^{2p} n^{-\frac{p}{2}} v^{-2^\nu p} \mathbf{E}(\operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv})^{(2^\nu-1)p} \Big(\frac{1}{n} \sum_{q\in\mathbb{T}_{\mathbb{J},k}} |R_{qq}^{(\mathbb{J},k)}|^p\Big), \quad (5.22)$$

where $C_3 = 3C_1C_2$. Applying Cauchy–Schwartz inequality, we may rewrite the last inequality in the form

$$\begin{aligned}\mathbf{E}|Q_\nu^{(\mathbb{J},k)}|^p &\le C_3^p p^p \mathbf{E}|Q_{\nu+1}^{(\mathbb{J},k)}|^{\frac{p}{2}} + C_3^p \mathbf{E}(\operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv})^{(2^\nu-1)p} v^{-(2^\nu-1)p} \\ &+ C_3^p p^{2p} v^{-2^\nu p} n^{-\frac{p}{4}} \mathbf{E}^{\frac12}(\operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv})^{(2^\nu-1)p} \mathbf{E}^{\frac14}(\frac{1}{n}\sum_{q\in\mathbb{T}_{\mathbb{J},k}} |R_{qq}^{(\mathbb{J},k)}|^{2p}) \\ &+ C_3^p p^{2p} n^{-\frac{p}{2}} v^{-2^\nu p} \mathbf{E}^{\frac12}(\operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv})^{(2^\nu-1)2p} \mathbf{E}^{\frac12}(\frac{1}{n}\sum_{q\in\mathbb{T}_{\mathbb{J},k}} |R_{qq}^{(\mathbb{J},k)}|^{2p}).\end{aligned} \tag{5.23}$$

Introduce the notation

$$\Gamma_p(z) := \mathbf{E}^{\frac12}\left(\frac{1}{n}\sum_{q\in\mathbb{T}_{\mathbb{J},k}} |R_{qq}^{(\mathbb{J},k)}|^{2p}\right).$$

We rewrite the inequality (5.23) using $\Gamma_p(z)$ and the notations of (5.10) as follows

$$\begin{aligned}T_{\nu,p}^{(\mathbb{J},k)} &\le (C_3 p)^p T_{\nu+1,p/2}^{(\mathbb{J},k)} + C_3^p A_{\nu,p}^{(\mathbb{J})} v^{-(2^\nu-1)p} \\ &+ (C_3 p^2)^p \Big(v^{-2^\nu p} n^{-\frac{p}{4}} (A_{\nu,p/2}^{(\mathbb{J})})^{\frac12} \Gamma_p^{\frac12}(z) + v^{-2^\nu p} n^{-\frac{p}{2}} A_{\nu,p}^{(\mathbb{J})} \Gamma_p(z)\Big).\end{aligned} \tag{5.24}$$

Note that

$$A_{0,p}^{(\mathbb{J})} = 1, \quad A_{\nu,p/2^\nu}^{(\mathbb{J})} \le \sqrt{1 + \mathbf{E}(\Psi^{(\mathbb{J})})^{2p}} \le 1 + \mathbf{E}^{\frac14}(\Psi^{(\mathbb{J})})^{4p},$$

where $\Psi^{(\mathbb{J})} = \operatorname{Im} m_n^{(\mathbb{J})}(z) + \frac{1}{nv}$. Furthermore,

$$\Gamma_{p/2^\nu} \le \Gamma_p^{\frac{1}{2^\nu}}.$$

Without loss of generality we may assume $p = 2^L$ and $\nu = 0, \dots, L$. We may write

$$T_{0,p}^{(\mathbb{J},k)} \le (C_3 p)^p T_{1,p/2}^{(\mathbb{J},k)} + C_3^p + (C_3 p^2)^p v^{-p}\Big(n^{-\frac{p}{4}} \Gamma_p^{\frac12}(z) + n^{-\frac{p}{2}} \Gamma_p(z)\Big).$$

By induction we get

$$T_{0,p}^{(\mathbb{J},k)} \le \prod_{\nu=0}^{L}(C_3p/2^\nu)^{p/2^\nu}T_{L,1}^{(\mathbb{J},k)} + A_p^{(\mathbb{J})}\sum_{l=1}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)v^{-(2^l-1)p/2^l}$$
$$+ A_p^{(\mathbb{J})}v^{-p}\sum_{l=1}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)(n^{-p}\Gamma_p^2)^{\frac{1}{2^{l+1}}}$$
$$+ A_p^{(\mathbb{J})}\sum_{l=1}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)(n^{-p}{\Gamma_p}^2)^{\frac{1}{2^l}}. \tag{5.25}$$

It is straightforward to check that

$$\sum_{\nu=1}^{l-1}\frac{\nu}{2^\nu} = 2(1-\frac{l+1}{2^l}).$$

Note that, for $l \ge 1$,

$$\prod_{\nu=0}^{l-1}\big(C_3(p/2^\nu)\big)^{p/2^\nu} = \frac{(C_3p)^{2p(1-2^{-l})}}{2^{2p(1-\frac{l+1}{2^l})}} = 2^{2p\frac{l}{2^l}}\Big(\frac{C_3p}{2}\Big)^{2p(1-2^{-l})}. \tag{5.26}$$

Applying this relation, we get

$$A_p^{(\mathbb{J})}\sum_{l=0}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)v^{-(2^l-1)p/2^l} \le A_p^{(\mathbb{J})}(\frac{C_3p}{2})^{2p}v^{-p}\sum_{l=0}^{L-1}2^{\frac{2pl}{2^l}}\Big(\frac{4v}{C_3^2p^2}\Big)^{\frac{p}{2^l}}.$$

Note that for $l \ge 0$, $\frac{l}{2^l} \le \frac{1}{2}$ and recall that $p = 2^L$. Using this observation, we get

$$A_p^{(\mathbb{J})}\sum_{l=0}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)v^{-(2^l-1)p/2^l} \le A_p^{(\mathbb{J})}(\frac{C_3p}{2})^{2p}v^{-p}2^p\sum_{l=0}^{L-1}\Big(\frac{4v}{C_3^2p^2}\Big)^{2^{L-l}}.$$

This implies that for $\frac{4v}{C_3^2p^2} \le \frac{1}{2}$,

$$A_p^{(\mathbb{J})}\sum_{l=1}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)v^{-(2^l-1)p/2^l} \le (C_3p)^{2p}A_p^{(\mathbb{J})}v^{-p}.$$

Furthermore, by definition of $T_{\nu,p}$, we have

$$T_{L,1}^{(\mathbb{J},k)} = \mathbf{E}Q_L^{(\mathbb{J},k)} \le \mathbf{E}\sum_{q,r\in\mathbb{T}_k}(a_{qr}^{(\mathbb{J},k,L)})^2.$$

Applying Corollary 5.2 and Hölder's inequality, we get

$$T_{L,1}^{(\mathbb{J},k)} \le E(v^{-1}\Psi^{(\mathbb{J})})^p \le v^{-p}A_p^{(\mathbb{J})}. \tag{5.27}$$

By condition (5.11), we have

$$\Gamma_p := \Gamma_p(u+iv) \le s_0^{2p}C_0^{2p}.$$

Using this inequality, we get,

$$A_p^{(\mathbb{J})}v^{-p}\sum_{l=1}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)n^{-\frac{p}{2^{l+1}}}\Gamma_p^{\frac{2}{2^{l+1}}}$$
$$\le A_p^{(\mathbb{J})}v^{-p}\sum_{l=0}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)(s_0^4C_0^4n^{-1})^{\frac{p}{2^{l+1}}}. \tag{5.28}$$

Applying relation (5.26), we obtain

$$A_p^{(\mathbb{J})}v^{-p}\sum_{l=1}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)n^{-\frac{p}{2^{l+1}}}\Gamma_p^{\frac{1}{2^l}}$$
$$\le \Big(\frac{C_3p}{2}\Big)^{2p}A_p^{(\mathbb{J})}v^{-p}\sum_{l=1}^{L}2^{2p\frac{l}{2^l}}\Big(\frac{C_3p}{2}\Big)^{-\frac{2p}{2^l}}(s_0^4C_0^4n^{-1})^{\frac{p}{2^{l+1}}}$$
$$= \Big(\frac{C_3p}{2}\Big)^{2p}A_p^{(\mathbb{J})}v^{-p}\sum_{l=1}^{L}2^{p\frac{l}{2^{l-1}}}\Big(\frac{C_3p}{2}\Big)^{-\frac{2p}{2^{l-1}}}((s_0^4C_0^4n^{-1})^{\frac14})^{\frac{p}{2^{l-1}}}$$
$$= \Big(\frac{C_3p}{\sqrt{2}}\Big)^{2p}A_p^{(\mathbb{J})}v^{-p}\sum_{l=1}^{L}\Big(\frac{(s_0C_0)}{C_3pn^{\frac14}}\Big)^{2^{L-l+1}}.$$

Without loss of generality we may assume that $C_3 \ge 2(C_0s_0)$. Then we get

$$A_p^{(\mathbb{J})}v^{-p}\sum_{l=0}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)n^{-\frac{p}{2^{l+1}}}\Gamma_p^{\frac{1}{2^l}} \le (C_3p)^{2p}A_p^{(\mathbb{J})}v^{-p}.$$

Analogously we get

$$A_p^{(\mathbb{J})}v^{-p}\sum_{l=1}^{L}\Big(\prod_{\nu=0}^{l-1}(C_3p/2^\nu)^{p/2^\nu}\Big)n^{-\frac{p}{2^l}}\Gamma_p^{\frac{1}{2^{l-1}}} \le (C_3p)^{2p}v^{-p}A_p^{(\mathbb{J})}. \tag{5.29}$$

Combining inequalities (5.25), (5.27), (5.28), (5.29), we finally arrive at

$$T_{0,p}^{(\mathbb{J},k)} \le 6(C_3p)^{2p}v^{-p}A_p^{(\mathbb{J})}. \tag{5.30}$$

Thus, Lemma 5.4 is proved. □

5.2 Diagonal Entries of the Resolvent Matrix

We shall use the representation, for any $j \in \mathbb{T}_{\mathbb{J}}$,

$$R_{jj}^{(\mathbb{J})} = \frac{1}{-z + \frac{1}{\sqrt{n}} X_{jj} - \frac{1}{n}\sum_{k,l \in \mathbb{T}_{\mathbb{J},j}} X_{jk} X_{jl} R_{kl}^{(\mathbb{J},j)}},$$

(see, for example, equality (4.6) in [13]). Recall the following relations (compare (3.2),(3.7))

$$R_{jj}^{(\mathbb{J})} = -\frac{1}{z + m_n^{(\mathbb{J})}(z)} + \frac{1}{z + m_n^{(\mathbb{J})}(z)} \varepsilon_j R_{jj}^{(\mathbb{J})}, \tag{5.31}$$

or

$$R_{jj}^{(\mathbb{J})} = s(z) - s(z)\Lambda_n^{(\mathbb{J})} R_{jj}^{(\mathbb{J})} - s(z)\varepsilon_j^{(\mathbb{J})} R_{jj}^{(\mathbb{J})}, \tag{5.32}$$

where $\varepsilon_j^{(\mathbb{J})} := \varepsilon_{j1}^{(\mathbb{J})} + \varepsilon_{j2}^{(\mathbb{J})} + \varepsilon_{j3}^{(\mathbb{J})} + \varepsilon_{j4}^{(\mathbb{J})}$ with

$$\varepsilon_{j1}^{(\mathbb{J})} := \frac{1}{\sqrt{n}} X_{jj},\ \varepsilon_{j2}^{(\mathbb{J})} := -\frac{1}{n} \sum_{k \neq l \in \mathbb{T}_{\mathbb{J},j}} X_{jk} X_{jl} R_{kl}^{(\mathbb{J},j)},\ \varepsilon_{j3}^{(\mathbb{J})} := -\frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J},j}} (X_{jk}^2 - 1) R_{kk}^{(\mathbb{J},j)},$$
$$\varepsilon_{j4}^{(\mathbb{J})} := \frac{1}{n}(\operatorname{Tr}\mathbf{R}^{(\mathbb{J})} - \operatorname{Tr}\mathbf{R}^{(\mathbb{J},j)}), \Lambda_n^{(\mathbb{J})} := m_n^{(\mathbb{J})}(z) - s(z) = \frac{1}{n}\operatorname{Tr}\mathbf{R}^{(\mathbb{J})} - s(z). \tag{5.33}$$

Since $|s(z)| \le 1$, the representation (5.32) yields, for any $p \ge 1$,

$$|R_{jj}^{(\mathbb{J})}|^p \le 3^p + 3^p |\varepsilon_j^{(\mathbb{J})}|^p |R_{jj}^{(\mathbb{J})}|^p + 3^p |\Lambda_n^{(\mathbb{J})}|^p |R_{jj}^{(\mathbb{J})}|^p. \tag{5.34}$$

Applying the Cauchy – Schwartz inequality, we get

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le 3^p + 3^p \mathbf{E}^{\frac12} |\varepsilon_j^{(\mathbb{J})}|^{2p} \mathbf{E}^{\frac12} |R_{jj}^{(\mathbb{J})}|^{2p} + 3^p \mathbf{E}^{\frac12} |\Lambda_n^{(\mathbb{J})}|^{2p} \mathbf{E}^{\frac12} |R_{jj}^{(\mathbb{J})}|^{2p}. \tag{5.35}$$

We shall investigate now the behavior of $\mathbf{E}|\varepsilon_j^{(\mathbb{J})}|^{2p}$ and $\mathbf{E}|\Lambda_n^{(\mathbb{J})}|^{2p}$. First we note,

$$\mathbf{E}|\varepsilon_j^{(\mathbb{J})}|^{2p} \le 4^{2p} \sum_{\nu=1}^{4} \mathbf{E}|\varepsilon_{j\nu}^{(\mathbb{J})}|^{2p}.$$

Lemma 5.5. *Assuming the conditions of Theorem 1.1 we have, for any $p \ge 1$, and for any $z = u + iv \in \mathbb{C}_+$,*

$$\mathbf{E}|\varepsilon_{j1}^{(\mathbb{J})}|^{2p} \le \frac{\mu_4}{n^{\frac{p}{2}+1}}.$$

Proof. The proof follows immediately from the definition of ε_{j1} and condition (1.4). $\square$

Lemma 5.6. *Assuming the conditions of Theorem 1.3 we have, for any $p \ge 1$, and for any $z = u + iv \in \mathbb{C}_+$,*

$$\mathbf{E}|\varepsilon_{j4}^{(\mathbb{J})}|^{2p} \le \frac{1}{n^{2p}v^{2p}}.$$

Proof. For a proof of this Lemma see [13, Lemma 4.1]. $\square$

Let $A_1 > 0$ and $0 \le v_1 \le 4$ be a fixed.

Lemma 5.7. *Assuming the conditions of Theorem 1.1, and assuming for all $\mathbb{J} \subset T$ with $|\mathbb{J}| \le L$ and all $l \in \mathbb{T}_{\mathbb{J}}$*

$$\mathbf{E}|R_{ll}^{(\mathbb{J})}|^q \le C_0^q, \text{ for } 1 \le q \le A_1(nv)^{\frac{1}{4}} \text{ and for } v \ge v_1, \tag{5.36}$$

we have, for all $v \ge v_1/s_0$, and for all $\mathbb{J} \subset \mathbb{T}$ with $|\mathbb{J}| \le L-1$,

$$\mathbf{E}|\varepsilon_{j3}^{(\mathbb{J})}|^{2p} \le (C_1 p)^{2p} n^{-p} s_0^{2p} C_0^{4p}, \text{ for } 1 \le p \le A_1(nv)^{\frac{1}{4}}.$$

Proof. Recall that $s_0 = 2^4$ and note that if $p \le A_1(nv)^{\frac{1}{4}}$ for $v \ge v_1/s_0$ then $q = 2p \le A_1(nv)^{\frac{1}{4}}$ for $v \ge v_1$. Let $v' := vs_0$. If $v \ge v_1/s_0$ then $v' \ge v_1$. We have

$$q = 2p \le 2A_1(nv)^{\frac{1}{4}} = 2A_1(nv's_0^{-1})^{\frac{1}{4}} = A_1(nv')^{\frac{1}{4}}. \tag{5.37}$$

We apply now Rosenthal's inequality for the moments of sums of independent random variables and get

$$\mathbf{E}|\varepsilon_{j3}^{(\mathbb{J})}|^{2p} \le (C_1 p)^{2p} n^{-2p} \Big(\mathbf{E}\big(\sum_{l\in\mathbb{T}_{\mathbb{J},j}} |R_{ll}^{(\mathbb{J},j)}|^2\big)^p + \mathbf{E}|X_{jl}|^{4p} \sum_{l\in\mathbb{T}_{\mathbb{J},j}} \mathbf{E}|R_{ll}^{(\mathbb{J},j)}|^{2p}\Big).$$

According to inequality (5.37) we may apply Lemma 7.7 and condition (5.36) for $q = 2p$. We get, for $v \ge v_1/s_0$,

$$\mathbf{E}|\varepsilon_{j3}^{(\mathbb{J})}|^{2p} \le (C_1 p)^{2p} n^{-p} s_0^{2p} C_0^{2p}.$$

We use as well that by the conditions of Theorem 1.1, $\mathbf{E}|X_{jl}|^{4p} \le D_0^{4p-4} n^{p-1}\mu_4$, and by Jensen's inequality, $(\frac{1}{n}\sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^2)^p \le \frac{1}{n}\sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^{2p}$. Thus, Lemma 5.7 is proved. $\square$

Lemma 5.8. *Assuming the conditions of Theorem 1.1, condition* (5.36)*, for* $v \geq v_1$ *and* $p \leq A_1(nv)^{\frac{1}{4}}$*, we have, for any* $v \geq v_1/s_0$ *and* $p \leq A_1(nv)^{\frac{1}{4}}$*,*

$$\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2p} \leq 6(C_3p)^{4p}n^{-p}v^{-p}A_p^{(\mathbb{J})} + 2(C_3p)^{4p}n^{-p}v^{-p}(C_0s_0)^p.$$

