An exponential estimate for the extinction time of the branching random walk on a cube

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Abstract

We prove the exponential estimate

$$P\{s < \tau < \infty\} \le Ce^{-qs}, \quad s \ge 0,$$

where $C, q > 0$ are constants and τ is the extinction time of the supercritical branching random walk (BRW) on a cube. We cover both the discrete-space and continuous-space BRWs.

Mathematics subject classification: 60K35, 60J80.

1 Introduction

In this short paper we prove an exponential estimate for the extinction time of a branching random walk on a cube. We treat both the discrete-space and continuous-space models. Time is continuous in both models. A detailed description of them can be found in Section 2.

More specifically, we prove the exponential estimate

$$P\{s < \tau < \infty\} \le Ce^{-qs}, \quad s \ge 0, \tag{1}$$

where $C, q > 0$ are some constants and τ is the extinction time. For supercritical spatial random structures, first estimates of this type have probably been obtained for the oriented percolation process in two dimensions, see Durrett [Dur84]; for the supercritical contact process, see e.g. Theorem 2.30 in Liggett [Lig99].

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This work relies on results of Mountford and Schinazi [MS05] and Bertacchi and Zucca [BZ09] (see also [BZ15]), who proved in discrete-space settings that the supercritical branching random walk survives on large finite cubes with positive probability. We adapt their result to the continuous-space case.

Our proof of (1) relies on renormalization and comparison with oriented percolation. This scheme has been carried out for the contact process, see e.g. Bezuidenhout and Grimmett [BG90], Durrett [Dur91] or Liggett [Lig99]. Since in our case the geographic space is bounded but the spin space is unbounded, we use a different approach based on the genealogical structure.

The paper is organized as follows. In Section 2 we describe the model and give our assumptions and results. Sections 3 to 5 are devoted to proofs.

2 The model, assumptions and results

Description. The evolution of the system admits the following description. Each particle "lives" in $\mathbb{Z}^d$ (the discrete-space case) or $\mathbb{R}^d$ (the continuous-space case) and has two exponential clocks with parameters 1 and λ, $\lambda > 1$. When the first clock rings, the particle is deleted from the system ("death"). When the second clock rings, the particle gives a birth to a new particle. After that the clocks are reset. The offspring is distributed according to some radially symmetric dispersal kernel a. Births outside of some cube $\mathbf{B}$ are suppressed, and there are no particles outside $\mathbf{B}$ at the beginning.

In the discrete-space case the state space of the process is $\mathbb{Z}_+^{\mathbf{B}}$, in the continuous-space case it is the collection of finite subsets of $\mathbf{B}$: $\{\eta \subset \mathbf{B} : |\eta \cap \mathbf{B}| < \infty\}$. In either case we denote the state space by $\mathcal{X}$.

The heuristic generator is given by

$$LF(\eta) = \sum_{x \in \eta} \big\{ F(\eta \setminus \{x\}) - F(\eta) \big\} + \lambda \sum_{x \in \eta} \int_{y \in X : x - y \in \mathbf{B}} a(y - x) \big\{ F(\eta \cup \{y\}) - F(\eta) \big\} \nu(dy),$$

where $\lambda > 0$ is the branching rate, $F : \mathcal{X} \to \mathbb{R}_+$ is some function from an appropriate domain, $X = \mathbb{R}^d$ and ν is the Lebesgue measure, or $X = \mathbb{Z}^d$ and ν is the counting measure. In both cases,

