THE PRIME SPECTRA OF RELATIVE STABLE MODULE CATEGORIES

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To Dave Benson on the occasion of his 60th birthday

Abstract. For a finite group G and an arbitrary commutative ring R, Benson, Iyengar and Krause have defined a Frobenius exact structure on the category of finitely generated RG-modules by letting the exact sequences be those that split on restriction to the trivial subgroup. The corresponding stable category has a tensor triangulated structure. In this paper we examine the case where the coefficient ring R is $\mathbb{Z}/p^n$, showing that the prime ideal spectrum (in the sense of Balmer) of the relative stable category of RG is a disjoint union of n copies of that for $\mathbb{F}_pG$.

Contents

1. Introduction

Let G be a finite group and k a field whose characteristic divides the order of G. One of the main goals in modular representation theory is to try to understand the stable module category $\mathsf{stmod}\, kG$ of kG. The objects in $\mathsf{stmod}\, kG$ are the same as those in the category $\mathsf{mod}\, kG$ of finitely generated kG-modules. If M and N are finitely generated kG-modules, then the morphisms from M to N in $\mathsf{stmod}\, kG$ are elements of the quotient

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N),$$

where $\mathrm{PHom}_{kG}(M,N)$ denotes the set of kG-module homomorphisms $M \to N$ that factor through some projective kG-module. In this case $\mathsf{mod}\, kG$ is a *Frobenius category,* and so $\mathsf{stmod}\, kG$ is triangulated. Recall that the suspension of an object M in $\mathsf{stmod}\, kG$ is defined

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to be the cosyzygy $\Omega^{-1}M$, i.e., the cokernel of the inclusion of M into its injective hull. Distinguished triangles

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow \Omega^{-1}M'$$

in $\mathsf{stmod}\, kG$ are, by definition, induced by short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathsf{mod}\, kG$. Further details may be found in [6], for example. Moreover, for every object N in $\mathsf{stmod}\, kG$, the functor $- \otimes_k N$ takes exact triangles to exact triangles. The symmetric monoidal structure induced by tensoring over k therefore gives $\mathsf{stmod}\, kG$ the structure of a *tensor triangulated* category.

Now suppose that we replace k with an arbitrary commutative ring R. Can one hope to study the category $\mathsf{mod}\, RG$ by mimicking the above setup? The first obstruction to doing so is the fact that $\mathsf{mod}\, RG$ may no longer be Frobenius, in which case $\mathsf{stmod}\, RG$ would fail to be triangulated in the usual way. Even if $\mathsf{mod}\, RG$ were Frobenius, there would still be no guarantee that tensoring over R would be exact, that is, $\mathsf{stmod}\, RG$ might not be tensor triangulated.

Recently, Benson, Iyengar and Krause [5] have examined an alternative exact structure on $\mathsf{mod}\, RG$. Let $\iota\colon R \hookrightarrow RG$ denote the inclusion of the ground ring R and define a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of RG-modules to be admissible if the restriction

$$0 \longrightarrow \iota_*M' \longrightarrow \iota_*M \longrightarrow \iota_*M'' \longrightarrow 0$$

is a split short exact sequence of R-modules. In this paper we will denote the category of finitely generated RG-modules endowed with this relatively split exact structure by $\mathsf{rel}\, RG$. It was shown in [5] that $\mathsf{rel}\, RG$ *is* a Frobenius category, so its stable category $\mathsf{strel}\, RG$ is triangulated. It turns out that the injective/projective objects in $\mathsf{rel}\, RG$ are precisely the direct summands of those RG-modules lying in the image of the induction functor

$$\iota^*\colon\ \mathsf{mod}\, R \longrightarrow \mathsf{mod}\, RG.$$

As in [5], we shall call such modules *weakly projective.* The suspension of an object M in $\mathsf{strel}\, RG$ is the cokernel ΣM in the R-split short exact sequence

$$0 \longrightarrow M \longrightarrow \iota^*\iota_*M \longrightarrow \Sigma M \longrightarrow 0,$$

where the left hand map $M \hookrightarrow \iota^*\iota_*M = RG \otimes_R M$ is given by $m \mapsto \sum_{g\in G} g \otimes g^{-1}m$. A nice feature of the relative stable category that follows directly from the definition is that in the special case where $R = k$ is a field, $\mathsf{strel}\, kG$ and $\mathsf{stmod}\, kG$ coincide.

Another motivation for appealing to this relative exact structure is that tensoring over R preserves R-split short exact sequences, so $\mathsf{strel}\, RG$ is tensor triangulated with unit the trivial module R. The relative stable category therefore provides a setting in which one may exploit a monoidal structure to study representations of G over arbitrary commutative rings.

With the relative stable category in mind, we now recall that Balmer [1] has developed a general framework for studying the coarse structure of essentially small tensor triangulated categories in terms of supports, namely the yoga of *tensor triangular geometry.* Let $(\mathcal{K}, \otimes, \mathbf{1})$ be an essentially small tensor triangulated category. A thick subcategory $\mathcal{I}$ of $\mathcal{K}$ is a *tensor ideal* of $\mathcal{K}$ if $\mathcal{K} \otimes \mathcal{I} \subseteq \mathcal{I}$. Continuing the analogy with commutative algebra, Balmer defines a proper tensor ideal $\mathcal{P}$ of $\mathcal{K}$ to be *prime* if whenever x and y are objects in $\mathcal{K}$ satisfying $x \otimes y \in \mathcal{P}$, then $x \in \mathcal{P}$ or $y \in \mathcal{P}$. The collection of prime ideals of $\mathcal{K}$ is called the *(prime ideal) spectrum* of $\mathcal{K}$, denoted $\operatorname{Spc} \mathcal{K}$.

The prime ideal spectrum has a *Zariski topology* whose closed basis elements are of the form

$$\operatorname{supp}(x) = \{\mathcal{P} \in \operatorname{Spc} \mathcal{K} \mid x \notin \mathcal{P}\} \qquad \text{for all } x \text{ in } \mathcal{K}.$$

The closed subset $\operatorname{supp}(x)$ is called the *support* of x. The topological space $\operatorname{Spc} \mathcal{K}$, along with the assignment $x \mapsto \operatorname{supp}(x)$, is something Balmer calls a *universal support data* for $\mathcal{K}$. Roughly speaking, this means that $\operatorname{Spc} \mathcal{K}$ is the best abstract setting in which to ask support theoretic questions about the category $\mathcal{K}$. For example, one may express the universality of $\operatorname{Spc} \mathcal{K}$ by interpreting it as the space dual to the lattice of (radical) thick tensor ideals; therefore any question about the structure of the lattice of tensor ideals is really a question about $\operatorname{Spc} \mathcal{K}$.

