

LOCAL SEMICIRCLE LAW UNDER MOMENT CONDITIONS. PART I: THE STIELTJES TRANSFORM

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ABSTRACT. We consider a random symmetric matrix $\mathbf{X} = [X_{jk}]_{j,k=1}^n$ in which the upper triangular entries are independent identically distributed random variables with mean zero and unit variance. We additionally suppose that $\mathbb{E}|X_{11}|^{4+\delta} =: \mu_{4+\delta} < \infty$ for some $\delta > 0$. Under these conditions we show that the typical distance between the Stieltjes transform of the empirical spectral distribution (ESD) of the matrix $n^{-\frac{1}{2}}\mathbf{X}$ and Wigner's semicircle law is of order $(nv)^{-1}$, where v is the distance in the complex plane to the real line. Furthermore we outline applications which are deferred to a subsequent paper, such as the rate of convergence in probability of the ESD to the distribution function of the semicircle law, rigidity of the eigenvalues and eigenvector delocalization.

1. INTRODUCTION AND MAIN RESULT

We consider a random symmetric matrix $\mathbf{X} = [X_{jk}]_{j,k=1}^n$ where the upper triangular entries are independent random variables with mean zero and unit variance. We will be mostly interested in limiting laws for the eigenvalues and eigenvectors of large $n \times n$ symmetric random matrices in the asymptotic limit as n goes to infinity.

For the symmetric matrix $\mathbf{W} := \frac{1}{\sqrt{n}}\mathbf{X}$ we denote its n eigenvalues in the increasing order as

$$\lambda_1(\mathbf{W}) \leq \dots \leq \lambda_n(\mathbf{W})$$

and introduce the eigenvalue counting function

$$N_I(\mathbf{W}) := |\{1 \leq k \leq n : \lambda_k(\mathbf{W}) \in I\}|$$

for any interval $I \subset \mathbb{R}$, where $|A|$ denotes the number of elements in the set A . Note that we shall sometimes omit \mathbf{W} from the notation of $\lambda_j(\mathbf{W})$.

It is well known since the pioneering work of E. Wigner [32] that for any interval $I \subset \mathbb{R}$ of fixed length and independent of n

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} N_I(\mathbf{W}) = \int_I g_{sc}(\lambda) d\lambda, \quad (1.1)$$

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where

$$g_{sc}(\lambda) := \frac{1}{2\pi} \sqrt{4 - \lambda^2} \mathbb{1}[|\lambda| \leq 2]$$

is the density function of Wigner's semicircle law. Here and in what follows we denote by $\mathbb{1}[A]$ the indicator function of the set A . Wigner considered the special case when all X_{jk} take only two values ± 1 with equal probabilities. To prove (1.1) he used the moment method which may be described as follows. Since g_{sc} is compactly supported it is uniquely determined by the sequence of its moments given by

$$\beta_k = \begin{cases} \frac{1}{m+1} \binom{2m}{m}, & k = 2m, \\ 0, & k = 2m + 1. \end{cases}, \quad k \geq 1.$$

We remark here that $\beta_{2m}, m \geq 1$, are Catalan numbers. To establish the convergence (1.1) one needs to show that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \lambda^k dF_n(\lambda) = \int_{-2}^2 \lambda^k g_{sc}(\lambda) d\lambda,$$

where $F_n(\lambda) := \frac{1}{n} N_{(-\infty, \lambda]}(\mathbf{W})$ is the empirical spectral distribution function. Further details may be found in [3].

Wigner's semicircle law has been extended in various aspects. For example, L. Arnold in [1] proved almost sure (a.s.) convergence of F_n to the semicircle law but under additional moment assumptions on the matrix entries. More general conditions of convergence to Wigner's semicircle law were established by L. Pastur in [27]. For all $\tau > 0$ we define Lindeberg's ratio for the random matrix \mathbf{X} by the relation

$$L_n(\tau) := \frac{1}{n^2} \sum_{j,k=1}^n \mathbb{E} X_{jk}^2 \mathbb{1}[|X_{jk}| \geq \tau \sqrt{n}]. \quad (1.2)$$

Pastur has shown that the convergence of Lindeberg's ratio to zero is sufficient for the convergence in probability to the semicircle law. V. Girko in [15], [14] extended Pastur's result to a.s. convergence and stated that (1.2) is also necessary condition. This result, in particular, implies that if $X_{jk}, 1 \leq j \leq k \leq n$, are independent identically distributed (i.i.d.) random variables and have zero mean and unit variance, then F_n converges a.s. to Wigner's semicircle law. We remark that all these results were established for symmetric random matrices with independent, not necessary identically distributed entries, but assuming that $\mathbb{E} X_{jk}^2 = 1$ for all $1 \leq j \leq k \leq n$. This limitation has been overcome in a sequence of papers, see, for example, [29], [26] and [16]. In the last years there has been increasing interest in random matrices with dependent entries. For some models it has been shown that Wigner's semicircle law holds as well for the matrices with dependent entries, see, for example, [20], [16], [4].

All these results hold for intervals I of fixed length, independent of n , which typically contain a macroscopically large number of eigenvalues, which means a number of order n . Unfortunately for smaller intervals where the number of eigenvalues cease to be

macroscopically large the moment method does not apply and one needs the Stieltjes transform of the empirical spectral distribution function F_n , which is given by

$$m_n(z) := \int_{-\infty}^{\infty} \frac{dF_n(\lambda)}{\lambda - z} = \frac{1}{n} \operatorname{Tr}(\mathbf{W} - z\mathbf{I})^{-1} = \frac{1}{n} \sum_{j=1}^n \frac{1}{\lambda_j(\mathbf{W}) - z},$$

where $z = u + iv$, $v \geq 0$. The Stieltjes transform is an appropriate tool to study spectral densities, since taking the imaginary part of $m_n(z)$ we get

$$\operatorname{Im} m_n(u + iv) = \int_{-\infty}^{\infty} \frac{v}{(\lambda - u)^2 + v^2} dF_n(\lambda) = \frac{1}{v} \int_{-\infty}^{\infty} K\left(\frac{u - \lambda}{v}\right) dF_n(\lambda)$$

which is the kernel density estimator with kernel K and bandwidth v . For a meaningful estimator of the spectral density we cannot allow the distance v to the real line, that is the bandwidth of the kernel density estimator, to be smaller than the typical $\frac{1}{n}$ -distance between eigenvalues. Hence, in what follows we shall be mostly interested in the situations when $v \gg \frac{1}{n}$.

Under rather general conditions, like convergence of Lindeberg's ratio (1.2), to zero for fixed $v > 0$ one may establish the convergence of $m_n(z)$ to the the Stieltjes transform of Wigner's semicircle law which is given by

$$s(z) = \int_{-\infty}^{\infty} \frac{g_{sc}(\lambda) d\lambda}{\lambda - z}$$

and may be calculated explicitly by

$$s(z) = -\frac{z}{2} + \sqrt{\frac{z^2}{4} - 1}, \quad (1.3)$$

see, for example, [3] for a simple explanation.

It is much more difficult to establish the convergence in the region $1 \gg v \gg \frac{1}{n}$. Significant progress in that direction was recently made in a series of results by L. Erdős, B. Schlein, H.-T. Yau and et al., [11], [10], [12], [8], showing that with high probability uniformly in $u \in \mathbb{R}$

$$|m_n(u + iv) - s(u + iv)| \leq \frac{\log^c n}{nv}, \quad (1.4)$$

which they called *local semicircle law*. It means that the fluctuations of $m_n(z)$ around $s(z)$ are of order $(nv)^{-1}$ (up to a logarithmic factor). To prove (1.4) in those papers [11], [10], [12] it was assumed that the distribution of X_{jk} for all $1 \leq j, k \leq n$ has sub-exponential tails. Moreover in [8] this assumption had been relaxed to requiring $\mathbb{E} |X_{jk}|^p \leq \mu_p$ for all $p \geq 1$, where μ_p are some constants. Since there is meanwhile an extensive literature on the local semicircle law we refrain from providing a complete list here and refer the reader to the surveys of L. Erdős [6] and T. Tao, V. Vu, [30].

Combining the results and arguments of the papers [9], [7] with the more recent results of [25] it follows that (1.4) holds under the condition that $\mathbb{E} |X_{jk}|^{4+\delta} =: \mu_{4+\delta} < \infty$. (L. Erdős and H.-T. Yau private communication). To explain it in more details, assume that $1 \leq j, k \leq n$ $|X_{jk}| \leq n^{\frac{1}{2}} q^{-1}$ with probability larger than $1 - e^{-nc}$ for some $c > 0$.

Here, q may depend on n and usually $n^\phi \leq q \leq n^{1/2} \log^{-1} n$ for some $\phi > 0$. This means that all X_{jk} are bounded in absolute value by some quantity from $\log n$ to $n^{1/2-\phi}$. Define the following region in the complex plane

$$\mathbb{S}(C) = \{z = u + iv \in \mathbb{C} : |u| \leq 5, 10 \geq v \geq \varphi^C n^{-1}\},$$

where $\varphi := (\log n)^{\log \log n}$ and $C > 0$ and let

$$\gamma := \gamma(u) := ||u| - 2|. \quad (1.5)$$

In [9][Theorem 2.8] it is shown that there exist positive constants C and c such that for all $q \geq \varphi^C$

$$\mathbb{P} \left(\bigcap_{z \in \mathbb{S}(C)} \left\{ |m_n(z) - s(z)| \leq \varphi^C \left(\min \left(\frac{1}{q^2 \sqrt{\gamma + v}}, \frac{1}{q} \right) + \frac{1}{nv} \right) \right\} \right) \geq 1 - n^C e^{-c\varphi^3}. \quad (1.6)$$

Assume now that $\mathbb{E} |X_{jk}|^{4+\delta} < C$. In [7] (Lemma 7.6 and 7.7) the initial matrix has been replaced by a matrix $\tilde{\mathbf{X}}$ matching the first moments of \mathbf{X} but having sub-exponential decaying tails. In the recent paper [25][Lemma 5.1] it has been shown in particular that there exist such a matrix $\tilde{\mathbf{X}}$ such that $\mathbb{E} X_{jk}^s = \mathbb{E} \tilde{X}_{jk}^s$ for $s = 1, \dots, 4$ and $|\tilde{X}_{jk}| \leq C \log n$ (this means that $q = O(n^{1/2} \log^{-1} n)$). Finally, it remains to estimate the difference $m_n(z) - s(z)$ in terms of $\tilde{m}_n(z) - s(z)$, where $\tilde{m}_n(z)$ is the Stieltjes transform corresponding to $\tilde{\mathbf{X}}$.

The aim of this paper is to give self-contained proof of (1.4) assuming that $\mathbb{E} |X_{jk}|^{4+\delta} =: \mu_{4+\delta} < \infty$. We apply different techniques, which allow to reduce the power of $\log n$ (see the definition of φ above) to some constant with precise dependence on δ . We also extend the recent results of F. Götze and A. Tikhomirov in [21] and [18], with $\delta = 4$, where we required 8 moments together with u lying in the support of the semicircle law. Our work and many details of the proof were motivated by a recent paper of C. Cacciapuoti, A. Maltsev and B. Schlein, [5], where the authors improved the log-factor dependence in (1.4) in the sub-Gaussian case. We mention that in the latter case one has $\mathbb{E} |X_{jk}|^p \leq (C\sqrt{p})^p$ for all $p \geq 1$.

To control the distance between $m_n(z)$ and $s(z)$ one may estimate $\mathbb{E} |m_n(z) - s(z)|^p$. Instead of a combinatorial approach to deal with the last quantity we apply the method developed in [21], [18] which is a Stein type method. In addition we apply the descent method for the estimation of the moments of diagonal entries of the resolvent using a few *multiplicative steps* introduced in [5]. In earlier work of L. Erdős, B. Schlein, H.-T. Yau and et al. mentioned above a larger number of *additive steps* of descent had been used. For the details of our technique see Section 1.3.

1.1. Main result. We consider a random symmetric matrix $\mathbf{X} = [X_{jk}]_{j,k=1}^n$ in which $X_{jk}, 1 \leq j \leq k \leq n$, are independent identically distributed random variables with $\mathbb{E} X_{11} = 0$ and $\mathbb{E} X_{11}^2 = 1$. We additionally suppose that

$$\mathbb{E} |X_{11}|^{4+\delta} < C.$$

for some $\delta > 0$ and some constant $C > 0$. In this case we say that the matrix \mathbf{X} satisfies the conditions **(C0)**.

We introduce the following quantity depending on δ

$$\alpha := \alpha(\delta) = \frac{2}{4 + \delta},$$

which will control the level of truncation of the matrix entries.

Without loss of generality we may assume that $\delta \leq 4$ since otherwise all bounds will be independent of α . This means that we may assume that

$$\frac{1}{4} \leq \alpha < \frac{1}{2}.$$

The following theorem about approximation of Stieltjes transforms is the main result of this paper.

Theorem 1.1. *Assume that the conditions **(C0)** hold and let $V > 0$ be some constant.*

(i) *There exist positive constants A_0, A_1 and C depending on α and V such that*

$$\mathbb{E} |m_n(z) - s(z)|^p \leq \left(\frac{Cp^2}{nv} \right)^p,$$

for all $1 \leq p \leq A_1(nv)^{\frac{1-2\alpha}{2}}$, $V \geq v \geq A_0n^{-1}$ and $|u| \leq 2 + v$.

(ii) *For any $u_0 > 0$ there exist positive constants A_0, A_1 and C depending on α, u_0 and V such that*

$$\mathbb{E} |\operatorname{Im} m_n(z) - \operatorname{Im} s(z)|^p \leq \left(\frac{Cp^2}{nv} \right)^p,$$

for all $1 \leq p \leq A_1(nv)^{\frac{1-2\alpha}{2}}$, $V \geq v \geq A_0n^{-1}$ and $|u| \leq u_0$.

Remark. Let us complement the results stated above by the following remarks.

1. The methods used in our proof (see section 1.3 below) differ from those used in [9], [7] and [25] which are outlined above. In particular, applying Markov's inequality and taking p of order $\log n$ we may also rewrite our result in terms of probability bounds with an estimate $n^{-c \log \log n}$. We also reduce the power of $\log n$ (see definition of φ above) to some constant with precise dependence on δ .
2. We conjecture that the result of Theorem 1.1 holds with $\delta = 0$ which corresponds to the case of finite fourth moment only.
3. The case of non-identically distributed X_{jk} , can be dealt with by our methods, but the details are more involved and we omit its proof here.
4. Note that it is possible to reduce the power $\frac{1-2\alpha}{2}$ assuming that at least eight moments of the matrix entries are finite. This situation corresponds to $\delta = 4$ and $\alpha = \frac{1}{4}$.
5. On the right hand side of the estimates in (i) and (ii) the dependence on the power p is given sharpened to p^p compared to p^{p^2} in the main theorem of [5].

Applications of Theorem 1.1 outside the limit spectral interval, that is for $u \geq 2$, require stronger bounds on $\operatorname{Im} \Lambda_n$. We say that the set of conditions **(C1)** holds if **(C0)**

are satisfied and $|X_{jk}| \leq Dn^\alpha$, $1 \leq j, k \leq n$, where $D := D(\alpha)$ is some positive constant depending on α only.

Theorem 1.2. *Assume that the conditions (C1) hold and $u_0 > 2$ and $V > 0$. There exist positive constants A_0, A_1 and C depending on α, u_0 and V such that*

$$\mathbb{E} |\operatorname{Im} m_n(z) - \operatorname{Im} s(z)|^p \leq \frac{C^p p^p}{n^p (\gamma + v)^p} + \frac{C^p p^{3p}}{(nv)^{2p} (\gamma + v)^{\frac{p}{2}}} + \frac{C^p}{n^p v^{\frac{p}{2}} (\gamma + v)^{\frac{p}{2}}} + \frac{C^p p^p}{(nv)^{\frac{3p}{2}} (\gamma + v)^{\frac{p}{4}}},$$

for all $1 \leq p \leq A_1(nv)^{\frac{1-2\alpha}{2}}$, $V \geq v \geq A_0 n^{-1}$ and $2 \leq |u| \leq u_0$.

1.2. Applications of the main result. In a subsequent paper we shall apply Theorem 1.1 to prove a series of results which we will formulate and discuss now. We start with the rate of convergence in probability. Let us denote by

$$\Delta_n^* := \sup_{x \in \mathbb{R}} |F_n(x) - G_{sc}(x)|,$$

where $G_{sc}(x) := \int_{-\infty}^x g_{sc}(\lambda) d\lambda$. It was proved by F. Götze and A. Tikhomirov in [19] that assuming $\max_{1 \leq j, k \leq n} \mathbb{E} |X_{jk}|^{12} =: \mu_{12} < \infty$, one may obtain the following estimate

$$\mathbb{E} \Delta_n^* \leq \mu_{12}^{\frac{1}{6}} n^{-\frac{1}{2}}.$$

In particular this estimate implies by Markov's inequality that

$$\mathbb{P}(\Delta_n^* \geq K) \leq \frac{\mu_{12}^{\frac{1}{6}}}{K n^{\frac{1}{2}}}. \quad (1.7)$$

It is easy to see from the previous bound that one may take $K \gg n^{-\frac{1}{2}}$. This result has been extended by Bai and et al., see [2], showing that instead of the twelfth finite moments it suffices to require finiteness of six moments. Applying Theorem 1.1 we may obtain the following stronger bound. Namely, assuming the conditions (C0) for $1 \leq p \leq c(\alpha) \log n$ we get an error bound of next order

$$\mathbb{E}^{\frac{1}{p}} [\Delta_n^*]^p \leq n^{-1} \log^{\frac{2}{1-2\alpha}} n.$$

Similarly to (1.7) this inequality implies that

$$\mathbb{P}(\Delta_n^* \geq K) \leq \frac{C^p \log^{\frac{2p}{1-2\alpha}} n}{K^p n^p}, \quad (1.8)$$

which is optimal up to a power of logarithm since one may take $K \gg n^{-1}$ (see also [17], [31] and [18] using additional assumptions). A direct corollary of the last bound is the following estimate

$$\mathbb{P} \left(\left| \frac{N[x - \frac{\xi}{2n}; x + \frac{\xi}{2n}]}{\xi} - g_{sc}(x) \right| \geq \frac{K}{\xi} \right) \leq \frac{C^p \log^{\frac{2p}{1-2\alpha}} n}{K^p n^p},$$

valid for all $\xi > 0$, where $N[I] := N_I(\mathbf{W})$ with $I := [E - \frac{\xi}{2n}; E + \frac{\xi}{2n}]$. This means the semicircle law holds on a short scale as well.

Another application of Theorem 1.1 is the following result which shows the rigidity of the eigenvalues. Let us define the quantile position of the j -th eigenvalue by

$$\gamma_j := \int_{-\infty}^{\gamma_j} g_{sc}(\lambda) d\lambda = \frac{j}{n}, \quad 1 \leq j \leq n.$$

We will show that under conditions **(C0)** with high probability for all $1 \leq j \leq n$ the following inequality holds

$$|\lambda_j - \gamma_j| \leq K[\min(j, n - j + 1)]^{-\frac{1}{3}} n^{-\frac{2}{3}}, \quad (1.9)$$

where K is of logarithmic order. It is easy to see that up to a logarithmic factor the distance between λ_j and γ_j is of order n^{-1} in the bulk of spectrum and of order $n^{-\frac{2}{3}}$ at the edges. To prove (1.9) we shall apply Theorem 1.2 as well. For previous results we refer the interested reader to [22], [8][Theorem 7.6], [9][Theorem 2.13], [17][Remark 1.2], [25][Theorem 3.6] and [5][Theorem 4].

We may also show delocalization of eigenvectors of \mathbf{W} . Let us denote by $\mathbf{u}_j := (u_{j1}, \dots, u_{jn})$ the eigenvectors of \mathbf{W} corresponding to the eigenvalue λ_j . Assuming condition **(C0)** we have that with high probability

$$\max_{1 \leq j, k \leq n} |u_{jk}|^2 \leq \frac{K}{n}.$$

For previous and related results we refer the interested reader to the corresponding theorems in [11] [17], [9] and [7].

1.3. Sketch of the proof. Let $\Lambda_n(z) := m_n(z) - s(z)$. Applying Lemma B.1 (see [5][Proposition 2.2]) it is shown in Section 2 that one may estimate $\mathbb{E} |\Lambda_n(z)|^p$ and $\mathbb{E} |\operatorname{Im} \Lambda_n(z)|^p$ via $\mathbb{E} |T_n(z)|^p$ (see definition (2.4)) choosing one of the bounds depending on whether z is near the edge of the spectrum or away from it.

To estimate $\mathbb{E} |T_n(z)|^p$ we extend the methods developed in [18][Theorem 1.2] and [21]. The bound for $\mathbb{E} |T_n(z)|^p$ is given in Theorem 2.1. The crucial step in [21], [18] was to show that finiteness of eight moments suffices to show that $\max_{1 \leq j \leq n} \mathbb{E} |\mathbf{R}_{jj}(z)|^p \leq C_0^p$ for all $1 \leq p \leq C(nv)^{\frac{1}{4}}$ and $v \geq v_0$. The proof of this fact was based on the descent method developed in [5][Lemma 3.4] (see Lemma 4.1 below), but used in proving bounds for moments of the diagonal entries $\mathbf{R}_{jj}(z)$ only. In this paper we develop this approach to estimate the off-diagonal entries of the resolvent as well assuming **(C1)** (see Lemma 4.3 below). This provides improved bounds for $\mathbb{E} |\varepsilon_{j2}|^q$, where ε_{j2} is the quadratic form defined in (2.2). We would like to emphasize that this estimate is crucial for the proof of Theorem 1.1, assuming conditions **(C0)**. Similarly we establish a bound $\max_{1 \leq j \leq n} \mathbb{E} |\mathbf{R}_{jj}(z)|^p \leq C_0^p$, which is valid on the whole real line rather than on the support of the semicircle law only as in [21], [18]. The details may be found in Section 4, Lemma 4.1.

Note that $\mathbb{E} |T_n(z)|^p$ is bounded in terms of $\mathbb{E} \operatorname{Im}^p \mathbf{R}_{jj}$. In Section 5, Lemma 5.1 we show that $\max_{1 \leq j \leq n} \mathbb{E} \operatorname{Im}^q \mathbf{R}_{jj}(z)$ may be estimated by $\operatorname{Im}^q s(z)$ with some additional correction term (see definition (5.1) of $\Psi(z)$). Since we can derive explicit bounds for $\operatorname{Im} s(z)$ inside as well as outside the limit spectrum we are able to control the size for

$\mathbb{E}|T_n(z)|^q$ as well as $\mathbb{E}|\operatorname{Im}\Lambda_n(z)|^p$ on the whole real line in terms of the quantity γ (see (1.5)). This is another key fact for the proof of Theorem 1.2.

1.4. Notations. Throughout the paper we will use the following notations. We assume that all random variables are defined on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by \mathbb{E} the mathematical expectation with respect to \mathbb{P} .

We denote by \mathbb{R} and \mathbb{C} the set of all real and complex numbers. We also define $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. Let $\mathbb{T} = [1, \dots, n]$ denotes the set of the first n positive integers. For any $\mathbb{J} \subset \mathbb{T}$ introduce $\mathbb{T}_{\mathbb{J}} := \mathbb{T} \setminus \mathbb{J}$.

We shall systematically use for any matrix \mathbf{W} together with its resolvent \mathbf{R} and Stieltjes transform m_n the corresponding quantities $\mathbf{W}^{(\mathbb{J})}$, $\mathbf{R}^{(\mathbb{J})}$, $m_n^{(\mathbb{J})}$ for the corresponding sub matrix with entries X_{jk} , $j, k \in \mathbb{T} \setminus \mathbb{J}$.