Proof. We apply Burkholder's inequality for quadratic forms. See Lemma 7.3 in the Appendix. We obtain

$$\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2p} \leq (C_1p)^{2p}n^{-2p}\Bigg(\mathbf{E}\big(\sum_{l\in\mathbb{T}_{\mathbb{J},j}}|\sum_{r\in\mathbb{T}_{\mathbb{J},j}\cap\{1,\ldots,l-1\}}X_{jr}R_{lr}^{(\mathbb{J},j)}|^2\big)^p$$
$$+\max_{j,k}\mathbf{E}|X_{jk}|^{2p}\sum_{l\in\mathbb{T}_{\mathbb{J},j}}\mathbf{E}|\sum_{r\in\mathbb{T}_{\mathbb{J},j}\cap\{1,\ldots,l-1\}}X_{jr}R_{lr}^{(\mathbb{J},j)}|^{2p}\Bigg).$$

Using now the quantity $Q_0^{(\mathbb{J},j)}$ for the first term and Rosenthal's inequality and condition (1.4) for the second term, we obtain with Lemma 7.6, inequality (7.12), in the Appendix and $C_3 = C_1C_2$

$$\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2p} \leq (C_2p)^{2p}n^{-p}\mathbf{E}|Q_0^{(\mathbb{J},j)}|^p$$
$$+ (C_3p)^{4p}n^{-\frac{3p}{2}}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J},j}}\mathbf{E}\big(\sum_{r\in\mathbb{T}_{\mathbb{J},j}\cap\{1,\ldots,l-1\}}|R_{lr}^{(\mathbb{J},j)}|^2\big)^p$$
$$+ (C_3p)^{4p}n^{-p}\frac{1}{n^2}\sum_{l\in\mathbb{T}_{\mathbb{J},j}}\sum_{r\in\mathbb{T}_{\mathbb{J},j}\cap\{1,\ldots,l-1\}}\mathbf{E}|R_{lr}^{(\mathbb{J},j)}|^{2p}$$
$$\leq (C_2p)^{2p}n^{-p}\mathbf{E}|Q_0^{(\mathbb{J},j)}|^p + (C_3p)^{4p}n^{-\frac{3p}{2}}v^{-p}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J},j}}\mathbf{E}|R_{ll}^{(\mathbb{J},j)}|^p$$
$$+ (C_3p)^{4p}n^{-p}v^{-p}\frac{1}{n^2}\sum_{l\in\mathbb{T}_{\mathbb{J},j}}\mathbf{E}|R_{ll}^{(\mathbb{J},j)}|^p.$$

By Lemma 7.7 and condition (5.36), we get

$$\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2p} \leq (C_3p)^{2p}n^{-p}\mathbf{E}|Q_0^{(\mathbb{J},j)}|^p + 2(C_3p)^{3p}n^{-p}v^{-p}(C_0s_0)^p.$$

Applying now Lemma 5.4, we get the claim. Thus, Lemma 5.8 is proved. $\square$

Recall that

$$\Lambda_n^{(\mathbb{J})} = \frac{1}{n}\operatorname{Tr}\mathbf{R}^{(\mathbb{J})} - s(z), \quad \text{and} \quad T_n^{(\mathbb{J})}(z) = \frac{1}{n}\sum_{j\in\mathbb{T}_{\mathbb{J}}}\varepsilon_j^{(\mathbb{J})}R_{jj}^{(\mathbb{J})}.$$

Lemma 5.9. *Assuming the conditions of Theorem 1.1, we have*

$$|\Lambda_n^{(\mathbb{J})}| \le C(\sqrt{|T_n^{(\mathbb{J})}(z)|} + \frac{\sqrt{|\mathbb{J}|}}{\sqrt{n}}).$$

Proof. See e. g. inequality (2.10) in [17]. For completeness we include short proof here. Obviously

$$\Lambda_n^{(\mathbb{J})}(z) == \frac{s(z)(T_n^{(\mathbb{J})}(z) - \frac{|\mathbb{J}|}{n})}{z + 2s(z) + \Lambda_n^{(\mathbb{J})}(z)}. \tag{5.38}$$

Denote by

$$\widetilde{T}_n(z) = s(z)(T_n^{(\mathbb{J})}(z) - \frac{|\mathbb{J}|}{n}).$$

First we assume that $|z+2s(z)+\Lambda_n^{(\mathbb{J})}| > \sqrt{|\widetilde{T}_n(z)|}$ Then the claim of Lemma 5.9 holds. In the case $|z + 2s(z) + \Lambda_n^{(\mathbb{J})}| \le \sqrt{|\widetilde{T}_n(z)|}$, we assume $|\Lambda_n^{(\mathbb{J})}| > 2\sqrt{|\widetilde{T}_n(z)|}$. Otherwise the Lemma is proved. Under this assumptions we have

$$|z + 2s(z)| \ge |\Lambda^{(\mathbb{J})}| - |z + 2s(z) + \Lambda_n^{(\mathbb{J})}| \ge \sqrt{|\widetilde{T}_n(z)|} \tag{5.39}$$

On the other hand

$$|z + s(z) + m_n(z)| \ge \operatorname{Im}|z + s(z)| \ge \frac{1}{2}\operatorname{Im} z + 2s(z) = \frac{1}{2}\sqrt{z^2 - 4}. \tag{5.40}$$

We take here the branch of $\sqrt{z}$ such that $\operatorname{Im}\sqrt{z} \ge 0$. Note that for $z \in [-2, 2]$ we have $\operatorname{Re} z^2 - 4 \le 0$. This implies that

$$\operatorname{Im} z^2 - 4 \ge \frac{\sqrt{2}}{2}|z^2 - 4|^{\frac{1}{2}} = \frac{\sqrt{2}}{2}|z + 2s(z)|. \tag{5.41}$$

Inequalities (5.39), (5.40) and (5.41) together imply

$$|z + s(z) + m_n^{(\mathbb{J})}(z)| \ge \frac{1}{2\sqrt{2}}|z + 2s(z)| \ge \frac{\sqrt{2}}{2}\sqrt{|\widetilde{T}_n(z)}. \tag{5.42}$$

The last inequality and Eq. (5.38) complete the proof of lemma 5.9.

□

Lemma 5.10. *Assuming the conditions of Theorem 1.1 and condition* (5.36), *we obtain, for* $|\mathbb{J}| \le Cn^{\frac{1}{2}}$

$$\mathbf{E}|\Lambda_n^{(\mathbb{J})}|^{2p} \le \frac{C^p}{n^{\frac{p}{4}}} + \Big(\frac{\mu_4}{n^{\frac{p}{2}+1}} + \frac{1}{n^{2p}v^{2p}} + (C_1p)^{2p}n^{-p}s_0^{2p}C_0^{4p}$$
$$+ 6(C_3p)^{4p}n^{-p}v^{-p}A_p^{(\mathbb{J})} + 2(C_3p)^{4p}n^{-p}v^{-p}(C_0s_0)^p\Big)^{\frac{1}{2}}(C_0s_0)^p.$$

Proof. By Lemma 5.8, we have

$$\mathbf{E}|\Lambda_n^{(\mathbb{J})}|^{2p} \le C^p\mathbf{E}|T_n^{(\mathbb{J})}(z)|^p + \frac{|\mathbb{J}|^{\frac{p}{2}}}{n^{\frac{p}{2}}} \le C^p\mathbf{E}|T_n^{(\mathbb{J})}(z)|^p + \frac{C}{n^{\frac{p}{4}}}.$$

Furthermore,

$$\mathbf{E}|T_n^{(\mathbb{J})}(z)|^p \le \Big(\frac{1}{n}\sum_{j\in\mathbb{T}}\mathbf{E}|\varepsilon_j^{(\mathbb{J})}|^{2p}\Big)^{\frac{1}{2}}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}}\mathbf{E}|R_{jj}^{(\mathbb{J})}|^{2p}\Big)^{\frac{1}{2}}.$$

Lemmas 5.5 – 5.8 together with Lemma 7.7 imply

$$\begin{aligned}\frac{1}{n}\sum_{j\in\mathbb{T}}\mathbf{E}|\varepsilon_j^{(\mathbb{J})}|^{2p} \le & 4^{2p-1}\Big(\frac{\mu_4}{n^{\frac{p}{2}+1}} + \frac{1}{n^{2p}v^{2p}} + (C_1p)^{2p}n^{-p}s_0^{2p}C_0^{4p} \\ & + 6(C_3p)^{4p}n^{-p}v^{-p}A_p^{(\mathbb{J})} + 2(C_3p)^{4p}n^{-p}v^{-p}(C_0s_0)^p\Big).\end{aligned} \tag{5.43}$$

Thus, Lemma 5.10 is proved. □

Lemma 5.11. *Assuming the conditions of Theorem 1.1 and condition* (5.36)*, there exists an absolute constant* C_4 *such that, for* $p \le A_1(nv)^{\frac{1}{4}}$ *and* $v \ge v_1/s_0$*, we have, uniformly in* $\mathbb{J}\subset\mathbb{T}$ *such that* $|\mathbb{J}| \le Cn^{\frac{1}{2}}$*,*

$$A_p^{(\mathbb{J})} \le C_4^p.$$

Proof. We start from the obvious inequality, using $|s(z)| \le 1$,

$$\mathbf{E}|\Psi^{(\mathbb{J})}|^{2p} \le 3^{2p}(1 + (nv)^{-2p} + \mathbf{E}|\Lambda_n^{(\mathbb{J})}(z)|^{2p}).$$

Furthermore, applying Lemma 5.10, we get

$$\begin{aligned}\mathbf{E}|\Psi^{(\mathbb{J})}|^{2p} \le 3^{2p}\Bigg(& 1 + (nv)^{-p} + \Big(\frac{\mu_4}{n^{\frac{p}{2}+1}} + \frac{1}{n^{2p}v^{2p}} + (C_1p)^{2p}n^{-p}s_0^{2p}C_0^{4p} \\ & + 6(C_3p)^{4p}n^{-p}v^{-p}A_p^{(\mathbb{J})} + 2(C_3p)^{4p}n^{-p}v^{-p}(C_0s_0)^p\Big)^{\frac{1}{2}}(C_0s_0)^p\Bigg).\end{aligned} \tag{5.44}$$

By definition,

$$A_p^{(\mathbb{J})} \le 1 + \mathbf{E}^{\frac{1}{2}}(\Psi^{(\mathbb{J})})^{2p}. \tag{5.45}$$

Inequalities (5.45) and (5.44) together imply

$$A_p^{(\mathbb{J})} \le 1 + 3^p \Bigg(1 + (nv)^{-\frac{p}{2}} + (C_0 s_0)^{\frac{p}{2}} \Big(\mu_4^{\frac{1}{4}} n^{-\frac{p}{8}} + \frac{1}{n^{\frac{p}{2}} v^{\frac{p}{2}}} + (C_1 p)^{\frac{p}{2}} n^{-\frac{p}{4}} s_0^{\frac{p}{2}} C_0^p$$
$$+ 3(C_3 p)^p n^{-\frac{p}{4}} v^{-\frac{p}{4}} (A_p^{(\mathbb{J})})^{\frac{1}{4}} + 2(C_3 p)^p n^{-\frac{p}{4}} v^{-\frac{p}{4}} (C_0 s_0)^{\frac{p}{4}}\Big)\Bigg).$$

Let $C' = s_0 \max\{9, C_3^{\frac{1}{2}}, C_0^{\frac{3}{2}} C_1^{\frac{1}{2}}, 3C_3 C_0^{\frac{1}{2}}, C_3 C_0^{\frac{3}{4}}\}$. Using Lemma 7.4 with $x = (A_p^{(\mathbb{J})})^{\frac{1}{4}}$, $t = 4$, $r = 1$, we get

$$A_p^{(\mathbb{J})} \le C'^p \Big(1 + (nv)^{-\frac{p}{2}} + \mu_4^{\frac{1}{2}} n^{-\frac{p}{8}} + \frac{1}{n^{\frac{p}{2}} v^{\frac{p}{2}}}$$
$$+ p^{\frac{p}{2}} n^{-\frac{p}{4}} + p^p n^{-\frac{p}{4}} v^{-\frac{p}{4}} + p^{\frac{4p}{3}} n^{-\frac{p}{3}} v^{-\frac{p}{3}}\Big).$$

For $p \le A_1 (nv)^{\frac{1}{4}}$, we get, for $z \in \mathbb{G}$,

$$A_p^{(\mathbb{J})} \le C_4^p,$$

where C_4 is some absolute constant. We may take $C_4 = 2C'$. □

Corollary 5.12. *Assuming the conditions of Theorem 1.1 and condition* (5.36)*, we have , for $v \ge v_1/s_0$, and for any $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \le \sqrt{n}$*

$$\mathbf{E}|\Lambda_n^{(\mathbb{J})}|^{2p} \le C_0^{2p} \Big(\frac{4^{\frac{p}{4}} \mu_4^{\frac{1}{2}} s_0^p}{n^{\frac{p}{4}} v^{\frac{p}{4}}} + \frac{s_0^{\frac{p}{2}}}{n^p v^p} + \frac{C_5^p p^{2p}}{n^{\frac{p}{2}} v^{\frac{p}{2}}} \Big), \tag{5.46}$$

where

$$C_5 := 4C_1^2 s_0^4 + 6^{\frac{1}{p}} C_3^4 C_4 + 2^{\frac{1}{p}} C_3^4 s_0^3.$$

Proof. Without loss of generality we may assume that $C_0 > 1$. The bound (5.46) follows now from Lemmas 5.10 and 5.11. □

Lemma 5.13. *Assuming the conditions of Theorem 1.1 and condition* (5.36) *for $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \le L \le \sqrt{n}$, there exist positive constant A_0, C_0, A_1 depending on μ_4, D_0 only , such that we have, for $p \le A_1 (nv)^{\frac{1}{4}}$ and $v \ge v_1/s_0$ uniformly in $\mathbb{J}$ and v_1*

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le C_0^p$$

with $|\mathbb{J}| \le L - 1$.

Proof. According to inequality (5.35), we have

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le 4^p(1+(\mathbf{E}^{\frac12}|\Lambda_n^{(\mathbb{J})}|^{2p}+\mathbf{E}^{\frac12}|\varepsilon_j^{(\mathbb{J})}|^{2p})\mathbf{E}^{\frac12}|R_{jj}^{(\mathbb{J})}|^{2p}).$$

Applying condition (5.36), we get

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le 4^p(1+(\mathbf{E}^{\frac12}|\Lambda_n^{(\mathbb{J})}|^{2p}+\mathbf{E}^{\frac12}|\varepsilon_{j1}^{(\mathbb{J})}|^{2p}+\cdots+\mathbf{E}^{\frac12}|\varepsilon_{j4}^{(\mathbb{J})}|^{2p})s_0^pC_0^p).$$

Combining results of Lemmas 5.5 – 5.8 and Corollary 5.12, we obtain

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le 5^p\Bigg(1+s_0^pC_0^{2p}\Big(\frac{4^{\frac p4}\mu_4^{\frac14}s_0^p}{n^{\frac p4}v^{\frac p4}}+\frac{s_0^p}{n^pv^p}+\frac{C_5^pp^{2p}}{n^{\frac p2}v^{\frac p2}}\Big)^{\frac12}$$
$$+s_0^pC_0^{3p}\Big(\frac{4^{\frac p4}\mu_4^{\frac14}}{n^{\frac p4}v^{\frac p4}}+\frac{s_0^p}{n^pv^p}+\frac{C_5^pp^{2p}}{n^{\frac p2}v^{\frac p2}}\Big)\Bigg).$$

We may rewrite the last inequality as follows

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le C_0^p\Big(\frac{5^p}{C_0^p}+\frac{\widehat C_1^{\frac p8}}{(nv)^{\frac p8}}+\frac{(\widehat C_2p^4)^{\frac p4}}{(nv)^{\frac p4}}+\frac{(\widehat C_3p^4)^{\frac p2}}{(nv)^{\frac p2}}+\frac{\widehat C_4^p}{(nv)^p}\Big),$$

where

$$\widehat C_1 = 5^8s_0^{12}C_0^8\mu_4^{\frac1p},$$
$$\widehat C_2 = 5^4s_0^4C_0^4C_5^2(1+2C_0^4\mu_4^{\frac2p}),$$
$$\widehat C_3 = 5^2s_0^2C_0^2(s_0+C_5^2),$$
$$\widehat C_4 = 5C_0^2s_0^2.$$

Note that for

$$A_0 \ge 2^8A_1^4\max\{\widehat C_1,\dots,\widehat C_4\} \tag{5.47}$$

and $C_0 \ge 25$, we obtain that

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le C_0^p.$$

Thus Lemma 5.13 is proved. □

Corollary 5.14. *Assuming the conditions of Theorem 1.1, we have, for $p \le 8$ and $v \ge v_0 = A_0n^{-1}$ there exist a constant $C_0 > 0$ depending on μ_4 and D_0 only such that for all $1 \le j \le n$ and all $z \in \mathbb{G}$*

$$\mathbf{E}|R_{jj}|^p \le C_0^p, \tag{5.48}$$

and

$$\mathbf{E}\frac{1}{|z+m_n(z)|^p} \le C_0^p. \tag{5.49}$$

Proof. Let $L = [-\log_{s_0} v_0] + 1$. Note that $s_0^{-L} \le v_0$ and $A_1 \frac{n^{\frac14}}{s_0^{\frac{L}{4}}} \ge A_1 (n v_0)^{\frac14}$. We may choose $C_0 = 25$ and A_0, A_1 such that (5.47) holds and

$$A_1 (nv)^{\frac14} \ge 8.$$

Then, for $v = 1$, and for any $p \ge 1$, for any set $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \le L$

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^p \le C_0^p. \tag{5.50}$$

By Lemma 5.13, inequality (5.50) holds for $v \ge 1/s_0$ and for $p \le A_1 n^{\frac14}/s_0^{\frac14}$ and for $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \le L - 1$. After repeated application of Lemma 5.13 (with (5.50) as assumption valid for $v \ge 1/s_0$) we arrive at the conclusion that the inequality (5.50) holds for $v \ge 1/s_0^2$, $p \le A_1 n^{\frac14}/s_0^{\frac12}$ and all $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \le L - 2$. Continuing this iteration inequality (5.50) finally holds for $v \ge A_0 n^{-1}$, $p \le 8$ and $\mathbb{J} = \emptyset$.
The proof of inequality of (5.49) is similar: We have by (1.9)

$$\frac{1}{|z + m_n(z)|} \le \frac{1}{|s(z) + z|} + \frac{|\Lambda_n|}{|z + m_n(z)||z + s(z)|} \le |s(z)|(1 + \frac{|\Lambda_n|}{|z + m_n(z)|}). \tag{5.51}$$

Furthermore, using that $|m_n'(z)| \le |\frac1n \operatorname{Tr} \mathbf{R}^2| \le \frac1n \sum_{j=1}^n \mathbf{E}|R_{jj}|^2$ and Lemma 7.6 inequality (7.11) in the Appendix, we get

$$|\frac{d}{dz} \log(z + m_n(z))| \le \frac{|1 + m_n'(z)|}{|z + m_n(z)|} \le \frac1v \frac{v + \operatorname{Im} m_n(z)}{|z + m_n(z)|} \le \frac1v.$$

By integration, this implies that (see the proof of Lemma 7.7)

$$\frac{1}{|(u + iv/s_0) + m_n(u + iv/s_0)|} \le \frac{s_0}{|(u + iv) + m_n(u + iv)|}. \tag{5.52}$$

Inequality (5.51) and the Cauchy–Schwartz inequality together imply

$$\mathbf{E}\frac{1}{|z + m_n(z)|^p} \le 2^p |s(z)|^p (1 + \mathbf{E}^{\frac12} |\Lambda_n|^{2p} \mathbf{E}^{\frac12} \frac{1}{|z + m_n(z)|^{2p}}).$$

Applying inequality (5.52), we obtain

$$\mathbf{E}\frac{1}{|z + m_n(z)|^p} \le 2^p |s(z)|^p (1 + \mathbf{E}^{\frac12} |\Lambda_n|^{2p} s_0^p C_0^p).$$

Using Corollary 5.12, we get, for $v \ge 1/s_0$

$$\mathbf{E}\frac{1}{|z+m_n(z)|^p} \le 2^p|s(z)|^p\Big(1+\Big(\frac{4^{\frac{p}{8}}\mu_4^{\frac14}s_0^{\frac{p}{2}}}{n^{\frac{p}{8}}v^{\frac{p}{8}}}+\frac{s_0^{\frac{p}{4}}}{n^{\frac{p}{2}}v^{\frac{p}{2}}}+\frac{C_5^pp^p}{n^{\frac{p}{4}}v^{\frac{p}{4}}}\Big)s_0^pC_0^{2p}\Big).$$

Thus inequality (5.49) holds for $v \ge 1/s_0$ as well. Repeating this argument inductively with $A_0, A_1, C_)$ satisfying (5.47) for the regions $v \ge s_0^{-\nu}$, for $\nu = 1, \ldots, L$ and $z \in \mathbb{G}$, we get the claim. Thus, Corollary 5.14 is proved.