To see this in action, we recall a famous result due to Benson, Carlson and Rickard [4], which states that the thick tensor ideals of $\mathsf{stmod}\, kG$ are in one to one correspondence with the specialisation closed subsets of $\mathsf{Proj}\, H^*(G, k)$, where $H^*(G, k)$ denotes group cohomology with coefficients in k, i.e., $\operatorname{Ext}^*_{kG}(k, k)$. Statements that relate the structure of $\mathsf{stmod}\, kG$ to the geometry of $\mathsf{Proj}\, H^*(G, k)$ constitute the realm of *support theory.* For example, the above correspondence assigns to each thick tensor ideal $\mathcal{I}$ in $\mathsf{stmod}\, kG$ the subset $\bigcup_{M \in \mathcal{I}} V_G(M)$ of $\mathsf{Proj}\, H^*(G, k)$, where $V_G(M)$ is the *support variety* of M. (See [3, Chapter 5] for details.) Viewed in Balmer's framework, this result may be reinterpreted as saying that $(\mathsf{Proj}\, H^*(G, k), V_G(-))$ is the *classifying support data* for $\mathsf{stmod}\, kG$. (See [1, Section 5].) In particular, $\operatorname{Spc}(\mathsf{stmod}\, kG)$ is homeomorphic to $\mathsf{Proj}\, H^*(G, k)$.

We remark that little is known about relative stable categories in general. The goal of this paper is to determine the prime ideal spectrum of $\mathsf{strel}\, RG$ in perhaps the most basic non-trivial case, namely that in which R is the ring $\mathbb{Z}/p^n$, where p is a prime number and $n \geq 0$. Our main result is the following.

Theorem 1.1. *Letting R_n denote the ring $\mathbb{Z}/p^n$, there is a decomposition of spectra*

$$\operatorname{Spc}(\mathsf{strel}\, R_n G) \cong \coprod_{i=1}^{n} \operatorname{Spc}(\mathsf{strel}\, \mathbb{F}_p G).$$

In other words, the prime ideal spectrum of $\mathsf{strel}\, R_n G$ is homeomorphic to n disjoint copies of the prime ideal spectrum of $\mathsf{strel}\, \mathbb{F}_p G = \mathsf{stmod}\, \mathbb{F}_p G$.

As mentioned above, the spectrum of $\mathsf{stmod}\, \mathbb{F}_p G$ is known, so the theorem gives a complete description of the spectrum of $\mathsf{strel}\, R_n G$.

This computation is also valuable from the point of view of more abstract tensor triangular geometry. There are many examples in which the spectrum has been computed, but they all tend to share a common feature—the tensor triangulated category in question is rigid. It is known that relative stable categories tend not to be rigid, hence they provide a new family of examples that can be fed back into the abstract theory. For instance, if the spectrum of a rigid tensor triangulated category is a disjoint union of subspaces, then the category itself decomposes into a direct sum of subcategories indexed by those subspaces. However, in our example the relative stable category is indecomposable. The information encoded in the triviality of the topology of the spectrum must therefore manifest in more subtle ways that would be interesting to explore.

2. Notation and preliminary calculations

Let p be a prime number, $n \geq 1$ and $R_n = \mathbb{Z}/p^n$. Throughout this paper G will denote a finite group. For convenience, we denote the group algebra R_nG by A_n. Keeping the notation from the introduction, we denote the inclusion of the base ring by $\iota\colon R_n \hookrightarrow A_n$. If $i \leq n$, then the canonical surjection $A_n \to A_i$ makes A_i into an A_n-module. We write Ω_i for the syzygy, with respect to the usual abelian structure, taken in $\mathsf{mod}\, A_i$. We remark that, in general, this is not the suspension in the stable category $\mathsf{strel}\, A_i$.

We shall not recall many details concerning the relative stable categories beyond those given in the introduction (in fact we shall not need much technology). Further details on these categories may be found in [5].

We begin by determining the injective/projective objects in $\mathsf{rel}\, A_n$, i.e., the weakly projective modules in $\mathsf{mod}\, A_n$.

Proposition 2.1. *Every weakly projective A_n-module is a direct sum of objects in*

$$\bigcup_{i=1}^{n} \mathsf{proj}\, A_i,$$

where $\mathsf{proj}\, A_i$ is the full subcategory of finitely generated projective A_i-modules.

Proof. As mentioned in the introduction, the weakly projective A_n-modules are the direct summands of the modules in

$$\{\iota^* N \mid N \in \mathsf{mod}\, R_n\}.$$

(Recall that $\iota^* N$ is the induced module $A_n \otimes_{R_n} N$.) The indecomposable R_n-modules are of the form $R_i = \mathbb{Z}/p^i$ for $1 \leq i \leq n$. The result follows by noting that $A_n \otimes_{R_n} R_i = A_i$. □

We remark that since each A_i is R_i-free, every object in $\mathsf{proj}\, A_i$ is also R_i-free.

Now let k denote the field $\mathbb{F}_p$, in other words, the ring $\mathbb{Z}/p$. The main object of focus in the sequel will be the cosyzygy $\Omega_n^{-1} k$, which is the cokernel in the short exact sequence of A_n-modules

$$0 \longrightarrow k \longrightarrow A_n \longrightarrow \Omega_n^{-1} k \longrightarrow 0,$$

the left hand map being given by $1 \mapsto p^{n-1} \sum_{g \in G} g$. We first study the behaviour of $\Omega_n^{-1} k$ under base change.

Lemma 2.2. $\Omega_n^{-1} k \otimes_{R_n} R_{n-1} \cong A_{n-1}$.

Proof. By the third isomorphism theorem we have

$$\frac{\Omega_n^{-1}k}{p^{n-1}\Omega_n^{-1}k} \cong \frac{A_n/k}{p^{n-1}A_n/k} \cong \frac{A_n}{p^{n-1}A_n} \cong A_{n-1}. \qquad \square$$

The following computational ingredient will ensure that a very special triangle exists in $\mathsf{strel}\, A_n$.