$$\int_{y \in X} a(y) \nu(dy) = 1.$$

The process can be constructed in the following way. Take a rooted tree $\mathbf{E}$ as in Figure 2. To a vertex $\mathbf{e}$ we assign an independent vector $(b_{\mathbf{e}}, d_{\mathbf{e}}, s_{\mathbf{e}})$ with values in $\mathbb{R}_+ \times \mathbb{R}_+ \times X$, where $X = \mathbb{Z}^d$ or $X = \mathbb{R}^d$. We take $b_{\mathbf{e}}$ and $d_{\mathbf{e}}$ to be exponentials with parameters λ and 1 respectively, and $s_{\mathbf{e}}$ to be distributed according to a. Assume that the particle to which $\mathbf{e}$ is assigned is born
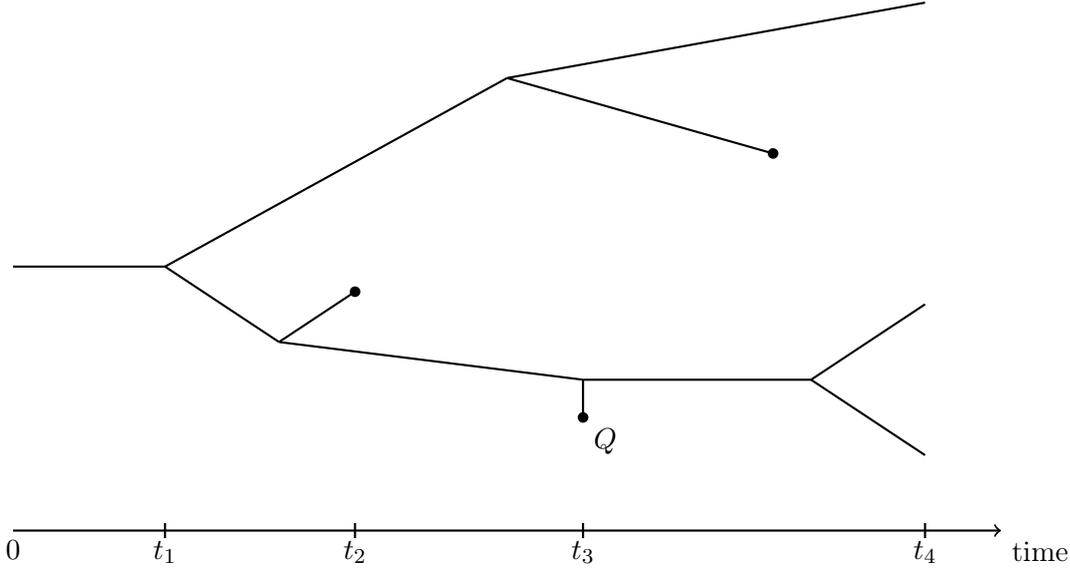
Figure 1: Genealogical structure of the process. The first birth occurs at t_1, the first death at t_2. The newly born at t_3 particle is outside B, so it dies instantly, hence the vertical line. There are 3 particles alive at t_4.

at time $t_{\mathbf{e}}$ at $x \in X$. If $d_{\mathbf{e}} < b_{\mathbf{e}}$, the particle dies at time $t_{\mathbf{e}} + d_{\mathbf{e}}$, otherwise the particle produces an offspring at time $t_{\mathbf{e}} + b_{\mathbf{e}}$. The position of the offspring is $s_{\mathbf{e}} + x$. The offspring is removed instantly if it is born outside $\mathbf{B}$. The initial particle is assigned to the root of the tree. This construction naturally allows us to endow the process with the genealogical structure.

If β is some collection of particles of the BRW alive at time s, we denote by $(\eta_t^{s,\beta})_{t \geq 0}$ the process starting from β at s. Clearly, if $\alpha \subset \beta$, then $\eta_t^{s,\alpha} \subset \eta_t^{s,\beta}$ for all $t \geq s$. The process started from a single particle at $x \in \mathbf{B}$ is denoted by $(\eta_t^{0,x})_{t \geq 0}$.

We write $(\eta_t^{0,x})$ as a shorthand for $(\eta_t^{0,x})_{t \geq 0}$, meaning the whole trajectory of the process. We say that the BRW *survives on* $\mathbf{B}$ *with positive probability*, if there is an $x \in \mathbf{B}$ such that $P\{(\eta_t^{0,x}) \text{ survives}\} > 0$. Note that if the BRW survives on some cube with positive probability, it also does so on a larger cube.