By C and c we denote some positive constants, which may depend on α, u and V , but not on n .

For an arbitrary matrix \mathbf{A} taking values in $\mathbb{C}^{n \times n}$ we define the operator norm by $\|\mathbf{A}\| := \sup_{x \in \mathbb{R}^n: \|x\|=1} \|\mathbf{A}x\|_2$, where $\|x\|_2 := \sum_{j=1}^n |x_j|^2$. We also define the Hilbert-Schmidt norm by $\|\mathbf{A}\|_2 := \operatorname{Tr} \mathbf{A} \mathbf{A}^* = \sum_{j,k=1}^n |\mathbf{A}_{jk}|^2$.

1.5. Acknowledgment. We would like to thank L. Erdős and H.-T. Yau for drawing our attention to relevant previous results and papers in connection with the results of this paper, in particular, [7], [8], [9] and [25].

2. PROOF OF THE MAIN RESULT

We start this section with the recursive representation of the diagonal entries $\mathbf{R}_{jj}(z) = (\mathbf{W} - z\mathbf{I})^{-1}$ of the resolvent. As noted before we shall systematically use for any matrix \mathbf{W} together with its resolvent \mathbf{R} , Stieltjes transform m_n and etc. the corresponding quantities $\mathbf{W}^{(\mathbb{J})}$, $\mathbf{R}^{(\mathbb{J})}$, $m_n^{(\mathbb{J})}$ and etc. for the corresponding sub matrix with entries X_{jk} , $j, k \in \mathbb{T} \setminus \mathbb{J}$. We will often omit the argument z from $\mathbf{R}(z)$ and write \mathbf{R} instead. We may express \mathbf{R}_{jj} in the following way

$$\mathbf{R}_{jj} = \frac{1}{-z + \frac{X_{jj}}{\sqrt{n}} - \frac{1}{n} \sum_{l,k \in \mathbb{T}_j} X_{jk} X_{jl} \mathbf{R}_{kl}^{(j)}}. \quad (2.1)$$

Let $\varepsilon_j := \varepsilon_{1j} + \varepsilon_{2j} + \varepsilon_{3j} + \varepsilon_{4j}$, where

$$\varepsilon_{1j} = \frac{1}{\sqrt{n}} X_{jj}, \quad \varepsilon_{2j} = -\frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} X_{jk} X_{jl} \mathbf{R}_{kl}^{(j)}, \quad \varepsilon_{3j} = -\frac{1}{n} \sum_{k \in \mathbb{T}_j} (X_{jk}^2 - 1) \mathbf{R}_{kk}^{(j)},$$

$$\varepsilon_{4j} = \frac{1}{n} (\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(j)}).$$

Using these notations we may rewrite (2.1) as follows

$$\mathbf{R}_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)} \varepsilon_j \mathbf{R}_{jj}. \quad (2.2)$$

Introduce

$$\Lambda_n := m_n(z) - s(z), \quad b(z) := z + 2s(z), \quad b_n(z) = b(z) + \Lambda_n, \quad (2.3)$$

and

$$T_n := \frac{1}{n} \sum_{j=1}^n \varepsilon_j \mathbf{R}_{jj}, \quad (2.4)$$

Applying (2.2) we arrive at the following representation for Λ_n in terms of T_n and b_n

$$\Lambda_n = \frac{T_n}{z + m_n(z) + s(z)} = \frac{T_n}{b_n(z)}.$$

From Lemma B.1 of the Appendix it follows that for all $v > 0$ and $u \leq 2 + v$ (using the quantities (2.3))

$$|\Lambda_n| \leq C \min \left\{ \frac{|T_n|}{|b(z)|}, \sqrt{|T_n|} \right\}. \quad (2.5)$$

Moreover, for all $v > 0$ and $|u| \leq u_0$

$$|\operatorname{Im} \Lambda_n| \leq C \min \left\{ \frac{|T_n|}{|b(z)|}, \sqrt{|T_n|} \right\}. \quad (2.6)$$

This means that in order to bound $\mathbb{E} |\Lambda_n|^p$ (or $\mathbb{E} |\operatorname{Im} \Lambda|^p$ respectively) it is enough to estimate $\mathbb{E} |T_n|^p$.

Let us introduce the following region in the complex plane:

$$\mathbb{D} := \{z = u + iv \in \mathbb{C} : |u| \leq u_0, V \geq v \geq v_0 := A_0 n^{-1}\}, \quad (2.7)$$

where u_0, V are arbitrary fixed positive real numbers and A_0 is some large constant defined below.

The following theorem provides a general bound for $\mathbb{E} |T_n|^p$ for all $z \in \mathbb{D}$ in terms of diagonal resolvent entries. To formulate the result of the theorem we need to introduce additional notations. Let

$$\mathcal{A}(q) := \max_{|\mathbb{J}| \leq 1} \max_{j \in \mathbb{T}_{\mathbb{J}}} \mathbb{E}^{\frac{1}{q}} \operatorname{Im}^q \mathbf{R}_{jj}^{(\mathbb{J})}, \quad (2.8)$$

where \mathbb{J} may be an empty set or one point set. We also denote

$$\mathcal{E}_p := \frac{p^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \frac{p^{2p}}{(nv)^{2p}} + \frac{|b(z)|^{\frac{p}{2}} \mathcal{A}^{\frac{p}{2}}(\kappa p)}{(nv)^p}, \quad (2.9)$$

where $\kappa = \frac{16}{1-2\alpha}$.

Theorem 2.1. *Assume that the conditions (C1) hold and $u_0 > 2$ and $V > 0$. There exist positive constants A_0, A_1 and C depending on α, u_0 and V such that for all $z \in \mathbb{D}$ we have*

$$\mathbb{E} |T_n|^p \leq C^p \mathcal{E}_p, \quad (2.10)$$

where $1 \leq p \leq A_1 (nv)^{\frac{1-2\alpha}{2}}$.

The proof of Theorem 2.1 is one of the crucial steps in the proof of the main result and will be given in the next section. Since (2.10) is an estimate in terms of the imaginary part of diagonal resolvent entries we refer to it as the main 'general' bound. We finish this section with the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. From Lemmas D.1– D.3 of the Appendix it follows that we may assume that

$$|X_{jk}| \leq Dn^\alpha \text{ for all } 1 \leq j, k \leq n$$

and some $D := D(\alpha) > 0$.

Applying Theorem 2.1 we will show in the section 5, Lemma 5.1, that there exist constants H_0 depending on u_0, V and A_0, A_1 depending on α and H_0 such that

$$\mathcal{A}^p(\kappa p) \leq H_0^p \operatorname{Im}^p s(z) + \frac{H_0^p p^{2p}}{(nv)^p}. \quad (2.11)$$

for all $1 \leq p \leq A_1(nv)^{\frac{1-2\alpha}{2}}$ and $z \in \mathbb{D}$. This inequality and Theorem 2.1 together imply that

$$\mathbb{E} |T_n|^p \leq \frac{C^p p^p \operatorname{Im}^p s(z)}{(nv)^p} + \frac{C^p p^{3p}}{(nv)^{2p}} + \frac{C^p |b(z)|^{\frac{p}{2}} \operatorname{Im}^{\frac{p}{2}} s(z)}{(nv)^p} + \frac{C^p |b(z)|^{\frac{p}{2}} p^p}{(nv)^{\frac{3p}{2}}}. \quad (2.12)$$

with some new constant C which depends on H_0 . To estimate $\mathbb{E} |\operatorname{Im} \Lambda_n|^p$ we may choose one of the bounds (2.6), depending on whether z is near the edge of the spectrum or away from it. If $|b(z)|^p \geq \frac{C^p p^p}{(nv)^p}$ then we may take the bound

$$\mathbb{E} |\operatorname{Im} \Lambda_n|^p \leq \frac{C^p \mathbb{E} |T_n|^p}{|b(z)|^p}.$$

The r.h.s. of the last inequality may be estimated applying (2.12). We get

$$\mathbb{E} |\operatorname{Im} \Lambda_n|^p \leq \frac{C^p p^p \operatorname{Im}^p s(z)}{(nv)^p |b(z)|^p} + \frac{C^p p^{3p}}{(nv)^{2p} |b(z)|^p} + \frac{C^p \operatorname{Im}^{\frac{p}{2}} s(z)}{(nv)^p |b(z)|^{\frac{p}{2}}} + \frac{C^p p^p}{(nv)^{\frac{3p}{2}} |b(z)|^{\frac{p}{2}}}.$$

Since $|b(z)|^p \geq \frac{C^p p^p}{(nv)^p}$ the last inequality may be rewritten in the following way

$$\mathbb{E} |\operatorname{Im} \Lambda_n|^p \leq \frac{C^p p^p \operatorname{Im}^p s(z)}{(nv)^p |b(z)|^p} + \frac{C^p \operatorname{Im}^{\frac{p}{2}} s(z)}{(nv)^p |b(z)|^{\frac{p}{2}}} + \frac{C^p p^{2p}}{(nv)^p}.$$

It remains to estimate the imaginary part of $s(z)$. Since

$$\operatorname{Im}^p s(z) \leq c^p |b(z)|^p \text{ for } |u| \leq 2 \text{ and } \operatorname{Im}^p s(z) \leq \frac{c^p v^p}{|b(z)|^p} \text{ otherwise}$$

both inequalities combined yield

$$\mathbb{E} |\operatorname{Im} \Lambda_n|^p \leq \left(\frac{C p^2}{nv} \right)^p, \quad (2.13)$$

where we have used as well the fact that $c\sqrt{\gamma+v} \leq |b(z)| \leq C\sqrt{\gamma+v}$ for all $|u| \leq u_0$, $0 < v \leq v_1$. If $|b(z)|^p \leq \frac{C^p p^p}{(nv)^p}$ and $\operatorname{Im}^p s(z) \geq \frac{C^p p^{2p}}{(nv)^p}$ then we may repeat the calculations

above and get the bound (2.13). In the case $\text{Im}^p s(z) \leq \frac{C^p p^{2p}}{(nv)^p}$ we take the bound proportional to $|T_n|^{\frac{1}{2}}$ and obtain the following inequality

$$\mathbb{E} |\text{Im} \Lambda_n|^p \leq \mathbb{E} |T_n|^{\frac{p}{2}} \leq \left(\frac{C p^2}{nv} \right)^p.$$

Similar arguments are applicable to $\mathbb{E} |\Lambda_n|^p$. \square

Proof of Theorem 1.2. From Theorem 2.1 we may conclude that

$$\mathbb{E} |T_n|^p \leq \frac{C^p p^p \text{Im}^p s(z)}{(nv)^p} + \frac{C^p p^{3p}}{(nv)^{3p}} + \frac{C^p |b(z)|^{\frac{p}{2}} \text{Im}^{\frac{p}{2}} s(z)}{(nv)^p} + \frac{C^p |b(z)|^{\frac{p}{2}} p^p}{(nv)^{\frac{3p}{2}}}. \quad (2.14)$$

Applying Lemma B.1 we get

$$\mathbb{E} |\text{Im} \Lambda_n|^p \leq \frac{\mathbb{E} |T_n|^p}{|b(z)|^p}.$$

This inequality together with (2.12) leads to

$$\mathbb{E} |\text{Im} \Lambda_n|^p \leq \frac{C^p p^p \text{Im}^p s(z)}{(nv)^p |b(z)|^p} + \frac{C^p p^{3p}}{(nv)^{2p} |b(z)|^p} + \frac{C^p \text{Im}^{\frac{p}{2}} s(z)}{(nv)^p |b(z)|^{\frac{p}{2}}} + \frac{C^p p^p}{(nv)^{\frac{3p}{2}} |b(z)|^{\frac{p}{2}}}. \quad (2.15)$$

Since $c\sqrt{\gamma+v} \leq |b(z)| \leq C\sqrt{\gamma+v}$ for all $|u| \leq u_0$, $0 < v \leq v_1$ and

$$\frac{cv}{\sqrt{\gamma+v}} \leq \text{Im} s(z) \leq \frac{cv}{\sqrt{\gamma+v}} \quad \text{for all } 2 \leq |u| \leq u_0, 0 < v \leq v_1,$$

we finally get

$$\mathbb{E} |\text{Im} \Lambda_n|^p \leq \frac{C^p p^p}{n^p (\gamma+v)^p} + \frac{C^p p^{3p}}{(nv)^{2p} (\gamma+v)^{\frac{p}{2}}} + \frac{C^p}{n^p v^{\frac{p}{2}} (\gamma+v)^{\frac{p}{2}}} + \frac{C^p p^p}{(nv)^{\frac{3p}{2}} (\gamma+v)^{\frac{p}{4}}}. \quad (2.16)$$

This bound concludes the proof of the theorem. \square

3. PROOF OF THE GENERAL BOUND OF THEOREM 2.1

In the next section we will show that there exist positive constants C_0, A_0 and A_1 (explicit dependence on α, u and \mathbf{V} will be given later) such that for all $z \in \mathbb{D}$ and $1 \leq p \leq A_1 (nv)^{\frac{1-2\alpha}{2}}$ the following bound holds

$$\max_{j \in \mathbb{T}} \mathbb{E} |\mathbf{R}_{jj}(z)|^p \leq C_0^p. \quad (3.1)$$

Similarly we prove that

$$\mathbb{E} \frac{1}{|z + m_n(z)|^p} \leq C_0^p. \quad (3.2)$$

The proof is given in Lemma 4.1. In the current section we will assume that (3.1) and (3.2) hold. The rest of this section is devoted to the proof of Theorem 2.1.

Proof of Theorem 2.1. For the proof we will apply the techniques developed in [18] and [21]. Recalling the definition of T_n (see (2.4)) we may rewrite it in the following way

$$T_n = \frac{1}{n} \sum_{j=1}^n \varepsilon_{4j} \mathbf{R}_{jj} + \frac{1}{n} \sum_{\nu=1}^3 \sum_{j=1}^n \varepsilon_{\nu j} \mathbf{R}_{jj}.$$

Since

$$\sum_{j=1}^n \varepsilon_{4j} \mathbf{R}_{jj} = \frac{1}{n} \operatorname{Tr} \mathbf{R}^2 = m'_n(z) \quad (3.3)$$

we get that

$$T_n = \frac{m'_n(z)}{n} + \frac{1}{n} \sum_{\nu=1}^3 \sum_{j=1}^n \varepsilon_{\nu j} \mathbf{R}_{jj} = \frac{m'_n(z)}{n} + \widehat{T}_n,$$

where we denoted

$$\widehat{T}_n := \frac{1}{n} \sum_{\nu=1}^3 \sum_{j=1}^n \varepsilon_{\nu j} \mathbf{R}_{jj}.$$

Let us introduce the function $\varphi(z) = \bar{z}|z|^{p-2}$. Then

$$\mathbb{E} |T_n|^p = \mathbb{E} T_n \varphi(T_n) = \frac{1}{n} \mathbb{E} m'_n(z) \varphi(T_n) + \mathbb{E} \widehat{T}_n \varphi(T_n).$$

Applying the result of Lemma C.5 we estimate the first term on the r.h.s. of the previous equation via

$$\frac{1}{n} \left| \mathbb{E} m'_n(z) \varphi(T_n) \right| \leq \frac{1}{nv} \mathbb{E}^{\frac{1}{p}} \operatorname{Im}^p m_n(z) \mathbb{E}^{\frac{p-1}{p}} |T_n|^p \leq \frac{\mathcal{A}(p)}{nv} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p. \quad (3.4)$$

It follows that

$$\mathbb{E} |T_n|^p \leq \left| \mathbb{E} \widehat{T}_n \varphi(T_n) \right| + \frac{\mathcal{A}(p)}{nv} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p. \quad (3.5)$$

To simplify all calculations we shall systematically apply the recursion inequality of Lemma B.4 which states that if $0 < q_1 \leq q_2 \leq \dots \leq q_k < p$ and $c_j, j = 0, \dots, k$ are positive numbers such that

$$x^p \leq c_0 + c_1 x^{q_1} + c_2 x^{q_2} + \dots + c_k x^{q_k}$$

then

$$x^p \leq \beta \left[c_0 + c_1^{\frac{p}{p-q_1}} + c_2^{\frac{p}{p-q_2}} + \dots + c_k^{\frac{p}{p-q_k}} \right],$$

where

$$\beta := \prod_{\nu=1}^k 2^{\frac{p}{p-q_\nu}} \leq 2^{\frac{kp}{p-q_k}}.$$

We now apply Lemma B.4 to inequality (3.5) with $c_0 = |\mathbb{E} \widehat{T}_n \varphi(T_n)|$, $c_1 = \frac{\mathcal{A}(p)}{nv}$ and $q_1 = p - 1$, obtaining

$$\mathbb{E} |T_n|^p \leq C^p |\mathbb{E} \widehat{T}_n \varphi(T_n)| + \frac{C^p \mathcal{A}^p(p)}{(nv)^p}, \quad (3.6)$$

with some absolute constant $C > 0$. Now we consider the term $\mathbb{E} \widehat{T}_n \varphi(T_n)$. We split it into three parts with respect to $\varepsilon_{\nu j}$, $\nu = 1, 2, 3$, obtaining

$$\mathbb{E} \widehat{T}_n \varphi(T_n) = \frac{1}{n} \sum_{\nu=1}^3 \sum_{j=1}^n \mathbb{E} \varepsilon_{\nu j} \mathbf{R}_{jj} \varphi(T_n) = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.$$

We may rewrite \mathcal{A}_ν as a sum of two terms of the general form

$$\begin{aligned} \mathcal{A}_{\nu 1} &:= -\frac{1}{n} \mathbb{E} \sum_{j=1}^n \varepsilon_{\nu j} a_n^{(j)} \varphi(T_n), \\ \mathcal{A}_{\nu 2} &:= \frac{1}{n} \sum_{j=1}^n \mathbb{E} \varepsilon_{\nu j} [\mathbf{R}_{jj} + a_n^{(j)}] \varphi(T_n), \end{aligned}$$

where

$$a_n(z) = \frac{1}{z + m_n(z)} \quad \text{and} \quad a_n^{(j)}(z) = \frac{1}{z + m_n^{(j)}(z)}.$$

3.1. Bound for $\mathcal{A}_{\nu 1}$, $\nu = 1, 2, 3$. For $\nu = 1$ the bound is a direct application of Rosenthal's inequality. The estimates for $\nu = 2, 3$ are more involved.

3.1.1. Bound for \mathcal{A}_{11} . We decompose \mathcal{A}_{11} into a sum of two terms

$$\begin{aligned} \mathcal{B}_{11} &:= -\frac{1}{n} \mathbb{E} \sum_{j=1}^n \varepsilon_{1j} a_n \varphi(T_n), \\ \mathcal{B}_{12} &:= \frac{1}{n} \mathbb{E} \sum_{j=1}^n \varepsilon_{1j} [a_n - a_n^{(j)}] \varphi(T_n), \end{aligned}$$

From Hölder's inequality and Lemma A.4 with $q = 1$ it follows that

$$|\mathcal{B}_{11}| \leq C \mathbb{E}^{\frac{1}{2p}} \left| \frac{1}{n} \sum_{j=1}^n \varepsilon_{1j} \right|^{2p} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p \leq \frac{Cp}{n} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p. \quad (3.7)$$

To estimate \mathcal{B}_{12} we first need to bound $[a_n - a_n^{(j)}]$. Applying the Schur complement formula (see, for example, [18][Lemma 7.23] or [19][Lemma 3.3]) we get

$$\mathrm{Tr} \mathbf{R} - \mathrm{Tr} \mathbf{R}^{(j)} = \left(1 + \frac{1}{n} \sum_{k,l \in \mathbb{T}_j} X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{kl} \right) \mathbf{R}_{jj} = \mathbf{R}_{jj}^{-1} \frac{d\mathbf{R}_{jj}}{dz}. \quad (3.8)$$

This equation and Lemma C.4[Inequality (C.8)] yield that

$$|a_n - a_n^{(j)}| \leq |m_n - m_n^{(j)}| |a_n a_n^{(j)}| \leq \frac{1}{nv} \operatorname{Im} \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{-1} |a_n a_n^{(j)}|.$$

We have applying Hölder's inequality and Lemma A.4 with $q = 2$

$$\begin{aligned} |\mathcal{B}_{12}| &\leq \frac{1}{nv} \mathbb{E} \left| \frac{1}{n} \sum_{j=1}^n \varepsilon_{1j}^2 \right|^{\frac{1}{2}} \left| \frac{1}{n} \sum_{j=1}^n \operatorname{Im}^2 \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{-2} |a_n a_n^{(j)}|^2 \right|^{\frac{1}{2}} |T_n|^{p-1} \\ &\leq \frac{Cp^{\frac{1}{2}}}{n^{\frac{3}{2}v}} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p \mathbb{E}^{\frac{1}{2p}} \left| \frac{1}{n} \sum_{j=1}^n \operatorname{Im}^2 \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{-2} |a_n a_n^{(j)}|^2 \right|^p. \end{aligned}$$

In view of the definition of $\mathcal{A}(q)$ (see (2.8)) and again Hölder's inequality we obtain

$$|\mathcal{B}_{12}| \leq \frac{Cp^{\frac{1}{2}}}{n^{\frac{3}{2}v}} \mathcal{A}(4p) \mathbb{E}^{\frac{p-1}{p}} |T_n|^p.$$

Finally

$$\mathcal{A}_{11} \leq \left[\frac{Cp}{n} + \frac{Cp^{\frac{1}{2}}}{n^{\frac{3}{2}v}} \mathcal{A}(4p) \right] \mathbb{E}^{\frac{p-1}{p}} |T_n|^p. \quad (3.9)$$

3.1.2. *Bound for \mathcal{A}_{21} and \mathcal{A}_{31} .* Let us introduce the following notation

$$\tilde{T}_n^{(j)} := \mathbb{E}(T_n | \mathfrak{M}^{(j)}),$$

where $\mathfrak{M}^{(j)} := \sigma\{X_{lk}, l, k \in \mathbb{T}_j\}$. Since $\mathbb{E}(\varepsilon_{\nu j} | \mathfrak{M}^{(j)}) = 0$ for $\nu = 2, 3$ it is easy to see that

$$\mathcal{A}_{\nu 1} = \frac{1}{n} \sum_{j=1}^n \mathbb{E} \varepsilon_{\nu j} a_n^{(j)} [\varphi(T_n) - \varphi(\tilde{T}_n^{(j)})].$$

Applying the Newton-Leibniz formula (see Lemma B.2 for details) and the simple inequality $(x + y)^q \leq ex^q + (q + 1)^q y^q$, $x, y > 0$, $q \geq 1$ we get with $q = p - 2$

$$|\mathcal{A}_{\nu 1}| \leq \mathcal{B}_{\nu 1} + \mathcal{B}_{\nu 2},$$

where

$$\mathcal{B}_{\nu 1} := \frac{ep}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j} a_n^{(j)}| |T_n - \tilde{T}_n^{(j)}| |\tilde{T}_n^{(j)}|^{p-2}, \quad (3.10)$$

$$\mathcal{B}_{\nu 2} := \frac{p^{p-2}}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j} a_n^{(j)}| |T_n - \tilde{T}_n^{(j)}|^{p-1}. \quad (3.11)$$

Since the derivation of estimates of these terms are rather involved we need to split them into several subsections.