Lemma 2.3. *There exists an R_n-split short exact sequence of A_n-modules*

$$0 \longrightarrow \Omega_n R_n \longrightarrow \Omega_n k \longrightarrow R_{n-1} \longrightarrow 0. \tag{1}$$

Proof. The submodule $\Omega_n k$ of A_n is the collection of elements $\sum_{g\in G} r_g g$ satisfying

$$\sum_{g\in G} r_g \in pR_n.$$

Consider the surjective A_n-module homomorphism $\phi\colon \Omega_n k \to R_{n-1}$ given by

$$\sum_{g\in G} r_g g \longmapsto \frac{1}{p}\sum_{g\in G} r_g \pmod{p^{n-1}}.$$

The kernel of ϕ is the collection of elements $\sum_{g\in G} r_g g$ satisfying $\sum_{g\in G} r_g = 0$, which we identify with $\Omega_n R_n$. This establishes the existence of the short exact sequence (1). The fact that it is R_n-split follows from the fact that $\Omega_n R_n$ is injective as an R_n-module. $\square$

Prior to giving our next result, we remark that the duality functor $\mathrm{Hom}_{R_n}(-, R_n)$ preserves R_n-split short exact sequences in $\mathsf{mod}\, A_n$.

Lemma 2.4. *There exists an R_n-split short exact sequence of A_n-modules*

$$0 \longrightarrow R_{n-1} \longrightarrow \Omega_n^{-1} k \longrightarrow \Omega_n^{-1} R_n \longrightarrow 0. \tag{2}$$

Proof. Apply $\mathrm{Hom}_{R_n}(-, R_n)$ to the R_n-split short exact sequence (1). $\square$

3. Localisation sequences and splitting of spectra

We now turn our attention to the stable category $\mathsf{strel}\, A_n$, which we shall denote by $\mathcal{D}_n$ for the sake of brevity. We will denote the tensor unit $R_n \in \mathcal{D}_n$ by $\mathbf{1}_n$, and the suspension in $\mathcal{D}_n$ will be denoted by Σ. Observe that it is not necessary to specify in which category Σ is suspension. Indeed, if $i \le n$ and $X \in \mathcal{D}_i$, then the suspension of X in $\mathcal{D}_i$ is equal to that in $\mathcal{D}_n$. This follows from the fact that a short exact sequence in $\mathsf{mod}\, A_i$ is R_i-split if and only if it is R_n-split.

An essential tool used in proving Theorem 1.1 will be that of a semi-orthogonal decomposition of a tensor triangulated category. Recall that a *localisation sequence*

$$\mathcal{R} \underset{\psi^!}{\overset{\psi_*}{\rightleftarrows}} \mathcal{S} \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{T}.$$

of triangulated categories is a couple of adjoint pairs

$$\psi_* \dashv \psi^! \qquad \text{and} \qquad \phi^* \dashv \phi_*$$

such that ψ_* and ϕ_* are fully faithful exact functors satisfying

$$\phi_*\mathcal{T} = (\psi_*\mathcal{R})^\perp \qquad \text{and} \qquad \psi_*\mathcal{R} = {}^\perp(\phi_*\mathcal{T}).$$

This is sometimes called a *semi-orthogonal decomposition* of $\mathcal{S}$.

For our purposes we also require that $\mathcal{S}$ is an essentially small tensor triangulated category and that $\mathcal{R}$ and $\mathcal{T}$ are tensor ideals of $\mathcal{S}$ under ψ_* and ϕ_*, respectively. In this very special situation one also obtains a decomposition of the prime ideal spectrum of $\mathcal{S}$. If $\mathcal{C}$ is any subcategory of $\mathcal{S}$, we define the *support* of $\mathcal{C}$ to be the specialisation closed subset

$$\operatorname{supp}_{\mathcal{S}} \mathcal{C} = \bigcup_{x\in\mathcal{C}} \operatorname{supp}_{\mathcal{S}} x$$

of $\operatorname{Spc}\mathcal{S}$.

Theorem 3.1 ([5], Theorem A.5)**.** *The subsets* $\operatorname{Spc}\mathcal{R} = \operatorname{supp}_{\mathcal{S}}\mathcal{R}$ *and* $\operatorname{Spc}\mathcal{T} = \operatorname{supp}_{\mathcal{S}}\mathcal{T}$ *of* $\operatorname{Spc}\mathcal{S}$ *are open and closed, and there is a decomposition*

$$\operatorname{Spc}\mathcal{S} = \operatorname{Spc}\mathcal{R} \coprod \operatorname{Spc}\mathcal{T}.$$

We now return to the categories $\mathcal{D}_i = \mathsf{strel}\, A_i$, $i \geq 1$. As explained in [5, Remark 6.10], each ring epimorphism $\phi_i \colon A_i \to A_{i-1}$ induces a localisation sequence

$$\mathcal{K}_i \underset{{\psi_i}^!}{\overset{{\psi_i}_*}{\rightleftarrows}} \mathcal{D}_i \underset{{\phi_i}_*}{\overset{{\phi_i}^*}{\rightleftarrows}} \mathcal{D}_{i-1}$$

where ${\phi_i}^* = - \otimes_{R_i} R_{i-1}$ and ${\phi_i}_*$ is restriction of scalars. Note that the functors ${\phi_i}^*$ and ${\psi_i}^!$ are strong monoidal, hence ${\psi_i}_*\mathcal{K}_i$ and ${\phi_i}_*\mathcal{D}_{i-1}$ are thick tensor ideals in $\mathcal{D}_i$. As usual, we identify $\mathcal{K}_i$ and $\mathcal{D}_{i-1}$ with their images under these fully faithful embeddings. A slight generalisation of the conclusion of [5, Remark 6.10] is the following.

Theorem 3.2. *Setting* $\mathcal{K}_1 = \mathcal{D}_1$*, there is a decomposition of spectra*

$$\operatorname{Spc}\mathcal{D}_n = \coprod_{i=1}^{n} \operatorname{Spc}\mathcal{K}_i.$$

Proof. This follows by induction on n and the appropriate use of Theorem 3.1. □

In order to prove Theorem 1.1, it therefore suffices to show that each $\operatorname{Spc}\mathcal{K}_i$ is homeomorphic to $\operatorname{Spc}\mathcal{D}_1$. To establish this fact, we first show that each $\mathcal{K}_i$ is equal to the thick tensor ideal

$$\mathsf{thick}^{\otimes}_{\mathcal{D}_i}(\Omega_i^{-1}k)$$

of $\mathcal{D}_i$ generated by $\Omega_i^{-1}k$. We then exhibit a monoidal equivalence between $\mathsf{thick}^{\otimes}_{\mathcal{D}_i}(\Omega_i^{-1}k)$ and $\mathcal{D}_1$.