Assumptions and results. Let $a^{(n)}$ be the n-time convolution of a, or the n-step transition function/density. In the discrete-space case we say that a is elliptic if (cf. [BZ09]) for any $y \in \mathbb{Z}^d$

$$a^{(n)}(y) > 0 \quad \text{for some } n \in \mathbb{N}. \tag{2}$$

In the continuous-space case we say that a is elliptic if for any $y \in \mathbb{R}^d$ and $r > 0$,

$$\inf_{z \in B(y,r)} a^{(n)}(z) > 0 \quad \text{for some } n \in \mathbb{N}, \tag{3}$$

where $B(y, r)$ is the ball of radius r around y.

We assume that a is continuous (in discrete-space settings it amounts to no assumption) and elliptic. Note that for the survival on a cube we need some kind of ellipticity of a: for example,

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if d $= 1$ and the support of a is contained by $[1, \infty)$, then the BRW dies out for every $\mathbf{B}$ and $\lambda > 0$. In the discrete-space settings, the survival of the supercritical BRW ($\lambda > 1$) on large cubes has been proven by Mountford and Schinazi [MS05], for the BRW corresponding to the simple random walk, and by Bertacchi and Zucca [BZ09, Section 3], under conditions similar to (2) for a BRW on a general connected graph of bounded degree. The following theorem extends these results to continuous-space settings.

Theorem 2.1. *In the continuous-space case, the BRW survives on* $\mathbf{B}$ *with positive probability provided that* $\mathbf{B}$ *is sufficiently large.*

Let τ be the moment of extinction, with convention that $\tau = \infty$ if the process survives. Assume that $\mathbf{B}$ is sufficiently large so that the process survives with positive probability.

For technical reasons, in the continuous-space case we will impose stronger conditions than (3). Let $\mathbf{0}$ be the origin in $\mathbb{R}_d$, Δ a 'cemetery' state, and $\tilde{a}_{\mathbf{B}} : (\mathbf{B} \cup \{\Delta\}) \times \mathscr{B} (\mathbf{B} \cup \{\Delta\}) \to [0, \infty)$ be the transition function given by

$$\tilde{a}_{\mathbf{B}}(x, B) = \int_{y \in B} a(y - x), \quad x, y \in \mathbf{B}, B \in \mathscr{B}(\mathbf{B}),$$

$\tilde{a}_{\mathbf{B}}(x, \{\Delta\}) = \int_{y \notin \mathbf{B}} a(y - x)$, and $\tilde{a}_{\mathbf{B}}(\Delta, \cdot) \equiv 0$. Here $\mathscr{B}(\mathbf{B})$ is the collection of Borel subsets of $\mathbf{B}$.

First, assume that $P\{(\eta_t^{0,\mathbf{0}})$ survives$\} > 0$. We further assume that for every $r > 0$ there exist $N \in \mathbb{N}$ and $\tilde{\delta} > 0$ such that

$$\forall x \in \mathbf{B} \quad \sum_{n=1}^{N} \tilde{a}_{\mathbf{B}}^{(n)}(x, B(\mathbf{0}, r)) \geq \tilde{\delta}. \tag{4}$$

and that there is a small ball $B(\mathbf{0}, \bar{r})$ such that for any $y \in B(\mathbf{0}, \bar{r})$ and $\bar{\delta} > 0$,

$$P\{(\eta_t^{0,y}) \text{ survives}\} > \bar{\delta}. \tag{5}$$

Combining (4) and (5) gives the existence of $\delta > 0$ such that

$$\forall y \in \mathbf{B} \quad P\{(\eta_t^{0,y}) \text{ survives}\} > \delta. \tag{6}$$

The following theorem is the main result of this paper.