□

6 Proof of Theorem 1.3

We return now to the representation (5.32) which implies that

$$s_n(z) = \frac{1}{n}\sum_{j=1}^n \mathbf{E}R_{jj} = s(z) + \mathbf{E}\Lambda_n = s(z) + \mathbf{E}\frac{T_n(z)}{z+s(z)+m_n(z)}. \tag{6.1}$$

We may continue the last equality as follows

$$s_n(z) = s(z) + \mathbf{E}\frac{\frac1n\sum_{j=1}^n \varepsilon_{j4}R_{jj}}{z+s(z)+m_n(z)} + \mathbf{E}\frac{\widehat{T}_n(z)}{z+s(z)+m_n(z)}, \tag{6.2}$$

where

$$\widehat{T}_n = \sum_{\nu=1}^3 \frac{1}{n}\sum_{j=1}^n \varepsilon_{j\nu}R_{jj}.$$

Note that the definition of ε_{j4} in (5.32) and equality (7.34) together imply

$$\frac1n\sum_{j=1}^n \varepsilon_{j4}R_{jj} = \frac1n \operatorname{Tr}\mathbf{R}^2 = \frac1n\frac{dm_n(z)}{dz}. \tag{6.3}$$

Thus we may rewrite (6.2) as

$$s_n(z) = s(z) + \frac1n\mathbf{E}\frac{m_n'(z)}{z+s(z)+m_n(z)} + \mathbf{E}\frac{\widehat{T}_n(z)}{z+s(z)+m_n(z)}. \tag{6.4}$$

Denote by

$$\mathfrak{T} = \mathbf{E}\frac{\widehat{T}_n(z)}{z+s(z)+m_n(z)}. \tag{6.5}$$

6.1 Estimation of $\mathfrak{T}$

We represent $\mathfrak{T}$

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2,$$

where

$$\mathfrak{T}_1 = -\frac{1}{n}\sum_{j=1}^{n}\sum_{\nu=1}^{3}\mathbf{E}\frac{\varepsilon_{j\nu}\frac{1}{z+m_n^{(j)}(z)}}{z+m_n(z)+s(z)},$$

$$\mathfrak{T}_2 = \frac{1}{n}\sum_{j=1}^{n}\sum_{\nu=1}^{3}\mathbf{E}\frac{\varepsilon_{j\nu}(R_{jj}+\frac{1}{z+m_n^{(j)}(z)})}{z+m_n(z)+s(z)}.$$

6.1.1 Estimation of $\mathfrak{T}_1$

We may decompose $\mathfrak{T}_1$ as

$$\mathfrak{T}_1 = \mathfrak{T}_{11} + \mathfrak{T}_{12}, \tag{6.6}$$

where

$$\mathfrak{T}_{11} = -\frac{1}{n}\sum_{j=1}^{n}\sum_{\nu=1}^{3}\mathbf{E}\frac{\varepsilon_{j\nu}\frac{1}{z+m_n^{(j)}(z)}}{z+m_n^{(j)}(z)+s(z)},$$

$$\mathfrak{T}_{12} = -\frac{1}{n}\sum_{j=1}^{n}\sum_{\nu=1}^{3}\mathbf{E}\frac{\varepsilon_{j\nu}\varepsilon_{j4}\frac{1}{z+m_n^{(j)}(z)}}{(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))}.$$

It is easy to see that, by conditional expectation

$$\mathfrak{T}_{11} = 0. \tag{6.7}$$

Applying the Cauchy–Schwartz inequality, for $\nu = 1, 2, 3$, we get

$$\left|\mathbf{E}\frac{\varepsilon_{j\nu}\varepsilon_{j4}\frac{1}{z+m_n^{(j)}(z)}}{(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))}\right|$$
$$\leq \mathbf{E}^{\frac{1}{2}}\left|\frac{\varepsilon_{j\nu}}{(z+m_n^{(j)}(z))(z+m_n^{(j)}(z)+s(z))}\right|^2\mathbf{E}^{\frac{1}{2}}\left|\frac{\varepsilon_{j4}}{z+m_n(z)+s(z)}\right|^2. \tag{6.8}$$

Applying the Cauchy – Schwartz inequality again, we get

$$\mathbf{E}^{\frac{1}{2}}\left|\frac{\varepsilon_{j\nu}}{(z+m_n^{(j)}(z))(z+m_n^{(j)}(z)+s(z))}\right|^2 \leq \mathbf{E}^{\frac{1}{4}}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)+s(z)|^4}\mathbf{E}^{\frac{1}{4}}\frac{1}{|z+m_n^{(j)}(z)|^4}. \tag{6.9}$$

Inequalities (6.8), (6.9), Corollaries 7.23 and 5.14 together imply

$$\left|\mathbf{E}\frac{\varepsilon_{j\nu}\varepsilon_{j4}\frac{1}{z+m_n^{(j)}(z)}}{(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))}\right| \leq \frac{C}{nv}\mathbf{E}^{\frac{1}{4}}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)+s(z)|^4}. \tag{6.10}$$

By Lemmas 7.14 and 7.5 we have for $\nu = 1$

$$\mathbf{E}^{\frac{1}{4}}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)+s(z)|^4} \leq \frac{C}{\sqrt{n}\sqrt{|z^2-4|}}. \tag{6.11}$$

By Corollary 7.17, inequality (7.31) with $\alpha = 0$ and $\beta = 4$ in the Appendix we have for $\nu = 2, 3$

$$\mathbf{E}^{\frac{1}{4}}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)+s(z)|^4} \leq \frac{C}{\sqrt{nv}|z^2-4|^{\frac{1}{4}}}. \tag{6.12}$$

with some constant $C > 0$ depending on μ_4 and D_0 only. Using that, by Lemma 7.13, for $z \in \mathbb{G}$,

$$\frac{1}{\sqrt{n}\sqrt{|z^2-4|}} \leq \frac{\sqrt{v}}{\sqrt{nv}\sqrt{|z^2-4|}} \leq \frac{C}{\sqrt{nv}|z^2-4|^{\frac{1}{4}}} \tag{6.13}$$

we get from (6.10), (6.11) and (6.12) that for $z \in \mathbb{G}$,

$$|\mathfrak{T}_1| \leq \frac{C}{(nv)^{\frac{3}{2}}|z^2-4|^{\frac{1}{4}}} \tag{6.14}$$

with some constant $C > 0$ depending on μ_4 and D_0 only.

6.1.2 Estimation of $\mathfrak{T}_2$

Using the representation (5.32), we write

$$\mathfrak{T}_2 = \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\widetilde{\varepsilon}_j^2 R_{jj}}{(z+m^{(j)}(z))(z+s(z)+m_n(z))}.$$

Furthermore we note that

$$\widetilde{\varepsilon}_j^2 = \varepsilon_{j2}^2 + \eta_j,$$

where

$$\eta_j = (\varepsilon_{j1} + \varepsilon_{j3})^2 + 2(\varepsilon_{j1} + \varepsilon_{j3})\varepsilon_{j2}.$$

We now decompose $\mathfrak{T}_2$ as follows

$$\mathfrak{T}_2 = \mathfrak{T}_{21} + \mathfrak{T}_{22} + \mathfrak{T}_{23}, \tag{6.15}$$

where

$$\begin{aligned}
\mathfrak{T}_{21} &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{\varepsilon_{j2}^2 R_{jj}}{(z + m^{(j)}(z))(z + s(z) + m_n(z))},\\
\mathfrak{T}_{22} &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{\eta_j R_{jj}}{(z + m^{(j)}(z))(z + s(z) + m_n^{(j)}(z))},\\
\mathfrak{T}_{23} &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{\eta_j R_{jj}\varepsilon_{j4}}{(z + m^{(j)}(z))(z + s(z) + m_n^{(j)}(z))(z + s(z) + m_n(z))}.
\end{aligned}$$

Applying the Cauchy – Schwartz inequality, we obtain

$$|\mathfrak{T}_{22}| \le \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}^{\frac{1}{2}}\frac{|\eta_j|^2}{|z + m^{(j)}(z)|^2|z + s(z) + m_n^{(j)}(z)|^2}\mathbf{E}^{\frac{1}{2}}|R_{jj}|^2. \tag{6.16}$$

In what follows we denote by C a generic constant depending on μ_4 and D_0 only. We note that

$$\begin{aligned}
\mathbf{E}\{|\eta_j|^2\Big|\mathfrak{M}^{(j)}\} &\le C(\mathbf{E}\{|\varepsilon_{j1}|^4\Big|\mathfrak{M}^{(j)}\} + \mathbf{E}\{|\varepsilon_{j3}|^4\Big|\mathfrak{M}^{(j)}\})\\
&\quad + C\Big(\mathbf{E}\{|\varepsilon_{j1}|^4\Big|\mathfrak{M}^{(j)}\} + \mathbf{E}\{|\varepsilon_{j3}|^4\Big|\mathfrak{M}^{(j)}\}\Big)^{\frac{1}{2}}\mathbf{E}^{\frac{1}{2}}\{|\varepsilon_{j2}|^4\Big|\mathfrak{M}^{(j)}\}.
\end{aligned}$$

Using Lemmas 7.14, 7.15, 7.16, we get

$$\begin{aligned}
\mathbf{E}\{|\eta_j|^2\Big|\mathfrak{M}^{(j)}\} &\le \frac{C}{n^2} + \frac{C}{n^2}\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^4\\
&\quad + (\frac{C}{n} + \frac{C}{n}(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^4)^{\frac{1}{2}})\frac{C}{nv}\mathrm{Im}\, m_n^{(j)}(z).
\end{aligned} \tag{6.17}$$

This inequality and Lemma 7.5 together imply

$$\mathbf{E}^{\frac12}\frac{|\eta_j|^2}{|z+m^{(j)}(z)|^2|z+s(z)+m_n^{(j)}(z)|^2}\le\frac{C}{n\sqrt{|z^2-4|}}\mathbf{E}^{\frac12}\frac{1}{|z+m_n^{(j)}(z)|^2}$$
$$+\frac{C}{n\sqrt{|z^2-4|}}\Big(\frac1n\sum_{l\in\mathbb{T}_j}\mathbf{E}|R_{ll}^{(j)}|^4\Big)^{\frac14}\mathbf{E}^{\frac14}\frac{1}{|z+m_n^{(j)}(z)|^4}$$
$$+\frac{C}{n\sqrt{v}|z^2-4|^{\frac14}}\Big(1+\Big(\frac1n\sum_{l\in\mathbb{T}_j}\mathbf{E}|R_{ll}^{(j)}|^4\Big)^{\frac14}\Big)\mathbf{E}^{\frac14}\frac{1}{|z+m_n^{(j)}(z)|^4} \qquad (6.18)$$

Inequality (6.18) and Corollary 5.14 together imply

$$|\mathfrak{T}_{22}|\le\frac{C}{n\sqrt{|z^2-4|}}+\frac{C}{n\sqrt{v}|z^2-4|^{\frac14}}.$$

Applying Lemma (7.13) for $z\in\mathbb{G}$, we get

$$|\mathfrak{T}_{22}|\le\frac{C}{n\sqrt{v}|z^2-4|^{\frac14}}\le\frac{C}{nv^{\frac34}}. \qquad (6.19)$$

By Hölder's inequality, we have

$$|\mathfrak{T}_{23}|\le\frac1n\sum_{j=1}^n\mathbf{E}^{\frac12}\frac{|\eta_j|^2}{|z+m^{(j)}(z)|^2|z+s(z)+m_n^{(j)}(z)|^2}$$
$$\times\,\mathbf{E}^{\frac14}\frac{|\varepsilon_{j4}|^4}{|z+s(z)+m_n(z)|^4}\mathbf{E}^{\frac14}|R_{jj}|^4.$$

Using now Lemmas 7.22, 7.13 and Corollary 5.14, we may write, for $z\in\mathbb{G}$,

$$|\mathfrak{T}_{23}|\le\frac{C}{n\sqrt{v}|z^2-4|^{\frac14}}\le\frac{C}{nv^{\frac34}}. \qquad (6.20)$$

We continue now with $\mathfrak{T}_{21}$. We represent it in the form

$$\mathfrak{T}_{21}=H_1+H_2, \qquad (6.21)$$

where

$$H_1=-\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_{j2}^2}{(z+m^{(j)}(z))^2(z+s(z)+m_n(z))},$$
$$H_2=\frac1n\sum_{j=1}^n\mathbf{E}\frac{\varepsilon_{j2}^2(R_{jj}+\frac{1}{z+m_n^{(j)}})}{(z+m^{(j)}(z))(z+s(z)+m_n(z))}.$$

Furthermore, using the representation

$$R_{jj} = -\frac{1}{z + m_n^{(j)}(z)} + \frac{1}{z + m_n^{(j)}(z)}(\varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3})R_{jj} \tag{6.22}$$

(compare with (3.2)), we bound H_2 in the following way

$$|H_2| \le H_{21} + H_{22} + H_{23},$$

where

$$H_{21} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{4|\varepsilon_{j1}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n(z)|},$$
$$H_{22} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{2|\varepsilon_{j2}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n(z)|},$$
$$H_{23} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{2|\varepsilon_{j3}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n(z)|}.$$

Using inequality (7.42) in the Appendix and Hölder inequality, we get, for $\nu = 1, 2, 3$

$$\begin{aligned} H_{2\nu} &\le \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{4|\varepsilon_{j\nu}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n^{(j)}(z)|} \\ &+ \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\frac{4|\varepsilon_{j\nu}|^3|R_{jj}||\varepsilon_{j4}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n^{(j)}(z)||z + s(z) + m_n(z)|} \\ &\le \frac{C}{n}\sum_{j=1}^{n} \mathbf{E}^{\frac{3}{4}}\frac{|\varepsilon_{j\nu}|^4}{|z + m^{(j)}(z)|^{\frac{8}{3}}|z + s(z) + m_n^{(j)}(z)|^{\frac{4}{3}}}\mathbf{E}^{\frac{1}{4}}|R_{jj}|^4. \end{aligned} \tag{6.23}$$

Applying Corollary 7.17 with $\beta = \frac{4}{3}$ and $\alpha = \frac{8}{3}$, we obtain, for $z \in \mathbb{G}$, and for $\nu = 1, 2, 3$

$$\mathbf{E}^{\frac{3}{4}}\frac{|\varepsilon_{j\nu}|^4}{|z + m^{(j)}(z)|^{\frac{8}{3}}|z + s(z) + m_n^{(j)}(z)|^{\frac{4}{3}}} \le \frac{C}{(nv)^{\frac{3}{2}}}.$$

This yields together with Corollary 5.14 and inequality (6.23)

$$H_2 \le \frac{C}{(nv)^{\frac{3}{2}}}. \tag{6.24}$$

Consider now H_1. Using the equality

$$\frac{1}{z+m_n(z)+s(z)} = \frac{1}{z+2s(z)} - \frac{\Lambda_n(z)}{(z+2s(z))(z+m_n(z)+s(z))}$$

and

$$\Lambda_n = \Lambda_n^{(j)} + \varepsilon_{j4}, \tag{6.25}$$

we represent it in the form

$$H_1 = H_{11} + H_{12} + H_{13}, \tag{6.26}$$

where

$$\begin{aligned} H_{11} &= -\frac{1}{(z+s(z))^2}\frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j2}^2}{z+s(z)+m_n(z)} \\ &= -s^2(z)\frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j2}^2}{z+s(z)+m_n(z)}, \\ H_{12} &= -\frac{1}{(z+s(z))}\frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j2}^2\Lambda_n^{(j)}}{(z+m_n^{(j)}(z))^2(z+s(z)+m_n(z))}, \\ H_{13} &= -\frac{1}{(z+s(z))^2}\frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j2}^2\Lambda_n^{(j)}}{(z+m_n^{(j)}(z))(z+s(z)+m_n(z))}. \end{aligned}$$