Representation of $T_n - \tilde{T}_n^{(j)}$. By definition we may write the following representation

$$T_n - T_n^{(j)} = (\Lambda_n - \Lambda_n^{(j)})(b(z) + 2\Lambda_n^{(j)}) + (\Lambda_n - \Lambda_n^{(j)})^2,$$

where in our notations $T_n^{(j)}$ is T_n with the matrix \mathbf{X} replaced by its corresponding submatrix. Let us denote

$$K^{(j)} := K^{(j)}(z) := b(z) + 2\Lambda_n^{(j)}.$$

Since

$$T_n - \tilde{T}_n^{(j)} = T_n - T_n^{(j)} - \mathbb{E}(T_n - T_n^{(j)} | \mathfrak{M}^{(j)})$$

we obtain the inequality

$$|T_n - \tilde{T}_n^{(j)}| \leq |K^{(j)}| |\Lambda_n - \tilde{\Lambda}_n^{(j)}| + |\Lambda_n - \Lambda_n^{(j)}|^2 + \mathbb{E}(|\Lambda_n - \Lambda_n^{(j)}|^2 | \mathfrak{M}^{(j)}), \quad (3.12)$$

where $\tilde{\Lambda}_n^{(j)} := \mathbb{E}(\Lambda_n | \mathfrak{M}^{(j)})$. The equation (3.8) and Lemma C.4[Inequality (C.8)] yield that

$$|\Lambda_n - \Lambda_n^{(j)}| \leq \frac{1}{nv} \frac{\text{Im} \mathbf{R}_{jj}}{|\mathbf{R}_{jj}|}. \quad (3.13)$$

For simplicity we denote the quadratic form in (3.8) by

$$\eta_j := \frac{1}{n} \sum_{k,l \in \mathbb{T}_j} X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{kl} \quad (3.14)$$

and rewrite it as a sum of the three terms

$$\eta_j = \eta_{0j} + \eta_{1j} + \eta_{2j},$$

where

$$\eta_{0j} := \frac{1}{n} \sum_{k \in \mathbb{T}_j} [(\mathbf{R}^{(j)})^2]_{kk} = (m_n^{(j)}(z))', \quad \eta_{1j} := \frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{kl}, \quad (3.15)$$

$$\eta_{2j} := \frac{1}{n} \sum_{k \in \mathbb{T}_j} [X_{jk}^2 - 1] [(\mathbf{R}^{(j)})^2]_{kk}. \quad (3.16)$$

It follows from (3.8) and $\Lambda_n - \tilde{\Lambda}_n^{(j)} = \Lambda_n - \Lambda_n^{(j)} - \mathbb{E}(\Lambda_n - \Lambda_n^{(j)} | \mathfrak{M}^{(j)})$ that

$$\Lambda_n - \tilde{\Lambda}_n^{(j)} = \frac{1 + \eta_{j0}}{n} [\mathbf{R}_{jj} - \mathbb{E}(\mathbf{R}_{jj} | \mathfrak{M}^{(j)})] + \frac{\eta_{1j} + \eta_{2j}}{n} \mathbf{R}_{jj} - \frac{1}{n} \mathbb{E}((\eta_{j1} + \eta_{j2}) \mathbf{R}_{jj} | \mathfrak{M}^{(j)}).$$

It is easy to see that

$$|\mathbf{R}_{jj} - \mathbb{E}(\mathbf{R}_{jj} | \mathfrak{M}^{(j)})| \leq |a_n^{(j)}| (|\hat{\varepsilon}_j \mathbf{R}_{jj}| + \mathbb{E}(|\hat{\varepsilon}_j \mathbf{R}_{jj}| | \mathfrak{M}^{(j)})),$$

where $\hat{\varepsilon}_j = \sum_1^3 \varepsilon_{\nu j}$. Applying this inequality and Lemma C.5 we may write

$$\begin{aligned} |\Lambda_n - \tilde{\Lambda}_n^{(j)}| &\leq \frac{1 + v^{-1} \text{Im} m_n^{(j)}(z)}{n} |a_n^{(j)}| (|\hat{\varepsilon}_j \mathbf{R}_{jj}| + \mathbb{E}(|\hat{\varepsilon}_j \mathbf{R}_{jj}| | \mathfrak{M}^{(j)})) \\ &\quad + \frac{|\eta_{1j} + \eta_{2j}|}{n} |\mathbf{R}_{jj}| + \frac{1}{n} \mathbb{E}(|\eta_{j1} + \eta_{j2}| |\mathbf{R}_{jj}| | \mathfrak{M}^{(j)}). \end{aligned}$$

Let us introduce the following quantity

$$\beta := \frac{1}{2\alpha}, \quad (3.17)$$

which will be used many times during the proof. It is easy to see that $\beta > 1$. Denote by ζ an arbitrary random variable such that $\mathbb{E}|\zeta|^{\frac{4\beta}{\beta-1}}$ exists. Then

$$\mathbb{E}(\varepsilon_{\nu j}|T_n - \tilde{T}_n^{(j)}||\zeta||\mathfrak{M}^{(j)}) \leq |K^{(j)}|[B_1 + \dots + B_6] + B_7 + B_8,$$

where

$$\begin{aligned} B_1 &:= \frac{1 + v^{-1} \operatorname{Im} m_n^{(j)}(z)}{n} |a_n^{(j)}| \mathbb{E}(|\varepsilon_{\nu j} \hat{\varepsilon}_j \mathbf{R}_{jj} \zeta| | \mathfrak{M}^{(j)}), \\ B_2 &:= \frac{1 + v^{-1} \operatorname{Im} m_n^{(j)}(z)}{n} |a_n^{(j)}| \mathbb{E}(|\hat{\varepsilon}_j \mathbf{R}_{jj}| | \mathfrak{M}^{(j)}) \mathbb{E}(|\varepsilon_{\nu j} \zeta| | \mathfrak{M}^{(j)}), \\ B_3 &:= \frac{1}{n} \mathbb{E}(|\varepsilon_{\nu j} \eta_{1j}| | \mathbf{R}_{jj} \zeta| | \mathfrak{M}^{(j)}), \\ B_4 &:= \frac{1}{n} \mathbb{E}(|\varepsilon_{\nu j} \eta_{2j}| | \mathbf{R}_{jj} \zeta| | \mathfrak{M}^{(j)}), \\ B_5 &:= \frac{1}{n} \mathbb{E}(|\eta_{1j}| | \mathbf{R}_{jj} | | \mathfrak{M}^{(j)}) \mathbb{E}(|\varepsilon_{\nu j} \zeta| | \mathfrak{M}^{(j)}), \\ B_6 &:= \frac{1}{n} \mathbb{E}(|\eta_{2j}| | \mathbf{R}_{jj} | | \mathfrak{M}^{(j)}) \mathbb{E}(|\varepsilon_{\nu j} \zeta| | \mathfrak{M}^{(j)}), \\ B_7 &:= \frac{1}{n^2 v^2} \mathbb{E}(|\varepsilon_{\nu j}| \operatorname{Im}^2 \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{-2} |\zeta| | \mathfrak{M}^{(j)}), \\ B_8 &:= \frac{1}{n^2 v^2} \mathbb{E}(|\varepsilon_{\nu j} \zeta| | \mathfrak{M}^{(j)}) \mathbb{E}(|\operatorname{Im}^2 \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{-2} | \mathfrak{M}^{(j)}), \end{aligned}$$

where B_7 and B_8 are the result of an application of (3.13). Let us consider the first term B_1 . By definition $B_1 \leq B_{11} + B_{12} + B_{13}$, where

$$B_{1\mu} := \frac{1 + v^{-1} \operatorname{Im} m_n^{(j)}(z)}{n} |a_n^{(j)}| \mathbb{E}(|\varepsilon_{\mu j}|^2 |\mathbf{R}_{jj} \zeta| | \mathfrak{M}^{(j)}), \quad \mu = 1, 2, 3.$$

For $\mu = 1$ we may apply Hölder's inequality, Lemma A.3 with $p = 4$ and obtain

$$B_{11} \leq \frac{1 + v^{-1} \operatorname{Im} m_n^{(j)}(z)}{n^2} |a_n^{(j)}| \mathbb{E}^{\frac{1}{2}}(|\mathbf{R}_{jj} \zeta|^2 | \mathfrak{M}^{(j)}). \quad (3.18)$$

For $\mu = 2$ we may proceed in a similar way and apply Lemma A.6 to get

$$B_{12} \leq \frac{\operatorname{Im} m_n^{(j)} + v^{-1} \operatorname{Im}^2 m_n^{(j)}(z)}{n^2 v} |a_n^{(j)}| \mathbb{E}^{\frac{1}{2}}(|\mathbf{R}_{jj} \zeta|^2 | \mathfrak{M}^{(j)}). \quad (3.19)$$

Applying Lemma A.8 for $\mu = 3$ we obtain

$$B_{13} \leq \frac{1 + v^{-1} \operatorname{Im} m_n^{(j)}(z)}{n^2} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{\beta}} |a_n^{(j)}| \mathbb{E}^{\frac{\beta-1}{\beta}}(|\mathbf{R}_{jj} \zeta|^{\frac{\beta}{\beta-1}} | \mathfrak{M}^{(j)}). \quad (3.20)$$

Combining inequalities (3.18)–(3.20) we get the following bound for B_1

$$B_1 \leq \frac{1 + v^{-1} \operatorname{Im} m_n^{(j)}(z)}{n^2} \left(1 + \frac{\operatorname{Im} m_n^{(j)}(z)}{v} + \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{\beta}} \right) \\ \times |a_n^{(j)}| \mathbb{E}^{\frac{\beta-1}{2\beta}} (|\mathbf{R}_{jj}|^{\frac{2\beta}{\beta-1}} |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{\beta-1}{2\beta}} (|\zeta|^{\frac{2\beta}{\beta-1}} |\mathfrak{M}^{(j)}|).$$

The term B_2 may be estimated by the same arguments as B_1 . We write

$$B_2 \leq \frac{1 + v^{-1} \operatorname{Im} m_n^{(j)}(z)}{n^2} \left(1 + \frac{\operatorname{Im} m_n^{(j)}(z)}{v} + \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{\beta}} \right) \\ \times |a_n^{(j)}| \mathbb{E}^{\frac{\beta-1}{2\beta}} (|\mathbf{R}_{jj}|^{\frac{2\beta}{\beta-1}} |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{\beta-1}{2\beta}} (|\zeta|^{\frac{2\beta}{\beta-1}} |\mathfrak{M}^{(j)}|).$$

We now consider the term B_3 . If $\nu = 2$ we apply Hölder's inequality, Lemmas A.10, A.6, obtaining

$$B_3 \leq \frac{1}{n} \mathbb{E}^{\frac{1}{4}} (\varepsilon_{2j}^4 |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{1}{2}} (\eta_{1j}^2 |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{1}{4}} (|\mathbf{R}_{jj} \zeta|^4 |\mathfrak{M}^{(j)}|) \\ \leq \frac{\operatorname{Im}^{\frac{1}{2}} m_n^{(j)}}{n^2 v^{\frac{1}{2}}} \left(\frac{1}{n} \operatorname{Tr} |\mathbf{R}^{(j)}|^4 \right)^{\frac{1}{2}} \mathbb{E}^{\frac{1}{8}} (|\mathbf{R}_{jj}|^8 |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{1}{8}} (|\zeta|^8 |\mathfrak{M}^{(j)}|).$$

By the same reasoning an application of Lemma A.8 with $\nu = 3$ will lead us to

$$B_3 \leq \frac{1}{n^2} \left(\frac{1}{n} \operatorname{Tr} |\mathbf{R}^{(j)}|^4 \right)^{\frac{1}{2}} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\mathbf{R}_{jj}|^{\frac{4\beta}{\beta-1}} |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\zeta|^{\frac{4\beta}{\beta-1}} |\mathfrak{M}^{(j)}|).$$

Combining the last two inequalities we get the following general estimate for B_3

$$B_3 \leq \frac{1}{n^2} \left(\frac{1}{n} \operatorname{Tr} |\mathbf{R}^{(j)}|^4 \right)^{\frac{1}{2}} \left[\left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} + \frac{\operatorname{Im}^{\frac{1}{2}} m_n^{(j)}(z)}{v^{\frac{1}{2}}} \right] \\ \times \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\mathbf{R}_{jj}|^{\frac{4\beta}{\beta-1}} |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\zeta|^{\frac{4\beta}{\beta-1}} |\mathfrak{M}^{(j)}|).$$

Let us consider the term B_4 . For $\nu = 2$ we apply Lemmas A.12, A.6 and obtain

$$B_4 \leq \frac{1}{n} \mathbb{E}^{\frac{1}{4}} (\varepsilon_{2j}^4 |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{1}{2}} (\eta_{2j}^2 |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{1}{4}} (|\mathbf{R}_{jj} \zeta|^4 |\mathfrak{M}^{(j)}|) \\ \leq \frac{\operatorname{Im}^{\frac{1}{2}} m_n^{(j)}}{n^2 v^{\frac{3}{2}}} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} \operatorname{Im}^2 \mathbf{R}_{kk}^{(j)} \right)^{\frac{1}{2}} \mathbb{E}^{\frac{1}{8}} (|\mathbf{R}_{jj}|^8 |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{1}{8}} (|\zeta|^8 |\mathfrak{M}^{(j)}|).$$

By the same arguments as before we get the following estimate for the case $\nu = 3$

$$B_4 \leq \frac{1}{n^2 v} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} \text{Im}^2 \mathbf{R}_{kk}^{(j)} \right)^{\frac{1}{2}} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\mathbf{R}_{jj}|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\zeta|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}).$$

Finally, the bound for B_4 has the form

$$B_4 \leq \frac{1}{n^2 v} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} \text{Im}^2 \mathbf{R}_{kk}^{(j)} \right)^{\frac{1}{2}} \left[\frac{\text{Im}^{\frac{1}{2}} m_n^{(j)}}{v^{\frac{1}{2}}} + \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} \right] \\ \times \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\mathbf{R}_{jj}|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\zeta|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}).$$

Obviously, the estimates of B_5 and B_6 are similar to those for B_3 and B_4 respectively. The same arguments yield that B_7 may be estimated as follows

$$B_7 \leq \frac{1}{n^{\frac{5}{2}} v^2} \left[\left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} + \frac{\text{Im}^{\frac{1}{2}} m_n^{(j)}(z)}{v^{\frac{1}{2}}} \right] \\ \times \mathbb{E}^{\frac{1}{4}} (\text{Im}^8 \mathbf{R}_{jj} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{1}{8}} (|\mathbf{R}_{jj}|^{-16} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{1}{8}} (|\zeta|^8 | \mathfrak{M}^{(j)}).$$

Since bound for B_8 will be an analog of B_7 , it is omitted. We finally collect all bounds for $B_\mu, \mu = 1, \dots, 7$ and write

$$\mathbb{E}(\varepsilon_{\nu j} | T_n - \tilde{T}_n^{(j)} | |\zeta| | \mathfrak{M}^{(j)}) \leq \left[|K^{(j)}| [|a_n^{(j)}| \Gamma_1 + \Gamma_2] \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\mathbf{R}_{jj}|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \right. \\ \left. + \Gamma_3 \mathbb{E}^{\frac{1}{8}} (|\mathbf{R}_{jj}|^{-16} | \mathfrak{M}^{(j)}) \right] \mathbb{E}^{\frac{\beta-1}{4\beta}} (|\zeta|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}), \quad (3.21)$$

where we denoted

$$\Gamma_1 := \frac{1 + v^{-1} \text{Im} m_n^{(j)}(z)}{n^2} \left[1 + \frac{\text{Im} m_n^{(j)}(z)}{v} + \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} \right], \\ \Gamma_2 := \frac{1}{n^2} \left[\left(\frac{1}{n} \text{Tr} |\mathbf{R}^{(j)}|^4 \right)^{\frac{1}{2}} + \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} \text{Im}^2 \mathbf{R}_{kk}^{(j)} \right)^{\frac{1}{2}} \right] \left[\left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} + \frac{\text{Im}^{\frac{1}{2}} m_n^{(j)}(z)}{v^{\frac{1}{2}}} \right], \\ \Gamma_3 := \frac{1}{n^{\frac{5}{2}} v^2} \left[\left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^{2\beta} \right)^{\frac{1}{2\beta}} + \frac{\text{Im}^{\frac{1}{2}} m_n^{(j)}}{v^{\frac{1}{2}}} \right] \mathbb{E}^{\frac{1}{4}} (\text{Im}^8 \mathbf{R}_{jj} | \mathfrak{M}^{(j)}).$$

We may now estimate the terms $\mathcal{B}_{\nu 1}$ and $\mathcal{B}_{\nu 2}$, defined in (3.10) and (3.11), by applying the representation (3.21) and choosing appropriate random variables ζ .

Bound for $\mathcal{B}_{\nu 1}$. Recall that

$$\mathcal{B}_{\nu 1} := \frac{ep}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j} a_n^{(j)}| |T_n - \tilde{T}_n^{(j)}| |\tilde{T}_n^{(j)}|^{p-2}.$$

Taking $\zeta = 1$ in (3.21) we get

$$\begin{aligned} \mathcal{B}_{\nu 1} &= \frac{ep}{n} \sum_{j=1}^n \mathbb{E} |\tilde{T}_n^{(j)}|^{p-2} \mathbb{E} (|\varepsilon_{\nu j}| |T_n - \tilde{T}_n^{(j)}| | \mathfrak{M}^{(j)}) \\ &\leq \frac{ep}{n} \sum_{j=1}^n \mathbb{E} |\tilde{T}_n^{(j)}|^{p-2} |K^{(j)}| \Gamma_1 \mathbb{E}^{\frac{\beta-1}{4\beta}} (|a_n^{(j)}|^{\frac{8\beta}{\beta-1}} |\mathbf{R}_{jj}|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \\ &\quad + \frac{ep}{n} \sum_{j=1}^n \mathbb{E} |\tilde{T}_n^{(j)}|^{p-2} |K^{(j)}| \Gamma_2 \mathbb{E}^{\frac{\beta-1}{4\beta}} (|a_n^{(j)}|^{\frac{4\beta}{\beta-1}} |\mathbf{R}_{jj}|^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \\ &\quad + \frac{ep}{n} \sum_{j=1}^n \mathbb{E} |\tilde{T}_n^{(j)}|^{p-2} \Gamma_3 \mathbb{E}^{\frac{\beta-1}{4\beta}} (|a_n^{(j)}|^{\frac{4\beta}{\beta-1}} |\mathbf{R}_{jj}|^{-\frac{8\beta}{\beta-1}} | \mathfrak{M}^{(j)}) =: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3. \end{aligned}$$

Applying Hölder's inequality we obtain

$$\mathcal{T}_1 \leq \frac{ep}{n} \sum_{j=1}^n \mathbb{E}^{\frac{p-2}{p}} |T_n|^p \mathbb{E}^{\frac{1}{p}} \Gamma_1^p \mathbb{E}^{\frac{1}{2p}} |K^{(j)}|^{2p}. \quad (3.22)$$

To finish the estimate of \mathcal{T}_1 it remains to bound $\mathbb{E}^{\frac{1}{2p}} |K^{(j)}|^{2p}$ and $\mathbb{E}^{\frac{1}{p}} \Gamma_1^p$. Recall that

$$K^{(j)} = b(z) + 2\Lambda_n^{(j)} = b(z) + 2(\Lambda_n^{(j)} - \Lambda_n) + 2\Lambda_n.$$

We claim that

$$\mathbb{E}^{\frac{1}{2p}} |K^{(j)}|^{2p} \leq C|b(z)| + C \mathbb{E}^{\frac{1}{2p}} |T_n|^p + \frac{C\mathcal{A}(4p)}{nv}. \quad (3.23)$$

Indeed, applying (3.13) we obtain

$$\mathbb{E}^{\frac{1}{2p}} |K^{(j)}|^{2p} \leq 2 \mathbb{E}^{\frac{1}{2p}} |b(z) + 2\Lambda_n|^{2p} + \frac{2\mathcal{A}(4p)}{nv}.$$

If $|b(z)| \geq |\Lambda_n|/2$ then (3.23) is obvious. On the other hand, if the opposite inequality holds, we find

$$|\Lambda_n| \leq \frac{|\sqrt{b^2(z) + 4T_n} - b(z)|}{2} \leq |b(z)| + |T_n|^{\frac{1}{2}}.$$

Consequently, $|\Lambda_n| \leq 2|T_n|^{\frac{1}{2}}$ and we conclude (3.23). Calculating the p -th moment of Γ_1 we get

$$\begin{aligned} \mathbb{E}^{\frac{1}{p}} \Gamma_1^p &\leq C \left(\frac{1}{n^2} + \frac{\mathbb{E}^{\frac{1}{p}} \text{Im}^{2p} m_n^{(j)}(z)}{(nv)^2} + \frac{\mathbb{E}^{\frac{1}{2p}} \text{Im}^{2p} m_n^{(j)}(z)}{n^2 v} \right) \\ &\leq C \left(\frac{1}{n^2} + \frac{\mathcal{A}^2(2p)}{(nv)^2} + \frac{\mathcal{A}(2p)}{n^2 v} \right) \leq C \left(\frac{1}{n^2} + \frac{\mathcal{A}^2(2p)}{(nv)^2} \right), \end{aligned} \quad (3.24)$$

where we have applied the well known Young inequality valid for all $a, b \geq 0$ and positive s, t such that $\frac{1}{s} + \frac{1}{t} = 1$,

$$ab \leq \frac{a^s}{s} + \frac{b^t}{t}. \quad (3.25)$$

The estimate (3.24) may be simplified further. Indeed, from Lemma C.3 we conclude that $v \leq \text{Im } \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{-2}$ for all $j \in \mathbb{T}$ and arrive at the following inequality

$$v \leq C \mathbb{E}^{\frac{1}{2p}} \text{Im}^{2p} \mathbf{R}_{jj} \leq C \mathcal{A}(2p). \quad (3.26)$$

The inequalities (3.23), (3.24) and (3.26) together imply that

$$\mathcal{T}_1 \leq C \mathbb{E}^{\frac{p-2}{p}} |T_n|^p \left[|b(z)| + \mathbb{E}^{\frac{1}{2p}} |T_n|^p + \frac{\mathcal{A}(4p)}{nv} \right] \frac{\mathcal{A}^2(2p)}{(nv)^2}. \quad (3.27)$$

Analogously to (3.22) we derive a bound for the term \mathcal{T}_2

$$\mathcal{T}_2 \leq \frac{ep}{n} \sum_{j=1}^n \mathbb{E}^{\frac{p-2}{p}} |T_n|^p \mathbb{E}^{\frac{1}{p}} \Gamma_2^p \mathbb{E}^{\frac{1}{2p}} |K^{(j)}|^{2p}. \quad (3.28)$$

Applying (3.25) and (3.26) the reader will have no difficulty in showing that

$$\mathbb{E}^{\frac{1}{p}} \Gamma_2^p \leq C \left(\frac{\mathcal{A}^{\frac{1}{2}}(2p)}{n^2 v^{\frac{3}{2}}} + \frac{\mathcal{A}(2p)}{n^2 v^2} + \frac{\mathcal{A}(2p)}{n^2 v} + \frac{\mathcal{A}^{\frac{3}{2}}(2p)}{n^2 v^{\frac{3}{2}}} \right) \leq \frac{C \mathcal{A}(2p)}{n^2 v^2}.$$

The last inequality and (3.28) imply the following bound

$$\mathcal{T}_2 \leq C \mathbb{E}^{\frac{p-2}{p}} |T_n|^p \left[|b(z)| + \mathbb{E}^{\frac{1}{2p}} |T_n|^p + \frac{\mathcal{A}(4p)}{nv} \right] \frac{\mathcal{A}(2p)}{n^2 v^2}.$$

By the same reasoning as before

$$\mathcal{T}_3 \leq \mathbb{E}^{\frac{p-2}{p}} |T_n|^p \left[\frac{\mathcal{A}^2(4p)}{n^{\frac{5}{2}} v^2} + \frac{\mathcal{A}^{\frac{5}{2}}(4p)}{(nv)^{\frac{5}{2}}} \right] \leq \frac{C \mathcal{A}^2(4p) \mathbb{E}^{\frac{p-2}{p}} |T_n|^p}{n^2 v^2}.$$