Theorem 2.2. *Under the above assumptions,* (1) *holds.*

Remark 2.3. Assumption (5) is not very restrictive due to the following observation. Assume that $P\{(\eta_t^{0,\mathbf{0}})$ survives$\} = p_{\mathbf{B}} > 0$ and let l be the length of an edge of $\mathbf{B}$. Then for a cube $\mathbf{B}^{\varepsilon}$ with the edge length $l + 2\varepsilon$, $\varepsilon > 0$, and for all $y \in (-\varepsilon, \varepsilon)^{\mathrm{d}}$

$$P\{(\eta_t^{0,y}) \text{ survives}\} \geq p_{\mathbf{B}}.$$

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Remark 2.4. For the supercritical process on the whole space, $\mathbb{Z}^{\mathrm{d}}$ or $\mathbb{R}^{\mathrm{d}}$, (1) comes down to the corresponding estimate for the Galton–Watson process, since $X_t := |\eta_t|$ is a birth-death process with transition rates

$$n \to n+1 \quad \text{at rate } \lambda n,$$

$$n \to n-1 \quad \text{at rate } n.$$

3 Proof of Theorem 2.1

The idea is to couple a continuous-space supercritical BRW with a discrete-space one and then use the result of [BZ09]. With no loss of generality we assume that the length of an edge of $\mathbf{B}$ is a natural number.

For $n \in \mathbb{N}$ and $j = (j_1, ..., j_{\mathrm{d}}) \in \frac{1}{2^n}\mathbb{Z}^{\mathrm{d}} \cap \mathrm{int}(\mathbf{B})$, where $\mathrm{int}(\mathbf{B})$ is the interior of $\mathbf{B}$, we define

$$a_n(j) = \frac{1}{2^{n\mathrm{d}}} \inf\{a(x-y) : x \in [-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})^{\mathrm{d}}, y \in \prod_{k=1}^{\mathrm{d}} [j_k - \frac{1}{2^{n+1}}, j_k + \frac{1}{2^{n+1}})\}.$$

Note that a_n is elliptic. Since a is continuous, we have

$$\sum_{j \in \frac{1}{2^n}\mathbb{Z}^{\mathrm{d}} \cap \mathrm{int}(\mathbf{B})} a_n(j) \to \int_{\mathbb{R}^{\mathrm{d}}} a(x)dx, \tag{7}$$

therefore $\sum_{j \in \frac{1}{2^n}\mathbb{Z}^{\mathrm{d}}} a_n(j) > 1$ for sufficiently large n. We will choose such an $n \in \mathbb{N}$ and couple the given continuous-space BRW (η_t) with discrete-space BRW $(\eta_t^{(n)})$ on $\frac{1}{2^n}\mathbb{Z}^{\mathrm{d}}$ with kernel a_n as follows. Each particle q from $(\eta_t^{(n)})$ is associated to a particle $s(q)$ from (η_t), and no particle from (η_t) may have two particles from $(\eta_t^{(n)})$ associated to it, so that $s : \eta_t^{(n)} \to \eta_t$ is an injection for each t. We consider (η_t) started from one particle at the origin. We let $\eta_0^{(n)}$ to have one particle at the origin of $\frac{1}{2^n}\mathbb{Z}^{\mathrm{d}}$, which we associate to the initial particle of (η_t).

If a particle $s(q)$ at x gives birth to a new particle at y at a time s, where $x \in [j_k^x - \frac{1}{2^{n+1}}, j_k^x + \frac{1}{2^{n+1}})$, $y \in [j_k^y - \frac{1}{2^{n+1}}, j_k^y + \frac{1}{2^{n+1}})$ for some $j^x, j^y \in \frac{1}{2^n}\mathbb{Z}^{\mathrm{d}}$, then the associated to the parent particle q at j^x gives birth to a new particle at j^y with probability $\frac{a_n(j^y - j^x)}{a(y-x)}$, provided that the particle $s(q)$ exists and is alive. We associate the newborn particles to each other. Also, associated particles die simultaneously.

It is clear that $|\eta_t^{(n)}| \le |\eta_t|$ for all $t \ge 0$; in particular, if $(\eta_t^{(n)})$ survives, then so does (η_t). It remains to note that from [BZ09, Theorem 3.1] we know that $(\eta_t^{(n)})$ survives on a sufficiently large finite cube with positive probability. $\qquad\square$

4 Proof of Theorem 2.2

We prove Theorem 2.2 concurrently in discrete and continuous settings, because the ideas involved are very similar. We endow our system with the genealogical structure, so that we can

talk about ancestors and descendants. Without loss of generality we assume that $\mathbf{B}$ is centered at the origin. Furthermore, we assume without loss of generality that in the discrete-space case the random walk on $\mathbf{B}$ with the kernel $a_{\mathbf{B}}$ is irreducible. Here for $x, y \in \mathbf{B}$

$$a_{\mathbf{B}}(y, x) = a(y - x) + I\{x = y\} \sum_{z \neq \mathbf{B}} a(z - x).$$

Concerning the last assumption, see Remark 4.2.