In order to apply conditional independence, we write

$$H_{11} = H_{111} + H_{112},$$

where

$$\begin{aligned} H_{111} &= -s^2(z)\frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j2}^2}{z+m_n^{(j)}(z)+s(z)}, \\ H_{112} &= s^2(z)\frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j2}^2\varepsilon_{j4}}{(z+s(z)+m_n(z))(z+m_n^{(j)}(z)+s(z))}. \end{aligned}$$

It is straightforward to check that

$$\mathbf{E}\{\varepsilon_{j2}^2|\mathfrak{M}^{(j)}\} = \frac{1}{n^2}\mathrm{Tr}\,(\mathbf{R}^{(j)})^2 - \frac{1}{n^2}\sum_{l\in\mathbb{T}_j}(R_{ll}^{(j)})^2.$$

Using equality (6.3) for $m_n'(z)$ and the corresponding relation for ${m_n^{(j)}}'(z)$, we may write

$$H_{111} = L_1 + L_2 + L_3 + L_4,$$

where

$$L_1 = -s^2(z)\frac{1}{n}\mathbf{E}\frac{m_n'(z)}{z + m_n(z) + s(z)},$$

$$L_2 = s^2(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\frac{1}{n^2}\sum_{l\in\mathbb{T}_j}(R_{ll}^{(j)})^2}{z + m_n^{(j)}(z) + s(z)}, \tag{6.27}$$

$$L_3 = s^2(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\frac{1}{n}((m_n^{(j)}(z))' - m_n'(z))}{z + m_n^{(j)}(z) + s(z)}, \tag{6.28}$$

$$L_4 = s^2(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\frac{1}{n}((m_n^{(j)}(z))' - m_n'(z))\varepsilon_{j4}}{(z + m_n(z) + s(z))(z + m_n^{(j)}(z) + s(z))}.$$

Using Lemmas 7.5, 7.18, 7.26, and Corollary 5.14, it is straightforward to check that

$$|L_2| \le \frac{C}{n\sqrt{|z^2-4|}},$$

$$|L_3| \le \frac{C}{n^2v^2\sqrt{|z^2-4|}},$$

$$|L_4| \le \frac{C}{n^3v^3|z^2-4|}.$$

Applying inequality (7.42), we may write

$$|H_{12}| \le \frac{C}{n}\sum_{j=1}^{n}\mathbf{E}\frac{|\varepsilon_{j2}|^2|\Lambda_n^{(j)}|}{|z + m_n^{(j)}(z)||z + m_n^{(j)}(z) + s(z)|}.$$

Conditioning on $\mathfrak{M}^{(j)}$ and applying Lemma 7.15, Lemma 7.5, inequality (7.8), Corollary 5.14 and equality (6.25), we get

$$|H_{12}| \le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{|\Lambda_n^{(j)}|}{|z + m_n^{(j)}(z)|} \le \frac{C}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 + \frac{C}{n^2v^2}.$$

By Lemma 7.24, we get

$$|H_{12}| \le \frac{C}{n^2v^2}. \tag{6.29}$$

Similar we get
$$|H_{13}| \le \frac{C}{n^2v^2}. \tag{6.30}$$

We rewrite now the equations (6.2) and (6.4) as follows, using the remainder term $\mathfrak{T}_3$, which is bounded by means of inequalities (6.14), (6.19), (6.20), (6.24).

$$\mathbf{E}\Lambda_n(z) = \mathbf{E}m_n(z) - s(z) = \frac{(1-s^2(z))}{n}\mathbf{E}\frac{m_n'(z)}{z+m_n(z)+s(z)} + \mathfrak{T}_3, \tag{6.31}$$

where
$$|\mathfrak{T}_3| \le \frac{C}{n\sqrt{v}^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|z^2-4|^{\frac{1}{4}}}.$$

Note that
$$1 - s^2(z) = s(z)\sqrt{z^2-4}.$$

In (6.31) we estimate now the remaining quantity
$$\mathfrak{T}_4 = -\frac{s(z)\sqrt{z^2-4}}{n}\mathbf{E}\frac{m_n'(z)}{z+m_n(z)+s(z)}.$$

6.2 Estimation of $\mathfrak{T}_4$

Using that $\Lambda_n = m_n(z) - s(z)$ we rewrite $\mathfrak{T}_4$ as
$$\mathfrak{T}_4 = \mathfrak{T}_{41} + \mathfrak{T}_{42} + \mathfrak{T}_{43},$$

where
$$\mathfrak{T}_{41} = -\frac{s(z)s'(z)}{n},$$
$$\mathfrak{T}_{42} = \frac{s(z)\sqrt{z^2-4}}{n}\mathbf{E}\frac{m_n'(z)-s'(z)}{z+m_n(z)+s(z)},$$
$$\mathfrak{T}_{43} = \frac{s(z)}{n}\mathbf{E}\frac{(m_n'(z)-s'(z))\Lambda_n}{z+m_n(z)+s(z)}.$$

6.2.1 Estimation of $\mathfrak{T}_{42}$

First we investigate $m_n'(z)$. The following equality holds
$$m_n'(z) = \frac{1}{n}\operatorname{Tr} R^2 = \sum_{j=1}^n \varepsilon_{j4}R_{jj} = s^2(z)\sum_{j=1}^n \varepsilon_{j4}R_{jj}^{-1} + D_1, \tag{6.32}$$

where

$$D_1 = \sum_{j=1}^{n} \varepsilon_{j4}(R_{jj} - s(z))(1 + R_{jj}^{-1} s(z)). \tag{6.33}$$

Using equality (7.34), we may write

$$m_n'(z) = \frac{s^2(z)}{n} \sum_{j=1}^{n} (1 + \frac{1}{n} \sum_{l,k \in \mathbb{T}_j} X_{jl} X_{jk} [(R^{(j)})^2]_{lk}) + D_1.$$

Denote by

$$\begin{aligned}
\beta_{j1} &= \frac{1}{n} \sum_{l \in \mathbb{T}_j} [(R^{(j)})^2]_{ll} - \frac{1}{n} \sum_{l=1}^{n} [(R)^2]_{ll} = \frac{1}{n} \sum_{l \in \mathbb{T}_j} [(R^{(j)})^2]_{ll} - m_n'(z) \\
&= \frac{1}{n} \frac{d}{dz} (\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(j)}), \\
\beta_{j2} &= \frac{1}{n} \sum_{l \in \mathbb{T}_j} (X_{jl}^2 - 1) [(R^{(j)})^2]_{ll}, \\
\beta_{j3} &= \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} X_{jl} X_{jk} [(R^{(j)})^2]_{lk}.
\end{aligned} \tag{6.34}$$

Using these notation we may write

$$m_n'(z) = s^2(z)(1 + m_n'(z)) + \frac{s^2(z)}{n} \sum_{j=1}^{n} (\beta_{j1} + \beta_{j2} + \beta_{j3}) + D_1.$$

Solving this equation with respect to $m_n'(z)$ we obtain

$$m_n'(z) = \frac{s^2(z)}{1 - s^2(z)} + \frac{1}{1 - s^2(z)} (D_1 + D_2), \tag{6.35}$$

where

$$D_2 = \frac{s^2(z)}{n} \sum_{j=1}^{n} (\beta_{j1} + \beta_{j2} + \beta_{j3}).$$

Note that for the semi-circular law

$$\frac{s^2(z)}{1 - s^2(z)} = \frac{s^2(z)}{1 + \frac{s(z)}{z + s(z)}} = -\frac{s(z)}{z + 2s(z)} = s'(z).$$

Applying this relation we rewrite equality (6.35) as

$$m_n'(z) - s'(z) = \frac{1}{s(z)(z+2s(z))}(D_1 + D_2). \tag{6.36}$$

Using the last equality, we may represent $\mathfrak{T}_{42}$ now as follows

$$\mathfrak{T}_{42} = \mathfrak{T}_{421} + \mathfrak{T}_{422},$$

where

$$\mathfrak{T}_{421} = \frac{1}{n}\mathbf{E}\frac{D_1}{z+m_n(z)+s(z)},$$
$$\mathfrak{T}_{422} = \frac{1}{n}\mathbf{E}\frac{D_2}{z+m_n(z)+s(z)}.$$

Recall that, by (6.33),

$$\mathfrak{T}_{421} = -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\varepsilon_{j4}(R_{jj}-s(z))(1+R_{jj}^{-1}s(z))}{(z+s(z)+m_n(z))}. \tag{6.37}$$

Applying the Cauchy – Schwartz inequality, we get for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{421}| \le \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac12}|R_{jj}-s(z)|^2\mathbf{E}^{\frac14}\frac{|\varepsilon_{j4}|^4}{|z+s(z)+m_n(z)|^4}(1+|s(z)|\mathbf{E}^{\frac14}|R_{jj}|^{-4}).$$

Using Corollary 7.23 and Corollary 5.14, we get

$$|\mathfrak{T}_{421}| \le \frac{C}{n^{\frac32}v^{\frac32}}. \tag{6.38}$$

6.2.2 Estimation of $\mathfrak{T}_{422}$

We represent now $\mathfrak{T}_{421}$ in the form

$$\mathfrak{T}_{422} = \mathfrak{T}_{51} + \mathfrak{T}_{52} + \mathfrak{T}_{53},$$

where

$$\mathfrak{T}_{5\nu} = \frac{1}{n^2}\sum_{j=1}^{n}\mathbf{E}\frac{\beta_{j\nu}}{z+m_n(z)+s(z)}, \quad \text{for} \quad \nu = 1,2,3.$$

At first we investigate $\mathfrak{T}_{53}$. Note that, by Lemma 7.26,

$$|\beta_{j1}| \le \frac{C}{nv^2}.$$

Therefore, for $z \in \mathbb{G}$, using Lemma 7.5, inequality (7.9), and Lemma 7.13,

$$|\mathfrak{T}_{51}| \leq \frac{C}{n^2 v^2 \sqrt{|z^2-4|}} \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2-4|^{\frac{1}{4}}}. \tag{6.39}$$

Furthermore, we consider the quantity $\mathfrak{T}_{5\nu}$, for $\nu = 2, 3$. Applying the Cauchy-Schwartz inequality and inequality (7.42) in the Appendix as well, we get

$$|\mathfrak{T}_{5\nu}| \leq \frac{C}{n^2} \sum_{j=1}^{n} \mathbf{E}^{\frac{1}{2}} \frac{|\beta_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)|^2}.$$

By Lemma 7.25 together with Lemma 7.5 in the Appendix, we obtain

$$\mathbf{E}^{\frac{1}{2}} \frac{|\beta_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)|^2} \leq \frac{C}{n^{\frac{1}{2}} v^{\frac{3}{2}} |z^2-4|^{\frac{1}{4}}}.$$

This implies that

$$|\mathfrak{T}_{5\nu}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2-4|^{\frac{1}{4}}}. \tag{6.40}$$

Inequalities (6.39) and (6.40) yield

$$|\mathfrak{T}_{422}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2-4|^{\frac{1}{4}}}.$$

Combining (6.38) and (6.41), we get, for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{42}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2-4|^{\frac{1}{4}}}. \tag{6.41}$$

6.2.3 Estimation of $\mathfrak{T}_{43}$

Recall that

$$\mathfrak{T}_{43} = \frac{s(z)}{n} \mathbf{E} \frac{(m_n'(z) - s'(z))\Lambda_n}{z + m_n(z) + s(z)}.$$

Applying equality (6.36), we obtain

$$\mathfrak{T}_{43} = \mathfrak{T}_{431} + \mathfrak{T}_{432},$$

where

$$\mathfrak{T}_{431} = \frac{1}{n(z+2s(z))}\mathbf{E}\frac{D_1\Lambda_n}{z+m_n(z)+s(z)}, \tag{6.42}$$
$$\mathfrak{T}_{432} = \frac{1}{n(z+2s(z))}\mathbf{E}\frac{D_2\Lambda_n}{z+m_n(z)+s(z)}.$$

Applying the Cauchy – Schwartz inequality, we get

$$|\mathfrak{T}_{431}| \le \frac{1}{n(z+2s(z))}\mathbf{E}^{\frac12}\frac{|D_1|^2}{|z+m_n(z)+s(z)|^2}\mathbf{E}^{\frac12}|\Lambda_n|^2.$$

By definition of D_1 and Lemmas 7.24 and 7.18, we get

$$|\mathfrak{T}_{431}| \le \frac{C}{n^3v^2|z^2-4|}\frac{1}{n}\sum_{j=1}^n(1+|s(z)|\mathbf{E}^{\frac14}|R_{jj}|^{-4})\mathbf{E}^{\frac14}|R_{jj}-s(z)|^4.$$

Applying now Corollary 5.14 and Lemma 7.22, we get

$$|\mathfrak{T}_{431}| \le \frac{4}{n^3v^2|z^2-4|^{\frac12}}.$$

For $z \in \mathbb{G}$ this yields

$$|\mathfrak{T}_{431}| \le \frac{4}{n^{\frac32}v^{\frac32}|z^2-4|^{\frac14}}.$$

Applying again the Cauchy – Schwartz inequality, we get for $\mathfrak{T}_{432}$ accordingly

$$|\mathfrak{T}_{432}| \le \frac{C}{n|z^2-4|^{\frac12}}\mathbf{E}^{\frac12}|D_2|^2\mathbf{E}^{\frac12}|\Lambda_n|^2.$$

By Lemma 7.24, we have

$$|\mathfrak{T}_{432}| \le \frac{C}{n^2v|z^2-4|^{\frac12}}\mathbf{E}^{\frac12}|D_2|^2. \tag{6.43}$$

By definition of D_2,

$$\mathbf{E}|D_2|^2 \le \frac{1}{n}\sum_{j=1}^n(\mathbf{E}|\beta_{j1}|^2+\mathbf{E}|\beta_{j2}|^2+\mathbf{E}|\beta_{j3}|^2).$$

Applying Lemmas 7.25 with $\nu = 2, 3$, and 7.26, we get

$$\mathbf{E}|D_2|^2 \le \frac{C}{n^2v^4}+\frac{C}{nv^3}. \tag{6.44}$$

Inequalities (6.43) and (6.44) together imply, for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{432}| \le \frac{C}{n^3 v^3 |z^2-4|^{\frac{1}{2}}} + \frac{C}{n^{\frac{5}{2}} v^{\frac{5}{2}} |z^2-4|^{\frac{1}{2}}} \le \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2-4|^{\frac{1}{4}}}.$$

Finally we observe that

$$s'(z) = -\frac{s(z)}{\sqrt{z^2-4}}$$

and, therefore

$$|\mathfrak{T}_{41}| \le \frac{C}{n|z^2-4|^{\frac{1}{2}}}.$$

For $z \in \mathbb{G}$ we may rewrite it

$$|\mathfrak{T}_{41}| \le \frac{C}{n\sqrt{v}}. \tag{6.45}$$

Combining now relations (6.31), (6.26), (6.24), (6.39), (6.41), (6.45), we get for $z \in \mathbb{G}$,

$$|\mathbf{E}\Lambda_n| \le \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |z^2-4|^{\frac{1}{4}}}.$$

The last inequality completes the proof of Theorem 1.3.

7 Appendix

7.1 Rosenthal's and Burkholder's inequalities

In this subsection we state the Rosenthal and Burkholder inequalities, starting with Rosenthal's inequality. Let $\xi_1, \dots, \xi_n$ be independent random variables with $\mathbf{E}\xi_j = 0$, $\mathbf{E}\xi_j^2 = 1$ and for $p \ge 1$ $\mathbf{E}|\xi_j|^p \le \mu_p$ for $j = 1, \dots, n$.

Lemma 7.1. (Rosenthal's inequality)
There exists an absolute constant C_1 *such that*

$$\mathbf{E}|\sum_{j=1}^n a_j\xi_j|^p \le C_1^p p^p ((\sum_{j=1}^p |a_j|^2)^{\frac{p}{2}} + \mu_p \sum_{j=1}^p |a_j|^p).$$

Proof. For a proof see [19, Theorem 3] and [16, inequality (A)]. □

Let $\xi_1, \dots \xi_n$ be a martingale-difference with respect to the σ-algebras $\mathfrak{M}_j = \sigma(\xi_1, \dots, \xi_{j-1})$. Assume that $\mathbf{E}\xi_j^2 = 1$ and $\mathbf{E}|\xi_j|^p < \infty$.

Lemma 7.2. (Burkholder's inequality) *There exist an absolute constant* C_2 *such that*

$$\mathbf{E}|\sum_{j=1}^{n}\xi_j|^p \le C_2^p p^p\Big(\mathbf{E}(\sum_{k=1}^{n}\mathbf{E}\{\xi_k^2|\mathfrak{M}_{k-1}\})^{\frac{p}{2}} + \sum_{k=1}^{p}\mathbf{E}|\xi_k|^p\Big).$$

Proof. For a proof of this inequality see [5, Theorem 3.2] and [15, Theorem 4.1]. □

We rewrite the Burkholder inequality for quadratic forms in independent random variables. Let $\zeta_1, \dots, \zeta_n$ be independent random variables such that $\mathbf{E}\zeta_j = 0$, $\mathbf{E}|\zeta_j|^2 = 1$ and $\mathbf{E}|\zeta_j|^p \le \mu_p$. Let $a_{ij} = a_{ji}$ for all $i, j = 1, \dots n$. Consider the quadratic form

$$Q = \sum_{1\le j\ne k\le n} a_{jk}\zeta_j\zeta_k.$$

Lemma 7.3. *There exists an absolute constant* C_2 *such that*

$$\mathbf{E}|Q|^p \le C_2^p\Big(\mathbf{E}(\sum_{j=2}^{n}(\sum_{k=1}^{j-1}a_{jk}\zeta_k)^2)^{\frac{p}{2}} + \mu_p\sum_{j=2}^{n}\mathbf{E}|\sum_{k=1}^{j-1}a_{jk}\zeta_k|^p\Big).$$

Proof. Introduce the random variables, for $j = 2, \dots, n$,

$$\xi_j = \zeta_j\sum_{k=1}^{j-1}a_{jk}\zeta_k.$$

It is straightforward to check that

$$\mathbf{E}\{\xi_j|\mathfrak{M}_{j-1}\} = 0,$$

and that ξ_j are $\mathfrak{M}_j$ measurable. That means that $\xi_1, \dots, \xi_n$ are martingale-differences. We may write

$$Q = 2\sum_{j=2}^{n}\xi_j.$$

Applying now Lemma 7.2 and using

$$\mathbf{E}\{|\xi_j|^2|\mathfrak{M}_{j-1}\} = (\sum_{k=1}^{j-1}a_{jk}\zeta_k)^2\mathbf{E}\zeta_j^2,$$

$$\mathbf{E}|\xi_j|^p = \mathbf{E}|\zeta_j|^p\mathbf{E}|\sum_{k=1}^{j-1}a_{jk}\zeta_j|^p, \tag{7.1}$$

we get the claim. Thus, Lemma 7.3 is proved.