Bound for $\mathcal{B}_{\nu 2}$. Recall that

$$\mathcal{B}_{\nu 2} := \frac{p^{p-2}}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j} a_n^{(j)}| |T_n - \tilde{T}_n^{(j)}|^{p-1}.$$

Taking in (3.21) $\zeta = |T_n - \tilde{T}_n^{(j)}|^{p-2}$ we get

$$\begin{aligned}
\mathcal{B}_{\nu 2} &= \frac{p^{p-2}}{n} \sum_{j=1}^n \mathbb{E} |a_n^{(j)}| \mathbb{E} (|\varepsilon_{\nu j}| |T_n - \tilde{T}_n^{(j)}|^{p-1} | \mathfrak{M}^{(j)}) \\
&\leq \frac{p^{p-2}}{n} \sum_{j=1}^n \mathbb{E} |K^{(j)}| \Gamma_1 \mathbb{E}^{\frac{\beta-1}{4\beta}} (|a_n^{(j)}|^{\frac{8\beta}{\beta-1}} | \mathbf{R}_{jj} |^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{\beta-1}{4\beta}} (|T_n - \tilde{T}_n^{(j)}|^{\frac{4\beta(p-2)}{\beta-1}} | \mathfrak{M}^{(j)}) \\
&\quad + \frac{p^{p-2}}{n} \sum_{j=1}^n \mathbb{E} |K^{(j)}| \Gamma_2 \mathbb{E}^{\frac{\beta-1}{4\beta}} (|a_n^{(j)}|^{\frac{4\beta}{\beta-1}} | \mathbf{R}_{jj} |^{\frac{4\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{\beta-1}{4\beta}} (|T_n - \tilde{T}_n^{(j)}|^{\frac{4\beta(p-2)}{\beta-1}} | \mathfrak{M}^{(j)}) \\
&\quad + \frac{p^{p-2}}{n} \sum_{j=1}^n \mathbb{E} \Gamma_3 \mathbb{E}^{\frac{\beta-1}{4\beta}} (|a_n^{(j)}|^{\frac{4\beta}{\beta-1}} | \mathbf{R}_{jj} |^{-\frac{8\beta}{\beta-1}} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{\beta-1}{4\beta}} (|T_n - \tilde{T}_n^{(j)}|^{\frac{4\beta(p-2)}{\beta-1}} | \mathfrak{M}^{(j)}) \\
&=: \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6.
\end{aligned}$$

It follows from (3.12) that

$$\begin{aligned}
\mathbb{E}^{\frac{\beta-1}{4\beta}} (|T_n - \tilde{T}_n^{(j)}|^{\frac{4\beta(p-2)}{\beta-1}} | \mathfrak{M}^{(j)}) &\leq \frac{C^p |K^{(j)}|^{p-2}}{(nv)^{p-2}} \mathbb{E}^{\frac{\beta-1}{8\beta}} (\text{Im}^{\frac{8\beta(p-2)}{\beta-1}} \mathbf{R}_{jj} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{\beta-1}{8\beta}} (\mathbf{R}_{jj}^{\frac{-8\beta(p-2)}{\beta-1}} | \mathfrak{M}^{(j)}) \\
&\quad + \frac{C^p}{(nv)^{2(p-2)}} \mathbb{E}^{\frac{\beta-1}{8\beta}} (\text{Im}^{\frac{16\beta(p-2)}{\beta-1}} \mathbf{R}_{jj} | \mathfrak{M}^{(j)}) \mathbb{E}^{\frac{\beta-1}{8\beta}} (\mathbf{R}_{jj}^{\frac{-16\beta(p-2)}{\beta-1}} | \mathfrak{M}^{(j)}).
\end{aligned}$$

Hence we get

$$\begin{aligned}
\mathcal{T}_4 &\leq \frac{C^p p^{p-2}}{(nv)^{p-2} n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^{2(p-1)} \mathbb{E}^{\frac{1}{4}} \Gamma_1^4 \mathbb{E}^{\frac{\beta-1}{8\beta}} \text{Im}^{\frac{8\beta(p-2)}{\beta-1}} \mathbf{R}_{jj} \\
&\quad + \frac{C^p p^{p-2}}{(nv)^{2(p-2)} n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^2 \mathbb{E}^{\frac{1}{4}} \Gamma_1^4 \mathbb{E}^{\frac{\beta-1}{8\beta}} \text{Im}^{\frac{16\beta(p-2)}{\beta-1}} \mathbf{R}_{jj} \\
&\leq \frac{C^p p^{p-2}}{(nv)^{p-2} n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^{2(p-1)} \mathbb{E}^{\frac{1}{4}} \Gamma_1^4 \mathcal{A}^{2(p-2)}(\kappa p) \\
&\quad + \frac{C^p p^{p-2}}{(nv)^{2(p-2)} n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^2 \mathbb{E}^{\frac{1}{4}} \Gamma_1^4 \mathcal{A}^{2(p-2)}(\kappa p),
\end{aligned}$$

where (as introduced in (2.9))

$$\kappa = \frac{16\beta}{\beta-1}.$$

Similarly as in the previous bounds of \mathcal{T}_1 we get

$$\begin{aligned}
\mathcal{T}_4 &\leq \frac{C^p p^{p-2} \mathcal{A}^2(2p) \mathcal{A}^{p-2}(\kappa p)}{(nv)^2 (nv)^{p-2}} \left[|b(z)|^{p-1} + \mathbb{E}^{\frac{p-1}{2p}} |T_n|^p + \frac{\mathcal{A}^{p-1}(4p)}{(nv)^{p-1}} \right] \\
&\quad + \frac{C^p p^{p-2} \mathcal{A}^2(2p) \mathcal{A}^{2(p-2)}(\kappa p)}{(nv)^2 (nv)^{2(p-2)}} \left[|b(z)| + \mathbb{E}^{\frac{1}{2p}} |T_n|^p + \frac{\mathcal{A}(4p)}{nv} \right].
\end{aligned}$$

We may now apply inequality (3.25) and obtain

$$\begin{aligned} \mathcal{T}_4 &\leq \frac{C^p p^p |b(z)|^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \frac{C^p |b(z)|^{\frac{p}{2}} \mathcal{A}^p(2p)}{(nv)^p} + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \mathbb{E}^{\frac{1}{2}} |T_n|^p \\ &\quad + \frac{C^p \mathcal{A}^p(2p)}{(nv)^p} \mathbb{E}^{\frac{1}{4}} |T_n|^p + \frac{C^p p^p \mathcal{A}^{2p}(\kappa p)}{(nv)^{2p}} + \frac{C^p \mathcal{A}^{\frac{3p}{2}}(2p)}{(nv)^{\frac{3p}{2}}}. \end{aligned}$$

The last inequality may be simplified as follows

$$\mathcal{T}_4 \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \mathbb{E}^{\frac{1}{2}} |T_n|^p + \frac{C^p \mathcal{A}^p(2p)}{(nv)^p} \mathbb{E}^{\frac{1}{4}} |T_n|^p.$$

By the same argument we obtain the estimate for \mathcal{T}_5

$$\begin{aligned} \mathcal{T}_5 &\leq \frac{C^p p^p |b(z)|^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \frac{C^p |b(z)|^{\frac{p}{2}} \mathcal{A}^{\frac{p}{2}}(2p)}{n^p v^p} + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \mathbb{E}^{\frac{1}{2}} |T_n|^p \\ &\quad + \frac{C^p \mathcal{A}^{\frac{p}{2}}(2p)}{n^p v^p} \mathbb{E}^{\frac{1}{4}} |T_n|^p + \frac{C^p p^p \mathcal{A}^{2p}(\kappa p)}{(nv)^{2p}} + \frac{C^p \mathcal{A}^p(2p)}{(nv)^{\frac{3p}{2}}}. \end{aligned}$$

Finally, the routine check that \mathcal{T}_6 may be estimated as follows is left to the reader

$$\mathcal{T}_6 \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \left[\frac{\mathcal{A}^p(4p)}{n^{\frac{5p}{4}} v^p} + \frac{\mathcal{A}^{\frac{5p}{4}}(4p)}{(nv)^{\frac{5p}{4}}} \right] \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p}.$$

3.1.3. Combining bounds for $\mathcal{A}_{\nu 1}$. We may now collect all bounds for $\mathcal{T}_\nu, \nu = 1, \dots, 6$ and (3.9) and insert them into (3.6) and apply Lemma B.4 to conclude

$$\mathbb{E} |T_n|^p \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \frac{C^p p^{2p}}{(nv)^{2p}} + \frac{C^p |b(z)|^{\frac{p}{2}} \mathcal{A}^{\frac{p}{2}}(\kappa p)}{(nv)^p} + C^p \sum_{\nu=1}^3 \mathcal{A}_{\nu 2}. \quad (3.29)$$

To finish the proof of the theorem it remains to estimate $\sum_{\nu=1}^3 \mathcal{A}_{\nu 2}$.

3.2. Bound for $\mathcal{A}_{\nu 2}, \nu = 1, 2, 3$. Recall that

$$\mathcal{A}_{\nu 2} := \frac{1}{n} \sum_{j=1}^n \mathbb{E} \varepsilon_{\nu j} [\mathbf{R}_{jj} + a_n^{(j)}] \varphi(T_n).$$

From the representation $\mathbf{R}_{jj} + a_n^{(j)} = \hat{\varepsilon}_j a_n^{(j)} \mathbf{R}_{jj}$ (see (2.2) with a_n, ε_j replaced by $a_n^{(j)}$ and $\hat{\varepsilon}_j$ respectively) it follows that

$$\mathcal{A}_{\nu 2} = \frac{1}{n} \sum_{j=1}^n \mathbb{E} \varepsilon_{\nu j} \hat{\varepsilon}_j a_n^{(j)} \mathbf{R}_{jj} \varphi(T_n).$$

Using the obvious inequality $2\varepsilon_{\nu j}\hat{\varepsilon}_j \leq \varepsilon_{1j}^2 + \varepsilon_{2j}^2 + \varepsilon_{3j}^2$, $\nu = 1, 2, 3$, we may bound $\mathcal{A}_{\nu 2}$, $\nu = 1, 2, 3$, by the sum of two terms (up to some constant) $\mathcal{N}_{\nu,1}$ and $\mathcal{N}_{\nu,2}$, $\nu = 1, 2, 3$, where

$$\begin{aligned}\mathcal{N}_{\nu 1} &:= \frac{e}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j}|^2 |a_n^{(j)} \mathbf{R}_{jj}| |\tilde{T}_n^{(j)}|^{p-1}, \\ \mathcal{N}_{\nu 2} &:= \frac{p^{p-1}}{n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j}|^2 |a_n^{(j)} \mathbf{R}_{jj}| |T_n - \tilde{T}_n^{(j)}|^{p-1}.\end{aligned}$$

Let us consider $\mathcal{N}_{\nu 1}$. Applying Hölder's inequality we obtain that

$$\mathcal{N}_{\nu 1} \leq \frac{C}{n} \sum_{j=1}^n \mathbb{E}^{\frac{p-1}{p}} |\tilde{T}_n^{(j)}|^p \mathbb{E}^{\frac{1}{2p}} \mathbb{E}^{\frac{2p}{\beta}} (|\varepsilon_{\nu j}|^{2\beta} |\mathfrak{M}^{(j)}|) \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2p}} \mathbb{E}^{\frac{2p}{\beta}} (|\varepsilon_{\nu j}|^{2\beta} |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{p-1}{p}} |T_n|^p.$$

Calculating conditional expectations and applying (3.26) we conclude

$$\begin{aligned}\sum_{\nu=1}^3 \mathcal{N}_{\nu 1} &\leq \frac{C}{n} \left[1 + \frac{\max_j \mathbb{E}^{\frac{1}{2p}} \operatorname{Im}^{2p} m_n^{(j)}(z)}{v} \right] \mathbb{E}^{\frac{p-1}{p}} |T_n|^p \\ &\leq \frac{C}{n} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p + \frac{\mathcal{A}(2p)}{nv} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p \leq \frac{C\mathcal{A}(2p)}{nv} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p.\end{aligned}$$

We now turn our attention to the second term $\mathcal{N}_{\nu 2}$. Applying (3.13) and (3.12) we may write

$$\begin{aligned}|T_n - \tilde{T}_n^{(j)}| &\leq \frac{|K^{(j)}| \operatorname{Im} \mathbf{R}_{jj}}{nv} + \frac{C \operatorname{Im}^2 \mathbf{R}_{jj}}{n^2 v^2 |\mathbf{R}_{jj}|^2} \\ &\quad + \frac{|K^{(j)}|}{nv} \mathbb{E} \left(\frac{\operatorname{Im} \mathbf{R}_{jj}}{|\mathbf{R}_{jj}|} \middle| \mathfrak{M}^{(j)} \right) + \frac{1}{n^2 v^2} \mathbb{E} \left(\frac{\operatorname{Im}^2 \mathbf{R}_{jj}}{|\mathbf{R}_{jj}|^2} \middle| \mathfrak{M}^{(j)} \right).\end{aligned}\tag{3.30}$$

Substitution of this inequality in $\mathcal{N}_{\nu 2}$ will give us

$$\begin{aligned}\mathcal{N}_{\nu 2} &\leq \frac{C^p p^{p-1}}{(nv)^{p-1} n} \sum_{j=1}^n \mathbb{E} |K^{(j)}|^{p-1} |\varepsilon_{\nu j}|^2 \operatorname{Im}^{p-1} \mathbf{R}_{jj} |a_n^{(j)}| |\mathbf{R}_{jj}|^{2-p} \\ &\quad + \frac{C^p p^{p-1}}{(nv)^{2(p-1)} n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j}|^2 \operatorname{Im}^{2(p-1)} \mathbf{R}_{jj} |a_n^{(j)}| |\mathbf{R}_{jj}|^{3-2p} \\ &\quad + \frac{C^p p^{p-1}}{(nv)^{p-1} n} \sum_{j=1}^n \mathbb{E} |K^{(j)}|^{p-1} |\varepsilon_{\nu j}|^2 \mathbb{E} (\operatorname{Im}^{p-1} \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{1-p} \middle| \mathfrak{M}^{(j)}) |a_n^{(j)}| |\mathbf{R}_{jj}| \\ &\quad + \frac{C^p p^{p-1}}{(nv)^{2(p-1)} n} \sum_{j=1}^n \mathbb{E} |\varepsilon_{\nu j}|^2 \mathbb{E} (\operatorname{Im}^{2(p-1)} \mathbf{R}_{jj} |\mathbf{R}_{jj}|^{2(p-1)} \middle| \mathfrak{M}^{(j)}) |a_n^{(j)}| |\mathbf{R}_{jj}| \\ &=: \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4.\end{aligned}$$

Let us consider the term \mathcal{L}_1 . We get

$$\begin{aligned} \mathcal{L}_1 &\leq \frac{C^p p^{p-1}}{(nv)^{p-1} n} \sum_{j=1}^n \mathbb{E} |K^{(j)}|^{p-1} \mathbb{E}^{\frac{1}{\beta}} (|\varepsilon_{\nu j}|^{2\beta} |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{\beta-1}{\beta}} (\text{Im}^{p-1} \mathbf{R}_{jj} |a_n^{(j)}| |\mathbf{R}_{jj}|^{2-p} |\mathfrak{M}^{(j)}|) \\ &\leq \frac{C^p p^{p-1}}{(nv)^{p-1} n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^{2(p-1)} \mathbb{E}^{\frac{1}{4}} \mathbb{E}^{\frac{4}{\beta}} (|\varepsilon_{\nu j}|^{2\beta} |\mathfrak{M}^{(j)}|) \mathbb{E}^{\frac{1}{8}} \mathbb{E}^{\frac{4(\beta-1)}{\beta}} (\text{Im}^{\frac{2\beta(p-1)}{\beta-1}} \mathbf{R}_{jj} |\mathfrak{M}^{(j)}|). \end{aligned}$$

We distinguish two cases. If $4(\beta-1)\beta^{-1} \leq 1$, then applying Lyapunov's inequality we obtain

$$\mathbb{E}^{\frac{1}{8}} \mathbb{E}^{\frac{4(\beta-1)}{\beta}} (\text{Im}^{\frac{2\beta(p-1)}{\beta-1}} \mathbf{R}_{jj} |\mathfrak{M}^{(j)}|) \leq \mathbb{E}^{\frac{(\beta-1)}{2\beta}} \text{Im}^{\frac{2\beta(p-1)}{\beta-1}} \mathbf{R}_{jj}.$$

In the opposite case use Jensen's inequality to get the following estimate

$$\mathbb{E}^{\frac{1}{8}} \mathbb{E}^{\frac{4(\beta-1)}{\beta}} (\text{Im}^{\frac{2\beta(p-1)}{\beta-1}} \mathbf{R}_{jj} |\mathfrak{M}^{(j)}|) \leq \mathbb{E}^{\frac{1}{8}} \text{Im}^{8(p-1)} \mathbf{R}_{jj}.$$

Both inequalities lead to the bound

$$\mathbb{E}^{\frac{1}{8}} \mathbb{E}^{\frac{4(\beta-1)}{\beta}} (\text{Im}^{\frac{2\beta(p-1)}{\beta-1}} \mathbf{R}_{jj} |\mathfrak{M}^{(j)}|) \leq \mathcal{A}^{p-1}(\kappa p).$$

Applying this inequality we arrive at a bound for \mathcal{L}_1

$$\mathcal{L}_1 \leq \frac{C^p p^{p-1}}{(nv)^{p-1} n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^{2(p-1)} \mathbb{E}^{\frac{1}{4}} \mathbb{E}^{\frac{4}{\beta}} (|\varepsilon_{\nu j}|^{2\beta} |\mathfrak{M}^{(j)}|) \mathcal{A}^{p-1}(\kappa p).$$

Lemmas A.6 and A.8 together imply that \mathcal{L}_1 is bounded by the sum of the following terms

$$\begin{aligned} \mathcal{L}_{11} &:= \frac{C^p p^{p-1}}{n^p v^{p-1} n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^{2(p-1)} \mathcal{A}^{p-1}(\kappa p), \\ \mathcal{L}_{12} &:= \frac{C^p p^{p-1}}{(nv)^p n} \sum_{j=1}^n \mathbb{E}^{\frac{1}{2}} |K^{(j)}|^{2(p-1)} \mathcal{A}^p(\kappa p). \end{aligned}$$

Since (see (3.23) for details)

$$\mathbb{E}^{\frac{1}{2}} |K^{(j)}|^{2(p-1)} \leq C \mathbb{E}^{\frac{1}{2}} |T_n|^{p-1} + |b(z)|^{p-1} + \frac{1}{(nv)^{p-1}}$$

we get

$$\mathcal{L}_{11} \leq \frac{C^p p^{p-1}}{n^p v^{p-1}} \mathbb{E}^{\frac{p-1}{2p}} |T_n|^p \mathcal{A}^{p-1}(\kappa p) + \frac{C^p p^{p-1} |b(z)|^{p-1}}{n^p v^{p-1}} \mathcal{A}^{p-1}(\kappa p) + \frac{C^p p^{p-1}}{n^{2p-1} v^{2(p-1)}}.$$

It remains to apply (3.25) and (3.26). Finally we obtain

$$\mathcal{L}_{11} \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \mathbb{E}^{\frac{1}{2}} |T_n|^p + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p}.$$

The term \mathcal{L}_{12} is estimated as follows

$$\begin{aligned} \mathcal{L}_{12} &\leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \mathbb{E}^{\frac{1}{2}} |T_n|^p + \frac{C^p p^p |b(z)|^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^{2p}} + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \\ &\leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \mathbb{E}^{\frac{1}{2}} |T_n|^p + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p}. \end{aligned}$$

The bound for \mathcal{L}_3 may be derived in a similar way. It remains to estimate the terms \mathcal{L}_2 and \mathcal{L}_3 . Similarly as before we obtain

$$\mathcal{L}_2 \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^{2p}} + \frac{C^p}{n^p} + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p}.$$

The same estimate holds for \mathcal{L}_4 . Finally we arrive at the following inequality for the sum of $\mathcal{A}_{\nu 2}$, $\nu = 1, 2, 3$

$$\sum_{\nu=1}^3 \mathcal{A}_{\nu 2} \leq \frac{\mathcal{A}(2p)}{nv} \mathbb{E}^{\frac{p-1}{p}} |T_n|^p + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} \mathbb{E}^{\frac{1}{2}} |T_n|^p + \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p}. \quad (3.31)$$

3.3. Combining bounds. We now combine the bounds (3.29), (3.31) and apply Lemma B.4, obtaining

$$\mathbb{E} |T_n|^p \leq \frac{C^p p^p \mathcal{A}^p(\kappa p)}{(nv)^p} + \frac{C^p p^{2p}}{(nv)^{2p}} + \frac{C^p |b(z)|^{\frac{p}{2}} \mathcal{A}^{\frac{p}{2}}(\kappa p)}{(nv)^p}.$$

In view of the definition of \mathcal{E}_p this concludes the proof of the theorem. \square

4. BOUNDS FOR MOMENTS OF DIAGONAL ENTRIES OF THE RESOLVENT

The main result of this section is the following lemma which provides a bound for moments of the diagonal entries of the resolvent. Recall that (see the definition (2.7))

$$\mathbb{D} := \{z = u + iv \in \mathbb{C} : |u| \leq u_0, V \geq v \geq v_0 := A_0 n^{-1}\},$$

where $u_0, V > 0$ are any fixed real numbers and A_0 is some large constant determined below.

Lemma 4.1. *Assuming the conditions (C1) there exist a positive constant C_0 depending on u_0, V and positive constants A_0, A_1 depending on C_0, α such that for all $z \in \mathbb{D}$ and $1 \leq p \leq A_1 (nv)^{\frac{1-2\alpha}{2}}$ we have*

$$\max_{j \in \mathbb{T}} \mathbb{E} |\mathbf{R}_{jj}(z)|^p \leq C_0^p \quad (4.1)$$

and

$$\mathbb{E} \frac{1}{|z + m_n(z)|^p} \leq C_0^p. \quad (4.2)$$

The proof of Lemma 4.1 is based on several auxiliary results and will be given at the end of this section. In this proof we shall use ideas from [18] and [21]. One of the main ingredients of the proof is the descent method for \mathbf{R}_{jj} which is based on Lemma 4.2 below and Lemma C.1 in the Appendix, which in this form appeared in [5].

Since u is fixed and $|u| \leq u_0$ we shall omit u from the notation of the resolvent and denote $\mathbf{R}(v) := \mathbf{R}(z)$. Sometimes in order to simplify notations we shall also omit the argument v in $\mathbf{R}(v)$ and just write \mathbf{R} . For any $j \in \mathbb{T}_{\mathbb{J}}$ we may express $\mathbf{R}_{jj}^{(\mathbb{J})}$ in the following way (compare (2.1))

$$\mathbf{R}_{jj}^{(\mathbb{J})} = \frac{1}{-z + \frac{X_{jj}}{\sqrt{n}} - \frac{1}{n} \sum_{l,k \in \mathbb{T}_{\mathbb{J},j}} X_{jk} X_{jl} \mathbf{R}_{lk}^{(\mathbb{J},j)}}. \quad (4.3)$$

Let $\varepsilon_j^{(\mathbb{J})} := \varepsilon_{1j}^{(\mathbb{J})} + \varepsilon_{2j}^{(\mathbb{J})} + \varepsilon_{3j}^{(\mathbb{J})} + \varepsilon_{4j}^{(\mathbb{J})}$, where

$$\begin{aligned} \varepsilon_{1j}^{(\mathbb{J})} &= \frac{1}{\sqrt{n}} X_{jj}, & \varepsilon_{2j}^{(\mathbb{J})} &= -\frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} X_{jk} X_{jl} \mathbf{R}_{kl}^{(\mathbb{J},j)}, & \varepsilon_{3j}^{(\mathbb{J})} &= -\frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J},j}} (X_{jk}^2 - 1) \mathbf{R}_{kk}^{(\mathbb{J})}(z), \\ \varepsilon_{4j}^{(\mathbb{J})} &= \frac{1}{n} (\text{Tr } \mathbf{R}^{(\mathbb{J})} - \text{Tr } \mathbf{R}^{(\mathbb{J},j)}(z)). \end{aligned}$$

We also introduce the quantities $\Lambda_n^{(\mathbb{J})}(z) := m_n^{(\mathbb{J})}(z) - s(z)$ and

$$T_n^{(\mathbb{J})} := \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \varepsilon_j^{(\mathbb{J})} \mathbf{R}_{jj}^{(\mathbb{J})}.$$

The following lemma, Lemma 4.2, allows to recursively estimate the moments of the diagonal entries of the resolvent. The proof of the first part of this lemma may be found in [5] and it is included here for the readers convenience.