Lemma 4.1. *In the discrete-space case, for any $\varepsilon > 0$ there are $T > 0$ and $M \in \mathbb{N}$ such that*

$$c_{ij} := P\{\eta_T^{0, MI_{A_i}} \geq MI_{A_j}\} \geq 1 - \varepsilon, \quad i, j = 1, 2, \tag{8}$$

where $A_1 = \{(x_1, ..., x_d) \in \mathbf{B} \mid x_1 \geq 0\}$ and $A_2 = \mathbf{B} \setminus A_1$.

Proof. The BRW can be considered as a continuous-time Markov chain on $\mathbf{B}^{\mathbb{Z}_+}$. Since zero state is a trap that can be reached from any state, any finite subset of $\mathbf{B}^{\mathbb{Z}_+}$ is transient. In particular, for any $L > 0$

$$P\{\tau = \infty, \max_{x \in \mathbf{B}} \eta_t(x) \leq L\} \to 0, \quad t \to \infty. \tag{9}$$

Let us choose M so large that

$$P\{\eta^{0, MI_{A_i}} \text{ dies out}\} \leq 1 - \frac{\varepsilon}{4}, \quad i = 1, 2.$$

Proceeding further, let us choose L so large that the following is satisfied: for any $x \in \mathbf{B}$, process started at 0 from L particles in x has at time 1 at least M particles everywhere on $\mathbf{B}$ with probability larger than $1 - \frac{\varepsilon}{4}$. Choosing now T so large that

$$P\{\max_{x \in \mathbf{B}} \eta_{T-1}^{0, MI_{A_i}}(x) \geq L\} \geq 1 - \frac{\varepsilon}{2}, \quad i = 1, 2.$$

completes the proof. $\qquad\square$

Remark 4.2. It can be that the random walk with transition function $a_{\mathbf{B}}$ is not irreducible on $\mathbf{B}$. As an example, let us take $d = 1$, $\mathbf{B} = \{-2, ..., 2\}$ and $a(x) = \frac{1}{2}I\{|x| = 5\}$ and note that the corresponding BRW survives with positive probability if $\lambda > 2$. If this is the case, there is a component $\bar{\mathbf{B}} \subset \mathbf{B}$ such that the BRW started from a single particle in $\bar{\mathbf{B}} \subset \mathbf{B}$ survives with positive probability within $\bar{\mathbf{B}}$ (that is, with births outside $\bar{\mathbf{B}}$ being suppressed; in the above example $\bar{\mathbf{B}}$ would be $\{-2, 2\}$). The above lemma still holds provided that A_i is replaced by $A_i \cap \bar{\mathbf{B}}$, $i = 1, 2$.

Define

$$Q_+ = \mathbf{B} \cap \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = (x_1, ..., x_d) \text{ with } x_1 \geq 0\right\}$$

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and

$$Q_- = \mathbf{B} \cap \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{d}} : \mathbf{x} = (x_1, ..., x_{\mathrm{d}}) \text{ with } x_1 < 0 \right\}.$$

For $M \in \mathbb{N}$, let

$$A_+^M = \{ \eta \in \Gamma_0(\mathbf{B}) : |\eta \cap Q_+| > M \}$$

and

$$A_-^M = \{ \eta \in \Gamma_0(\mathbf{B}) : |\eta \cap Q_-| > M \}.$$

Lemma 4.3. *In the continuous-space case, for any $\varepsilon > 0$ there are $T > 0$ and $M \in \mathbb{N}$ such that*

$$P\{\eta_T^{0,\eta_0} \in A_j^M\} \geq 1 - \varepsilon \tag{10}$$

for any $\eta_0 \in A_i^M$. Here each of the indices i and j can be either $+$ or $-$.