4. The tensor ideal $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$

Given its central role in the rest of the paper, we now denote the object $\Omega_n^{-1}k$ in $\mathcal{D}_n$ by X_n for the sake of brevity.

Lemma 4.1. *The thick tensor ideal $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ is equal to the kernel of the base change functor $-\otimes \mathbf{1}_{n-1}\colon \mathcal{D}_n \to \mathcal{D}_{n-1}$. That is, in the notation of Section 3 we have*

$$\mathcal{K}_n = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n).$$

Proof. By Lemma 2.2 we have $X_n \otimes \mathbf{1}_{n-1} = 0$ in $\mathcal{D}_n$ so that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n) \subseteq \mathcal{K}_n$.

Conversely, let $X \in \mathcal{K}_n$. The R_n-split short exact sequence (2) induces a triangle

$$\mathbf{1}_{n-1} \longrightarrow X_n \longrightarrow \Sigma\mathbf{1}_n \longrightarrow$$

in $\mathcal{D}_n$. Since $X \in \mathcal{K}_n$, tensoring this triangle with X yields a triangle of the form

$$0 \longrightarrow X \otimes X_n \longrightarrow \Sigma X \longrightarrow \ .$$

It follows that $\Sigma X \cong X \otimes X_n$ so that $X \in \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$. □

Lemma 4.2. *If X lies in the kernel of $-\otimes \mathbf{1}_{n-1}$, then X is isomorphic in $\mathcal{D}_n$ to an object in $\mathsf{mod}\, A_n$ whose abelian group decomposition contains only summands of the form R_n and R_{n-1}.*

Proof. Viewing X as an A_n-module, write $\iota_* X \cong \bigoplus_{i=1}^{n} R_i^{\oplus r_i}$. Let $S = R_n^{\oplus r_n} \oplus R_{n-1}^{\oplus r_{n-1}}$ and $T = \bigoplus_{i=1}^{n-2} R_i^{\oplus r_i}$ so that $\iota_* X = S \oplus T$. Set $K = \ker \pi$, where $\pi\colon X \to X \otimes_{R_n} R_{n-1}$ is the base change homomorphism. Note that $K \subseteq S$ since p^{n-1} annihilates T. We thus have

$$\iota_*(X \otimes_{R_n} R_{n-1}) = (S/K) \oplus T.$$

By assumption, $X \otimes \mathbf{1}_{n-1} = 0$ in $\mathcal{D}_n$, so $X \otimes_{R_n} R_{n-1}$ is weakly projective. Proposition 2.1 implies that

$$X \otimes_{R_n} R_{n-1} \cong Y \oplus Z,$$

where $Y \in \mathsf{proj}\, A_{n-1}$ and Z is a direct sum of objects in $\bigcup_{i=1}^{n-2} \mathsf{proj}\, A_i$. Comparing abelian group structures, we must have

$$\iota_* Y \cong S/K \qquad \text{and} \qquad \iota_* Z \cong T.$$

These abelian group isomorphisms allow us to place A_n-module structures $\widetilde{S/K}$ and $\widetilde{T}$ on S/K and T, respectively, through which $X \otimes_{R_n} R_{n-1} = \widetilde{S/K} \oplus \widetilde{T}$.

Now consider the short exact sequence of A_n-modules

$$0 \longrightarrow X' \longrightarrow X \longrightarrow \widetilde{T} \longrightarrow 0, \tag{3}$$

where the right hand map is the composition

$$X \xrightarrow{\ \pi\ } X \otimes_{R_n} R_{n-1} \xrightarrow{\mathrm{pr}_2} \widetilde{T}.$$

Observe that we have $X' = \{m \in X \mid \pi(m) \in \ker \mathrm{pr}_2\} = S$ as abelian groups. Applying ι_* to the sequence (3) therefore yields a split short exact sequence of abelian groups

$$0 \longrightarrow S \longrightarrow \iota_* X \longrightarrow T \longrightarrow 0.$$

This means that (3) gives rise to a triangle

$$X' \longrightarrow X \longrightarrow \widetilde{T} \longrightarrow$$

in $\mathcal{D}_n$. Because $\widetilde{T} \cong Z$ is weakly projective in $\mathsf{mod}\, A_n$, we have $\widetilde{T} = 0$ in $\mathcal{D}_n$. It follows that $X' \cong X$ in $\mathcal{D}_n$. □

Recall that $\mathcal{D}_1$ is just the usual stable module category $\mathsf{stmod}\, kG$ of kG.

Lemma 4.3. *If X lies in $\mathcal{K}_n = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$, then $\Omega_n X$ lies in $\mathcal{D}_1 = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(k)$.*

Proof. We wish to show that multiplication by p annihilates $\Omega_n X$. By Lemma 4.2 we can assume that the abelian group decomposition of $\iota_* X$ is of the form $R_n^{\oplus r} \oplus R_{n-1}^{\oplus s}$ for some non-negative integers r and s. We then have

$$\iota_*(X \otimes_{R_n} R_{n-1}) \cong R_{n-1}^{\oplus(r+s)}. \tag{4}$$

On the other hand, $X \otimes_{R_n} R_{n-1}$ is known to be weakly projective. The decomposition (4) reveals that $X \otimes_{R_n} R_{n-1} \in \mathsf{proj}\, A_{n-1}$. We thus have $X \otimes_{R_n} R_{n-1} \cong \bigoplus_{j=1}^{t} A_{n-1} e_j$ for some idempotents $e_j \in A_{n-1}$. It is well known (see [7, Theorem 21.28]) that since the extension of scalars $A_n \to A_{n-1}$ is a surjective ring homomorphism with nilpotent kernel, the e_j lift to idempotents $f_j \in A_n$.