Lemma 4.2. *For an arbitrary set $\mathbb{J} \subset \mathbb{T}$ and all $j \in \mathbb{T}_{\mathbb{J}}$ there exist a positive constant C_1 depending on u_0, V only such that for all $z = u + iv$ with $V \geq v > 0$ and $|u| \leq u_0$ we have*

$$|\mathbf{R}_{jj}^{(\mathbb{J})}| \leq C_1 \left(1 + |T_n^{(\mathbb{J})}|^{\frac{1}{2}} |\mathbf{R}_{jj}^{(\mathbb{J})}| + |\varepsilon_j^{(\mathbb{J})}| |\mathbf{R}_{jj}^{(\mathbb{J})}| \right) \quad (4.4)$$

and

$$\frac{1}{|z + m_n^{(\mathbb{J})}(z)|} \leq C_1 \left(1 + \frac{|T_n^{(\mathbb{J})}|^{\frac{1}{2}}}{|z + m_n^{(\mathbb{J})}(z)|} \right). \quad (4.5)$$

Proof. We first prove (4.4). We may rewrite (4.3) in the following way

$$\mathbf{R}_{jj}^{(\mathbb{J})} = -\frac{1}{z + m_n^{(\mathbb{J})}(z)} + \frac{1}{z + m_n^{(\mathbb{J})}(z)} \varepsilon_j^{(\mathbb{J})} \mathbf{R}_{jj}^{(\mathbb{J})}.$$

Applying the definition of $\Lambda_n^{(j)}$ we rewrite the previous equation as

$$\mathbf{R}_{jj}^{(\mathbb{J})} = s(z) + s(z) (\Lambda_n^{(\mathbb{J})} - \varepsilon_j^{(\mathbb{J})}) \mathbf{R}_{jj}^{(\mathbb{J})}. \quad (4.6)$$

Since $|s(z)| \leq 1$ we get

$$|\mathbf{R}_{jj}^{(\mathbb{J})}| \leq 1 + (|\Lambda_n^{(\mathbb{J})}| + |\varepsilon_j^{(\mathbb{J})}|)|\mathbf{R}_{jj}^{(\mathbb{J})}|. \quad (4.7)$$

Rewriting (4.6) in a

$$\mathbf{R}_{jj}^{(\mathbb{J})} = s(z) + s(z)(b_n^{(\mathbb{J})}(z) - \varepsilon_j^{(\mathbb{J})})\mathbf{R}_{jj}^{(\mathbb{J})} - s(z)b(z)\mathbf{R}_{jj}^{(\mathbb{J})}.$$

Since $1 + zs(z) + s^2(z) = 0$ we get the following inequality

$$|\mathbf{R}_{jj}^{(\mathbb{J})}| \leq \frac{1}{|s(z)|} + \frac{|b_n^{(\mathbb{J})}||\mathbf{R}_{jj}^{(\mathbb{J})}|}{|s(z)|} + \frac{|\varepsilon_j^{(\mathbb{J})}||\mathbf{R}_{jj}^{(\mathbb{J})}|}{|s(z)|}. \quad (4.8)$$

From $|s(z)|^{-1} \leq 1 + |z|$, (4.7), (4.8) and Lemma B.1[Inequality (B.4)] we conclude that there exists a positive constant C_1 such that

$$\begin{aligned} |\mathbf{R}_{jj}^{(\mathbb{J})}| &\leq C_1(1 + \min(|b_n^{(\mathbb{J})}|, |\Lambda_n^{(\mathbb{J})}|)|\mathbf{R}_{jj}^{(\mathbb{J})}| + |\varepsilon_j^{(\mathbb{J})}||\mathbf{R}_{jj}^{(\mathbb{J})}|) \\ &\leq C(1 + |T_n^{(\mathbb{J})}|^{\frac{1}{2}}|\mathbf{R}_{jj}^{(\mathbb{J})}| + |\varepsilon_j^{(\mathbb{J})}||\mathbf{R}_{jj}^{(\mathbb{J})}|). \end{aligned}$$

Consider now the second inequality (4.5). From the representation

$$\frac{1}{z + m_n^{(\mathbb{J})}(z)} = \frac{1}{z + s(z)} - \frac{\Lambda_n^{(\mathbb{J})}}{(z + m_n^{(\mathbb{J})}(z))(z + s(z))} \quad (4.9)$$

we conclude with $|z + s(z)| \geq 1$ that

$$\frac{1}{|z + m_n^{(\mathbb{J})}(z)|} \leq 1 + \frac{\Lambda_n^{(\mathbb{J})}}{|z + m_n^{(\mathbb{J})}(z)|}. \quad (4.10)$$

Rewriting (4.9) as follows we get

$$\frac{1}{z + m_n^{(\mathbb{J})}(z)} = \frac{1}{z + s(z)} - \frac{\Lambda_n^{(\mathbb{J})} + 2s(z) + z}{(z + m_n^{(\mathbb{J})}(z))(z + s(z))} + \frac{2s(z) + z}{(z + m_n^{(\mathbb{J})}(z))(z + s(z))}.$$

This equation may be rewritten as

$$-\frac{s(z)}{z + m_n^{(\mathbb{J})}(z)} = 1 - \frac{\Lambda_n^{(\mathbb{J})} + 2s(z) + z}{z + m_n^{(\mathbb{J})}(z)}.$$

Taking absolute values and applying the triangular inequality we get

$$\frac{1}{|z + m_n^{(\mathbb{J})}(z)|} \leq \frac{1}{|s(z)|} + \frac{1}{|s(z)|} \frac{|\Lambda_n^{(\mathbb{J})} + 2s(z) + z|}{|z + m_n^{(\mathbb{J})}(z)|}. \quad (4.11)$$

From $|s(z)|^{-1} \leq 1 + |z|$, (4.10), (4.11) and B.1[Inequality (B.4)] it follows

$$\frac{1}{|z + m_n^{(\mathbb{J})}(z)|} \leq C_1 \left(1 + \frac{|T_n^{(\mathbb{J})}|^{\frac{1}{2}}}{|z + m_n^{(\mathbb{J})}(z)|} \right).$$

□

Let us introduce the following quantities for an integer $K > 0$

$$\begin{aligned} A_{\nu,q} &:= A_{\nu,q}^{(K)}(v) := \max_{\mathbb{J}:|\mathbb{J}|\leq\nu+K} \max_{l\neq k\in\mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J},j)}(v)|^q, \\ F_{\nu,q} &:= F_{\nu,q}^{(K)}(v) := \max_{\mathbb{J}:|\mathbb{J}|\leq\nu+K} \max_{l\in\mathbb{T}_{\mathbb{J}}} \mathbb{E} \operatorname{Im}^q \mathbf{R}_{ll}^{(\mathbb{J},j)}(v). \end{aligned} \quad (4.12)$$

In the following lemma we show that $A_{1,q}$ is uniformly bounded with respect to v and n .

Lemma 4.3. *Let $\tilde{v} > 0$ be an arbitrary number and C_0 be some large positive constant. Suppose that the conditions **(C1)** hold. There exist positive constant s_0 depending on α and positive constants A_0, A_1 depending on C_0, s_0 such that assuming*

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K+2} \max_{l,k\in\mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J})}(v')|^q \leq C_0^q \quad (4.13)$$

for all $v' \geq \tilde{v}, |u| \leq u_0$ and $1 \leq q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$ we have

$$A_{1,q} \leq C_0^q$$

for all $v \geq \tilde{v}/s_0, |u| \leq u_0$ and $1 \leq q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$.

Proof. We start from the assumption that we have already chosen the value of s_0 . Let us fix an arbitrary $v \geq \tilde{v}/s_0, \mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq K+1$ and $j \in \mathbb{T}_{\mathbb{J}}$. We may express $\mathbf{R}_{jk}^{(\mathbb{J})}$ as follows

$$\mathbf{R}_{jk}^{(\mathbb{J})} = -\frac{1}{\sqrt{n}} \sum_{l \in \mathbb{T}_{\mathbb{J},j}} X_{jl} \mathbf{R}_{lk}^{(\mathbb{J},j)} \mathbf{R}_{jj}^{(\mathbb{J})}.$$

Applying Hölder's inequality we obtain

$$\mathbb{E} |\mathbf{R}_{jk}^{(\mathbb{J})}|^q \leq n^{-\frac{1}{2}} \mathbb{E}^{\frac{1}{2}} \left| \sum_{l \in \mathbb{T}_{\mathbb{J},j}} X_{jl} \mathbf{R}_{lk}^{(\mathbb{J},j)} \right|^{2q} \mathbb{E}^{\frac{1}{2}} |\mathbf{R}_{jj}^{(\mathbb{J})}|^{2q}.$$

From Lemma (C.1) we conclude that for $s_0 \geq 1$ the following relation holds

$$|\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^q \leq s_0^q |\mathbf{R}_{jj}^{(\mathbb{J})}(s_0 v)|^q. \quad (4.14)$$

We choose $s_0 := 2^{\frac{2}{1-2\alpha}}$. Since $v' := s_0 v \geq \tilde{v}$ and $2q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$ we may apply (4.14) and the assumption (4.13) to estimate $\mathbb{E}^{\frac{1}{2}} |\mathbf{R}_{jj}^{(\mathbb{J})}|^{2q} \leq (s_0 C_0)^q$. We get the following bound

$$\mathbb{E} |\mathbf{R}_{jk}^{(\mathbb{J})}|^q \leq n^{-\frac{q}{2}} (s_0 C_0)^q \mathbb{E}^{\frac{1}{2}} \left| \sum_{l \in \mathbb{T}_{\mathbb{J},j}} X_{jl} \mathbf{R}_{lk}^{(\mathbb{J},j)} \right|^{2q}. \quad (4.15)$$

Since $X_{jk}, k \in \mathbb{T}_{\mathbb{J},j}$, and $\mathbf{R}^{(\mathbb{J},j)}$ are independent we may apply Rosenthal's inequality and get

$$\mathbb{E} \left| \sum_{l \in \mathbb{T}_{\mathbb{J},j}'} X_{jl} \mathbf{R}_{lk}^{(\mathbb{J},j)} \right|^{2q} \leq C^q \left(q^q \mathbb{E} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j}'} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^q + \mu_{2q} q^{2q} \sum_{l \in \mathbb{T}_{\mathbb{J},j}'} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^{2q} \right).$$

From Lemma C.4 [Inequality (C.8)] and (4.14) we derive the following inequality

$$\mathbb{E} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j}'} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^q \leq \frac{1}{v^q} F_{2,q} \leq \frac{s_0^q C_0^q}{v^q}. \quad (4.16)$$

Since $|X_{jk}| \leq Dn^\alpha$ we get

$$\mu_{2q} \leq \mu_4 D^{2q-4} n^{\alpha(2q-4)}. \quad (4.17)$$

By definition

$$\sum_{l \in \mathbb{T}_{\mathbb{J},j}'} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^{2q} = \sum_{l \in \mathbb{T}_{\mathbb{J},j,k}'} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^{2q} + \mathbb{E} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^{2q}. \quad (4.18)$$

The second term on the right hand side of the previous equality can be bounded by (4.14) and the assumption (4.13). To the first term we may apply the following bound

$$\sum_{l \in \mathbb{T}_{\mathbb{J},j,k}'} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^{2q} \leq n A_{2,2q}. \quad (4.19)$$

It follows from (4.15), (4.16), (4.17) (4.18) and (4.19) that

$$A_{1,q} \leq (CC_0 s_0)^q \left(\frac{q^{\frac{q}{2}} (s_0 C_0)^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}} + \frac{q^q}{n^{\frac{q}{2}(1-2\alpha)}} A_{2,2q}^{\frac{1}{2}} + \frac{q^q (C_0 s_0)^q}{n^{\frac{q}{2}(1-2\alpha)}} \right). \quad (4.20)$$

To estimate $A_{2,2q}$ we apply the resolvent equality and obtain that for arbitrary $l \neq k \in \mathbb{T}_{\mathbb{J}}$

$$|\mathbf{R}_{lk}^{(\mathbb{J})}(v) - \mathbf{R}_{lk}^{(\mathbb{J})}(s_0 v)| \leq v(s_0 - 1) |[\mathbf{R}^{(\mathbb{J})}(v) \mathbf{R}^{(\mathbb{J})}(s_0 v)]_{lk}|.$$

The Cauchy-Schwartz inequality and Lemma C.4 [Inequality (C.8)] together imply that

$$|\mathbf{R}_{lk}^{(\mathbb{J})}(v) - \mathbf{R}_{lk}^{(\mathbb{J})}(s_0 v)| \leq \sqrt{s_0} \sqrt{|\mathbf{R}_{ll}^{(\mathbb{J})}(v)| |\mathbf{R}_{kk}^{(\mathbb{J})}(s_0 v)|}.$$

It remains to apply (4.14) and the assumption (4.13) to get

$$A_{2,2q} \leq 2^{2q} (s_0 C_0)^{2q} + 2^{2q} s_0^{3q} C_0^{2q}.$$

It follows from the last inequality and (4.20) that

$$A_{1,q} \leq (CC_0 s_0)^q \left(\frac{q^{\frac{q}{2}} (s_0 C_0)^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}} + \frac{q^q (C_0 s_0^{3/2})^q}{n^{\frac{q}{2}(1-2\alpha)}} \right).$$

We may choose the constants A_0 and A_1 depending on C_0, s_0 in such way that

$$A_{1,q} \leq C_0^q.$$

□

Lemma 4.4. *Let $\tilde{v} > 0$ be an arbitrary number and C_0 be some large positive constant. Suppose that the conditions **(C1)** hold. There exist positive constant s_0 depending on α and positive constants A_0, A_1 depending on C_0, s_0 such that assuming*

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K+2} \max_{l,k\in\mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J})}(v')|^q \leq C_0^q \quad (4.21)$$

for all $v' \geq \tilde{v}, |u| \leq u_0$ and $1 \leq q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$ we have

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K} \max_{j\in\mathbb{T}_{\mathbb{J}}} \mathbb{E} |\varepsilon_{2j}^{(\mathbb{J})}|^{2q} \leq \frac{(CC_0s_0)^q q^{4q}}{(nv)^{2q(1-2\alpha)}}$$

for all $v \geq \tilde{v}/s_0, |u| \leq u_0$ and $1 \leq q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$.

Proof. Let us fix an arbitrary $v \geq \tilde{v}/s_0, \mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq K$ and $j \in \mathbb{T}_{\mathbb{J}}$. Applying Lemma A.5 we get

$$\mathbb{E} |\varepsilon_{2j}^{(\mathbb{J})}|^{2q} \leq C^p \left(\frac{q^{2q}}{(nv)^q} \mathbb{E} [\text{Im } m_n^{(\mathbb{J},j)}(z)]^q + \frac{q^{3q}}{(nv)^q} F_{1,q} + \frac{q^{4q}}{n^{2q(1-2\alpha)}} A_{1,2q} \right).$$

To estimate $\mathbb{E} [\text{Im } m_n^{(\mathbb{J},j)}(z)]^q$ and $F_{1,q}$ we use Lemma (C.1) which states that for all $s_0 \geq 1$ the following relation holds

$$|\mathbf{R}_{kk}^{(\mathbb{J},j)}(v)|^q \leq s_0^q |\mathbf{R}_{kk}^{(\mathbb{J},j)}(s_0v)|^q. \quad (4.22)$$

We choose $s_0 := 2^{\frac{4}{1-2\alpha}}$. This choice implies that $v' := s_0v \geq \tilde{v}$ and $4q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$. We may apply (4.22) and (4.21) to estimate

$$\mathbb{E} [\text{Im } m_n^{(\mathbb{J},j)}(z)]^q \leq s_0^q C_0^q \text{ and } F_{1,q} \leq s_0^q C_0^q. \quad (4.23)$$

In view of these inequalities we may write

$$\mathbb{E} |\varepsilon_{2j}^{(\mathbb{J})}|^{2q} \leq \frac{(CC_0s_0)^q q^{3q}}{(nv)^q} + \frac{C^q q^{4q}}{n^{2q(1-2\alpha)}} A_{1,2q}. \quad (4.24)$$

Applying Lemma 4.3 we obtain that there exist some positive constants A_0 and A_1 depending on C_0, s_0 in such way that

$$A_{1,2q} \leq C_0^q. \quad (4.25)$$

Combining (4.24) and (4.25) we obtain

$$\mathbb{E} |\varepsilon_{2j}^{(\mathbb{J})}|^{2q} \leq \frac{(CC_0s_0)^q q^{4q}}{(nv)^{2q(1-2\alpha)}}.$$

□

Lemma 4.5. *Let $\tilde{v} > 0$ be an arbitrary number and A_0, A_1 and C_0 be some positive constants. Assume that the conditions **(C1)** hold and*

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K+2} \max_{l\in\mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{ll}^{(\mathbb{J})}(v')|^q \leq C_0^q \quad (4.26)$$

for all $v' \geq \tilde{v}$, $|u| \leq u_0$ and $1 \leq q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$. Then there exists $s_0 \geq 1$ depending on α such that for all $v \geq \tilde{v}/s_0$, $|u| \leq u_0$ and $1 \leq q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$ we have

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K} \max_{j \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\varepsilon_{3j}^{(\mathbb{J})}|^{2q} \leq \frac{(s_0 C C_0)^{2q} q^{2q}}{n^{2q(1-2\alpha)}}.$$

Proof. Let us take an arbitrary $v \geq \tilde{v}/s_0$, $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq K$ and $j \in \mathbb{T}_{\mathbb{J}}$. Applying Lemma A.7 we get

$$\mathbb{E} |\varepsilon_{3j}^{(\mathbb{J})}|^{2q} \leq C^q \left(\frac{q^q}{n^{2q}} \mathbb{E} \left(\sum_{k \in \mathbb{T}_{\mathbb{J}}} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^2 \right)^q + \frac{q^{2q}}{n^{2q(1-2\alpha)}} \frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^{2q} \right).$$

From Lemma (C.1) we get for $s_0 \geq 1$

$$|\mathbf{R}_{kk}^{(\mathbb{J},j)}(v)|^{2q} \leq s_0^{2q} |\mathbf{R}_{kk}^{(\mathbb{J},j)}(s_0 v)|^{2q}.$$

As in the previous lemmas we may take $s_0 := 2^{\frac{4}{1-2\alpha}}$ and set $v' := s_0 v \geq \tilde{v}$ (note that in this lemma it actually suffices to take $s_0 := 2^{\frac{2}{1-2\alpha}}$, but for simplicity we shall use the same value for s_0) and get that $2q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$. We may apply (4.26) to obtain the bound

$$\mathbb{E} |\mathbf{R}_{kk}^{(\mathbb{J},j)}(v)|^{2q} \leq s_0^{2q} C_0^{2q}.$$

It is easy to see that

$$\mathbb{E} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J}}} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^2 \right)^q \leq \frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J}}} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^{2q}.$$

The last two inequalities together imply that

$$\mathbb{E} |\varepsilon_{3j}^{(\mathbb{J})}|^{2q} \leq \frac{(s_0 C C_0)^{2q} q^{2q}}{n^{2q(1-2\alpha)}}.$$

□

Lemma 4.6. *Let $\tilde{v} > 0$ be an arbitrary number and C_0 be some large positive constant. Suppose that the conditions (C1) hold. There exist positive constant s_0 depending on α and positive constants A_0, A_1 depending on C_0, s_0 such that assuming*

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K+2} \max_{l, k \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J})}(v')|^q \leq C_0^q \quad (4.27)$$

for all $v' \geq \tilde{v}$, $|u| \leq u_0$ and $1 \leq q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$ we have

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K} \max_{l \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_l^{(\mathbb{J})}(v)|^q \leq C_0^q$$

for all $v \geq \tilde{v}/s_0$, $|u| \leq u_0$ and $1 \leq q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$.

Proof. Let us take an arbitrary $v \geq \tilde{v}/s_0$, $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq K$ and $j \in \mathbb{T}_{\mathbb{J}}$. From Lemma 4.2, inequality (4.4), it follows that

$$\mathbb{E} |\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^q \leq C_1^q (1 + \mathbb{E}^{\frac{1}{2}} |T_n^{(\mathbb{J})}|^q \mathbb{E}^{\frac{1}{2}} |\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^{2q} + \mathbb{E}^{\frac{1}{2}} |\varepsilon_j^{(\mathbb{J})}|^{2q} \mathbb{E}^{\frac{1}{2}} |\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^{2q}).$$

Let us choose again $s_0 := 2^{\frac{4}{1-2\alpha}}$. As shown in proof of the previous lemmas choosing $v' := s_0 v \geq v_1$ ensures $2q \leq A_1 (nv')^{\frac{1-2\alpha}{2}}$. We now apply Lemma (C.1) and the assumption (4.27) to estimate

$$\mathbb{E} |\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^{2q} \leq s_0^{2q} \mathbb{E} |\mathbf{R}_{jj}^{(\mathbb{J})}(s_0 v)|^{2q} \leq (C_0 s_0)^{2q}. \quad (4.28)$$

The Cauchy-Schwartz inequality implies that

$$\mathbb{E} |T_n^{(\mathbb{J})}|^q \leq \left(\frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\varepsilon_j^{(\mathbb{J})}|^{2q} \right)^{1/2} \left(\frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^{2q} \right)^{1/2}$$

From (4.28) it follows that

$$\left(\frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^{2q} \right)^{1/2} \leq s_0^q C_0^q.$$

By an obvious inequality we get

$$\mathbb{E} |\varepsilon_j^{(\mathbb{J})}|^{2q} \leq 4^{2q} (\mathbb{E} |\varepsilon_{1j}^{(\mathbb{J})}|^{2q} + \mathbb{E} |\varepsilon_{2j}^{(\mathbb{J})}|^{2q} + \mathbb{E} |\varepsilon_{3j}^{(\mathbb{J})}|^{2q} + \mathbb{E} |\varepsilon_{4j}^{(\mathbb{J})}|^{2q}).$$

Now we may use Lemmas 4.4, 4.5, A.3 and A.9 and obtain the following bound

$$\mathbb{E} |\varepsilon_j^{(\mathbb{J})}|^{2q} \leq 4^{2q} \left[\frac{\mu_4 D^{2q-4}}{n^{q(1-2\alpha)+4\alpha}} + \frac{(CC_0 s_0)^q q^{4q}}{(nv)^{2q(1-2\alpha)}} + \frac{(s_0 CC_0)^{2q} q^{2q}}{n^{2q(1-2\alpha)}} + \frac{1}{(nv)^{2q}} \right].$$

It is easy to see that since the estimates for $T_n^{(\mathbb{J})}$ and $\mathbf{R}_{jj}^{(\mathbb{J})}$ depend on $\mathbb{E} |\varepsilon_j^{(\mathbb{J})}|^{2q}$ we may choose the constants A_0 and A_1 (correcting the previous choice if needed) depending on C_0, s_0 in such way that

$$\mathbb{E} |\mathbf{R}_{jj}^{(\mathbb{J})}(v)|^q \leq C_0^q$$

for all $1 \leq q \leq A_1 (nv)^{\frac{1-2\alpha}{2}}$ and $z \in \mathbb{D}$. □

Proof of Lemma 4.1. We first prove (4.1). Let us take $v = 1$ and some large constant $C_0 \gg \max(1, C_1)$. We also take s_0, A_0 and A_1 as chosen in the previous Lemma 4.6. Set $L = \lceil -\log_{s_0} v_0 \rceil + 1$. Since $\|\mathbf{R}^{(\mathbb{J})}(v)\| \leq v^{-1} = 1$ we may write

$$\max_{\mathbb{J}: |\mathbb{J}| \leq 2L} \max_{l, k \in \mathbb{T}_{\mathbb{J}}} |\mathbf{R}_{lk}^{(\mathbb{J})}(v)| \leq C_0$$

and

$$\max_{\mathbb{J}: |\mathbb{J}| \leq 2L} \max_{l, k \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J})}(v)|^q \leq C_0^q$$

for $1 \leq q \leq A_1 n^{\frac{1-2\alpha}{2}}$. Taking $K := 2(L-1)$ and applying Lemma 4.6 we get

$$\max_{\mathbb{J}:|\mathbb{J}|\leq 2(L-1)} \max_{l,k \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J})}(v)|^q \leq C_0^q$$

for $1 \leq q \leq A_1 n^{\frac{1-2\alpha}{2}} s_0^{-\frac{1-2\alpha}{2}}$, $v \geq 1/s_0$. We may repeat this procedure L times and finally obtain

$$\max_{l,k \in \mathbb{T}} \mathbb{E} |\mathbf{R}_{lk}(v)|^q \leq C_0^q$$

for $1 \leq q \leq A_1 n^{\frac{1-2\alpha}{2}} s_0^{-L\frac{1-2\alpha}{2}} \leq A_1 (nv_0)^{\frac{1-2\alpha}{2}}$ and $v \geq 1/s_0^L \geq v_0$.