□

We prove as well the following simple Lemma

Lemma 7.4. *Let $t > r \geq 1$ and $a, b > 0$. Any $x > 0$ satisfying the inequality*

$$x^t \leq a + bx^r \tag{7.2}$$

is explicitly bounded as follows

$$x^t \leq \mathrm{e}a + \left(\frac{2t-r}{t-r}\right)^{\frac{t}{t-r}} b^{\frac{t}{t-r}}. \tag{7.3}$$

Proof. First assume that $x \leq a^{\frac{1}{t}}$. Then inequality (7.3) holds. If $x \geq a^{\frac{1}{t}}$, then according to inequality (7.2)

$$x^{t-r} \leq a^{\frac{t-r}{t}} + b,$$

or

$$x^t \leq (a^{\frac{t-r}{t}} + b)^{\frac{t}{t-r}}.$$

Using that for any $\alpha > 0$ and $a > 0, b > 0$

$$(a+b)^\alpha \leq (a + \frac{a}{\alpha})^\alpha + (b + \alpha b)^\alpha \leq \mathrm{e}a^\alpha + (1+\alpha)^\alpha b^\alpha,$$

we get the claim. □

In what follows we prove several lemmas about the resolvent matrices. Recall the equation, for $j \in \mathbb{T}_{\mathbb{J}}$, (compare with (3.2))

$$R_{jj}^{(\mathbb{J})} = -\frac{1}{z + m_n^{(\mathbb{J})}(z)} + \frac{1}{z + m_n^{(\mathbb{J})}(z)} \varepsilon_j^{(\mathbb{J})} R_{jj}^{(\mathbb{J})},$$

where

$$\varepsilon_{j1}^{(\mathbb{J})} = -\frac{X_{jj}}{\sqrt{n}}, \quad \varepsilon_{j2}^{(\mathbb{J})} = \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_{\mathbb{J},j}} X_{jl} X_{jk} R_{kl}^{(\mathbb{J},j)},$$
$$\varepsilon_{j3}^{(\mathbb{J})} = \frac{1}{n} \sum_{l \in \mathbb{T}_{\mathbb{J},j}} (X_{jl}^2 - 1) R_{ll}^{(\mathbb{J},j)}, \quad \varepsilon_{j4}^{(\mathbb{J})} = m_n^{(\mathbb{J})}(z) - m_n^{(\mathbb{J},j)}(z).$$

Summing these equations for $j \in \mathbb{T}_{\mathbb{J}}$, we get

$$m_n^{(\mathbb{J})}(z) = -\frac{n - |\mathbb{J}|}{n(z + m_n^{(\mathbb{J}))}(z)} + \frac{T_n^{(\mathbb{J})}}{z + m_n^{(\mathbb{J})}(z)}, \tag{7.4}$$

where

$$T_n^{(\mathbb{J})} = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j^{(\mathbb{J})} R_{jj}^{(\mathbb{J})}.$$

Note that

$$\frac{1}{z+m_n^{(\mathbb{J})}(z)} = \frac{1}{z+s(z)} - \frac{m_n^{(\mathbb{J})}(z)-s(z)}{(s(z)+z)(z+m_n^{(\mathbb{J})}(z))} = -s(z) + \frac{s(z)\Lambda_n^{(\mathbb{J})}(z)}{z+m_n^{(\mathbb{J})}(z)}, \tag{7.5}$$

where

$$\Lambda_n^{(\mathbb{J})} = \Lambda_n^{(\mathbb{J})}(z) = m_n^{(\mathbb{J})}(z) - s(z).$$

Equalities (7.4) and (7.5) together imply

$$\Lambda_n^{(\mathbb{J})} = -\frac{s(z)\Lambda_n^{(\mathbb{J})}}{z+m_n^{(\mathbb{J})}(z)} + \frac{T_n^{(\mathbb{J})}}{z+m_n^{(\mathbb{J})}(z)} + \frac{|\mathbb{J}|}{n(z+m_n^{(\mathbb{J})}(z))}. \tag{7.6}$$

Solving this with respect to $\Lambda_n^{(\mathbb{J})}$, we get

$$\Lambda_n^{(\mathbb{J})} = \frac{T_n^{(\mathbb{J})}}{z+m_n^{(\mathbb{J})}(z)+s(z)} + \frac{|\mathbb{J}|}{n(z+m_n^{(\mathbb{J})}(z)+s(z))}. \tag{7.7}$$

Lemma 7.5. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $c_0 > 0$ such that for $\mathbb{J} \subset \mathbb{T}$,*

$$|z+m_n^{(\mathbb{J})}(z)+s(z)| \ge \operatorname{Im} m_n^{(\mathbb{J})}(z), \tag{7.8}$$

moreover, for any $z \in \mathbb{G}$

$$|z+m_n^{(\mathbb{J})}(z)+s(z)| \ge c_0\sqrt{|z^2-4|}. \tag{7.9}$$

Proof. Firstly note that $\operatorname{Im}(z+s(z)) \ge 0$ and $\operatorname{Im} m_n^{(\mathbb{J})}(z) \ge 0$. Therefore

$$|z+m_n^{(\mathbb{J})}(z)+s(z)| \ge \operatorname{Im}(z+m_n^{(\mathbb{J})}(z)+s(z)) \ge \operatorname{Im} m_n^{(\mathbb{J})}(z).$$

Furthermore,

$$|z+m_n^{(\mathbb{J})}(z)+s(z)| \ge \operatorname{Im}(z+s(z)) = \frac{1}{2}\operatorname{Im}(z+\sqrt{z^2-4}) \ge \frac{1}{2}\operatorname{Im}\sqrt{z^2-4}.$$

Note that for $z \in \mathbb{G}$

$$\operatorname{Re}(z^2-4) < 0. \tag{7.10}$$

Therefore,

$$\operatorname{Im}\sqrt{z^2-4} \ge \frac{\sqrt{2}}{2}\sqrt{|z^2-4|}.$$

Thus Lemma 7.5 is proved. ∎

Lemma 7.6. *For any $z = u + iv$ with $v > 0$ and for any $\mathbb{J} \subset \mathbb{T}$, we have*

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}} |R^{(\mathbb{J})}_{kl}|^2 \le v^{-1}\operatorname{Im} m^{(\mathbb{J})}_n(z). \tag{7.11}$$

For any $l \in \mathbb{T}_{\mathbb{J}}$

$$\sum_{k\in\mathbb{T}_{\mathbb{J}}} |R^{(\mathbb{J})}_{kl}|^2 \le v^{-1}\operatorname{Im} R^{(\mathbb{J})}_{ll}. \tag{7.12}$$

and

$$\sum_{k\in\mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{kl}|^2 \le v^{-3}\operatorname{Im} R^{(\mathbb{J})}_{ll}. \tag{7.13}$$

Moreover, for any $\mathbb{J} \subset T$ and for any $l \in \mathbb{T}_{\mathbb{J}}$ we have

$$\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^2 \le v^{-3}\operatorname{Im} m^{(\mathbb{J})}_n(z), \tag{7.14}$$

and, for any $p \ge 1$

$$\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^p \le v^{-p}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}} \operatorname{Im}^p R^{(\mathbb{J})}_{ll}. \tag{7.15}$$

Finally,

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \le v^{-3}\operatorname{Im} m^{(\mathbb{J})}_n(z), \tag{7.16}$$

and

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^{2p} \le v^{-3p}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}} \operatorname{Im}^p R^{(\mathbb{J})}_{ll}, \tag{7.17}$$

We have as well

$$\frac{1}{n^2}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^{2p} \le v^{-2p}(\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}} \operatorname{Im}^p R^{(\mathbb{J})}_{ll})^2. \tag{7.18}$$

Proof. For $l \in \mathbb{T}_{\mathbb{J}}$ let us denote by $\lambda^{(\mathbb{J})}_l$ for $l \in \mathbb{T}_{\mathbb{J}}$ eigenvalues of matrix $\mathbf{W}^{(\mathbb{J})}$. Then we may write (compare (7.20))

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}} |R^{(\mathbb{J})}_{kl}|^2 \le \frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}} \frac{1}{|\lambda^{(\mathbb{J})}_l - z|^2}. \tag{7.19}$$

Note that, for any $x \in \mathbb{R}^1$

$$\operatorname{Im}\frac{1}{x-z} = \frac{v}{|x-z|^2}.$$

We may write

$$\frac{1}{|\lambda_l^{(\mathbb{J})}-z|^2} = v^{-1}\operatorname{Im}\frac{1}{\lambda_l^{(\mathbb{J})}-z}$$

and

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}}|R_{kl}^{(\mathbb{J})}|^2 \le v^{-1}\operatorname{Im}\big(\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}\frac{1}{\lambda_l^{(\mathbb{J})}-z}\big) = v^{-1}\operatorname{Im} m_n^{(\mathbb{J})}(z).$$

So, inequality (7.11) is proved. Let denote now by $\mathbf{u}_l^{(\mathbb{J})} = (u_{lk}^{(\mathbb{J})})_{k\in\mathbb{T}_{\mathbb{J}}}$ the eigenvector of matrix $\mathbf{W}^{(\mathbb{J})}$ corresponding to the eigenvalue $\lambda_l^{(\mathbb{J})}$. Using this notation we may write

$$R_{lk}^{(\mathbb{J})} = \sum_{q\in\mathbb{T}_{\mathbb{J}}}\frac{1}{\lambda_q^{(\mathbb{J})}-z}u_{lq}^{(\mathbb{J})}u_{kq}^{(\mathbb{J})}. \tag{7.20}$$

It is straightforward to check the following inequality

$$\begin{aligned}\sum_{k\in\mathbb{T}_{\mathbb{J}}}|R_{kl}^{(\mathbb{J})}|^2 &\le \sum_{q\in\mathbb{T}_{\mathbb{J}}}\frac{1}{|\lambda_q^{(\mathbb{J})}-z|^2}|u_{lq}^{(\mathbb{J})}|^2\\ &= v^{-1}\operatorname{Im}\Big(\sum_{q\in\mathbb{T}_{\mathbb{J}}}\frac{1}{\lambda_q^{(\mathbb{J})}-z}|u_{lq}^{(\mathbb{J})}|^2\Big) = v^{-1}\operatorname{Im} R_{ll}^{(\mathbb{J})}.\end{aligned} \tag{7.21}$$

Thus, inequality (7.12) is proved. Similar we get

$$\sum_{k\in\mathbb{T}_{\mathbb{J}}}|[(R^{(\mathbb{J})})^2]_{kl}|^2 \le \sum_{q\in\mathbb{T}_{\mathbb{J}}}\frac{1}{|\lambda_q^{(\mathbb{J})}-z|^4}|u_{lq}^{(\mathbb{J})}|^2 \le v^{-3}\operatorname{Im} R_{ll}^{(\mathbb{J})}. \tag{7.22}$$

This proves inequality (7.13). To prove inequality (7.14) we observe that

$$|[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}| \le \sum_{k\in\mathbb{T}_{\mathbb{J}}}|R_{lk}^{(\mathbb{J})}|^2. \tag{7.23}$$

This inequality implies

$$\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^2 \le \frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}\big(\sum_{k\in\mathbb{T}_{\mathbb{J}}}|R_{lk}^{(\mathbb{J})}|^2\big)^2.$$

Applying now inequality (7.12), we get

$$\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^2 \le v^{-2}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}\mathrm{Im}^2 R^{(\mathbb{J})}_{ll}.$$

This leads (using $|R^{(\mathbb{J})}_{ll}| \le v^{-1}$) to the following bound

$$\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^2 \le v^{-3}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}\mathrm{Im}\, R^{(\mathbb{J})}_{ll} = v^{-3}\mathrm{Im}\, m^{(\mathbb{J})}_n(z).$$

Thus inequality (7.14) is proved. Furthermore, applying inequality (7.23), we may write

$$\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^4 \le \frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}(\sum_{k\in\mathbb{T}_{\mathbb{J}}}|R^{(\mathbb{J})}_{lk}|^2)^4.$$

Applying (7.12), this inequality yields

$$\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^4 \le v^{-4}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}\mathrm{Im}^4 R^{(\mathbb{J})}_{ll}.$$

The last inequality proves inequality (7.15). Note that

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \le \frac{1}{n}\mathrm{Tr}\,|\mathbf{R}^{(\mathbb{J})}|^4 = \frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}\frac{1}{|\lambda^{(\mathbb{J})}_l - z|^4}$$
$$\le v^{-3}\mathrm{Im}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}\frac{1}{\lambda^{(\mathbb{J})}_l - z} = v^{-3}\mathrm{Im}\, m^{(\mathbb{J})}_n(z).$$

Thus, inequality (7.16) is proved. To finish we note that

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^4 \le \frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}(\sum_{k\in\mathbb{T}_{\mathbb{J}}}|[(R^{(\mathbb{J})})^2]_{lk}|^2)^2.$$

Applying inequality (7.13), we get

$$\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^4 \le v^{-6}\frac{1}{n}\sum_{l\in\mathbb{T}_{\mathbb{J}}}(\mathrm{Im}\, R^{(\mathbb{J})}_{ll})^2.$$

To prove inequality (7.18), we note

$$|[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \le (\sum_{q\in\mathbb{T}_{\mathbb{J}}}|R^{(\mathbb{J})}_{lq}|^2)(\sum_{q\in\mathbb{T}_{\mathbb{J}}}|R^{(\mathbb{J})}_{kq}|^2).$$

This inequality implies

$$\frac{1}{n^2}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}}|[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^{2p}\le(\frac{1}{n}\sum_{l,k\in\mathbb{T}_{\mathbb{J}}}(\sum_{q\in\mathbb{T}_{\mathbb{J}}}|R^{(\mathbb{J})}_{lq}|^2)^p)^2.$$

Applying inequality (7.11), we get the claim. Thus, Lemma 7.6 is proved. □

Lemma 7.7. *For any $s\ge 1$, and for any $z=u+iv$ and for any $\mathbb{J}\subset\mathbb{T}$, and $j\in\mathbb{T}_{\mathbb{J}}$,*

$$|R^{(\mathbb{J})}_{jj}(u+iv/s)|\le s|R^{(\mathbb{J})}_{jj}(u+iv)|.$$

Proof. See [17, Lemma 3.4]. For the readers convenience we include the short argument here. Note that, for any $j\in\mathbb{T}_{\mathbb{J}}$,

$$|\frac{d}{dv}\log R^{(\mathbb{J})}_{jj}(u+iv)|\le\frac{1}{|R^{(\mathbb{J})}_{jj}(u+iv)|}|\frac{d}{dv}R^{(\mathbb{J})}_{jj}(u+iv)|.$$

Furthermore,

$$\frac{d}{dv}R^{(\mathbb{J})}_{jj}(u+iv)=[(\mathbb{R}^{(\mathbb{J})})^2]_{jj}(u+iv)$$

and

$$|[(\mathbf{R}^{(\mathbb{J})})^2]_{jj}(u+iv)|\le v^{-1}\mathrm{Im}\,R^{(\mathbb{J})}_{jj}.$$

From here it follows that

$$|\frac{d}{dv}\log R^{(\mathbb{J})}_{jj}(u+iv)|\le v^{-1}.$$

We may write now

$$|\log R^{(\mathbb{J})}_{jj}(u+iv)-\log R^{(\mathbb{J})}_{jj}(u+iv/s)|\le\int_{v/s}^{v}\frac{du}{u}=\log s.$$

The last inequality yields the claim. Thus Lemma 7.7 follows. □

Lemma 7.8. *Assuming the conditions of Theorem 1.1, we get*

$$\mathbf{E}|\varepsilon_{j1}|^2\le\frac{C}{n}.$$

Proof. The proof follows immediately from the definition of ε_{j1} and conditions of Theorem 1.1. □

7.1.1 Some Auxiliary Bounds for Resolvent Matrices for $\operatorname{Im} z = 4$

We need the bound for the $\varepsilon_{j\nu}$, and η_j for $V = 4$.

Lemma 7.9. *Assuming the conditions of Theorem 1.1, we get*

$$\mathbf{E}|\varepsilon_{j2}|^2 \le \frac{C}{n}.$$

Proof. Conditioning on $\mathfrak{M}^{(j)}$, we get

$$\mathbf{E}|\varepsilon_{j2}|^2 \le n^{-2} \sum_{k,l\in\mathbb{T}_j} \mathbf{E}|R^{(j)}_{kl}|^2.$$

Applying now Lemma 7.6, inequality (7.12), we get with $\operatorname{Im} R^{(\mathbb{J})} \le \frac{1}{4}$,

$$\mathbf{E}|\varepsilon_{j2}|^2 \le \frac{1}{4} n^{-2} \sum_{l\in\mathbb{T}_j} \operatorname{Im} R^{(j)}_{ll} \le \frac{1}{16n}.$$

Thus Lemma 7.9 is proved. □

Lemma 7.10. *Assuming the conditions of Theorem 1.1, we get*

$$\mathbf{E}|\varepsilon_{j3}|^2 \le \frac{C}{n}.$$

Proof. Conditioning on $\mathfrak{M}^{(j)}$, we obtain as above

$$\mathbf{E}|\varepsilon_{j3}|^2 \le \frac{\mu_4}{n^2} \mathbf{E} \sum_{l\in\mathbb{T}_j} |R^{(j)}_{ll}|^2 \le \frac{\mu_4}{16n}.$$

Thus Lemma 7.10 is proved.