Proof. By a similar argument, for any $n \in \mathbb{N}$ the set $\{\eta \subset \mathbf{B} : |\eta| = n\}$ is transient in the sense that a.s. it is entered finitely many times only. The counterpart of (9) is

$$P\{\tau = \infty, |\eta_t(x)| \leq L\} \to 0, \quad t \to \infty.$$

By (6), the probability of survival is separated from 0. We can choose M so large that

$$P\{(\eta_t^{0,\eta_0}) \text{ dies out}\} \leq 1 - \frac{\varepsilon}{4}$$

for any $\eta_0 \in A_i^M$, $i = +, -$, then L so large that any process started from L particles at time 0 is in the intersection $A_+^M \cap A_-^M$ by time 1 with high probability ($1 - \frac{\varepsilon}{4}$ is sufficient), and finally we choose T so that

$$P\{|\eta_{T-1}^{0,\eta_0}| \geq L\} \leq 1 - \frac{\varepsilon}{2}$$

for any $\eta_0 \in A_i^M$, $i = +, -$, and the proof goes as in Lemma 4.1. $\qquad\square$

Let $G = \{(n, m) : n + m \text{ is even}\}$. We will use Lemmas 4.1 and 4.3 to make a comparison with the oriented percolation process on G. Let (n, m) be connected to $(n + 1, m + 1)$ and $(n - 1, m + 1)$. Each bond is open with probability p independently of the other bonds. We say that percolation occurs if there is an infinite path starting from the origin. The model is well-known, see e.g. Durrett [Dur84, Dur88].

Let p_c be the critical value for independent oriented percolation in two dimension, and let

$$\sigma = \min \big\{ m \in \mathbb{N} : \text{ there is no open path from } (0, 0) \text{ to } \{(k, m) \mid k \in \mathbb{Z}\} \big\},$$

the moment of extinction of the percolation process. We use the following estimate in the proof of Theorem 2.2.

Lemma 4.4 ([Dur84]). *Assume that $p > p_c$. Then there are $q_1, C_1 > 0$ such that*

$$P\{r < \sigma < \infty\} \leq C_1 e^{-q_1 r}, \quad r \geq 0. \tag{11}$$

Proof of Theorem 2.2 **in the discrete-space case**. Let us take M and T so large that Lemma 4.1 is satisfied with $1 - \varepsilon = p > p_c$. Let $(u_n)_{n \in G}$ be a sequence of independent random variables distributed uniformly on $[0, 1]$, independent of everything introduced so far. Denote also

$$c_{ij} = P\{\eta_T^{0, MI_{A_i}} \geq MI_{A_j}\} \geq p.$$

Let $\tau_1 = \tau \wedge \inf\{t : \eta_t \geq MI_{A_2}\}$. Since every particle alive at some time t_0 produces by the time $t_0 + 1$ so many particles as to dominate MI_{A_1} with positive probability separated from zero, τ_1 is dominated by a geometric random variable and has subexponential tails (see Section 5 for the precise meaning of "subexponential tails"). If the process does not die out at τ_1, then we build an oriented bond percolation process on G according to the following procedure.

Choose a collection of particles $\alpha_{(0,0)}$ alive at time τ_1 in such a way that $S(\alpha_{(0,0)}) = MI_{A_2}$. Here $S(\alpha_{(0,0)}) = MI_{A_2}$ means that $\alpha_{(0,0)}$ has exactly M particles at every site from A_2 and has no particles outside A_2. In our construction, $S(\alpha_{(n,m)}) = MI_{A_2}$ if $m \equiv n \mod 4$, and $S(\alpha_{(n,m)}) = MI_{A_1}$ if $m \equiv n + 2 \mod 4$.