To prove (4.2) it is enough to repeat all the previous steps and apply the inequality (C.2) from Lemma C.2. We omit the details. \square

5. IMAGINARY PART OF DIAGONAL ENTRIES OF THE RESOLVENT

In this section we estimate the moments of the imaginary part of diagonal entries of the resolvent. Let us introduce the following quantity

$$\Psi(z) := \operatorname{Im} s(z) + \frac{p^2}{nv}. \quad (5.1)$$

To simplify notations we will often write $\Psi(v)$ and Ψ instead of $\Psi(z)$. The main result of this section is the following lemma.

Lemma 5.1. *Assuming conditions (C1) there exist positive constants H_0 depending on u_0, V and positive constants A_0, A_1 depending on H_0, α such that for all $1 \leq p \leq A_1 (nv)^{\frac{1-2\alpha}{2}}$ and $z \in \mathbb{D}$ we get*

$$\max_{j \in \mathbb{T}} \mathbb{E} \operatorname{Im}^p \mathbf{R}_{jj}(z) \leq H_0^p \Psi^p(z).$$

We remark here that the values of A_0 and A_1 in this lemma are different from the values of respective quantities in Lemma 4.1, but for simplicity we shall use the same notations. Applying both Lemmas we shall restrict the upper limit of the moment q to the minimum of the two A_1 's and the lower end of the range of v to the maximum of the two A_0 's via $v \geq A_0 n^{-1}$. Throughout this section we shall assume that for all $1 \leq p \leq A_1 (nv)^{\frac{1-2\alpha}{2}}$ and $z \in \mathbb{D}$ we have

$$\mathbb{E} |\mathbf{R}_{jj}(u+iv)|^p \leq C_0^p, \quad (5.2)$$

where the value of C_0 is defined in Lemma 4.1.

The following lemma is the analogue of Lemma 4.2 and provides a recurrence relation for $\operatorname{Im} \mathbf{R}_{jj}$.

Lemma 5.2. *For any set \mathbb{J} and $j \in \mathbb{T}_{\mathbb{J}}$ there exists a positive constant C_1 depending on u_0, V such that for all $z = u+iv$ with $V \geq v > 0$ and $|u| \leq u_0$ we have*

$$\begin{aligned} \operatorname{Im} \mathbf{R}_{jj}^{(\mathbb{J})}(z) \leq C_1 \left[\operatorname{Im} s(z) (1 + (|\varepsilon_j^{(\mathbb{J})}| + |T_n^{(\mathbb{J})}|^{\frac{1}{2}}) |\mathbf{R}_{jj}^{(\mathbb{J})}(z)|) + |\operatorname{Im} \varepsilon_j^{(\mathbb{J})} + \operatorname{Im} \Lambda_n^{(\mathbb{J})}| |\mathbf{R}_{jj}^{(\mathbb{J})}(z)| \right. \\ \left. + (|\varepsilon_j^{(\mathbb{J})}| + |T_n^{(\mathbb{J})}|^{\frac{1}{2}}) \operatorname{Im} \mathbf{R}_{jj}^{(\mathbb{J})}(z) \right]. \end{aligned}$$

Proof. The proof is similar to the proof of Lemma 4.2 is omitted. \square

Lemma 5.3. *Let $\tilde{v} > 0$ be an arbitrary number and H_0 be some large positive constant. Assume that the conditions (C1) hold. There exist a positive constant s_0 depending on α and positive constants A_0, A_1 depending on H_0, s_0 such that assuming*

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K+M} \max_{l \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} \operatorname{Im}^q \mathbf{R}_{ll}^{(\mathbb{J})}(v') \leq H_0^q \Psi^q(v') \quad (5.3)$$

for all $v' \geq \tilde{v}, |u| \leq u_0$ and $1 \leq q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$ we have

$$\max_{\mathbb{J}:|\mathbb{J}|\leq K} \max_{l \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} \operatorname{Im}^q \mathbf{R}_{ll}^{(\mathbb{J})}(v) \leq H_0^q \Psi^q(v)$$

for all $v \geq \tilde{v}/s_0, |u| \leq u_0$ and $1 \leq q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$.

Proof. From Lemma 5.2 it follows that

$$\begin{aligned} \mathbb{E} \operatorname{Im}^q \mathbf{R}_{jj}^{(\mathbb{J})} &\leq (CC_0)^q \operatorname{Im}^q s(z) \mathbb{E}^{\frac{1}{2}} (1 + (|\varepsilon_j^{(\mathbb{J})}| + |T_n^{(\mathbb{J})}|^{\frac{1}{2}})^{2q} \\ &\quad + (CC_0)^q \mathbb{E}^{\frac{1}{2}} |\operatorname{Im} \varepsilon_j^{(\mathbb{J})} + \operatorname{Im} \Lambda_n^{(\mathbb{J})}|^{2q} \\ &\quad + C^q \mathbb{E}^{\frac{1}{2}} (|\varepsilon_j^{(\mathbb{J})}| + |T_n^{(\mathbb{J})}|^{\frac{1}{2}})^{2q} \mathbb{E}^{\frac{1}{2}} \operatorname{Im}^{2q} \mathbf{R}_{jj}^{(\mathbb{J})}. \end{aligned} \quad (5.4)$$

To estimate $\mathbb{E} |\varepsilon_j^{(\mathbb{J})}|^{2q}$ and $\mathbb{E} |T_n^{(\mathbb{J})}|^q$ we may proceed as in Lemma 4.6. We obtain the following inequalities

$$\mathbb{E} |\varepsilon_j^{(\mathbb{J})}|^{2q} \leq 4^{2q} \left[\frac{\mu_4 D^{2q-4}}{n^{q(1-2\alpha)+4\alpha}} + \frac{C^q q^{4q}}{(nv)^{2q(1-2\alpha)}} + \frac{C^{2q} q^{2q}}{n^{2q(1-2\alpha)}} + \frac{1}{(nv)^{2q}} \right]. \quad (5.5)$$

and

$$\mathbb{E} |T_n^{(\mathbb{J})}|^q \leq C_0^q \left(\frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\varepsilon_j^{(\mathbb{J})}|^{2q} \right)^{1/2}. \quad (5.6)$$

Take as before $s_0 := 2^{\frac{4}{1-2\alpha}}$. Choosing $v' := s_0 v \geq v_1$ we may show that $2q \leq A_1(nv')^{\frac{1-2\alpha}{2}}$. Applying Lemma C.2 and using the assumption (5.3) we get

$$\mathbb{E} \operatorname{Im}^{2q} \mathbf{R}_{jj}^{(\mathbb{J})}(v) \leq s_0^{2q} \mathbb{E} \operatorname{Im}^{2q} \mathbf{R}_{jj}^{(\mathbb{J})}(s_0 v) \leq s_0^{2q} H_0^{2q} \Psi^{2q}(s_0 v). \quad (5.7)$$

Since we need an estimate involving $\Psi^{2q}(v)$ instead of $\Psi^{2q}(s_0 v)$ on the r.h.s. of the previous inequality we need to perform a descent along the imaginary line from $s_0 v$ to v . To this purpose we again apply Lemma C.2. Choosing suitable constants A_0 and A_1 in (5.5) and (5.6) one may show that

$$\mathbb{E} \operatorname{Im}^q \mathbf{R}_{jj}^{(\mathbb{J})} \leq (CC_0)^q \operatorname{Im}^q s(z) + (CC_0)^q \mathbb{E}^{\frac{1}{2}} |\operatorname{Im} \varepsilon_j^{(\mathbb{J})} + \operatorname{Im} \Lambda_n^{(\mathbb{J})}|^{2q} + \frac{H_0^q}{3} \Psi^q. \quad (5.8)$$

Applying Lemmas A.13–A.14 we obtain

$$\mathbb{E} |\operatorname{Im} \varepsilon_j^{(\mathbb{J})}|^{2q} \leq 3^{2q} \left[C^q \left(\frac{q^{2q} \mathbb{E} \operatorname{Im}^q m_n^{(\mathbb{J},j)}(z)}{(nv)^q} + \frac{q^{4q}}{n^{2q(1-2\alpha)}} F_{1,2q} \right) + \frac{C^q q^{2q}}{n^{2q(1-2\alpha)}} F_{1,2q} + \frac{1}{(nv)^{2q}} \right].$$

We may write

$$\begin{aligned} (CC_0)^q \mathbb{E} |\operatorname{Im} \varepsilon_j^{(\mathbb{J})}|^{2q} &\leq C^q \left[\frac{q^{2q} s_0^{2q} H_0^q \Psi^q}{(nv)^q} + \frac{q^{4q} s_0^{4q} H_0^{2q} \Psi^{2q}}{n^{2q(1-2\alpha)}} + \frac{1}{(nv)^{2q}} \right] \\ &\leq \frac{H_0^{2q}}{C_2} \Psi^{2q} + \frac{(Cs_0)^{4q} q^{4q}}{(nv)^{2q}} + \frac{q^{4q} s_0^{4q} H_0^{2q} \Psi^{2q}}{n^{2q(1-2\alpha)}}, \end{aligned} \quad (5.9)$$

where C_2 will be chosen later. To estimate $\mathbb{E} |\operatorname{Im} \Lambda_n^{(\mathbb{J})}|^q$ we may proceed as in the proof of Theorem 1.1. We will apply Theorem 2.1 (one has to replace in the definition of (2.8) the maximum over $|\mathbb{J}| \leq 1$ by the maximum over $|\mathbb{J}| \leq L$) and assumption (5.3). Hence,

$$(CC_0)^q \mathbb{E} |\operatorname{Im} \Lambda_n^{(\mathbb{J})}|^{2q} \leq \frac{H_0^{2q}}{C_2} \Psi^{2q} + \frac{(Cs_0)^{4q} q^{4q}}{(nv)^{2q}}. \quad (5.10)$$

Combining the estimates (5.9) and (5.10) we may choose constants C_2, A_0 and A_1 (correcting the previous choice if needed) such that

$$(CC_0)^q \mathbb{E}^{\frac{1}{2}} |\operatorname{Im} \varepsilon_j^{(\mathbb{J})} + \operatorname{Im} \Lambda_n^{(\mathbb{J})}|^{2q} \leq \frac{H_0^q}{3} \Psi^q, \quad \text{and} \quad (CC_0)^q \operatorname{Im}^q s(z) \leq \frac{H_0^q}{3} \Psi^q.$$

The last two inequalities and (5.8) together imply the desired bound

$$\mathbb{E} \operatorname{Im}^q \mathbf{R}_{jj}^{(\mathbb{J})} \leq H_0^q \Psi^q.$$

□

Proof of Lemma 5.1. Let us take any $u_0 > 0$ and any $\hat{v} \geq 2 + u_0, |u| \leq u_0$. We also fix arbitrary $\mathbb{J} \subset \mathbb{T}$. We claim that

$$\operatorname{Im} s(u + i\hat{v}) \geq \frac{1}{2} \operatorname{Im} \mathbf{R}_{jj}^{(\mathbb{J})}(u + i\hat{v}). \quad (5.11)$$

Indeed, we first mention that for all u (and $|u| \leq u_0$ as well)

$$\operatorname{Im} \mathbf{R}_{jj}^{(\mathbb{J})}(u + i\hat{v}) \leq \frac{1}{\hat{v}}. \quad (5.12)$$

For all $|u| \leq u_0$ and $|x| \leq 2$ we obtain

$$\frac{\hat{v}}{(x-u)^2 + \hat{v}^2} \geq \frac{\hat{v}}{(2+u_0)^2 + \hat{v}^2} \geq \frac{1}{2\hat{v}}. \quad (5.13)$$

It follows from the last inequality that

$$\operatorname{Im} s(u + i\hat{v}) = \frac{1}{2\pi} \int_{-2}^2 \frac{\hat{v}}{(u-x)^2 + \hat{v}^2} \sqrt{4-x^2} dx \geq \frac{1}{2\hat{v}}. \quad (5.14)$$

Comparing (5.12) and (5.14) we arrive at (5.11).

We not take $v \geq \hat{v}$. Let H_0 be some large constant, $H_0 \geq \max(C', C''')$. We choose s_0, A_0 and A_1 as in the previous Lemma 5.3 obtaining

$$\max_{\mathbb{J}: |\mathbb{J}| \leq 2L} \max_{j \in \mathbb{T}_{\mathbb{J}}} \operatorname{Im}^q \mathbf{R}_{jj}^{(\mathbb{J})}(z) \leq H_0^q \Psi^q(z)$$

with $L = \lceil -\log_{s_0} v_0 \rceil + 1$. We may now proceed recursively in L steps and arrive at

$$\max_{j \in \mathbb{T}} \operatorname{Im}^q \mathbf{R}_{jj}(z) \leq H_0^q \Psi^q(z)$$

for $v \geq v_0$ and $1 \leq q \leq A_1(nv)^{\frac{1-2\alpha}{2}}$. \square

APPENDIX A. MOMENT INEQUALITIES FOR LINEAR AND QUADRATIC FORMS OF RANDOM VARIABLES

In this section we present some inequalities for linear and quadratic forms.

Theorem A.1 (Rosenthal type inequality). *Let $X_j, j = 1, \dots, n$, be independent random variables with $\mathbb{E} X_j = 0, \mathbb{E} X_j^2 = \sigma^2$ and $\mu_p := \max_j \mathbb{E} |X_j|^p < \infty$. Then there exists some absolute constant C such that for $p \geq 2$*

$$\mathbb{E} \left| \sum_{k=1}^n a_k X_k \right|^p \leq (Cp)^{p/2} \sigma^p \left(\sum_{k=1}^n a_k^2 \right)^{\frac{p}{2}} + \mu_p (Cp)^p \sum_{k=1}^n |a_k|^p.$$

Proof. For a proof see [28][Theorem 3] and [24][Inequality (A)]. \square

To estimate the moments of quadratic forms in our proof we shall use the following inequality due to Giné, Latała, Zinn [13]. Let us denote

$$Q := \sum_{1 \leq j \neq k \leq n} a_{jk} X_j X_k.$$

Theorem A.2. *Let $X_j, j = 1, \dots, n$, be independent random variables with $\mathbb{E} X_j = 0, \mathbb{E} X_j^2 = \sigma^2$ and $\mu_p := \max_j \mathbb{E} |X_j|^p < \infty$. Then for $p \geq 2$*

$$\mathbb{E} |Q|^p \leq C^p \left[p^p \left(\sum_{j=1}^n \sum_{k \in \mathbb{T}_j} |a_{jk}|^2 \right)^{\frac{p}{2}} + \mu_p p^{\frac{3p}{2}} \sum_{j=1}^n \left(\sum_{k \in \mathbb{T}_j} |a_{jk}|^2 \right)^{\frac{p}{2}} + \mu_p^2 p^{2p} \sum_{j=1}^n \sum_{k \in \mathbb{T}_j} |a_{jk}|^p \right].$$

Proof. See [13][Proposition 2.4] or [19][Lemma A.1]. \square

We remark here that the sequence of papers [5], [11], [10], [12] is relying on the Hanson–Wright large deviation inequality [23] for quadratic forms instead of this estimate.

The following result is trivial but we formulate it as a lemma since we shall use it many times during the proof of the main result.

Lemma A.3. *Assuming conditions (C1) for $p \geq 1$ we have*

$$\mathbb{E} |\varepsilon_{1j}|^{2p} \leq \frac{\mu_4 D^{2p-4}}{n^{p(1-2\alpha)+4\alpha}}.$$

Proof. The proof follows directly from the definition of $\varepsilon_{1j} := \frac{1}{\sqrt{n}} X_{jj}$. \square

The rest of this section is devoted to the proof of moment inequalities for linear and quadratic forms based on the entries of the resolvent $\mathbf{R}^{(\mathbb{J})}$ or some functions of it.

Lemma A.4. *Under conditions (C1) for $p \geq 1$ and $q = 1, 2$ we have*

$$\mathbb{E} \left| \frac{1}{n} \sum_{j=1}^n \varepsilon_{1j}^q \right|^p \leq \frac{(Cp)^p}{n^p}. \quad (\text{A.1})$$

Proof. From the definition $\varepsilon_{1j} := \frac{1}{\sqrt{n}} X_{jj}$. Let us introduce the following notations for moments $\beta_q := \mathbb{E} X^q$. In these notations $\beta_1 = 0$ and $\beta_2 = 1$. It is easy to see that

$$\mathbb{E} \left| \frac{1}{nn^{\frac{q}{2}}} \sum_{j=1}^n X_{jj}^q \right|^p \leq 2^p \mathbb{E} \left| \frac{1}{nn^{\frac{q}{2}}} \sum_{j=1}^n [X_{jj}^q - \beta_q] \right|^p + \frac{(2\beta_q)^p}{n^{\frac{qp}{2}}} \mathbf{1}[q = 2]. \quad (\text{A.2})$$

Applying Rosenthal's inequality A.1 we get

$$\mathbb{E} \left| \sum_{j=1}^n [X_{jj}^q - \beta_q] \right|^p \leq (Cp)^{\frac{p}{2}} (n \mathbb{E} |X_{jj}|^{2q})^{\frac{p}{2}} + \mathbb{E} |X_{jj}|^{qp} (Cp)^p n.$$

Since $\max_{1 \leq j \leq n} \mathbb{E} |X_{jj}|^4 \leq \mu_4 < \infty$ and $|X_{jj}| \leq Dn^\alpha$ it follows that

$$\mu_{qp} \leq \mu_4 D n^{\alpha(pq-4)}.$$

and we get

$$\mathbb{E} \left| \sum_{j=1}^n [X_{jj}^q - \beta_q] \right|^p \leq (Cp)^p n^{qp}. \quad (\text{A.3})$$

Inequalities (A.2) and (A.3) conclude the proof of the lemma. \square

Recall the definition of the following quantities given in (4.12)

$$A_{\nu,q} := A_{\nu,q}^{(K)} := \max_{\mathbb{J}; |\mathbb{J}| \leq \nu + K} \max_{l \neq k \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J})}|^q, \quad F_{\nu,q} := F_{\nu,q}^{(K)} := \max_{\mathbb{J}; |\mathbb{J}| \leq \nu + K} \max_{l \in \mathbb{T}_{\mathbb{J}}} \mathbb{E} \text{Im}^q \mathbf{R}_{ll}^{(\mathbb{J})},$$

where $K > 0$ is some integer.

Lemma A.5. *Assuming conditions (C1) for $p \geq 2$ and $|\mathbb{J}| \leq K$ we have*

$$\mathbb{E} |\varepsilon_{2j}^{(\mathbb{J})}|^p \leq C^p \left(\frac{p^p}{(nv)^{\frac{p}{2}}} \mathbb{E} [\text{Im} m_n^{(\mathbb{J},j)}(z)]^{\frac{p}{2}} + \frac{p^{\frac{3p}{2}}}{(nv)^{\frac{p}{2}}} F_{1,\frac{p}{2}} + \frac{p^{2p}}{n^{p(1-2\alpha)}} A_{1,p} \right).$$

Proof. By definition

$$\varepsilon_{2j}^{(\mathbb{J})} := -\frac{1}{n} \sum_{k \neq l \in \mathbb{T}_{\mathbb{J},j}} X_{jk} X_{jl} \mathbf{R}_{lk}^{(\mathbb{J},j)}.$$

Applying Lemma A.2 we get

$$\begin{aligned} \mathbb{E} |\varepsilon_{2j}^{(\mathbb{J})}|^p \leq \frac{C^p}{n^p} \left[p^p \mathbb{E} \left(\sum_{k \in \mathbb{T}_{\mathbb{J},j}} \sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} + \mu_p p^{\frac{3p}{2}} \sum_{k \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} \right. \\ \left. + \mu_p^2 p^{2p} \sum_{k \neq l \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^p \right]. \end{aligned} \quad (\text{A.4})$$

From Lemma C.4 [Inequality (C.7)] we get

$$\left(\sum_{k \in \mathbb{T}_{\mathbb{J},j}} \sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} \leq \left(\frac{n}{v} \operatorname{Im} m_n^{(\mathbb{J},j)}(z) \right)^{\frac{p}{2}}. \quad (\text{A.5})$$

Since $|X_{jk}| \leq Dn^\alpha$ we get

$$\mu_p \leq \mu_4 D^{p-4} n^{\alpha(p-4)}. \quad (\text{A.6})$$

From Lemma C.4 [Inequality (C.8)] we may conclude that

$$\sum_{k \in \mathbb{T}_{\mathbb{J},j}} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} \leq \sum_{k \in \mathbb{T}_{\mathbb{J},j}} \left(\frac{1}{v} \operatorname{Im} \mathbf{R}_{kk}^{(\mathbb{J},j)} \right)^{\frac{p}{2}}. \quad (\text{A.7})$$

Substituting (A.5)–(A.7) into (A.4) concludes the proof of the lemma. \square

For $p = 2$ and 4 we may give a better bound for the quadratic form ε_{2j} .