Now consider the projective module $Y = \bigoplus_{j=1}^{t} A_n f_j$. Let

$$\pi\colon X \longrightarrow X \otimes_{R_n} R_{n-1} \qquad \text{and} \qquad \phi\colon Y \longrightarrow X \otimes_{R_n} R_{n-1}$$

be the A_n-module homomorphisms induced by the extension of scalars from R_n to R_{n-1}. Because π is surjective and Y is projective, ϕ lifts to an A_n-module homomorphism

$$\psi\colon Y \longrightarrow X$$

satisfying $\pi \circ \psi = \phi$. Note that $\psi \otimes_{R_n} R_{n-1}$ is the identity on $X \otimes_{R_n} R_{n-1}$; in particular, it is surjective. It follows by Nakayama's lemma that ψ is surjective. We therefore have a short exact sequence

$$0 \longrightarrow \Omega_n X \longrightarrow Y \xrightarrow{\ \psi\ } X \longrightarrow 0.$$

But $\Omega_n X$ is also contained in the kernel of the composition $\pi \circ \psi = \phi$. The kernel of the latter is annihilated by p, hence the same is true of its submodule $\Omega_n X$. □

Lemma 4.4. *If X lies in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(k)$, then $\Omega_n^{-1} X$ lies in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$.*

Proof. Suppose that $X \in \mathcal{D}_1$ as in the statement and consider the short exact sequence

$$0 \longrightarrow X \xrightarrow{\ \phi\ } A_n^{\oplus r} \xrightarrow{\ \psi\ } \Omega_n^{-1} X \longrightarrow 0$$

defining a cosyzygy of X in $\mathsf{mod}\, A_n$. Since X lies in $\mathcal{D}_1$, we know that $\iota_* X$ is an $\mathbb{F}_p$-vector space, so the image of ϕ is contained in $p^{n-1}A_n^{\oplus r}$. This means that $\phi \otimes_{R_n} R_{n-1} = 0$, hence $\psi \otimes_{R_n} R_{n-1}$ is an isomorphism. We therefore have $\Omega_n^{-1} X \otimes_{R_n} R_{n-1} \cong A_{n-1}^{\oplus r}$, and the latter is weakly projective. This shows that $\Omega_n^{-1} X$ lies in $\mathcal{K}_n$ as claimed. □

The main theorem of this section follows by the observation that the syzygy Ω_n induces a (non-monoidal) exact autoequivalence of $\mathcal{D}_n$. Indeed, it is straightforward to check that Ω_n takes R_n-split short exact sequences in $\mathsf{mod}\, A_n$ to R_n-split short exact sequences. The same is then of course true of its quasi-inverse Ω_n^{-1}.

Theorem 4.5. *There is an equivalence of triangulated categories*

$$\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n) \cong \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(k) = \mathsf{stmod}\, \mathbb{F}_p G.$$

Proof. As mentioned prior to the theorem, Ω_n and Ω_n^{-1} are quasi-inverse autoequivalences of $\mathcal{D}_n$. The restriction of Ω_n to $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ is therefore an equivalence onto its essential image. By Lemmas 4.3 and 4.4, the latter is precisely $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(k)$. □

Although we have shown that $\mathcal{K}_n = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ is triangle equivalent to $\mathcal{D}_1$, we have not produced an equivalence of *tensor triangulated categories*. In fact, these categories are also equivalent as tensor triangulated categories, as we show in Theorem 5.6.

5. A monoidal equivalence

In Theorem 4.5 we learned that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ and $\mathcal{D}_1$ are equivalent as triangulated categories, and the latter is known to be monoidal. The localisation sequences in Section 3 show that $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n) = \mathcal{K}_n$ is equal to $\mathcal{D}_n/\mathcal{D}_{n-1}$ (as tensor ideals in $\mathcal{D}_n$), so $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ is also monoidal.

In this section we produce a monoidal exact equivalence between $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ and $\mathcal{D}_1$, namely the restriction of the functor

$$P\colon \mathcal{D}_n \longrightarrow \mathcal{D}_1$$

induced by multiplication by p^{n-1}.

Specifically, if M is an A_n-module, then as an abelian group, $P(M)$ is defined to be the A_n-submodule $p^{n-1}M$ of M. Identifying A_1 with $A_n/(p)$, we obtain an action of A_1 on $p^{n-1}M$ via

$$\overline{a}(p^{n-1}m) = p^{n-1}am \qquad \text{for all } a \in A_n.$$

This action is well defined since p annihilates $p^{n-1}M$. If $\phi\colon M \to N$ is a homomorphism of A_n-modules, then so is $\phi|_{p^{n-1}M}\colon p^{n-1}M \to N$. Moreover, for $m \in M$ we have

$$\phi(p^{n-1}m) = p^{n-1}\phi(m) \in p^{n-1}N,$$

hence $\phi|_{p^{n-1}M}$ induces a map $p^{n-1}M \to p^{n-1}N$. We therefore set $P(\phi) = \phi|_{p^{n-1}M}$.

Observe that multiplication by p^{n-1} preserves R_n-split short exact sequences in $\mathsf{mod}\, A_n$, so P is exact. We claim that P is also monoidal. To see this, let M and M' be A_n-modules and consider the map

$$\phi\colon P(M) \otimes_{R_1} P(M') \longrightarrow P(M \otimes_{R_n} M')$$

given by $p^{n-1}m \otimes p^{n-1}m' \mapsto p^{n-1}(m \otimes m')$. One readily verifies that ϕ is a well defined isomorphism of abelian groups and that the action of G commutes with ϕ.

Our strategy in proving that the restriction of P to $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ is an equivalence will be to show that the functor

$$F = \Omega_n^{-1}\Omega_1 \colon \mathcal{D}_1 \longrightarrow \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n).$$

is a quasi-inverse.

Lemma 5.1. *The composition $PF \colon \mathcal{D}_1 \to \mathcal{D}_1$ is naturally isomorphic to the identity functor.*

Proof. Let X be an object in $\mathcal{D}_1$ and consider the short exact sequence of A_1-modules

$$0 \longrightarrow \Omega_1 X \longrightarrow A_1^{\oplus r} \longrightarrow X \longrightarrow 0$$

defining a syzygy $\Omega_1 X$ in $\mathsf{mod}\, A_1$. A cosyzygy of $\Omega_1 X$ in $\mathsf{mod}\, A_n$ is then obtained via the short exact sequence

$$0 \longrightarrow \Omega_1 X \longrightarrow A_n^{\oplus r} \longrightarrow \Omega_n^{-1}\Omega_1 X \longrightarrow 0,$$

and we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_1 X & \longrightarrow & A_1^{\oplus r} & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_1 X & \longrightarrow & A_n^{\oplus r} & \longrightarrow & FX & \longrightarrow & 0, \end{array}$$

where the right two vertical arrows are those induced by multiplication by p^{n-1}. The right hand arrow therefore identifies X with the submodule PFX. $\square$

Corollary 5.2. *The functor $P \colon \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n) \to \mathcal{D}_1$ is essentially surjective and full.*

Proof. This is immediate from Lemma 5.1, combined with Lemma 4.4 which implies that the essential image of F is contained in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$. $\square$

Corollary 5.3. *The functor $P \colon \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n) \to \mathcal{D}_1$ is monoidal.*

Proof. We know from the discussion preceding Lemma 5.1 that P is monoidal from $\mathcal{D}_n$ to $\mathcal{D}_1$, thus its restriction to $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ respects tensor products. Any such functor that is also essentially surjective will automaticaly be monoidal. $\square$

Lemma 5.4. *The kernel of $P \colon \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n) \to \mathcal{D}_1$ is trivial.*

Proof. Let X be an object in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$ with PX weakly projective. By Lemma 4.2, we may assume that $\iota_* X \cong R_n^{\oplus r} \oplus R_{n-1}^{\oplus s}$ for some non-negative integers r and s. Because PX is weakly projective, we have $PX \cong \bigoplus_{j=1}^{t} A_1 e_j$ for some idempotents e_j in A_1.