We say the edge $\langle (0,0), (1,1) \rangle$ from $(0,0)$ to $(1,1)$ is open if both

$$\{\eta_{\tau_1 + T}^{\tau_1, \alpha_{(0,0)}} \geq MI_{A_2}\}$$

and

$$\{u_{\langle (0,0),(1,1) \rangle} < \frac{p}{c_{22}}\}$$

occur, and we say that the edge $\langle (0,0), (-1,1) \rangle$ is open if both $\{\eta_{\tau_1 + T}^{\tau_1, \alpha_{(0,0)}} \geq MI_{A_1}\}$ and $\{u_{\langle (0,0),(-1,1) \rangle} < \frac{p}{c_{21}}\}$ occur. If $\langle (0,0), (1,1) \rangle$ is open, then we choose $\alpha_{(1,1)}$ in such a way that $S(\alpha_{(1,1)}) = MI_{A_2}$ and that every particle from $\alpha_{(1,1)}$ is an descendant of a particle from $\alpha_{(0,0)}$ (here we consider a particle to be a descendant of itself provided that it is still alive). Similarly, if $\langle (0,0), (-1,1) \rangle$ is open, we choose $\alpha_{(-1,1)}$ in such a way that $S(\alpha_{(-1,1)}) = MI_{A_1}$ and that every particle from $\alpha_{(-1,1)}$ is an descendant of a particle from $\alpha_{(0,0)}$. Further proceeding, assume that there is an open path from the origin to (n, m), and a collection $\alpha_{(n,m)}$ of particles alive at $\tau_1 + mT$ is chosen, such that

$$S(\alpha_{(n,m)}) = \begin{cases} MI_{A_1} & \text{if } m \equiv n + 2 \mod 4, \\ MI_{A_2} & \text{if } m \equiv n \mod 4. \end{cases} \tag{12}$$

For $m \equiv n \mod 4$, we let $\langle (n,m), (n+1, m+1) \rangle$ be open if $\{\eta_{\tau_1 + (m+1)T}^{\tau_1 + mT, \alpha_{(n,m)}} \geq MI_{A_2}\}$ and $\{u_{\langle (n,m),(n+1,m+1) \rangle} < \frac{p}{c_{22}}\}$ occur, and $\langle (n,m), (n-1, m+1) \rangle$ is open if $\{\eta_{\tau_1 + (m+1)T}^{\tau_1 + mT, \alpha_{(n,m)}} \geq MI_{A_1}\}$ and $\{u_{\langle (n,m),(n-1,m+1) \rangle} < \frac{p}{c_{21}}\}$ do. Similarly, for $m \equiv n+2 \mod 4$, $\langle (n,m), (n+1, m+1) \rangle$ is open if $\{\eta_{\tau_1 + (m+1)T}^{\tau_1 + mT, \alpha_{(n,m)}} \geq MI_{A_1}\}$ and $\{u_{\langle (n,m),(n+1,m+1) \rangle} < \frac{p}{c_{11}}\}$ occur, and $\langle (n,m), (n-1, m+1) \rangle$ is open if $\{\eta_{\tau_1 + (m+1)T}^{\tau_1 + mT, \alpha_{(n,m)}} \geq MI_{A_2}\}$ and $\{u_{\langle (n,m),(n-1,m+1) \rangle} < \frac{p}{c_{12}}\}$ do. Furthermore, if $\langle (n,m), (n \pm$

$1, m + 1)\rangle$ is open, we choose $\alpha_{(n\pm 1, m+1)}$ in such a way that each particle from $\alpha_{(n\pm 1, m+1)}$ is a descendant from a particle from $\alpha_{(n,m)}$ and (12) is satisfied.

If there is no open path to (n, m), then $\alpha_{(n,m)}$ is not defined, and we may take $\langle (n, m), (n \pm 1, m + 1)\rangle$ to be open iff $u_{\langle (n,m),(n\pm 1,m+1)\rangle} < p$.