Lemma A.6. *Let $\mathfrak{M}^{(j)} := \sigma\{X_{lk}, l, k \in \mathbb{T}_j\}$. Assuming conditions (C0) for $q = 2$ and 4 we have*

$$\mathbb{E} (|\varepsilon_{2j}|^q | \mathfrak{M}^{(j)}) \leq \frac{C}{(nv)^{\frac{q}{2}}} \operatorname{Im}^{\frac{q}{2}} m_n^{(j)}(z).$$

Proof. Recall that

$$\varepsilon_{2j} := -\frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jk} X_{jl} \mathbf{R}_{lk}^{(j)}.$$

For $q = 2$ the proof follows immediately from Lemma C.4 [Inequality (C.7)] and

$$\sum_{k \in \mathbb{T}_j} \sum_{l \in \mathbb{T}_{j,k}} |\mathbf{R}_{lk}^{(j)}|^2 \leq \frac{n}{v} \operatorname{Im} m_n^{(j)}(z). \quad (\text{A.8})$$

For $q = 4$ we apply Lemma A.2 and get

$$\mathbb{E} (|\varepsilon_{2j}|^q | \mathfrak{M}^{(j)}) \leq \frac{C \mu_4^2}{n^q} \left(\sum_{k \in \mathbb{T}_j} \sum_{l \in \mathbb{T}_{j,k}} |\mathbf{R}_{lk}^{(j)}|^2 \right)^2.$$

and use (A.8). \square

Lemma A.7. *Assuming conditions (C0) for $p \geq 2$ we have*

$$\mathbb{E} |\varepsilon_{3j}^{(\mathbb{J})}|^p \leq C^p \left(\frac{p^{\frac{p}{2}}}{n^p} \mathbb{E} \left(\sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} + \frac{p^p}{n^{p(1-2\alpha)}} \frac{1}{n} \sum_{k \in \mathbb{T}_j} \mathbb{E} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^p \right).$$

Proof. By definition

$$\varepsilon_{3j}^{(\mathbb{J})} := -\frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J},j}} \mathbf{R}_{kk}^{(\mathbb{J},j)} [X_{jk}^2 - 1].$$

Applying Rosenthal's inequality A.1 we get

$$\mathbb{E} |\varepsilon_{3j}^{(\mathbb{J})}|^p \leq \frac{(Cp)^p}{n^p} \left(\mathbb{E} \left(\sum_{k \in \mathbb{T}_{\mathbb{J},j} } |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} + \mathbb{E} |X_{11}|^{2p} \sum_{k \in \mathbb{T}_{\mathbb{J},j} } \mathbb{E} |\mathbf{R}_{kk}^{(\mathbb{J},j)}|^p \right). \quad (\text{A.9})$$

Since $|X_{jk}| \leq Dn^\alpha$ we get

$$\mu_{2p} \leq \mu_4 D^{2p-4} n^{\alpha(2p-4)}. \quad (\text{A.10})$$

Substituting (A.10) to (A.9) we finish the proof of the lemma. \square

For small p that is $p \leq \frac{1}{\alpha}$ we may state a better bound for ε_{3j} .

Lemma A.8. *Assuming conditions (C1) for $2 \leq p \leq \frac{1}{\alpha}$ we have*

$$\mathbb{E} (|\varepsilon_{3j}|^p |\mathfrak{M}^{(j)}|) \leq \frac{C}{n^{\frac{p}{2}+1}} \sum_{k \in \mathbb{T}_j} \mathbb{E} |\mathbf{R}_{kk}^{(j)}|^p.$$

Proof. Applying Rosenthal's inequality A.1 and $|X_{jk}| \leq Dn^\alpha$ we get

$$\mathbb{E} (|\varepsilon_{3j}|^p |\mathfrak{M}^{(j)}|) \leq \frac{C}{n^{\frac{p}{2}}} \left(\mathbb{E} \left(\frac{1}{n} \sum_{k \in \mathbb{T}_j} |\mathbf{R}_{kk}^{(j)}|^2 \right)^{\frac{p}{2}} + \frac{1}{n} \sum_{k \in \mathbb{T}_j} \mathbb{E} |\mathbf{R}_{kk}^{(j)}|^p \right) \leq \frac{C}{n^{\frac{p}{2}+1}} \sum_{k \in \mathbb{T}_j} \mathbb{E} |\mathbf{R}_{kk}^{(j)}|^p,$$

where the last inequality holds since $2 \leq p \leq \frac{1}{\alpha}$. \square

Lemma A.9. *For $p \geq 2$ we have*

$$\mathbb{E} |\varepsilon_{4j}^{(\mathbb{J})}|^p \leq \frac{1}{(nv)^p}.$$

Proof. Applying the Schur complement formula, see [18][Lemma 7.23] or [19][Lemma 3.3], one may write

$$\varepsilon_{4j}^{(\mathbb{J})} = \frac{1}{n} (\text{Tr } \mathbf{R}^{(\mathbb{J})} - \text{Tr } \mathbf{R}^{(\mathbb{J},j)}) = \frac{1}{n} \left(1 + \frac{1}{n} \sum_{l,k \in \mathbb{T}_{\mathbb{J},j}} X_{jk} X_{jl} [(\mathbf{R}^{(\mathbb{J},j)})^2]_{kl} \right) \mathbf{R}_{jj}^{(\mathbb{J})} = \frac{(\mathbf{R}_{jj}^{(\mathbb{J})})^{-1}}{n} \frac{d\mathbf{R}_{jj}^{(\mathbb{J})}}{dz}.$$

Applying now Lemma C.4 concludes the proof of the lemma. \square

Recall the definition of the quantities $\eta_{\nu j}$, $\nu = 0, 1, 2$

$$\begin{aligned}\eta_{0j} &:= \frac{1}{n} \sum_{k \in \mathbb{T}_j} [(\mathbf{R}^{(j)})^2]_{kk}, & \eta_{1j} &:= \frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{kl}, \\ \eta_{2j} &:= \frac{1}{n} \sum_{k \in \mathbb{T}_j} [X_{jk}^2 - 1][(\mathbf{R}^{(j)})^2]_{kk}.\end{aligned}$$

Lemma A.10. *Assuming conditions (C0) for $2 \leq p \leq 4$ we have*

$$\mathbb{E}(|\eta_{1j}|^p | \mathfrak{M}^{(j)}) \leq \frac{C}{n^{\frac{p}{2}}} \left(\frac{1}{n} \text{Tr} |\mathbf{R}^{(j)}|^4 \right)^{\frac{p}{2}}.$$

Proof. Applying Lemma A.2 we get

$$\begin{aligned}\mathbb{E}(|\eta_{1j}|^p | \mathfrak{M}^{(j)}) &\leq \frac{C}{n^p} \left[\mathbb{E} \left(\sum_{k \in \mathbb{T}_j} \sum_{l \in \mathbb{T}_{j,k}} |[(\mathbf{R}^{(j)})^2]_{lk}|^2 \right)^{\frac{p}{2}} + \mu_p \sum_{k \in \mathbb{T}_j} \mathbb{E} \left(\sum_{l \in \mathbb{T}_{j,k}} |[(\mathbf{R}^{(j)})^2]_{lk}|^2 \right)^{\frac{p}{2}} \right. \\ &\quad \left. + \mu_p^2 \sum_{k \neq l \in \mathbb{T}_j} \mathbb{E} |[(\mathbf{R}^{(j)})^2]_{lk}|^p \right].\end{aligned}$$

Since $2 \leq p \leq 4$ we have that $\mu_p < \infty$ and

$$\mathbb{E}(|\eta_{1j}|^p | \mathfrak{M}^{(j)}) \leq \frac{C}{n^p} \mathbb{E} \left(\sum_{k \in \mathbb{T}_j} \sum_{l \in \mathbb{T}_{j,k}} |[(\mathbf{R}^{(j)})^2]_{lk}|^2 \right)^{\frac{p}{2}} \leq \frac{C}{n^{\frac{p}{2}}} \left(\frac{1}{n} \text{Tr} |\mathbf{R}^{(j)}|^4 \right)^{\frac{p}{2}}.$$

□

Lemma A.11. *Assuming conditions (C1) for $2 \leq p \leq \frac{1}{\alpha}$ we have*

$$\mathbb{E}(|\eta_{2j}|^p | \mathfrak{M}^{(j)}) \leq \frac{C}{n^{\frac{p}{2}+1} v^p} \sum_{k \in \mathbb{T}_j} \mathbb{E} |[(\mathbf{R}^{(j)})^2]_{kk}|^p.$$

Proof. Applying Rosenthal's inequality A.1 we get

$$\mathbb{E}(|\eta_{2j}|^p | \mathfrak{M}^{(j)}) \leq \frac{C}{n^p} \left(\mathbb{E} \left(\sum_{k \in \mathbb{T}_j} |[(\mathbf{R}^{(j)})^2]_{kk}|^2 \right)^{\frac{p}{2}} + \mathbb{E} |X_{11}|^{2p} \sum_{k \in \mathbb{T}_j} \mathbb{E} |[(\mathbf{R}^{(j)})^2]_{kk}|^p \right).$$

Applying Lemma C.5 we obtain

$$\sum_{k \in \mathbb{T}_j} \mathbb{E} |[(\mathbf{R}^{(j)})^2]_{kk}|^p \leq \frac{1}{v^p} \sum_{k \in \mathbb{T}_j} \mathbb{E} \text{Im}^p \mathbf{R}_{kk}^{(j)}.$$

Repeating the arguments in the proof of the previous lemma concludes the proof of the lemma. □

Lemma A.12. For $p \geq 2$ we have

$$|\eta_{0j}|^p \leq \frac{\operatorname{Im}^p m_n^{(j)}}{n^p v^p}.$$

Proof. The proof follows directly from Lemma C.5. \square

Lemma A.13. Assuming conditions (C1) for all $|\mathbb{J}| \leq K$ we have for $p \geq 2$

$$\mathbb{E} |\operatorname{Im} \varepsilon_{2j}^{(\mathbb{J})}|^p \leq C^p \left(\frac{p^p \mathbb{E} \operatorname{Im}^{\frac{p}{2}} m_n^{(\mathbb{J},j)}(z)}{(nv)^{\frac{p}{2}}} + \frac{p^{2p}}{n^{p(1-2\alpha)}} F_{\nu,p} \right).$$

Proof. Applying Lemma A.2 we get

$$\begin{aligned} \mathbb{E} |\operatorname{Im} \varepsilon_{2j}^{(\mathbb{J})}|^p &\leq \frac{C^p}{n^p} \left[p^p \mathbb{E} \left(\sum_{k \in \mathbb{T}_{\mathbb{J},j}} \sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\operatorname{Im} \mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} \right. \\ &\quad \left. + \mu_p p^{\frac{3p}{2}} \sum_{k \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\operatorname{Im} \mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} + \mu_p^2 p^{2p} \sum_{k \neq l \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} |\operatorname{Im} \mathbf{R}_{lk}^{(\mathbb{J},j)}|^p \right]. \end{aligned}$$

Since

$$\operatorname{Im} \mathbf{R}_{kl}^{(\mathbb{J},j)} = v [\mathbf{R}^{(\mathbb{J},j)} (\mathbf{R}^{(\mathbb{J},j)})^*]_{kl}$$

it follows that

$$\begin{aligned} \sum_{k \neq l \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} |\operatorname{Im} \mathbf{R}_{lk}^{(\mathbb{J},j)}|^p &\leq n \sum_{k \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} \operatorname{Im}^p \mathbf{R}_{kk}^{(\mathbb{J},j)}, \\ \mathbb{E} \left(\sum_{k \in \mathbb{T}_{\mathbb{J},j}} \sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\operatorname{Im} \mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} &\leq \frac{n^{\frac{p}{2}}}{v^{\frac{p}{2}}} \mathbb{E} \operatorname{Im}^{\frac{p}{2}} m_n^{(\mathbb{J},j)}(z) \end{aligned}$$

and

$$\sum_{k \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j,k}} |\operatorname{Im} \mathbf{R}_{lk}^{(\mathbb{J},j)}|^2 \right)^{\frac{p}{2}} \leq n^{\frac{p}{2}} \sum_{k \in \mathbb{T}_{\mathbb{J},j}} \mathbb{E} \operatorname{Im}^p \mathbf{R}_{kk}^{(\mathbb{J},j)}.$$

Since

$$\mu_p \leq \mu_4 D^{p-4} n^{\alpha(p-4)}$$

we get the statement of the lemma. \square

Lemma A.14. Assuming conditions (C0) we have for $p \geq 2$

$$\mathbb{E} |\operatorname{Im} \varepsilon_{3j}^{(\mathbb{J})}|^p \leq \frac{C^p p^p}{n^{p(1-2\alpha)}} F_{1,p}.$$

Proof. Recall that

$$\operatorname{Im} \varepsilon_{3j}^{(\mathbb{J})} = \frac{1}{n} \sum_{l \in \mathbb{T}_{\mathbb{J},j}} (X_{jl}^2 - 1) \operatorname{Im} \mathbf{R}_l^{(\mathbb{J},j)}.$$

Applying Rosenthal's inequality we obtain

$$\mathbb{E} |\operatorname{Im} \varepsilon_{3j}^{(\mathbb{J})}|^p \leq \frac{C^p}{n^p} \left(p^{\frac{p}{2}} \mathbb{E} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j}} \operatorname{Im}^2 \mathbf{R}_l^{(\mathbb{J},j)} \right)^{\frac{p}{2}} + p^p \mu_{2p} \sum_{l \in \mathbb{T}_{\mathbb{J},j} } \mathbb{E} \operatorname{Im}^p \mathbf{R}_l^{(\mathbb{J},j)} \right).$$

Since

$$\mu_{2p} \leq \mu_4 D^{2p-4} n^{\alpha(2p-4)}$$

we get that

$$\mathbb{E} |\operatorname{Im} \varepsilon_{3j}^{(\mathbb{J})}|^p \leq \frac{C^p}{n^p} \left(p^{\frac{p}{2}} \mathbb{E} \left(\sum_{l \in \mathbb{T}_{\mathbb{J},j} } \operatorname{Im}^2 \mathbf{R}_l^{(\mathbb{J},j)} \right)^{\frac{p}{2}} + p^p n^{\alpha(2p-4)} \sum_{l \in \mathbb{T}_{\mathbb{J},j} } \mathbb{E} \operatorname{Im}^p \mathbf{R}_l^{(\mathbb{J},j)} \right).$$

Thus we arrive at the following bound

$$\mathbb{E} |\operatorname{Im} \varepsilon_{3j}^{(\mathbb{J})}|^p \leq C^p F_{\nu,p} \left(\frac{p^{\frac{p}{2}}}{n^{\frac{p}{2}}} + \frac{p^p}{n^{p(1-2\alpha)}} \right).$$

□

APPENDIX B. AUXILIARY LEMMAS I

Recall the notations $\Lambda_n := \Lambda_n(z) := m_n(z) - s(z)$, $\Lambda_n^{(j)} := m_n^{(j)}(z) - s(z)$ and

$$T_n := \frac{1}{n} \sum_{j=1}^n \varepsilon_j \mathbf{R}_{jj},$$

where $\varepsilon_j = \varepsilon_{1j} + \varepsilon_{2j} + \varepsilon_{3j} + \varepsilon_{4j}$ and $\varepsilon_{\alpha j}$, $\alpha = 1, 2, 3, 4$ are defined in (2.2). Recall the identity

$$T_n(z) = (z + m_n(z) + s(z)) \Lambda_n(z), \tag{B.1}$$

as well as the notations

$$b(z) = z + 2s(z) \text{ and } b_n(z) = b(z) + \Lambda_n(z).$$

The following lemma plays a crucial role in the proof of Theorem 1.1. It has been proved in [5][Proposition 2.2]. For the readers convenience we include its short proof below.

Lemma B.1. *For all $v > 0$ and $u \leq 2 + v$*

$$|\Lambda_n| \leq C \min \left\{ \frac{|T_n|}{|b(z)|}, \sqrt{|T_n|} \right\}. \tag{B.2}$$

Moreover, for all $v > 0$ and $u \in \mathbb{R}$

$$|\operatorname{Im} \Lambda_n| \leq C \min \left\{ \frac{|T_n|}{|b(z)|}, \sqrt{|T_n|} \right\} \tag{B.3}$$

and

$$\min(|\Lambda_n|, |b_n(z)|) \leq C\sqrt{|T_n|}. \quad (\text{B.4})$$

Proof. We start the proof from the following identity, see representation (2.2),

$$m_n(z) = -\frac{1}{z + m_n(z)} + \frac{T_n}{z + m_n(z)}.$$

Solving this quadratic equation we get that

$$m_n(z) = -\frac{z}{2} + \sqrt{\frac{z^2}{4} - 1 + T_n}.$$

The Stieltjes transform of the semicircle law may be written explicitly

$$s(z) = -\frac{z}{2} + \sqrt{\frac{z^2}{4} - 1}.$$

From the last two equation we conclude that

$$\Lambda_n = \sqrt{\frac{z^2}{4} - 1 + T_n} - \sqrt{\frac{z^2}{4} - 1}.$$

In what follows we will use the the following additional notation

$$a := \frac{z^2}{4} - 1.$$

It is easy to see that $a = (s(z) + \frac{z}{2})^2 = b^2(z)/4$.

We start from the proof of (B.4) since it is trivial. If $|b_n(z)| \leq C\sqrt{|T_n|}$ there is nothing to prove. In the opposite case we get

$$|\Lambda_n| \leq \frac{|T_n|}{|b_n(z)|} \leq C\sqrt{|T_n|}.$$

Now we establish inequality (B.3). First we show that $|\text{Im } \Lambda_n| \leq C\sqrt{|T_n|}$. Let us consider several cases:

I) $|a| \leq 2|T_n|$. In this situation

$$|\text{Im } \sqrt{a + T_n} - \text{Im } \sqrt{a}| \leq \sqrt{|a + T_n|} + \sqrt{|a|} \leq (\sqrt{3} + \sqrt{2})\sqrt{|T_n|}. \quad (\text{B.5})$$

II) $|a| > 2|T_n|$. We split this case into several sub cases

II) 1. $\text{Re } a < 0$. Since we always take the branch with the positive imaginary part we may write

$$\text{Im } \sqrt{a + T_n} + \text{Im } \sqrt{a} \geq \text{Im } \sqrt{a} \geq \frac{\sqrt{2}}{2}|a|^{1/2} \geq \sqrt{|T_n|}. \quad (\text{B.6})$$

The last inequality implies that

$$|\operatorname{Im} \sqrt{a + T_n} - \operatorname{Im} \sqrt{a}| \leq \frac{|T_n|}{|\sqrt{a + T_n} + \sqrt{a}|} \leq \sqrt{|T_n|}. \quad (\text{B.7})$$

II) 2. $\operatorname{Re} a > 0$ and $\operatorname{Re}(a + T_n) < 0$. We have

$$\operatorname{Im} \sqrt{a + T_n} \geq \frac{\sqrt{2}}{2} \sqrt{|a + T_n|} \geq \frac{\sqrt{2}}{2} (|a| - |T_n|)^{1/2} \geq \frac{\sqrt{2}}{2} \sqrt{|T_n|}$$

and similarly to (B.7) we obtain

$$|\operatorname{Im} \sqrt{a + T_n} - \operatorname{Im} \sqrt{a}| \leq \sqrt{2} \sqrt{|T_n|}.$$

II) 3. $\operatorname{Re} a > 0$ and $\operatorname{Re}(a + T_n) > 0$. We again consider two cases, but both are similar.

II) 3. 1. $\operatorname{Im} a \operatorname{Im}(a + T_n) > 0$. In this situation

$$|\sqrt{a + T_n} + \sqrt{a}| \geq \sqrt{|a|} \geq \sqrt{2} \sqrt{|T_n|}. \quad (\text{B.8})$$

II) 3. 2. $\operatorname{Im} a \operatorname{Im}(a + T_n) < 0$. Since $\operatorname{Im} \sqrt{a + T_n} = \operatorname{Im} \sqrt{\overline{a + T_n}}$ we have

$$\operatorname{Im} a \operatorname{Im}(\overline{a + T_n}) < 0$$

and

$$|\sqrt{a + T_n} + \sqrt{a}| \geq \sqrt{|a|} \geq \sqrt{2} \sqrt{|T_n|}. \quad (\text{B.9})$$

Similarly to (B.7) we may conclude from (B.8) and (B.9) that

$$|\operatorname{Im} \sqrt{a + T_n} - \operatorname{Im} \sqrt{a}| \leq \sqrt{2} \sqrt{|T_n|}.$$

To finish the proof of (B.3) we need to show that

$$|\operatorname{Im} \Lambda_n| \leq C \frac{|T_n|}{\sqrt{|a|}}.$$

The proof follows by similar arguments as in the proof of $|\operatorname{Im} \Lambda_n| \leq C|T_n|^{\frac{1}{2}}$. We consider several cases:

I) $\operatorname{Re} a < 0$. In this situation one need to repeat the inequalities (B.6) and (B.7). We get

$$|\operatorname{Im} \sqrt{a + T_n} - \operatorname{Im} \sqrt{a}| \leq \frac{|T_n|}{\operatorname{Im} \sqrt{a}} \leq \sqrt{2} \frac{|T_n|}{\sqrt{|a|}}.$$

II) $\operatorname{Re} a > 0$ and $|a| < 2|T_n|$. Then

$$|\operatorname{Im} \sqrt{a + T_n} - \operatorname{Im} \sqrt{a}| \leq C \sqrt{|T_n|} = C \frac{|T_n|}{\sqrt{|T_n|}} \leq C' \frac{|T_n|}{\sqrt{|a|}}.$$

III) $\operatorname{Re} a > 0$ and $|a| > 2|T_n|$. We consider two sub cases

III) 1. $\operatorname{Re}(a + T_n) < 0$. Then

$$\operatorname{Im} \sqrt{a + T_n} \geq \frac{\sqrt{2}}{2} |a + T_n|^{1/2} \geq \frac{|a|^{1/2}}{2}$$

and it follows that

$$|\operatorname{Im} \sqrt{a + T_n} - \operatorname{Im} \sqrt{a}| \leq C \frac{|T_n|}{\sqrt{|a|}}.$$

III) 2. $\operatorname{Re}(a + T_n) > 0$. Then without loss of generality we may assume that

$$\operatorname{Im} a \operatorname{Im}(a + T_n) > 0.$$

Then

$$|\sqrt{a + T_n} + \sqrt{a}| \geq \sqrt{|a|}$$

and we conclude

$$|\operatorname{Im} \sqrt{a + T_n} - \operatorname{Im} \sqrt{a}| \leq C \frac{|T_n|}{\sqrt{|a|}}.$$

It remains to prove (B.2). Let us first suppose that $|\Lambda_n| \leq c\sqrt{|T_n|}$. Then

$$\begin{aligned} \Lambda_n &= \frac{T_n}{z + m_n(z) + s(z)} = \frac{T_n}{z + 2s(z)} + \frac{T_n \Lambda_n}{(z + 2s(z))(z + m_n(z) + s(z))} \\ &= \frac{T_n}{z + 2s(z)} + \frac{\Lambda_n^2}{z + 2s(z)} \end{aligned}$$

and we immediately get that

$$|\Lambda_n(z)| \leq C \frac{|T_n|}{\sqrt{|z^2 - 4|}}.$$

Finally it remains to prove the assumption $|\Lambda_n| \leq c\sqrt{|T_n|}$. There is nothing to prove if $|a| \leq 2|T_n|$, one need to apply the same inequalities as in (B.5) and get

$$|\sqrt{a + T_n} - \sqrt{a}| \leq \sqrt{|a + T_n|} + \sqrt{|a|} \leq (\sqrt{3} + \sqrt{2})|T_n|^{1/2}.$$

If $|a| \geq 2|T_n|$ we apply the fact that for $|u| \leq 2 + v$ there exists the constant $c > 0$ such that $|\operatorname{Im} a| \geq c \operatorname{Re} a$. Applying this fact we get

$$|\sqrt{a + T_n} - \sqrt{a}| \leq \frac{|T_n|}{|\sqrt{a + T_n} - \sqrt{a}|} \leq \frac{|T_n|}{\operatorname{Im} \sqrt{a}} \leq c' \frac{|T_n|}{|a|^{1/2}} \leq c'' |T_n|^{1/2}.$$

□

Recall that $\varphi(z) = \bar{z}|z|^{p-1}$. In the following lemma we estimate the difference between $\varphi(T_n)$ and $\varphi(\tilde{T}_n^{(j)})$.