The surjection $A_n \to A_1$ given by multiplication by p^{n-1} has kernel pA_n, thus it is isomorphic to the base change homomorphism $A_n \to A_n \otimes_{R_n} \mathbb{F}_p$. As in the proof of Lemma

4.3, there then exist idempotents f_j in A_n satisfying $e_j = p^{n-1}f_j$. Letting $Y = \bigoplus_{j=1}^t A_n f_j$, we obtain a natural embedding $\phi\colon PX \hookrightarrow Y$ mapping PX isomorphically onto $p^{n-1}Y$.

Note that since A_n is injective as a module over itself and Y is a direct summand of a free A_n-module, Y is also injective. (Actually, Y is the injective hull of PX.) This, along with the embedding $PX \hookrightarrow X$, allows us to extend ϕ to a morphism $\psi\colon X \to Y$.

Now let $\widetilde{\psi}$ denote the map of free R_n-modules obtained by restricting ψ to the R_n-free component $R_n^{\oplus r}$ of X. Then $p^{n-1}\widetilde{\psi} = \phi$, hence $\widetilde{\psi}$ is an isomorphism on socles. This shows that Y has rank r as a free R_n-module and that ψ is surjective. Because Y is projective, we may therefore split off a direct summand Y from X and assume that $\iota_* X \cong R_{n-1}^{\oplus s}$. We then have $X \otimes_{R_n} R_{n-1} \cong X$. But X lies in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n)$, the kernel of $- \otimes_{R_n} R_{n-1}$, hence $X \cong 0$ in $\mathcal{D}_n$. □

The final ingredient in proving that P is an equivalence is the following bit of folklore.

Lemma 5.5 ([2], Proposition 3.18)**.** *Let $G : \mathcal{K} \to \mathcal{L}$ be an essentially surjective full exact functor between triangulated categories having trivial kernel. Then G is an equivalence.*

Theorem 5.6. *The functor $P\colon \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(X_n) \to \mathcal{D}_1$ is a monoidal equivalence of triangulated categories.*

Proof. By Corollary 5.2 we know that P is full and essentially surjective. Lemma 5.4 also tells us that P has trivial kernel. It follows by Lemma 5.5 that P is an equivalence. It is monoidal by Corollary 5.3. □

We are now in a position to prove the main result.

Theorem (1.1)**.** *For every positive integer n there is a decomposition of spectra*

$$\operatorname{Spc} \mathcal{D}_n \cong \coprod_{i=1}^{n} \operatorname{Spc} \mathcal{D}_1.$$

Proof. Setting $\mathcal{K}_1 = \mathcal{D}_1$, Theorem 3.2 tells us that there is a decomposition of spectra

$$\operatorname{Spc} \mathcal{D}_n = \coprod_{i=1}^{n} \operatorname{Spc} \mathcal{K}_i.$$

By Lemma 4.1 we have $\mathcal{K}_i = \mathsf{thick}^{\otimes}_{\mathcal{D}_i}(X_i)$. Theorem 5.6 shows that the functor

$$P\colon\ \mathsf{thick}^{\otimes}_{\mathcal{D}_i}(X_i) \longrightarrow \mathcal{D}_1$$

induced by multiplication by p^{i-1} is an equivalence of tensor triangulated categories. Putting this all together, we have $\operatorname{Spc} \mathcal{K}_i \cong \operatorname{Spc} \mathcal{D}_1$ for all $1 \le i \le n$. □

The following corollary summarises the consequences of our results for $\mathcal{D}_n$.

Corollary 5.7. *The relative stable module category $\mathcal{D}_n = \mathsf{strel}\, R_n G$ admits a semi-orthogonal decomposition into n tensor ideals*

$$\mathsf{strel}\, R_n G = (\mathsf{stmod}\, \mathbb{F}_p G, \ldots, \mathsf{stmod}\, \mathbb{F}_p G),$$

where the ith copy embeds as $\mathcal{K}_i = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(\Omega_i^{-1} k)$.

Proof. This follows from the discussion in Section 3, along with Theorem 5.6 and Lemma 4.1. ☐

6. An example: cyclic groups of prime order

In this section we give a more detailed description of $\mathrm{Spc}(\mathsf{strel}\, R_n G)$ in the case where $G = C_p$, the cyclic group of prime order p. In particular, we give concrete generators for all of the prime tensor ideals.

The first handful of results actually hold for any finite group G. Recall that Ω_i denotes the syzygy taken in $\mathsf{mod}\, A_i = \mathsf{mod}\, R_i G$ under the usual abelian category structure.

Proposition 6.1. *If G is any finite group, then the A_n-modules $\Omega_i^{-1}k$ for $i \leq n$ generate $\mathcal{D}_n = \mathsf{strel}\, R_n G$ as a thick tensor ideal.*

Proof. This is immediate from Corollary 5.7. ☐

Lemma 6.2. *For all $1 \leq i < j \leq n$ we have $\Omega_i^{-1}k \otimes_{R_n} \Omega_j^{-1}k \cong A_{i-1} \oplus A_i^{\oplus(p-1)}$.*

Proof. Since p^j annihilates both R_i and R_j, we have

$$\Omega_i^{-1}k \otimes_{R_n} \Omega_j^{-1}k \cong \Omega_i^{-1}k \otimes_{R_j} \Omega_j^{-1}k.$$