□

Lemma 7.11. *Assuming the conditions of Theorem 1.1, we get*

$$\mathbf{E}|\eta_j|^2 \le \frac{C}{n}.$$

Proof. The proof is similar to proof of Lemma 7.9. We need to use that $|[(R^{(j)})^2]_{kl}| \le V^{-2} = \frac{1}{16}$ and $\sum_{l\in\mathbb{T}_j} |[(R^{(j)})^2]_{kl}|^2 \le V^{-4}$. □

Lemma 7.12. *Assuming the conditions of Theorem 1.1, we get*

$$\mathbf{E}|\varepsilon_{j4}|^q \le \frac{C^q}{n^q}.$$

k,

Proof. The result follows immediately from the bound

$$|\varepsilon_{j4}| \le \frac{1}{nv}, \text{ a. s.}$$

See for instance [10], Lemma 3.3. □

7.2 Some auxiliary bounds for resolvent matrices for $z \in \mathbb{G}$

Introduce now the region

$$\mathbb{G} := \{z = u + iv \in \mathbb{C}^+ : u \in \mathbb{J}_\varepsilon, v \ge v_0/\sqrt{\gamma}\}, \quad \text{where } v_0 = A_0 n^{-1}, \quad (7.24)$$
$$\mathbb{J}_\varepsilon = [-2+\varepsilon, 2-\varepsilon], \quad \varepsilon := c_1 n^{-\frac{2}{3}}, \quad \gamma = \gamma(u) = \min\{2-u, 2+u\}.$$

In the next lemma we state some useful inequalities for the region $\mathbb{G}$.

Lemma 7.13. *For any $z \in \mathbb{G}$ we have*

$$|z^2 - 4| \ge 2\max\{\gamma, v\}, \qquad nv\sqrt{|z^2-4|} \ge 2A_0.$$

Proof. We observe that

$$|z^2 - 4| = |z-2||z+2| \ge 2\sqrt{\gamma^2 + v^2}.$$

This inequality proves the Lemma. □

Lemma 7.14. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \dots, n$,*

$$\mathbf{E}\{|\varepsilon_{j1}|^4 \big| \mathfrak{M}^{(j)}\} \le \frac{C\mu_4}{n^2}. \quad (7.25)$$

Proof. The result follows immediately from the definition of ε_{j1}. □

Lemma 7.15. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \dots, n$,*

$$\mathbf{E}\{|\varepsilon_{j2}|^2 \big| \mathfrak{M}^{(j)}\} \le \frac{C}{nv} \operatorname{Im} m_n^{(j)}(z), \quad (7.26)$$

and

$$\mathbf{E}\{|\varepsilon_{j2}|^4 \big| \mathfrak{M}^{(j)}\} \le \frac{C\mu_4^2}{n^2 v^2} \operatorname{Im}^2 m_n^{(j)}(z). \quad (7.27)$$

Proof. Note that r.v.'s X_{jl}, for $l \in \mathbb{T}_j$ are independent of $\mathfrak{M}^{(j)}$ and that for $l, k \in \mathbb{T}_j$ $R^{(j)}_{lk}$ are measurable with respect to $\mathfrak{M}^{(j)}$. This implies that ε_{j2} is a quadratic form with coefficients $R^{(j)}_{lk}$ independent of X_{jl}. Thus its variance and fourth moment are easily available.

$$\mathbf{E}\{|\varepsilon_{j2}|^2 \big| \mathfrak{M}^{(j)}\} = \frac{1}{n^2} \sum_{l \neq k \in \mathbb{T}_j} |R^{(j)}_{lk}|^2 \le \frac{1}{n^2} \mathrm{Tr}\, |\mathbf{R}^{(j)}|^2.$$

Here we use the notation $|\mathbf{A}|^2 = \mathbf{A}\mathbf{A}^*$ for any matrix $\mathbf{A}$. Applying Lemma 7.6, inequality (7.11), we get equality (7.26).

Furthermore, direct calculations show that

$$\begin{aligned}\mathbf{E}\{|\varepsilon_{j2}|^4 \big| \mathfrak{M}^{(j)}\} &\le \frac{C}{n^2} (\frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |R^{(j)}_{lk}|^2)^2 + \frac{C\mu_4^2}{n^2} \frac{1}{n^2} \sum_{l \in \mathbb{T}_j} |R^{(j)}_{lk}|^4 \\ &\le \frac{C\mu_4^2}{n^2} (\frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |R^{(j)}_{lk}|^2)^2 \le \frac{C\mu_4^2}{n^2 v^2} (\mathrm{Im}\, m^{(j)}_n(z))^2.\end{aligned}$$

Here again we used Lemma 7.6, inequality (7.11). Thus Lemma 7.15 is proved. □

Lemma 7.16. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \ldots, n$,*

$$\mathbf{E}\{|\varepsilon_{j3}|^2 \big| \mathfrak{M}^{(j)}\} \le \frac{C\mu_4}{n} \frac{1}{n} \sum_{l \in \mathbb{T}_j} |R^{(j)}_{ll}|^2, \tag{7.28}$$

and

$$\mathbf{E}\{|\varepsilon_{j3}|^4 \big| \mathfrak{M}^{(j)}\} \le \frac{C}{n^2} (\frac{1}{n} \sum_{l \in \mathbb{T}_j} |R^{(j)}_{ll}|^2)^2 + \frac{C\mu_4}{n^2} \frac{1}{n} \sum_{l \in \mathbb{T}_j} |R^{(j)}_{ll}|^4. \tag{7.29}$$

Proof. The first inequality is easy. To prove the second, we apply Rosenthal's inequality. We obtain

$$\mathbf{E}\{|\varepsilon_{j3}|^4 \big| \mathfrak{M}^{(j)}\} \le \frac{C\mu_4}{n^2} (\frac{1}{n} \sum_{l \in \mathbb{T}_j} |R^{(j)}_{ll}|^2)^2 + \frac{C\mu_8}{n^3} \frac{1}{n} \sum_{l \in \mathbb{T}_j} |R^{(j)}_{ll}|^4.$$

Using $|X_{jl}| \le C n^{\frac{1}{4}}$ we get $\mu_8 \le C n \mu_4$ and the claim. Thus Lemma 7.16 is proved. □

Corollary 7.17. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C>0$, depending on μ_4 and D_0 only, such that for any $j=1,\ldots,n$, $\nu=1,2,3$ $z\in\mathbb{G}$, and $0\le\alpha\le\frac12A_1(nv)^{\frac14}$ and $\beta\ge1$*

$$\mathbf{E}\frac{|\varepsilon_{j\nu}|^2}{|z+m_n^{(j)}(z)+s(z)|^\beta|z+m_n^{(j)}(z)|^\alpha}\le\frac{C}{nv|z^2-4|^{\frac{\beta-1}{2}}}\tag{7.30}$$

and for $\beta\ge2$

$$\mathbf{E}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)+s(z)|^\beta|z+m_n^{(j)}(z)|^\alpha}\le\frac{C}{n^2v^2|z^2-4|^{\frac{\beta-2}{2}}}.\tag{7.31}$$

We have as well, for $\nu=2,3$, and $\beta\ge4$

$$\mathbf{E}\frac{|\varepsilon_{j\nu}|^8}{|z+m_n^{(j)}(z)+s(z)|^\beta|z+m_n^{(j)}(z)|^\alpha}\le\frac{C}{n^4v^4|z^2-4|^{\frac{\beta-4}{2}}}\tag{7.32}$$

Proof. For $\nu=1$, by Lemma 7.5, we have

$$\mathbf{E}\frac{|\varepsilon_{j\nu}|^2}{|z+m_n^{(j)}(z)+s(z)||z+m_n^{(j)}(z)|^\alpha}\le\frac{1}{n|z^2-4|^{\frac{\beta}{2}}}\mathbf{E}|X_{jj}|^4\mathbf{E}\frac{1}{|z+m_n^{(j)}(z)|^\alpha}.$$

Applying now Corollary 5.14 and lemma 7.13, we get the claim. The proof of the second inequality for $\nu=1$ is similar. For $\nu=2$ we apply Lemmas 7.15 and 7.5, inequality (7.26) and obtain, using inequality (7.8),

$$\begin{aligned}\mathbf{E}&\frac{|\varepsilon_{j2}|^2}{|z+m_n^{(j)}(z)+s(z)||z+m_n^{(j)}(z)|^\alpha}\\&\le\frac{1}{nv|z^2-4|^{\frac{\beta-1}{2}}}\mathbf{E}\frac{\operatorname{Im}m_n^{(j)}(z)}{|z+m_n^{(j)}(z)+s(z)||z+m_n^{(j)}(z)|^\alpha}\\&\le\frac{C}{nv}\mathbf{E}\frac{1}{|z+m_n^{(j)}(z)|^\alpha}.\end{aligned}$$

Similar, using Lemma 7.15, inequality (7.27), and inequality (7.8) we get

$$\begin{aligned}\mathbf{E}&\frac{|\varepsilon_{j2}|^4}{|z+m_n^{(j)}(z)+s(z)|^\beta|z+m_n^{(j)}(z)|^\alpha}\\&\le\frac{C}{n^2v^2|z^2-4|^{\frac{\beta-2}{2}}}\mathbf{E}\frac{\operatorname{Im}^2m_n^{(j)}(z)}{|z+m_n^{(j)}(z)+s(z)|^2|z+m_n^{(j)}(z)|^\alpha}\\&\le\frac{C}{n^2v^2|z^2-4|^{\frac{\beta-2}{2}}}\mathbf{E}\frac{1}{|z+m_n^{(j)}(z)|^\alpha}.\end{aligned}$$

Applying Corollary 5.14, we get the claim. For $\nu = 3$, we apply Lemma 7.16, inequalities (7.28) and (7.29) and Lemma 7.5. We get

$$\mathbf{E}\frac{|\varepsilon_{j3}|^2}{|z+m_n^{(j)}(z)+s(z)||z+m_n^{(j)}(z)|^\alpha} \le \frac{C}{n\sqrt{|z^2-4}}\mathbf{E}\frac{1}{|z+m_n^{(j)}(z)|^\alpha}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\Big), \tag{7.33}$$

and

$$\mathbf{E}\frac{|\varepsilon_{j3}|^4}{|z+m_n^{(j)}(z)+s(z)|^2|z+m_n^{(j)}(z)|^\alpha} \le \frac{C}{n^2|z^2-4|}\mathbf{E}\frac{1}{|z+m_n^{(j)}(z)|^\alpha}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^4\Big).$$

Using now the Cauchy – Schwartz inequality and Corollary 5.14, we get the claim. □

Lemma 7.18. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \dots, n$,*

$$|\varepsilon_{j4}| \le \frac{C}{nv} \quad a.s.$$

Proof. This inequality follows from

$$\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)} = (1 + \frac{1}{n}\sum_{l,k\in\mathbb{T}_j} X_{jl}X_{jk}[(R^{(j)})^2]_{kl})R_{jj} = R_{jj}^{-1}\frac{dR_{jj}}{dz}, \tag{7.34}$$

which may be obtained using the Schur complement formula. See, for instance [10], Lemma 3.3.

□

Corollary 7.19. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \dots, n$,*

$$\mathbf{E}|\varepsilon_j|^4 \le \frac{C}{n^2v^2}. \tag{7.35}$$

Proof. By definition of ε_j (see 3.2), we have

$$\mathbf{E}|\varepsilon_j|^4 \le 4^3(\mathbf{E}|\varepsilon_{j1}|^4 + \cdots + \mathbf{E}|\varepsilon_{j4}|^4).$$

Applying now Lemmas 7.14, 7.15 (inequality (7.27)), 7.16 (inequality (7.29)) and 7.18 and taking expectation where needed (7.35) follows. Thus the Corollary is proved. □

Introduce quantities

$$\eta_{j1} = \frac{1}{n}\sum_{l\in\mathbb{T}_j}[(R^{(j)})^2]_{ll} = \frac{1}{n}\mathrm{Tr}\,(\mathbf{R}^{(j)})^2,$$

$$\eta_{j2} = \frac{1}{n}\sum_{l\neq k\in\mathbb{T}_j} X_{jl}X_{jk}[(\mathbf{R}^{(j)})^2]_{lk},$$

$$\eta_{j3} = \frac{1}{n}\sum_{l\in\mathbb{T}_j}(X_{jl}^2-1)[(\mathbf{R}^{(j)})^2]_{ll}. \tag{7.36}$$

Lemma 7.20. *Assuming the conditions of Theorem 1.1, we have, for $z\in\mathbb{G}$, for any $j=1,\ldots,n$*

$$\mathbf{E}\{|\eta_{j3}|^4\big|\mathfrak{M}^{(j)}\} \le Cn^{-2}v^{-6}\mathrm{Im}^2 m_n^{(j)}(z) + Cn^{-2}v^{-4}\mu_4\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathrm{Im}^4 R_{ll}^{(j)}, \tag{7.37}$$

$$\mathbf{E}\{|\eta_{j3}|^8\big|\mathfrak{M}^{(j)}\} \le \frac{C\mu_4^4}{n^4v^{12}}\mathrm{Im}^4 m_n^{(j)}(z) + \frac{C\mu_4^4}{n^4v^8}\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathrm{Im}^8 R_{ll}^{(j)}. \tag{7.38}$$

Proof. Direct calculation shows

$$\mathbf{E}\{|\eta_{j3}|^4\big|\mathfrak{M}^{(j)}\} \le \frac{C\mu_4}{n^2}(\sum_{l\in\mathbb{T}_j}|[(R^{(j)})^2]_{ll}|^2)^2 + \frac{C\mu_8}{n^3}\frac{1}{n}\sum_{l\in\mathbb{T}_j}|[(R^{(j)})^2]_{ll}|^4.$$

Applying Lemma 7.6, the inequality (7.15) and the inequality $\mu_8 \le Cn\mu_4$, we get

$$\mathbf{E}\{|\eta_{j3}|^4\big|\mathfrak{M}^{(j)}\} \le Cn^{-2}v^{-6}\mathrm{Im}^2 m_n^{(j)}(z) + Cn^{-2}\mu_4\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathrm{Im}^4 R_{ll}^{(j)}.$$

Thus, inequality (7.37) is proved. Consider now the 8th moment of η_{j3}. Applying Rosenthal's inequality, we get

$$\mathbf{E}\{|\eta_{j3}|^8\big|\mathfrak{M}^{(j)}\} \le \frac{C\mu_4^4}{n^4}(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|[(R^{(j)})^2]_{ll}|^2)^4 + \frac{C\mu_{16}}{n^7}\frac{1}{n}\sum_{l\in\mathbb{T}_j}|[(R^{(j)})^2]_{ll}|^8.$$

Using now Lemma 7.6, inequalities (7.14) and (7.15), and that $\mu_{16}\le Cn^3\mu_4$, we obtain

$$\mathbf{E}\{|\eta_{j3}|^8\big|\mathfrak{M}^{(j)}\} \le \frac{C\mu_4^4}{n^4v^{12}}\mathrm{Im}^4 m_n^{(j)}(z) + \frac{C\mu_4^4}{n^4v^8}\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathrm{Im}^8 R_{ll}^{(j)}.$$

Thus, inequality (7.38) is proved. □

Lemma 7.21. *Assuming the conditions of Theorem 1.1, we have, for $z \in \mathbb{G}$, for any $j = 1, \ldots, n$*

$$\mathbf{E}\{|\eta_{j2}|^4 \big|\mathfrak{M}^{(j)}\} \le \frac{C\mathrm{Im}^2 m_n^{(j)}(z)}{n^2 v^6} + \frac{C\mu_4^2}{n^3 v^6}\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2, \tag{7.39}$$

$$\begin{aligned}\mathbf{E}\{|\eta_{j2}|^8 \big|\mathfrak{M}^{(j)}\} \le{}& \frac{C}{n^4 v^{12}}\big(\mathrm{Im}\, m_n^{(j)}(z)\big)^4 + \frac{C\mu_4^2}{n^4 v^8}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathrm{Im}^4 R_{ll}^{(j)}\Big)^2 \\ &+ \frac{C\mu_4^2}{n^4 v^8}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathrm{Im}^3 R_{ll}^{(j)}\Big)^2\big(\mathrm{Im}\, m_n^{(j)}(z)\big) + \frac{C\mu_4^4}{n^6 v^{12}}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}\big(\mathrm{Im}\, R_{ll}^{(j)}\big)^2\Big)^2 \\ &+ \frac{C\mu_4^2}{n^6 v^{12}}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}\big(\mathrm{Im}\, R_{ll}^{(j)}\big)^2\Big)\big(\mathrm{Im}\, m_n^{(j)}(z)\big)^2. \end{aligned}\tag{7.40}$$

Proof. Direct calculation shows that

$$\mathbf{E}\{|\eta_{j2}|^4 \big|\mathfrak{M}^{(j)}\} \le \frac{C}{n^2}(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2)^2 + \frac{C\mu_4^2}{n^4}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^4.$$

Applying inequalities (7.16) and (7.17) of Lemma 7.6, we get

$$\mathbf{E}\{|\eta_{j2}|^4 \big|\mathfrak{M}^{(j)}\} \le \frac{C\mathrm{Im}^2 m_n^{(j)}(z)}{n^2 v^6} + \frac{C\mu_4^2}{n^3 v^6}\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2.$$

Furthermore, we have

$$\begin{aligned}\mathbf{E}\{|\eta_{j2}|^8 \big|\mathfrak{M}^{(j)}\} \le{}& \frac{C}{n^4}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2\Big)^4 + \frac{C\mu_8^2}{n^8}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^8 \\ &+ \frac{C\mu_6^2}{n^6}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^6\Big)\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2\Big) \\ &+ \frac{C\mu_4^4}{n^6}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^4\Big)^2 \\ &+ \frac{C\mu_4^2}{n^6}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^4\Big)\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2\Big)^2.\end{aligned}$$

Note that

$$\mu_6 \le C\sqrt{n}\mu_4, \quad \mu_8 \le Cn\mu_4.$$

Using this relations, we get

$$\mathbf{E}\{|\eta_{j2}|^8\big|\mathfrak{M}^{(j)}\} \le \frac{C}{n^4}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2\Big)^4 + \frac{C\mu_4^2}{n^6}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}\Big|^8$$
$$+\frac{C\mu_4^2}{n^5}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^6\Big)\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2\Big)$$
$$+\frac{C\mu_4^4}{n^6}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^4\Big)^2$$
$$+\frac{C\mu_4^2}{n^6}\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^4\Big)\Big(\frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2\Big)^2.$$