Thus we get the desired percolation process, in which edges are open independently with probability p, and which is constructed in such a way that percolation implies survival of $(\eta_t)_{t\geq 0}$. Let σ_1 be the lifetime of the percolation process. If percolation doesn't occur but the BRW still lives, we start anew and on $\{\tau > \tau_1, \sigma_1 < \infty\}$ define τ_2 analogously to τ_1,

$$\tau_2 = \tau \wedge \inf\{t > \tau_1 + \sigma_1 T : \eta_t \geq MI_{A_2}\}.$$

If, after some time, the BRW dies out at some τ_i, then we use an independent collection of oriented percolation processes to define σ_i until the first time percolation occurs. Let $g \in \mathbb{N}$ the number of the first percolation process that survives, that is $\sigma_{g-1} < \infty$ and $\sigma_{g-1} = \infty$. Clearly, g has a geometric distribution. A.s. on $\{\tau < \infty\}$ we have

$$\tau \leq I\{g \neq 1\} \sum_{j=1}^{g-1} (\tau_j + \sigma_j T) + \tau_g,$$

where τ_j, σ_j have subexponential tails and g has a geometric distribution. It remains to apply two lemmas from Section 5. $\qquad\square$

Proof of Theorem 2.2 **in the continuous-space case**. We will use a similar percolation argument to prove Theorem 2.2 in continuous-space settings. Take $T > 0$ and $M \in \mathbb{N}$ so large that Lemma 4.3 is satisfied with $1 - \varepsilon = p \in (p_c, 1)$. Similarly to the discrete-space case, let $\tau_1 = \tau \wedge \inf\{t : \eta_t \in A_-^M\}$. If $\tau_1 \neq \tau$, choose a minimal $\alpha_{(0,0)}$ such that $\alpha_{(0,0)} \subset \eta_{\tau_1}$ and $\alpha_{(0,0)} \in A_-^M$.

Let $\bar{\alpha}_{(0,0)}$ be some collection of particles alive at time 0 and having spatial positions identical to particles from $\alpha_{(0,0)}$. We declare $\langle (0,0), (-1,1)\rangle$ to be open if $\{\eta_{\tau_1+T}^{\tau_1,\alpha_{(0,0)}} \in A_-^M\}$ and

$$u_{\langle (0,0),(-1,1)\rangle} < p\big(P\{\eta_T^{0,\bar{\alpha}_{(0,0)}} \in A_-^M\}\big)^{-1},$$

and so on, proceeding exactly as in the discrete-space case. That will yield the desired result. $\quad\square$

Remark 4.5. In the proof of Theorem 2.2 we tacitly assumed that the strong Markov property holds at $\tau_1, \tau_2, \dots$ We could prove that (η_t) has the strong Markov property, but in this case it is easier to replace τ_1 with

$$\tilde{\tau}_1 = \lceil \tau \rceil \wedge \min\{n \in \mathbb{N} : \eta_n \geq MI_{A_2}\},$$

where $\lceil \cdot \rceil$ is the ceiling function, and use the fact that the strong Markov property is satisfied for the stopping times which take countably many values only, see e.g. Kallenberg [Kal02, Proposition 8.9]. In a similar way we can replace σ_1, τ_2, and so on. The proof needs no further changes.

5 Subexponential tails

We say that a random variable X has subexponential tails if there are $C_X, q_X > 0$ such that

$$P\{X \geq x\} \leq C_X e^{-q_X x}, \quad x \geq 0.$$

Note that $Ee^{\theta X} < \infty$ if $\theta < q_X$.

Lemma 5.1. *Let X and Y be independent random variables with subexponential tails. Then their sum has subexponential tails too.*

    Proof. $P\{X + Y \geq 2z\} \leq P\{X \geq z\} + P\{Y \geq z\}.$ $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.2. *Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with subexponential tails, and let g be an independent random variable with a geometric distribution,*

$$P\{g = m\} = (1 - p)p^{m-1}, \quad m \in \mathbb{N},$$

where $p \in (0, 1)$. Then $S = \sum\limits_{j=1}^{g} X_j$ has subexponential tails.

    Proof. By the Lebesgue dominated convergence theorem there exists $\theta > 0$ such that $Ee^{\theta X_1} < \frac{1}{p}$. For such a θ,

$$Ee^{\theta S} = \sum_{m=1} P\{g = m\}(Ee^{\theta X_1})^m < \infty,$$

hence by Chebyshev's inequality

$$P\{S > x\} \leq Ee^{\theta S}e^{-\theta x}.$$

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