We therefore compute the right hand term in $\mathsf{mod}\, A_j$. Applying $- \otimes_{R_j} \Omega_i^{-1}k$ to the short exact sequence

$$0 \longrightarrow k \xrightarrow{p^{j-1}f} A_j \longrightarrow \Omega_j^{-1}k \longrightarrow 0$$

yields the exact sequence

$$k \otimes_{R_j} \Omega_i^{-1}k \xrightarrow{p^{j-1}f\otimes 1} A_j \otimes_{R_j} \Omega_i^{-1}k \longrightarrow \Omega_j^{-1}k \otimes_{R_j} \Omega_i^{-1}k \longrightarrow 0$$

so that $\Omega_j^{-1}k \otimes_{R_j} \Omega_i^{-1}k$ is isomorphic to $A_j \otimes_{R_j} \Omega_i^{-1}k$ modulo the image of $p^{j-1}f \otimes 1$. By Frobenius reciprocity and the short exact sequence of Lemma 2.4, one sees that

$$A_j \otimes_{R_j} \Omega_i^{-1}k \cong A_{i-1} \oplus A_i^{\oplus(p-1)}. \tag{5}$$

Since $i < j$, p^{j-1} annihilates the right hand side so that the image of $p^{j-1}f \otimes 1$ is zero and

$$\Omega_j^{-1}k \otimes_{R_j} \Omega_i^{-1}k \cong A_{i-1} \oplus A_i^{\oplus(p-1)}.$$ ☐

We again denote the objects $\Omega_i^{-1}k$ in $\mathcal{D}_n$ by X_i for the sake of brevity.

Proposition 6.3. *Any prime tensor ideal in $\mathrm{Spc}\,\mathcal{D}_n$ contains at least $n-1$ objects in the set $\{X_1, \ldots, X_n\}$.*

Proof. Let $\mathcal{P} \in \mathrm{Spc}\,\mathcal{D}_n$ and suppose that there are distinct objects X_i and X_j, both of which do not lie in $\mathcal{P}$. By Lemma 6.2 we have

$$X_i \otimes X_j = 0 \in \mathcal{P}.$$

Since $\mathcal{P}$ is prime, this forces $X_i \in \mathcal{P}$ or $X_j \in \mathcal{P}$, a contradiction. ☐

Motivated by this lemma, we now focus our attention on certain thick tensor ideals of $\mathcal{D}_n$. For $1 \le i \le n$, we let

$$\mathcal{P}_{i,n} = \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(\{X_1, \dots, X_n\} \setminus \{X_i\}).$$

Our goal will be to show that these are precisely the prime ideals in $\mathcal{D}_n$. (We know from Theorem 1.1 that the spectrum is a disjoint union of n points.)

Lemma 6.4. *For all $X \in \mathcal{P}_{i,n}$ we have $X \otimes \mathbf{1}_{n-1} \in \mathcal{P}_{i,n-1}$, i.e.,*

$$\phi_i{}^* \mathcal{P}_{i,n} \subseteq \mathcal{P}_{i,n-1}.$$

Proof. If $i \le n-1$ then $X_i \otimes \mathbf{1}_{n-1} = X_i$, whereas $X_n \otimes \mathbf{1}_{n-1} = 0$ by Lemma 2.2. Hence $\phi_i{}^*$ sends the generators of $\mathcal{P}_{i,n}$ into $\mathcal{P}_{i,n-1}$. Because $\mathcal{P}_{i,n-1}$ is thick and $\phi_i{}^*$ is exact, the lemma follows immediately. (The dubious reader may consult [8, Lemma 3.8].) □

Lemma 6.5. *Each tensor ideal $\mathcal{P}_{i,n}$ is proper in $\mathcal{D}_n$.*

Proof. We fix an i and proceed by induction on n, the base case being $n = i$. For this we need to show that $\mathcal{P}_{i,i} = \mathsf{thick}^{\otimes}_{\mathcal{D}_i}(X_1, \dots, X_{i-1})$ is proper. We saw in Section 3 that the restriction functor ϕ_{i-1*} embeds $\mathcal{D}_{i-1}$ as a proper tensor ideal in $\mathcal{D}_i$. For $1 \le j \le i-1$ we have $X_j \in \phi_{i-1*}\mathcal{D}_{i-1}$, so $\mathcal{P}_{i,i}$ is contained in $\phi_{i-1*}\mathcal{D}_{i-1}$ and $\mathcal{P}_{i,i}$ is proper in $\mathcal{D}_i$. (In fact, $\mathcal{P}_{i,i} = \phi_{i-1*}\mathcal{D}_{i-1}$ by Proposition 6.1.)

Now let $n > i$ and assume that $\mathcal{P}_{i,n-1}$ is proper in $\mathcal{D}_{n-1}$. For the sake of contradiction, suppose that $\mathcal{P}_{i,n}$ is not proper in $\mathcal{D}_n$ so that it contains the tensor unit $\mathbf{1}_n$. Then $\mathbf{1}_{n-1} = \mathbf{1}_n \otimes \mathbf{1}_{n-1}$ lies in $\mathcal{P}_{i,n-1}$ by Lemma 6.4, a contradiction. □

We are only now forced to specialise to the case where G is the cyclic group C_p.

Lemma 6.6. *If $G = C_p$, then each $\mathcal{P}_{i,n}$ is a maximal tensor ideal in $\mathcal{D}_n$.*

Proof. We fix i and proceed by induction on n. For the base case $n = i$ we need to show that $\mathsf{thick}^{\otimes}_{\mathcal{D}_i}(X_1, \dots, X_{i-1}) = \mathcal{D}_{i-1}$ is maximal in $\mathcal{D}_i$. Recall that the thick tensor ideals in $\mathcal{D}_i$ containing $\mathcal{D}_{i-1}$ are in bijection with those in the quotient $\mathcal{D}_i/\mathcal{D}_{i-1}$. By the discussion in Section 3, that quotient is tensor equivalent to $\mathcal{K}_i$, which in turn is equivalent to

$$\mathcal{D}_1 = \mathsf{strel}\,\mathbb{F}_p C_p = \mathsf{stmod}\,\mathbb{F}_p C_p$$

by Theorem 5.6. It is known that the rightmost category has precisely two tensor ideals, namely the zero ideal and the entire category. It follows that the only tensor ideal in $\mathcal{D}_i$ properly containing $\mathcal{D}_{i-1}$ is $\mathcal{D}_i$ itself, so $\mathcal{D}_{i-1}$ is maximal as claimed.