Applying now Lemma 7.6, we obtain

$$\mathbf{E}\{|\eta_{j2}|^8\big|\mathfrak{M}^{(j)}\} \le \frac{C}{n^4v^{12}}(\operatorname{Im} m_n^{(j)}(z))^4 + \frac{C\mu_4^2}{n^4v^8}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}(\operatorname{Im} R_{ll}^{(j)})^4\Big)^2$$
$$+\frac{C\mu_4^2}{n^5v^{12}}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}\operatorname{Im}^3 R_{ll}^{(j)}\Big)(\operatorname{Im} m_n^{(j)}(z)) + \frac{C\mu_4^4}{n^6v^{12}}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}(\operatorname{Im} R_{ll}^{(j)})^2\Big)^2$$
$$+\frac{C\mu_4^2}{n^6}\Big(\frac{1}{n}\sum_{l\in\mathbb{T}_j}(\operatorname{Im} R_{ll}^{(j)})^2\Big)(\operatorname{Im} m_n^{(j)}(z))^2.$$

Thus, the Lemma is proved. □

Lemma 7.22. *Assuming the conditions of Theorem 1.1, we have*

$$\mathbf{E}\frac{|\varepsilon_{j4}|^4}{|z+s(z)+m_n(z)|^4} \le \frac{C}{n^4v^4}.$$

Proof. Using the representations (7.34)-(7.36) we have

$$\varepsilon_{j4} = \frac{1}{n}(1+\frac{1}{n}\sum_{l,k\in\mathbb{T}_j}X_{jl}X_{jk}[(R^{(j)})^2]_{lk})R_{jj} = \frac{1}{n}(1+\eta_{j1}+\eta_{j2}+\eta_{j3})R_{jj}. \quad (7.41)$$

Applying the Cauchy–Schwartz inequality, we get

$$\mathbf{E}|\frac{\varepsilon_{j4}}{z+s(z)+m_n(z)}|^4$$
$$\le \frac{C}{n^4}\left(1+\mathbf{E}^{\frac12}\left(\frac{|\frac1n\sum_{l,k\in\mathbb{T}_j}X_{jl}X_{jk}[(R^{(j)})^2]_{lk}|}{|z+s(z)+m_n(z)|}\right)^8\right)\mathbf{E}^{\frac12}|R_{jj}|^8.$$

Using Corollary 5.14, we we may write

$$\mathbf{E}|\frac{\varepsilon_{j4}}{z+s(z)+m_n(z)}|^4 \\ \le \frac{C}{n^4}\Big(1+\mathbf{E}^{\frac12}\Big|\frac{\eta_{j1}}{z+m_n(z)+s(z)}\Big|^8+\mathbf{E}^{\frac12}\Big|\frac{\eta_{j2}}{z+m_n(z)+s(z)}\Big|^8 \\ +\mathbf{E}^{\frac12}\Big|\frac{\eta_{j3}}{z+m_n(z)+s(z)}\Big|^8\Big).$$

Observe that,

$$\frac{1}{|z+s(z)+m_n(z)|}\le\frac{1}{|z+s(z)+m_n^{(j)}(z)|}(1+\frac{|\varepsilon_{j4}|}{|z+s(z)+m_n(z)|}).$$

Therefore, by Lemmas 7.18, 7.5 and 7.13, for $z\in\mathbb{G}$,

$$\frac{1}{|z+s(z)+m_n(z)|}\le\frac{C}{|z+s(z)+m_n^{(j)}(z)|}. \tag{7.42}$$

Using (7.42) we get by definition of η_{j1}, Lemma 7.6, and inequality (7.11)

$$\mathbf{E}^{\frac12}\Big|\frac{\eta_{j1}}{z+m_n(z)+s(z)}\Big|^8\le C\mathbf{E}^{\frac12}\frac{v^{-8}\mathrm{Im}^8 m_n^{(j)}(z)}{|z+m_n^{(j)}(z)+s(z)|^8}\le v^{-4}.$$

Furthermore, applying inequality (7.42) again, we obtain

$$\mathbf{E}^{\frac12}\Big|\frac{\eta_{j2}}{z+m_n(z)+s(z)}\Big|^8\le\mathbf{E}^{\frac12}\Big|\frac{\eta_{j2}}{z+m_n^{(j)}(z)+s(z)}\Big|^8.$$

Conditioning with respect to $\mathfrak{M}^{(j)}$ and applying Lemma 7.20, we obtain

$$\mathbf{E}^{\frac12}\Big|\frac{\eta_{j2}}{z+m_n(z)+s(z)}\Big|^8\le\mathbf{E}^{\frac12}\frac{C}{|z+m_n^{(j)}(z)+s(z)|^8}\Big(\frac{1}{n^4v^{12}}(\mathrm{Im}\, m_n^{(j)}(z))^4 \\ +\frac{\mu_4^4}{n^4v^8}\frac1n\sum_{l\in\mathbb{T}_{\mathbb{J}}}(\mathrm{Im}\, R_{ll}^{(j)})^8\Big).$$

Using Lemma 7.5, inequality (7.8), together with Corollary 5.14 we get

$$\mathbf{E}^{\frac12}\Big|\frac{\eta_{j2}}{z+m_n(z)+s(z)}\Big|^8\le\frac{C}{n^2v^6|z^2-4|}+\frac{C\mu_4^2}{n^2v^4|z^2-4|^2}.$$

Applying inequality (7.42) and conditioning with respect to $\mathfrak{M}^{(j)}$ and applying Lemma 7.21, we get

$$\begin{aligned}\mathbf{E}^{\frac12}\Big|\frac{\eta_{j3}}{z+m_n(z)+s(z)}\Big|^8 \le \mathbf{E}^{\frac12}\frac{1}{|z+s(z)+m_n^{(j)}(z)|^8}\Bigg(\frac{C}{n^4v^{12}}(\operatorname{Im} m_n^{(j)}(z))^4\\ +\frac{C\mu_4^2}{n^4v^8}\Big(\frac1n\sum_{l\in\mathbb{T}_j}(\operatorname{Im} R_{ll}^{(j)})^4\Big)^2+\frac{C\mu_4^2}{n^5v^{12}}\Big(\frac1n\sum_{l\in\mathbb{T}_j}(\operatorname{Im} R_{ll}^{(j)})^3\Big)(\operatorname{Im} m_n^{(j)}(z))\\ +\frac{C\mu_4^4}{n^6v^{12}}\Big(\frac1n\sum_{l\in\mathbb{T}_j}(\operatorname{Im} R_{ll}^{(j)})^2\Big)^2+\frac{C\mu_4^2}{n^6v^{12}}\Big(\frac1n\sum_{l\in\mathbb{T}_j}(\operatorname{Im} R_{ll}^{(j)})^2\Big)(\operatorname{Im} m_n^{(j)}(z))^2\Bigg).\end{aligned} \tag{7.43}$$

Using that $|z+m_n^{(j)}(z)+s(z)|\ge \operatorname{Im} m_n^{(j)}(z)$ together with Lemma 7.6, we arrive at

$$\begin{aligned}\mathbf{E}^{\frac12}\Big|\frac{\eta_{j3}}{z+m_n(z)+s(z)}\Big|^8 \le \frac{C}{n^2v^6|z^2-4|}+\frac{C}{n^2v^4|z^2-4|^2}\\ +\frac{C}{n^{\frac52}v^6|z^2-4|^{\frac74}}+\frac{C}{n^3v^6|z^2-4|^2}+\frac{C}{n^3v^6|z^2-4|^{\frac32}}.\end{aligned}$$

Summarizing we may write now, for $z\in\mathbb{G}$,

$$\begin{aligned}\mathbf{E}\frac{|\varepsilon_{j4}|^4}{|z+s(z)+m_n(z)|^4} \le \frac{C}{n^4v^4}+\frac{C}{n^6v^6|z^2-4|}+\frac{C}{n^6v^4|z^2-4|^2}\\ +\frac{C}{n^{\frac{13}{2}}v^6|z^2-4|^{\frac74}}+\frac{C}{n^7v^6|z^2-4|^2}+\frac{C}{n^7v^6|z^2-4|^{\frac32}}.\end{aligned}$$

For $z\in\mathbb{G}$, see (7.24) and Lemma 7.13, this inequality may be simplified by means of the following bounds (with $v_0=A_0n^{-1}$)

$$\begin{aligned}n^{\frac52}v^2|z^2-4|^{\frac74}\ge n^{\frac52}v_0^2\gamma^{-1+\frac74}\ge C\sqrt{n}\gamma^{\frac34}\ge C,\\ n^3v^2|z^2-4|\ge C,\quad n^3v^2|z^2-4|^{\frac32}\ge C,\\ n^2|z^2-4|^2\ge C.\end{aligned} \tag{7.44}$$

Using these relation, we obtain

$$\mathbf{E}\frac{|\varepsilon_{j4}|^4}{|z+s(z)+m_n(z)|^4}\le\frac{C}{n^4v^4}.$$

Thus Lemma 7.22 is proved. □

Corollary 7.23. *Assuming the conditions of Theorem 1.1, we have*

$$\mathbf{E}\frac{|\varepsilon_{j4}|^2}{|z+s(z)+m_n(z)|^2} \le \frac{C}{n^2v^2}.$$

Proof. The result follows immediately from Lemma 7.22 and Jensen's inequality. ☐

Lemma 7.24. *Assuming the conditions of Theorem 1.1, we have, for $z \in \mathbb{G}$,*

$$\mathbf{E}|\Lambda_n|^2 \le \frac{C}{n^2v^2}. \tag{7.45}$$

Proof. We write

$$\mathbf{E}|\Lambda_n|^2 = \mathbf{E}\Lambda_n\overline{\Lambda}_n = \mathbf{E}\frac{T_n}{z+m_n(z)+s(z)}\overline{\Lambda}_n = \sum_{\nu=1}^{4}\mathbf{E}\frac{T_{n\nu}}{z+m_n(z)+s(z)}\overline{\Lambda}_n,$$

where

$$T_{n\nu} := \frac{1}{n}\sum_{j=1}^{n}\varepsilon_{j\nu}R_{jj}, \text{ for } \nu = 1,\ldots,4.$$

First we observe that by (7.34)

$$|T_{n4}| = \frac{1}{n}|m_n'(z)| \le \frac{1}{nv}\operatorname{Im} m_n(z).$$

Hence $|z+m_n^{(j)}(z)+s(z)| \ge \operatorname{Im} m_n^{(j)}(z)$ and Jensen's inequality yields

$$|\mathbf{E}\frac{T_{n4}}{z+m_n(z)+s(z)}\overline{\Lambda}_n| \le \frac{1}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2. \tag{7.46}$$

Furthermore, we represent T_{n1} as follows

$$T_{n1} = T_{n11} + T_{n12},$$

where

$$T_{n11} = -\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{j1}\frac{1}{z+m_n(z)},$$

$$T_{n12} = \frac{1}{n}\sum_{j=1}^{n}\varepsilon_{j1}(R_{jj}+\frac{1}{z+m_n(z)}).$$

Using these notations we may write

$$V_1 := \mathbf{E}\frac{T_{n11}}{z+m_n(z)+s(z)}\overline{\Lambda}_n = -\mathbf{E}\frac{(\frac{1}{n}\sum_{j=1}^n \varepsilon_{j1})}{(z+m_n(z))(z+s(z)+m_n(z))}\overline{\Lambda}_n.$$

Applying the Cauchy – Schwartz inequality twice and using the definition of ε_{j1} (see (3.2)), we get by Lemma 7.5

$$|V_1| \le \frac{1}{|z^2-4|^{\frac{1}{2}}}\mathbf{E}^{\frac{1}{4}}\Big|\frac{1}{n\sqrt{n}}\sum_{j=1}^n X_{jj}\Big|^4 \mathbf{E}^{\frac{1}{4}}|\frac{1}{z+m_n(z)|^4}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2. \qquad (7.47)$$

By Rosenthal's inequality, we have, for $z \in \mathbb{G}$

$$|V_1| \le \frac{C}{n|z^2-4|^{\frac{1}{2}}}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 \le \frac{1}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2.$$

Using (3.2) we rewrite T_{n12}, obtaining

$$V_2 := \mathbf{E}\frac{T_{n12}}{z+m_n(z)+s(z)}\overline{\Lambda}_n = \frac{1}{n\sqrt{n}}\sum_{j=1}^n \mathbf{E}\frac{X_{jj}\varepsilon_j R_{jj}}{(z+m_n(z))(z+m_n(z)+s(z))}\overline{\Lambda}_n.$$

By the Cauchy – Schwartz inequality, using the definition of ε_j (see representation 3.2), we obtain

$$|V_2| \le \frac{1}{\sqrt{n}}\sum_{\nu=1}^4 \mathbf{E}^{\frac{1}{2}}\frac{|\frac{1}{n}\sum_{j=1}^n \varepsilon_{j\nu}X_{jj}R_{jj}|^2}{|z+m_n(z)+s(z)|^2|z+m_n(z)|^2}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 =: \sum_{\nu=1}^4 V_{2\nu}. \qquad (7.48)$$

For $\nu = 1$, we have

$$V_{21} \le \frac{1}{n}\mathbf{E}^{\frac{1}{2}}\frac{\Big|\frac{1}{n}\sum_{j=1}^n X_{jj}^2 R_{jj}\Big|^2}{|z+m_n(z)+s(z)|^2|z+m_n(z)|^2}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2.$$

Applying the Cauchy – Schwartz inequality twice and Lemma 7.5, we arrive at

$$V_{21} \le \frac{1}{n\sqrt{|z^2-4|}}\mathbf{E}^{\frac{1}{4}}\Big(\frac{1}{n}\sum_{j=1}^n X_{jj}^4\Big)^2 \mathbf{E}^{\frac{1}{8}}\Big(\frac{1}{n}\sum_{j=1}^n |R_{jj}|^2\Big)^4 \mathbf{E}^{\frac{1}{8}}\frac{1}{|z+m_n(z)|^8}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2. \qquad (7.49)$$

Observe that

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j=1}^n X_{jj}^4\Big)^2 = \Big(\frac{1}{n}\sum_{j=1}^n \mathbf{E}X_{jj}^4\Big)^2 + \mathbf{E}\Big(\frac{1}{n}\sum_{j=1}^n (X_{jj}^4 - \mathbf{E}X_{jj}^4)\Big)^2$$
$$\leq \mu_4^2 + \frac{2}{n^2}\sum_{j=1}^n \mathbf{E}|X_{jj}|^8 \leq (\mu_4 + D_0^4)\mu_4 \leq C. \qquad (7.50)$$

The last inequality, inequality (7.49), Corollary 5.14 and Lemma 7.13 together imply

$$V_{21} \leq \frac{C}{n\sqrt{|z^2-4|}}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 \leq \frac{C}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2. \qquad (7.51)$$

Furthermore, for $\nu = 4$, by Lemma 7.18 we have

$$V_{24} \leq \frac{1}{nv\sqrt{n}}\mathbf{E}^{\frac{1}{2}}\frac{\frac{1}{n}\sum_{j=1}^n |X_{jj}|^2|R_{jj}|^2}{|z+m_n(z)+s(z)|^2|z+m_n(z)|^2}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2.$$

Applying the Cauchy – Schwartz inequality and Lemma 7.5, we get

$$V_{24} \leq \frac{1}{nv\sqrt{n}\sqrt{|z^2-4|}}\mathbf{E}^{\frac{1}{4}}\Big(\frac{1}{n}\sum_{j=1}^n X_{jj}^4\Big)^2\mathbf{E}^{\frac{1}{8}}\Big(\frac{1}{n}\sum_{j=1}^n |R_{jj}|^2\Big)^4\mathbf{E}^{\frac{1}{8}}\frac{1}{|z+m_n(z)|^8}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2.$$

Similar to inequality (7.51), applying Lemma 7.13, inequality (7.50) and Corollary 5.14, we get

$$V_{24} \leq \frac{C}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2. \qquad (7.52)$$

By Hölder's inequality, we have for $\nu = 2, 3$,

$$V_{2\nu} \leq \frac{1}{\sqrt{n}}\frac{1}{n}\sum_{j=1}^n \mathbf{E}^{\frac{1}{4}}\frac{|\varepsilon_{j\nu}|^4|X_{jj}|^4}{|z+m_n(z)+s(z)|^4}\mathbf{E}^{\frac{1}{8}}\frac{1}{|z+m_n(z)|^8}\mathbf{E}^{\frac{1}{8}}|R_{jj}|^8\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2.$$

Note that for $\nu = 2, 3$, r.v. X_{jj} doesn't depend on $\varepsilon_{j\nu}$ and on σ-algebra $\mathfrak{M}^{(j)}$. Using inequality (7.42) for $z \in \mathbb{G}$, we get

$$\mathbf{E}\frac{|\varepsilon_{j\nu}|^4|X_{jj}|^4}{|z+m_n(z)+s(z)|^4} \leq C\mathbf{E}\frac{|\varepsilon_{j\nu}|^4|X_{jj}|^4}{|z+m_n^{(j)}(z)+s(z)|^4} \leq C\mu_4\mathbf{E}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)+s(z)|^4}.$$

Applying now Lemmas 7.15, 7.16 and 7.13, arrive at

$$V_{2\nu} \leq \frac{C}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2, \text{ for } \nu = 2, 3. \qquad (7.53)$$

Inequalities (7.51), (7.52), (7.53) together imply

$$V_2 \leq \frac{C}{nv}\mathbf{E}|\Lambda_n|^2. \tag{7.54}$$

Consider now the quantity

$$Y_\nu := \mathbf{E}\frac{T_{n\nu}}{z+m_n(z)+s(z)}\overline{\Lambda}_n,$$

for $\nu = 2, 3$. We represent it as follows

$$Y_\nu = Y_{\nu 1} + Y_{\nu 2},$$

where

$$Y_{\nu 1} = -\frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j\nu}\overline{\Lambda}_n}{(z+m_n^{(j)}(z))(z+m_n(z)+s(z))},$$
$$Y_{\nu 2} = \frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j\nu}(R_{jj}+\frac{1}{z+m_n^{(j)}(z)})\overline{\Lambda}_n}{z+m_n(z)+s(z)}.$$