Lemma B.2. For $p \geq 2$ and arbitrary $j \in \mathbb{T}$ we have

$$|\varphi(T_n) - \varphi(\tilde{T}_n^{(j)})| \leq p \mathbb{E}_\tau |T_n - \tilde{T}_n^{(j)}| |\tilde{T}_n^{(j)} + \tau(T_n - \tilde{T}_n^{(j)})|^{p-2},$$

where \mathbb{E}_τ is a mathematical expectation with respect to the uniformly distributed on $[0, 1]$ random variable τ .

Proof. The proof follows from the Newton-Leibniz formula applied to

$$\hat{\varphi}(x) = \varphi(\tilde{T}_n^{(j)} + x(T_n - \tilde{T}_n^{(j)})), \quad x \in [0, 1],$$

and

$$|\hat{\varphi}'(x)| \leq p |\tilde{T}_n^{(j)} + x(T_n - \tilde{T}_n^{(j)})|^{p-2}.$$

□

The proofs of the following two lemmas are rather straightforward, but will be used many times in the proof of Theorem 2.1.

Lemma B.3. Let us assume that for all $p > q \geq 1$ and $a, b > 0$ the following inequality holds

$$x^p \leq a + bx^q. \tag{B.10}$$

Then

$$x^p \leq 2^{\frac{p}{p-q}} (a + b^{\frac{p}{p-q}}).$$

Proof. The proof is easy. We may assume that $x > a^{\frac{1}{p}}$ since in the opposite case the inequality is trivial. Dividing both parts of (B.10) by x^q we obtain

$$x^{p-q} \leq a^{\frac{p-q}{p}} + b.$$

Finally we get

$$x^p \leq 2^{\frac{p}{p-q}} (a + b^{\frac{p}{p-q}}).$$

□

Lemma B.4. Let $0 < q_1 \leq q_2 \leq \dots \leq q_k < p$ and $c_j, j = 0, \dots, k$ be positive numbers such that

$$x^p \leq c_0 + c_1 x^{q_1} + c_2 x^{q_2} + \dots + c_k x^{q_k}.$$

Then

$$x^p \leq \beta \left[c_0 + c_1^{\frac{p}{p-q_1}} + c_2^{\frac{p}{p-q_2}} + \dots + c_k^{\frac{p}{p-q_k}} \right],$$

where

$$\beta := \prod_{\nu=1}^k 2^{\frac{p}{p-q_\nu}} \leq 2^{\frac{kp}{p-q_k}}.$$

Proof. Let $a_1 := c_0 + c_2 x^{q_2} + \dots + c_k x^{q_k}$ and $b_1 := c_1$. We may apply Lemma B.3 and get

$$x^p \leq 2^{\frac{p}{p-q_1}} (a_1 + b_1^{\frac{p}{p-q_1}}).$$

Repeating this step $k - 1$ times we obtain

$$x^p \leq \beta \left[c_0 + c_1^{\frac{p}{p-q_1}} + c_2^{\frac{p}{p-q_2}} + \dots + c_k^{\frac{p}{p-q_k}} \right],$$

where β is defined above. □

APPENDIX C. AUXILIARY LEMMAS II

In this section we collect all inequalities for the resolvent of the matrix \mathbf{W} .

Lemma C.1. *For any $z = u + iv \in \mathbb{C}^+$ we have for any $s \geq 1$*

$$|\mathbf{R}_{jj}^{(\mathbb{J})}(u + iv/s)| \leq s |\mathbf{R}_{jj}^{(\mathbb{J})}(u + iv)| \quad (\text{C.1})$$

and

$$\frac{1}{|u + iv/s_0 + m_n^{(\mathbb{J})}(u + iv/s_0)|} \leq \frac{s_0}{|u + iv + m_n^{(\mathbb{J})}(u + iv)|}. \quad (\text{C.2})$$

Proof. The proof of (C.1) is given in [5], but for the readers convenience we will present it here. To simplify all formulas we shall omit the index \mathbb{J} from the notation of \mathbf{R}_{jj} . Since

$$\left| \frac{d}{dv} \log \mathbf{R}_{jj}(v) \right| \leq \frac{1}{|\mathbf{R}_{jj}(v)|} \left| \frac{d}{dv} \mathbf{R}_{jj}(v) \right|. \quad (\text{C.3})$$

Furthermore,

$$\frac{d}{dv} \mathbf{R}_{jj}(v) = [\mathbf{R}^2]_{jj}(v)$$

and

$$|[\mathbf{R}^2]_{jj}(v)| \leq v^{-1} \text{Im } \mathbf{R}_{jj}.$$

Applying this inequality to (C.3) we get

$$\left| \frac{d}{dv} \log \mathbf{R}_{jj}(v) \right| \leq \frac{1}{v}.$$

This inequality implies that

$$|\log \mathbf{R}_{jj}(v) - \log \mathbf{R}_{jj}(v/s)| \leq \left| \int_{v/s}^v \frac{d}{dv} \log \mathbf{R}_{jj}(\eta) d\eta \right| \leq \log s.$$

Since the last inequality holds for the real parts of the logarithm as well, we may conclude that

$$|\mathbf{R}_{jj}(u + iv/s)| \leq s |\mathbf{R}_{jj}(u + iv)|.$$

The proof of (C.2) is similar and we omit it. □

Lemma C.2. *Let $g(v) := g(u + iv)$ be the Stieltjes transform of some distribution function $G(x)$. Then for any $s \geq 1$*

$$\operatorname{Im} g(v/s) \leq s \operatorname{Im} g(v) \quad \text{and} \quad \operatorname{Im} g(v) \leq s \operatorname{Im} g(v/s). \quad (\text{C.4})$$

Proof. Recall that

$$\operatorname{Im} g(v) = \int_{-\infty}^{\infty} \frac{v}{(x-u)^2 + v^2} dG(x).$$

Hence,

$$\left| \frac{d \operatorname{Im} g(v)}{dv} \right| = \int_{-\infty}^{\infty} \frac{|(x-u)^2 - v^2|}{((x-u)^2 + v^2)^2} dG(x) \leq \frac{1}{v} \operatorname{Im} g(v).$$

We may conclude that

$$\left| \frac{d \operatorname{Im} g(v)}{dv} \right| \leq \frac{1}{v}.$$

We may repeat now the second part of the previous lemma and get the desired bounds. \square

Lemma C.3. *For any $z = u + iv \in \mathbb{C}^+$ there exists a constant $c = c(z) > 0$ such that*

$$v \leq c \operatorname{Im} s(z) \quad (\text{C.5})$$

and

$$v \leq \frac{\operatorname{Im} \mathbf{R}_{jj}}{|\mathbf{R}_{jj}|^2}. \quad (\text{C.6})$$

Proof. It is easy to see that

$$|s(z)|^2 \leq \int_{-2}^2 \frac{g_{sc}(\lambda)}{|\lambda - z|^2} d\lambda = \frac{1}{v} \operatorname{Im} s(z).$$

Since $|s(z)|^2 \geq c^{-1}$ for some $c = c(z)$ the inequality (C.5) follows. In order to prove (C.6) one should repeat the calculations above using the following spectral representation of \mathbf{R}_{jj}

$$\mathbf{R}_{jj} = \int_{-\infty}^{\infty} \frac{1}{z - \lambda} dF_{nj}(\lambda), \quad F_{nj}(\lambda) := \sum_{k=1}^n |u_{jk}|^2 \mathbf{1}[\lambda_j \leq \lambda].$$

\square

We finish this section with two lemmas. These lemmas are proved in [21][Lemma 7.10] and [18][Lemma 7.6], but for the readers convenience we include them here.

Lemma C.4. *For any $z = u + iv \in \mathbb{C}^+$ we have*

$$\frac{1}{n} \sum_{l, k \in \mathbb{T}_J} |\mathbf{R}_{kl}^{(J)}|^2 \leq \frac{1}{v} \operatorname{Im} m_n^{(J)}(z). \quad (\text{C.7})$$

For any $l \in \mathbb{T}_J$

$$\sum_{k \in \mathbb{T}_J} |\mathbf{R}_{kl}^{(J)}|^2 \leq \frac{1}{v} \operatorname{Im} \mathbf{R}_{ll}^{(J)}. \quad (\text{C.8})$$

Proof. We denote by $u_k^{(\mathbb{J})} := (u_{lk}^{(\mathbb{J})})_{l \in \mathbb{T}_{\mathbb{J}}}$ the eigenvector of $\mathbf{W}^{(\mathbb{J})}$ corresponding to the eigenvalue $\lambda_k^{(\mathbb{J})}$. It follows from the eigenvector decomposition that

$$\mathbf{R}_{kl}^{(\mathbb{J})} = \sum_{s \in \mathbb{T}_{\mathbb{J}}} \frac{1}{\lambda_s^{(\mathbb{J})} - z} u_{ks}^{(\mathbb{J})} u_{ls}^{(\mathbb{J})}. \quad (\text{C.9})$$

Since $\mathbf{U} := [u_{lk}^{(\mathbb{J})}]_{l,k \in \mathbb{T}_{\mathbb{J}}}$ is a unitary matrix we get

$$\frac{1}{n} \sum_{l,k \in \mathbb{T}_{\mathbb{J}}} |\mathbf{R}_{kl}^{(\mathbb{J})}|^2 \leq \frac{1}{n} \sum_{s \in \mathbb{T}_{\mathbb{J}}} \frac{1}{|\lambda_s^{(\mathbb{J})} - z|^2} \leq \frac{1}{v} \text{Im } m_n^{(\mathbb{J})}(z).$$

To prove (C.8) we may conclude from (C.9) that

$$\sum_{k \in \mathbb{T}_{\mathbb{J}}} |\mathbf{R}_{kl}^{(\mathbb{J})}|^2 \leq \sum_{s \in \mathbb{T}_{\mathbb{J}}} \frac{|u_{ls}^{(\mathbb{J})}|^2}{|\lambda_s^{(\mathbb{J})} - z|^2} = \frac{1}{v} \text{Im} \left(\sum_{s \in \mathbb{T}_{\mathbb{J}}} \frac{|u_{ls}^{(\mathbb{J})}|^2}{\lambda_s^{(\mathbb{J})} - z} \right) = \frac{1}{v} \text{Im } \mathbf{R}_{ll}^{(\mathbb{J})}.$$

□

Lemma C.5. *For any $z = u + iv \in \mathbb{C}^+$ we have*

$$\frac{1}{n} |\text{Tr}(\mathbf{R}^{(\mathbb{J})})^2| \leq \frac{1}{v} \text{Im } m_n^{(\mathbb{J})}(z), \quad (\text{C.10})$$

$$\frac{1}{n} \sum_{k,l \in \mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \leq \frac{1}{v^3} \text{Im } m_n^{(\mathbb{J})}(z), \quad (\text{C.11})$$

$$\frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{kk}|^2 \leq \frac{1}{v^3} \text{Im } m_n^{(\mathbb{J})}(z), \quad (\text{C.12})$$

$$\frac{1}{n} \sum_{k \in \mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{kk}|^p \leq \frac{1}{nv^p} \sum_{k \in \mathbb{T}_{\mathbb{J}}} \text{Im}^p \mathbf{R}_{kk}^{(\mathbb{J})}(z) \text{ for any } p \geq 1. \quad (\text{C.13})$$

For any $l \in \mathbb{T}_{\mathbb{J}}$

$$\sum_{k \in \mathbb{T}_{\mathbb{J}}} |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \leq \frac{1}{v^3} \text{Im } \mathbf{R}_{ll}^{(\mathbb{J})}. \quad (\text{C.14})$$

Proof. The proof of (C.10) follows from

$$\frac{1}{n} |\text{Tr}(\mathbf{R}^{(\mathbb{J})})^2| \leq \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \frac{1}{|\lambda_j^{(\mathbb{J})} - z|^2} \leq \frac{1}{v} \text{Im } m_n^{(\mathbb{J})}(z).$$

We denote by $u_k^{(\mathbb{J})} := (u_{lk}^{(\mathbb{J})})_{l \in \mathbb{T}_{\mathbb{J}}}$ the eigenvector of $\mathbf{W}^{(\mathbb{J})}$ corresponding to the eigenvalue $\lambda_k^{(\mathbb{J})}$. It follows from the eigenvector decomposition that

$$[(\mathbf{R}^{(\mathbb{J})})^2]_{kl} = \sum_{s \in \mathbb{T}_{\mathbb{J}}} \frac{1}{(\lambda_s^{(\mathbb{J})} - z)^2} u_{ks}^{(\mathbb{J})} u_{ls}^{(\mathbb{J})}. \quad (\text{C.15})$$

Since $\mathbf{U} := [u_{lk}^{(\mathbb{J})}]_{l,k \in \mathbb{T}_J}$ is a unitary matrix we get

$$\frac{1}{n} \sum_{l,k \in \mathbb{T}_J} |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \leq \frac{1}{n} \sum_{s \in \mathbb{T}_J} \frac{1}{|\lambda_s^{(\mathbb{J})} - z|^4} \leq \frac{1}{v^3} \operatorname{Im} \left(\frac{1}{n} \sum_{s \in \mathbb{T}_J} \frac{1}{\lambda_s^{(\mathbb{J})} - z} \right) = \frac{1}{v^3} \operatorname{Im} m_n^{(\mathbb{J})}(z).$$

This proves (C.11). Inequality (C.12) follows from (C.8) and observation that

$$[(\mathbf{R}^{(\mathbb{J})})^2]_{kk} = \sum_{l \in \mathbb{T}_J} (\mathbf{R}_{lk}^{(\mathbb{J})})^2.$$

The proof of (C.13) is similar. To prove (C.14) we may conclude from (C.15) that

$$\sum_{k \in \mathbb{T}_J} |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \leq \sum_{s \in \mathbb{T}_J} \frac{|u_{ls}^{(\mathbb{J})}|^2}{|\lambda_s^{(\mathbb{J})} - z|^4} \leq \frac{1}{v^3} \operatorname{Im} \mathbf{R}_{ll}^{(\mathbb{J})}.$$

□

APPENDIX D. TRUNCATION OF MATRIX ENTRIES

In this section we will show that the conditions **(C0)** allows us to assume that for all $1 \leq j, k \leq n$ we have $|X_{jk}| \leq Dn^\alpha$, where D is some positive constant and

$$\alpha = \frac{2}{4 + \delta}.$$

Let $\hat{X}_{jk} := X_{jk} \mathbf{1}[|X_{jk}| \leq Dn^\alpha]$, $\tilde{X}_{jk} := X_{jk} \mathbf{1}[|X_{jk}| \geq Dn^\alpha] - \mathbb{E} X_{jk} \mathbf{1}[|X_{jk}| \geq Dn^\alpha]$ and finally $\check{X}_{jk} := \tilde{X}_{jk} \sigma^{-1}$, where $\sigma^2 := \mathbb{E} |\tilde{X}_{11}|^2$. We denote symmetric random matrices by $\hat{\mathbf{X}}$, $\tilde{\mathbf{X}}$ and $\check{\mathbf{X}}$ formed from \hat{X}_{jk} , \tilde{X}_{jk} and \check{X}_{jk} respectively. Similar notations are used for the resolvent matrices and corresponding Stieltjes transforms.

Lemma D.1. *Assuming the conditions of Theorem 1.1 we have*

$$\mathbb{E} |m_n(z) - \hat{m}_n(z)|^p \leq \left(\frac{Cp}{nv} \right)^p.$$

Proof. From Bai's rank inequality (see [3][Theorem A.43]) we conclude that

$$\sup |F_n(x) - \hat{F}_n(x)| \leq \frac{1}{n} \operatorname{Rank}(\mathbf{X} - \hat{\mathbf{X}}) \leq \frac{1}{n} \sum_{j,k=1}^n \mathbf{1}[|X_{jk}| \geq Dn^\alpha].$$

Integrating by parts we get

$$\mathbb{E} |m_n(z) - \hat{m}_n(z)|^p \leq \frac{1}{(nv)^p} \mathbb{E} \left(\sum_{j,k=1}^n \mathbf{1}[|X_{jk}| \geq Dn^\alpha] \right)^p.$$

It is easy to see that

$$\left(\sum_{j,k=1}^n \mathbb{E} \mathbf{1}[|X_{jk}| \geq Dn^\alpha] \right)^p \leq C^p.$$

Applying Rosenthal's inequality (Theorem A.1) we get that

$$\begin{aligned} & \mathbb{E} \left(\sum_{j,k=1}^n [\mathbf{1}[|X_{jk}| \geq Dn^\alpha] - \mathbb{E} \mathbf{1}[|X_{jk}| \geq Dn^\alpha]] \right)^p \\ & \leq C^p p^p \left(\left(\frac{1}{n^2} \sum_{j,k=1}^n E|X_{jk}|^{4+\delta} \right)^{\frac{p}{2}} + \frac{1}{n^2} \sum_{j,k=1}^n E|X_{jk}|^{4+\delta} \right) \leq C^p p^p. \end{aligned}$$

From these inequalities we may conclude the statement of Lemma. \square

Lemma D.2. *Assuming the conditions of Theorem 1.1 we have*

$$\mathbb{E} |\tilde{m}_n(z) - \check{m}_n(z)|^p \leq \frac{C^p p^p \mathcal{A}^p(2p)}{(nv)^p}.$$

Proof. It is easy to see that

$$\tilde{\mathbf{R}}(z) = (\tilde{\mathbf{W}} - z\mathbf{I})^{-1} = \sigma^{-1}(\check{\mathbf{W}} - z\sigma^{-1}\mathbf{I})^{-1} = \sigma^{-1}\check{\mathbf{R}}(\sigma^{-1}z). \quad (\text{D.1})$$

Applying the resolvent equality we get

$$\check{\mathbf{R}}(z) - \check{\mathbf{R}}(\sigma^{-1}z) = (z - \sigma^{-1}z)\check{\mathbf{R}}(z)\check{\mathbf{R}}(\sigma^{-1}z). \quad (\text{D.2})$$

From (D.1) and (D.2) we may conclude

$$\begin{aligned} |\tilde{m}_n(z) - \check{m}_n(z)| &= \frac{1}{n} |\text{Tr } \tilde{\mathbf{R}}(z) - \text{Tr } \check{\mathbf{R}}(z)| = \frac{1}{n} |\sigma^{-1} \text{Tr } \check{\mathbf{R}}(\sigma^{-1}z) - \text{Tr } \check{\mathbf{R}}(z)| \\ &= \frac{1}{n} |\sigma^{-1} \text{Tr } \check{\mathbf{R}}(z) - \text{Tr } \check{\mathbf{R}}(z) - (z - \sigma^{-1}z) \text{Tr } \check{\mathbf{R}}(z)\check{\mathbf{R}}(\sigma^{-1}z)| \\ &\leq \frac{1}{n} (\sigma^{-1} - 1) |\text{Tr } \check{\mathbf{R}}(z)| + (\sigma^{-1} - 1) \frac{|z|}{n} |\text{Tr } \check{\mathbf{R}}(z)\check{\mathbf{R}}(\sigma^{-1}z)|. \end{aligned}$$

Taking the p -th power and mathematical expectation we get

$$\mathbb{E} |\tilde{m}_n(z) - \check{m}_n(z)|^p \leq \frac{1}{n^p} (\sigma^{-1} - 1)^p \mathbb{E} |\text{Tr } \check{\mathbf{R}}(z)|^p + (\sigma^{-1} - 1)^p \frac{C^p}{n^p} \mathbb{E} |\text{Tr } \check{\mathbf{R}}(z)\check{\mathbf{R}}(\sigma^{-1}z)|^p.$$

Since $\check{\mathbf{X}}$ satisfies conditions (C1) we may apply Lemma 4.1 and conclude

$$\frac{1}{n^p} \mathbb{E} |\text{Tr } \check{\mathbf{R}}(z)|^p \leq C_0^p.$$

We also have

$$\sigma^{-1} - 1 \leq \sigma^{-1}(1 - \sigma) \leq \sigma^{-1}(1 - \sigma^2) \leq \sigma^{-1} \mathbb{E} |X_{jk}|^2 \mathbf{1}[|X_{jk}| \geq Dn^\alpha] \leq \frac{C}{n}. \quad (\text{D.3})$$

To finish the proof it remains to estimate the term

$$\frac{1}{n^p} \mathbb{E} |\text{Tr } \check{\mathbf{R}}(z)\check{\mathbf{R}}(\sigma^{-1}z)|^p.$$

Applying the obvious inequality $|\operatorname{Tr} \mathbf{A}\mathbf{B}| \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$ we get

$$\begin{aligned} \frac{1}{n^p} \mathbb{E} |\operatorname{Tr} \check{\mathbf{R}}(z) \check{\mathbf{R}}(\sigma^{-1}z)|^p &\leq \frac{1}{n^p} \mathbb{E}^{\frac{1}{2}} \|\check{\mathbf{R}}(z)\|_2^{2p} \mathbb{E}^{\frac{1}{2}} \|\check{\mathbf{R}}(\sigma^{-1}z)\|_2^{2p} \\ &\leq \frac{\mathbb{E}^{\frac{1}{2}} \operatorname{Im}^p \check{m}_n(z) \mathbb{E}^{\frac{1}{2}} \operatorname{Im}^p \check{m}_n(\sigma^{-1}z)}{v^p}. \end{aligned}$$

From this inequality and (D.3) we conclude the statement of the lemma. \square

Lemma D.3. *Assuming the conditions of Theorem 1.1 we have*

$$\mathbb{E} |\tilde{m}_n(z) - \hat{m}_n(z)|^p \leq \left(\frac{C}{nv} \right)^{\frac{3p}{2}}.$$

Proof. It is easy to see that

$$\tilde{m}_n(z) - \hat{m}_n(z) = \frac{1}{n} \operatorname{Tr}(\tilde{\mathbf{W}} - \hat{\mathbf{W}}) \hat{\mathbf{R}} \tilde{\mathbf{R}}.$$

Applying the obvious inequalities $|\operatorname{Tr} \mathbf{A}\mathbf{B}| \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$ and $\|\mathbf{A}\mathbf{B}\|_2 \leq \|\mathbf{A}\| \|\mathbf{B}\|_2$ we get

$$|\tilde{m}_n(z) - \hat{m}_n(z)| \leq \|\tilde{\mathbf{W}} - \hat{\mathbf{W}}\|_2 \|\hat{\mathbf{R}}\|_2 \|\tilde{\mathbf{R}}\| = \|\mathbb{E} \hat{\mathbf{W}}\|_2 \|\tilde{\mathbf{R}}\|_2 \|\hat{\mathbf{R}}\|.$$

From

$$|\mathbb{E} \hat{X}_{jk}| = |\mathbb{E} X_{jk} \mathbf{1}[|X_{jk}| \geq Dn^\alpha]| \leq \frac{C}{n^{\frac{2(3+\delta)}{4+\delta}}}$$

we obtain

$$\|\mathbb{E} \hat{\mathbf{W}}\|_2 \leq \frac{C}{n^{\frac{8+3\delta}{2(4+\delta)}}}.$$

By Lemma D.2 we know $\mathbb{E} |\tilde{m}_n(z)|^p \leq C^p$. This implies that

$$\frac{1}{n^{\frac{p}{2}}} \mathbb{E} \|\tilde{\mathbf{R}}\|_2^p \leq \frac{C^p}{v^{\frac{p}{2}}}.$$

Finally

$$\mathbb{E} |\tilde{m}_n(z) - \hat{m}_n(z)|^p \leq \frac{C^p}{v^{\frac{3p}{2}} n^{\frac{p}{2}} n^{\frac{(8+3\delta)p}{2(4+\delta)}}} \leq \left(\frac{C}{nv} \right)^{\frac{3p}{2}}.$$

\square

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