Now let $n > i$, assume that $\mathcal{P}_{i,n-1}$ is maximal in $\mathcal{D}_{n-1}$ and choose $X \notin \mathcal{P}_{i,n}$. Tensoring the short exact sequence of Lemma 2.4 with X produces a triangle

$$X \otimes \mathbf{1}_{n-1} \longrightarrow X \otimes X_n \longrightarrow \Sigma X \longrightarrow$$

in $\mathcal{D}_n$. The middle term lies in $\mathcal{P}_{i,n}$, but the right hand term does not. This implies that $X \otimes \mathbf{1}_{n-1}$ cannot lie in $\mathcal{P}_{i,n}$. In particular, it cannot lie in $\mathcal{P}_{i,n-1}$ since the latter is contained in the former. By the inductive hypothesis on maximality, this means that

$$\mathbf{1}_{n-1} \in \mathsf{thick}^{\otimes}_{\mathcal{D}_{n-1}}(\{X_1, \dots, X_{n-1}, X \otimes \mathbf{1}_{n-1}\} \setminus \{X_i\}).$$

Now consider the triangle

$$\mathbf{1}_{n-1} \longrightarrow X_n \longrightarrow \Sigma\mathbf{1}_n \longrightarrow$$

in $\mathcal{D}_n$ induced by the short exact sequence of Lemma 2.4. By the above remarks, the left two terms lie in $\mathsf{thick}^{\otimes}_{\mathcal{D}_n}(\{X_1, \ldots, X_n, X\} \setminus \{X_i\})$, whence so does the right hand term. In other words

$$\mathbf{1}_n \in \mathsf{thick}^{\otimes}_{\mathcal{D}_n}(\{X_1, \ldots, X_n, X\} \setminus \{X_i\}),$$

proving that $\mathcal{P}_{i.n}$ is maximal. □

It now follows from [1, Proposition 2.3] that each $\mathcal{P}_{i,n}$ is a prime tensor ideal, i.e., gives a point in $\operatorname{Spc} \mathcal{D}_n$.

Lemma 6.7. *Each $\mathcal{P}_{i,n}$ is a minimal prime.*

Proof. Suppose there exists a prime ideal $\mathcal{P}$ properly contained in $\mathcal{P}_{i,n}$. Then there is an object in the set $\{X_1, \ldots, X_n\} \setminus \{X_i\}$ not contained in $\mathcal{P}$. Since $\mathcal{P}$ is prime, Proposition 6.3 forces us to have $X_i \in \mathcal{P}$. But this implies that $X_i \in \mathcal{P}_{i,n}$, so $\{X_1, \ldots, X_n\} \subseteq \mathcal{P}_{i,n}$. Proposition 6.1 now tells us that $\mathcal{P}_{i,n} = \mathcal{D}_n$, contradicting Lemma 6.5. □

We are now ready to give a direct computation of the spectrum, verifying Theorem 1.1 in this case.

Theorem 6.8. *The prime ideal spectrum of $\mathcal{D}_n = \mathsf{strel}\,(\mathbb{Z}/p^n)C_p$ is a disjoint union of n points.*

Proof. If $\mathcal{P}$ is an element of $\operatorname{Spc} \mathcal{D}_n$, then by Proposition 6.3 there is an integer $1 \leq i \leq n$ such that $\{X_1, \ldots, X_n\} \setminus \{X_i\} \subseteq \mathcal{P}$. We then have $\mathcal{P}_{i,n} \subseteq \mathcal{P}$. Since $\mathcal{P}_{i,n}$ is maximal, this implies that $\mathcal{P} = \mathcal{P}_{i,n}$. It follows that $\operatorname{Spc} \mathcal{D}_n = \{\mathcal{P}_{1,n}, \ldots, \mathcal{P}_{n,n}\}$ as a set. We now recall from [1, Proposition 2.9] that the closed points in the prime ideal spectrum are precisely the *minimal* primes. Lemma 6.7 therefore informs us that each point $\mathcal{P}_{i,n}$ is closed in the topology of $\operatorname{Spc} \mathcal{D}_n$. □

We now make a few observations based on and related to the theorem; all of the statements are, more or less, trivial consequences of what we have already done, but are worth making explicit.

Corollary 6.9. *Each $X_i = \Omega_i^{-1} k$ is supported at a single point*

$$\operatorname{supp} X_i = \{\mathcal{P}_{i,n}\}.$$

Proof. Given the computation of the spectrum and the definition of support, we have

$$\operatorname{supp} X_i = \{\mathcal{P} \in \operatorname{Spc} \mathcal{D}_n \mid X_i \notin \mathcal{P}\} = \{\mathcal{P}_{i,n}\}.$$ □

Corollary 6.10. *The base change functor*

$$- \otimes \mathbf{1}_m \colon \mathcal{D}_n \longrightarrow \mathcal{D}_m$$

induces an embedding $\operatorname{Spc} \mathcal{D}_m \hookrightarrow \operatorname{Spc} \mathcal{D}_n$ with image $\{\mathcal{P}_{i,n} \mid 1 \leq i \leq m\}$.

Proof. Letting ϕ^* denote the base change functor, it is a general fact (see [1, Proposition 3.11]) that $\mathrm{Spc}(\phi^*)$ is a homeomorphism onto its image. The latter set consists precisely of those prime ideals that contain the kernel of ϕ^*. It follows by repeated applications of Lemma 4.1 that the kernel of ϕ^* is generated by $\{X_i \mid m+1 \leq i \leq n\}$. The result follows immediately. □

In the above corollary, one might also consider the corresponding base change functors having source $\mathsf{strel}\,\mathbb{Z}C_p$. The result remains true, except for the description of the image. By invoking [5, Theorem A.5], one may write

$$\mathrm{Spc}(\mathsf{strel}\,\mathbb{Z}C_p) \cong Z_n \coprod \mathrm{Spc}\,\mathcal{D}_n$$

for all $n \geq 1$, where Z_n is the spectrum of the kernel. Thus for any $n \geq 1$ we can find a disjoint union of n points as an open and closed subspace of the spectrum of $\mathsf{strel}\,\mathbb{Z}C_p$. It would be interesting to fully understand the space $\mathrm{Spc}(\mathsf{strel}\,\mathbb{Z}C_p)$ and, in particular, how the spectra of the $\mathcal{D}_n$ sit inside of it. We note that since $\mathrm{Spc}(\mathsf{strel}\,\mathbb{Z}C_p)$ is quasi-compact, there must be some non-trivial topology on the union of these points in $\mathrm{Spc}(\mathsf{strel}\,\mathbb{Z}C_p)$.

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