By the representation (5.32), which is similar to (3.2) we have

$$Y_{\nu 2} = \sum_{\mu=1}^3 \frac{1}{n}\sum_{j=1}^n \mathbf{E}\frac{\varepsilon_{j\nu}\varepsilon_{j\mu}\overline{\Lambda}_n R_{jj}}{(z+m_n(z)+s(z))(z+m_n^{(j)}(z))}.$$

Using inequality (7.42), we may write, for $z \in \mathbb{G}$

$$|Y_{\nu 2}| \leq \sum_{\mu=1}^3 \frac{C}{n}\sum_{j=1}^n \mathbf{E}\frac{|\varepsilon_{j\nu}||\varepsilon_{j\mu}||\overline{\Lambda}_n||R_{jj}|}{|z+m_n^{(j)}(z)+s(z)||z+m_n^{(j)}(z)|}.$$

Applying the Cauchy – Schwartz inequality and the inequality $ab \leq \frac{1}{2}(a^2 +$

b^2), we get

$$\begin{aligned}|Y_{\nu 2}| &\le \sum_{\mu=1}^{3}\frac{C}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac12}\frac{|\varepsilon_{j\nu}|^2|\varepsilon_{j\mu}|^2|R_{jj}|^2}{|z+m_n^{(j)}(z)+s(z)|^2|z+m_n^{(j)}(z)|^2}\mathbf{E}^{\frac12}|\Lambda_n|^2\\
&\le \sum_{\mu=2}^{3}\frac{C}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac12}\frac{|\varepsilon_{j\mu}|^4|R_{jj}|^2}{|z+m_n^{(j)}(z)+s(z)|^2|z+m_n^{(j)}(z)|^2}\mathbf{E}^{\frac12}|\Lambda_n|^2\\
&+\frac{C}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac12}\frac{|\varepsilon_{j\nu}|^2|\varepsilon_{j1}|^2|R_{jj}|^2}{|z+m_n^{(j)}(z)+s(z)|^2|z+m_n^{(j)}(z)|^2}\mathbf{E}^{\frac12}|\Lambda_n|^2\\
&\le \sum_{\mu=2}^{3}\frac{C}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac14}\frac{|\varepsilon_{j\mu}|^8}{|z+m_n^{(j)}(z)+s(z)|^4|z+m_n^{(j)}(z)|^4}\mathbf{E}^{\frac12}|\Lambda_n|^2\mathbf{E}^{\frac14}|R_{jj}|^4\\
&+\frac{C}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac14}\frac{|\varepsilon_{j\nu}|^4|\varepsilon_{j1}|^4}{|z+m_n^{(j)}(z)+s(z)|^4|z+m_n^{(j)}(z)|^4}\mathbf{E}^{\frac12}|\Lambda_n|^2\mathbf{E}^{\frac14}|R_{jj}|^4.\end{aligned}$$

Using Corollary 7.17 with $\alpha=2$, we arrive at

$$|Y_{\nu 2}| \le \frac{C}{nv}\mathbf{E}^{\frac12}|\Lambda_n|^2. \tag{7.55}$$

In order to estimate $Y_{\nu 1}$ we introduce now the quantity

$$\Lambda_n^{(j1)} = \frac1n\operatorname{Tr}\mathbf{R}^{(j)} - s(z) + \frac{s(z)}{n} + \frac{1}{n^2}\operatorname{Tr}{\mathbf{R}^{(j)}}^2 s(z).$$

Recall that

$$\begin{aligned}&\eta_{j1} = \frac1n\sum_{l\in\mathbb{T}_j}[(\mathbf{R}^{(j)})^2]_{ll}, \quad \eta_{j2} = \frac1n\sum_{k\ne l\in\mathbb{T}_j}X_{jk}X_{jl}[(\mathbf{R}^{(j)})^2]_{l,k},\\
&\eta_{j3} = \frac1n\sum_{l\in\mathbb{T}_j}(X_{jl}^2-1)[(\mathbf{R}^{(j)})^2]_{ll}.\end{aligned}\tag{7.56}$$

Note that

$$|\eta_{j1}| \le \frac1n|\operatorname{Tr}(\mathbf{R}^{(j)})^2|. \tag{7.57}$$

We use that (see [11, Lemma 7.5])

$$\varepsilon_{j4} = \frac1n(1+\eta_{j1}+\eta_{j2}+\eta_{j3})R_{jj}. \tag{7.58}$$

Note that

$$\delta_{nj} = \Lambda_n - \widetilde{\Lambda}_n^{(j)} = -\varepsilon_{j4} - \frac{s(z)}{n} - \frac{1}{n}\eta_{j0}s(z)$$
$$= \frac{1}{n}(R_{jj} - s(z))(1+\eta_{j1}) + \frac{1}{n}(\eta_{j2}+\eta_{j3})R_{jj}.$$

This yields

$$|\delta_{nj}| \le \frac{1}{n}(1+|\eta_{j1}|)|R_{jj} - s(z)| + \frac{1}{n}|\eta_{j2}+\eta_{j3}||R_{jj}| \tag{7.59}$$

We represent $Y_{\nu1}$ in the form

$$Y_{\nu1} = Z_{\nu1} + Z_{\nu2} + Z_{\nu3} + Z_{\nu4},$$

where

$$Z_{\nu1} = -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\varepsilon_{j\nu}\overline{\Lambda}_n^{(j1)}}{(z+m_n^{(j)}(z))(z+m_n^{(j)}(z)+s(z))},$$
$$Z_{\nu2} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\varepsilon_{j\nu}\overline{\delta}_{nj}}{(z+m_n^{(j)}(z))(z+m_n(z)+s(z))},$$
$$Z_{\nu3} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\varepsilon_{j\nu}\overline{\Lambda}_n\varepsilon_{j4}}{(z+m_n^{(j)}(z))(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))},$$
$$Z_{\nu4} = -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\frac{\varepsilon_{j\nu}\overline{\delta}_{nj}\varepsilon_{j4}}{(z+m_n^{(j)}(z))(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))}.$$

First, note that by conditionally independence

$$Z_{\nu1} = 0. \tag{7.60}$$

Furthermore, applying Hölder's inequality, we get

$$|Z_{\nu3}| \le \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac14}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)|^4|z+m_n^{(j)}(z)+s(z)|^4}$$
$$\times\,\mathbf{E}^{\frac14}\frac{|\varepsilon_{j4}|^4}{|z+m_n(z)+s(z)|^4}\mathbf{E}^{\frac12}|\Lambda_n^{(j1)}|^2.$$

Using Corollary 7.17 with $\alpha = 4$ and Lemmas 7.22 and 7.5, we obtain

$$|Z_{\nu3}| \le \frac{C}{(nv)^{\frac32}|z^2-4|^{\frac14}}\mathbf{E}^{\frac12}|\Lambda_n|^2.$$

For $z \in \mathbb{G}$ we may rewrite this bound using Lemma 7.13

$$|Z_{\nu 3}| \le \frac{C}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2. \tag{7.61}$$

Furthermore, note that

$$|1+\eta_{j1}| \le v^{-1}\mathrm{Im}\{z+m_n^{(j)}(z)\} \le \mathrm{Im}\{z+m_n^{(j)}(z)+s(z)\}.$$

This inequality together with (7.59) implies that

$$|Z_{\nu 4}| \le \widetilde{Z}_{\nu 4} + \widehat{Z}_{\nu 4}, \tag{7.62}$$

where

$$\begin{aligned}\widetilde{Z}_{\nu 4} &= \frac{1}{n^2 v}\sum_{j=1}^n \mathbf{E}\frac{|\varepsilon_{j\nu}\varepsilon_{j4}||R_{jj}-s(z)|}{|z+m_n^{(j)}(z)||z+m_n(z)+s(z)|},\\ \widetilde{Z}_{\nu 4} &= \frac{1}{n^2 v}\sum_{j=1}^n \mathbf{E}\frac{|\varepsilon_{j\nu}\varepsilon_{j4}||\eta_{j2}+\eta_{j3}||R_{jj}-s(z)|}{|z+m_n^{(j)}(z)||z+m_n(z)+s(z)|}\end{aligned} \tag{7.63}$$

By representation (3.2), we have

$$|R_{jj}-s(z)| \le |\Lambda_n||R_{jj}| + |\varepsilon_j||R_{jj}|.$$

This implies that

$$\widetilde{Z}_{\nu 4} \le \widetilde{Z}_{\nu 41} + \widetilde{Z}_{\nu 42},$$

where

$$\begin{aligned}\widetilde{Z}_{\nu 41} &= \frac{1}{n^2 v}\sum_{j=1}^n \mathbf{E}\frac{|\varepsilon_{j\nu}\varepsilon_{j4}||\Lambda_n|}{|z+m_n^{(j)}(z)||z+m_n(z)+s(z)|},\\ \widetilde{Z}_{\nu 42} &= \frac{1}{n^2 v}\sum_{j=1}^n \mathbf{E}\frac{|\varepsilon_{j\nu}\varepsilon_{j4}||\varepsilon_j|}{|z+m_n^{(j)}(z)||z+m_n(z)+s(z)|}.\end{aligned}$$

Applying Hölder inequality, we get

$$\begin{aligned}\widetilde{Z}_{\nu 41} \le{}& \frac{1}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2\mathbf{E}^{\frac{1}{8}}\Big(\frac{1}{n}\sum_{j=1}^n|\varepsilon_{j\nu}|^4\Big)^2\mathbf{E}^{\frac{1}{16}}\Big(\frac{1}{n}\sum_{j=1}^n|R_{jj}|^{16}\Big)\\ &\times \mathbf{E}^{\frac{1}{16}}\Big(\frac{1}{n}\sum_{j=1}^n\frac{1}{|z+m_n^{(j)}(z)|^{16}}\Big)\Big(\frac{1}{n}\sum_{j=1}^n\mathbf{E}^{\frac{1}{4}}\frac{|\varepsilon_{j4}|^4}{|z+m_n(z)+s(z)|^4}\Big).\end{aligned}$$

By Lemma 7.2.1, inequality (7.39) and Corollary 5.14, we have

$$\widetilde{Z}_{\nu 41} \le \frac{C}{n^2 v^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \tag{7.64}$$

Futhermore, applying Hölder inequality again, we get

$$\begin{aligned}\widetilde{Z}_{\nu 42} \le \frac{1}{nv} \mathbf{E}^{\frac{1}{4}} \Big(\frac{1}{n} \sum_{j=1}^{n} |\varepsilon_{j\nu}|^4 \Big) \mathbf{E}^{\frac{1}{4}} \Big(\frac{1}{n} \sum_{j=1}^{n} |\varepsilon_j|^4 \Big) \mathbf{E}^{\frac{1}{4}} \Big(\frac{1}{n} \sum_{j=1}^{n} |R_{jj}|^8 \Big) \\ \times \mathbf{E}^{\frac{1}{8}} \Big(\frac{1}{n} \sum_{j=1}^{n} \frac{1}{|z + m_n^{(j)}(z)|^8} \Big) \mathbf{E}^{\frac{1}{4}} \Big(\frac{1}{n} \sum_{j=1}^{n} \frac{|\varepsilon_{j4}|^4}{|z + m_n(z) + s(z)|^4} \Big).\end{aligned} \tag{7.65}$$

The last inequality, Corollaries 5.14, 7.17, Lemma 7.22 together imply

$$\widetilde{Z}_{\nu 42} \le \frac{C}{n^2 v^2}. \tag{7.66}$$

Inequalities (7.64) and (7.66) together imply

$$|Z_{\nu 4}| \le \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \tag{7.67}$$

To bound $Z_{\nu 2}$ we first apply inequality (7.42) and obtain

$$|Z_{\nu 2}| \le \frac{C}{n} \sum_{j=1}^{n} \mathbf{E} \frac{|\varepsilon_{j\nu}||\delta_{nj}|}{|z + m_n^{(j)}(z)||z + m_n^{(j)}(z) + s(z)|}.$$

Furthermore, similar to bound $Z_{\nu 4}$ – inequality (7.62) – we amy write

$$|Z_{\nu 2}| \le \widetilde{Z}_{\nu 2} + \widehat{Z}_{\nu 2},$$

where

$$\widetilde{Z}_{\nu 2} = \frac{C}{n^2} \sum_{j=1}^{n} \mathbf{E} \frac{|\varepsilon_{j\nu}||R_{jj} - s(z)|}{|z + m_n^{(j)}(z)|},$$

$$\widehat{Z}_{\nu 2} = \frac{C}{n^2} \sum_{j=1}^{n} \mathbf{E} \frac{|\varepsilon_{j\nu}||\eta_{j2} + \eta_{j3}||R_{jj}|}{z + m_n^{(j)}(z)||z + m_n^{(j)}(z) + s(z)|}.$$

Applying inequality (7.63) and Cauchy – Schwartz inequality, we get

$$\begin{aligned}\widetilde{Z}_{\nu 2} \le \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 \frac{C}{n^2} \sum_{j=1}^{n} \mathbf{E}^{\frac{1}{4}} |\varepsilon_{j\nu}|^4 \mathbf{E}^{\frac{1}{8}} \frac{1}{|z + m_n^{(j)}(z)|^8} \mathbf{E}^{\frac{1}{8}} |R_{jj}|^8 \\ + \frac{C}{n^2} \sum_{j=1}^{n} \mathbf{E}^{\frac{1}{4}} |\varepsilon_{j\nu}|^4 \mathbf{E}^{\frac{1}{8}} \frac{1}{|z + m_n^{(j)}(z)|^8} \mathbf{E}^{\frac{1}{8}} |R_{jj}|^8 \mathbf{E}^{\frac{1}{4}} |\varepsilon_j|^4.\end{aligned} \tag{7.68}$$

Lemmas 7.15, 7.16, 7.22, inequality (7.39) and Corollary 5.14 together imply

$$\widetilde{Z}_{\nu 2} \leq \frac{C}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 + \frac{C}{n^2v^2}. \tag{7.69}$$

Applying now Hölder's inequality, we get

$$|\widehat{Z}_{\nu 2}| \leq \frac{C}{n^2}\sum_{j=1}^n \mathbf{E}^{\frac{1}{4}}\frac{|\varepsilon_{j\nu}|^4}{|z+m_n^{(j)}(z)+s(z)|^2}\mathbf{E}^{\frac{1}{4}}\frac{|\eta_{j2}+\eta_{j3}|^4}{|z+m_n^{(j)}(z)+s(z)|^2} \\ \times \mathbf{E}^{\frac{1}{4}}|R_{jj}|^4\mathbf{E}^{\frac{1}{4}}\frac{1}{|z+m_n^{(j)}(z)|^4}.$$

The last inequality together with Lemmas 7.22, 7.20, 7.21 and Corollaries 5.14, 7.17 implies

$$|\widehat{Z}_{\nu 2}| \leq \frac{C}{n^2v^2}. \tag{7.70}$$

Combining inequalities (7.46), (7.47), (7.54), (7.55), (7.60), (7.61), (7.67), (7.69), (7.70), we get

$$\mathbf{E}|\Lambda_n|^2 \leq \frac{C}{nv}\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 + \frac{C}{n^2v^2}. \tag{7.71}$$

Applying lemma 7.4 with $t = 2$, $r = 1$ completes the proof of Lemma 7.24.

□

We relabel η_{j2}, η_{j3} and introduce the following quantity

$$\beta_{j1} = \frac{1}{n}\sum_{l\in\mathbb{T}_j}[(R^{(j)})^2]_{ll} - \frac{1}{n}\sum_{l=1}^n[(R)^2]_{ll},$$

$$\beta_{j2} = \frac{1}{n}\sum_{l\neq k\in\mathbb{T}_j}X_{jl}X_{jk}[(R^{(j)})^2]_{lk} = \eta_{j2},$$

$$\beta_{j3} = \frac{1}{n}\sum_{l\in\mathbb{T}_j}(X_{jl}^2-1)[(R^{(j)})^2]_{ll} = \eta_{j3}.$$

Lemma 7.25. *Assuming the conditions of Theorem 1.1, we have, for $\nu = 2,3$,*

$$\mathbf{E}\{|\beta_{j\nu}|^2\Big|\mathfrak{M}^{(j)}\} \leq \frac{C}{nv^3}\mathrm{Im}\, m_n^{(j)}(z).$$

Proof. We recall that by C we denote the generic constant depending on μ_4 and D_0 only. By definition of $\beta_{j\nu}$ for $\nu = 2, 3$, conditioning on $\mathfrak{M}^{(j)}$, we get

$$\mathbf{E}\{|\beta_{j2}|^2\big|\mathfrak{M}^{(j)}\} \leq \frac{C}{n^2}\sum_{l\neq k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2 \leq \frac{C}{n^2}\sum_{l,k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2,$$
$$\mathbf{E}\{|\beta_{j3}|^2\big|\mathfrak{M}^{(j)}\} \leq \frac{C}{n^2}\sum_{l\in\mathbb{T}_j}|[(R^{(j)})^2]_{ll}|^2 \leq \frac{C}{n^2}\sum_{l,k\in\mathbb{T}_j}|[(R^{(j)})^2]_{kl}|^2.$$

Applying Lemma 7.6, we get the claim. Thus Lemma 7.25 is proved. □

Lemma 7.26. *Assuming the conditions of Theorem 1.1, we have, for* $j = 1, \ldots, n$,

$$\mathbf{E}|\beta_{j1}| \leq \frac{C}{nv^2}.$$

Proof. Let $\mathcal{F}_n^{(j)}(x)$ denote empirical spectral distribution function of matrix $\mathbf{W}^{(j)}$. According to *interlacing eigenvalues Theorem* (see [18], Theorem 4.38) we have

$$\sup_x |\mathcal{F}_n(x) - \mathcal{F}_n^{(j)}(x)| \leq \frac{C}{n}.$$

Furthermore, we represent

$$\beta_{j1} = \int_{-\infty}^{\infty}\frac{1}{(x-z)^2}d(\mathcal{F}_n(x) - \mathcal{F}_n^{(j)}(x)) + \frac{1}{n(n-1)}\mathrm{Tr}\,(\mathbf{R}^{(j)})^2.$$

Integrating by parts, we get the claim.

Thus Lemma 7.26 is proved. □

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