

DISCRETE TRIANGULATED CATEGORIES

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ABSTRACT. We introduce and study several homological notions which generalise the discrete derived categories of D. Vossieck. As an application, we show that Vossieck discrete algebras have this property with respect to all bounded t-structures. We give many examples of triangulated categories regarding these notions.

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INTRODUCTION

In this article, we investigate Hom-finite triangulated categories which are, in various senses, small. Our motivating examples are bounded derived categories of derived-discrete algebras, which were introduced and classified by D. Vossieck [21]. In previous work [12], we observed some special properties of these categories: the dimension of Hom spaces between indecomposables is bounded (by 1 or 2, depending on the algebra), and all hearts of bounded t-structures have only finitely many indecomposable objects.

We set out to introduce and compare abstract notions which apply to such examples. The three most relevant for this article are

- *cone finite*: any two objects admit only finitely many cones, up to isomorphism;
- *hom bounded*: universal bound on Hom dimension among indecomposable objects;
- *countable*: the category has only countably many objects, up to isomorphism.

We establish the following relations between these properties in [Theorem 1.2](#):

Theorem. (i) *A hom 1-bounded triangulated category is cone finite.*
(ii) *A cone finite triangulated category with a classical generator is countable.*

We give examples showing that countability doesn't imply cone finite or hom bounded, and that cone finite doesn't imply hom bounded. Vossieck's definition of discreteness does not generalise to abstract triangulated categories, as it invokes cohomology objects, i.e. a t-structure. Therefore, we study pairs (\mathbf{D}, \mathbf{H}) of a triangulated category together with the heart of a bounded t-structure. Moreover, the Grothendieck group $K_0(\mathbf{D}) = K_0(\mathbf{H})$ enters, generalising dimension vectors of modules. We introduce two notions characterising different aspects of smallness for abelian categories via their K-groups which we call *modular* and *abelian discrete* (see [Section 2](#)), and we prove in [Theorem 2.5](#):

Theorem. *Let (\mathbf{D}, \mathbf{H}) be a triangulated category with the heart of a bounded t-structure.*

- (i) *\mathbf{D} cone finite and \mathbf{H} abelian discrete $\implies \mathbf{D}$ is discrete with respect to \mathbf{H}*
- (ii) *\mathbf{D} is discrete with respect to $\mathbf{H} \implies \mathbf{H}$ abelian discrete.*
- (iii) *\mathbf{D} is discrete with respect to \mathbf{H} and \mathbf{H} modular $\implies \mathbf{D}$ cone finite.*

As one application of this result, we show that derived-discrete algebras are discrete with respect to any bounded t-structure, not just the standard one; see [Proposition 3.2](#).

In [Section 4](#), we introduce and study an analogous definition of discreteness with respect to a bounded co-t-structure with co-heart \mathcal{C} , assuming the existence of a silting object.

There are many interesting examples of triangulated categories having certain of the properties in question. Here, we list some of them. For a more elaborate version, with further properties and example classes, see the table on [page 12](#). In this table, Γ_{ADE} , A_∞ and \tilde{A}_1 denote any ADE quiver, the doubly infinite quiver of type A , and the Kronecker quiver, respectively. The column then refers to the bounded derived category of the path algebra. DDC stands for the bounded derived category of a derived-discrete algebra $\Lambda(r, n, m)$. Finally, \mathbb{T}_w is the triangulated category generated by a w -spherical object.

| | $\mathbf{k}\Gamma_{\text{ADE}}$ | $\mathbf{k}A_\infty$ | DDC | \mathbb{T}_1 | $\mathbb{T}_{>1}$ | $\mathbb{T}_{<1}$ | $\mathbb{Q}\tilde{A}_1$ | $\mathbb{F}_q\tilde{A}_1$ |
|-------------------|---------------------------------|----------------------|-----|----------------|-------------------|-------------------|-------------------------|---------------------------|
| H-discrete | ✓ | ✓ | ✓ | ✓ | ✓ | — | × | ✓ |
| C-discrete | ✓ | ✓ | ✓ | — | — | ✓ | × | ✓ |
| hom bounded | ✓ | ✓ | ✓ | × | ✓ | ✓ | × | × |
| cone finite | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | × | ✓ |
| countable objects | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| finite hearts | ✓ | × | ✓ | × | ✓ | — | × | × |

We mention other approaches for capturing the smallness of categories and algebras:

The Krull–Gabriel dimension of a triangulated category (or rather, of its abelianisation), is a numerical invariant. Small values do correspond to “small” categories. For example, by work of G. Bobiński and H. Krause [\[10\]](#), the Krull–Gabriel dimension of (perfect categories of) Dynkin quivers is 0, and that of derived-discrete algebras is either 1 or 2, depending on whether the algebra has infinite or finite global dimension. However, there are non-discrete-derived algebras whose perfect categories have Krull–Gabriel dimension 2; for the example of the Kronecker quiver, see [\[19, Proposition 1.8\]](#).

Another abstract concept for triangulated categories is that of a generic object, and its absence, generic triviality. In [\[15\]](#), Z. Han shows that generic triviality of a compactly generated triangulated category is equivalent to local finiteness of the compact subcategory, and to the abelianisation of the compact subcategory having Krull–Gabriel dimension 0 in the above sense. In particular, it seems unlikely that these notions are immediately useful in the study of the smallness notions investigated here.

Finally, we mention [\[14\]](#). In this work, Y. Han and C. Zhang characterise derived-discrete algebras as the finite-dimensional algebras of finite global cohomological length. Their approach depends on cohomology and modules, i.e. does not apply to abstract triangulated categories.

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1. PROPERTIES OF TRIANGULATED CATEGORIES: CONE FINITE, HOM BOUNDED

We define a number of properties that suitable \mathbf{k} -linear triangulated categories can enjoy, all of which capture certain aspects of ‘smallness’. Throughout, we assume that the class of objects of any category forms a set. Moreover, we apply the following abuse of

terminology: whenever we speak of a ‘set’ of objects defined by some property, we mean the class of such objects, up to isomorphism.

We let \mathbf{k} be any field. Recall that a \mathbf{k} -linear category is called Hom-finite if all homomorphism spaces are finite-dimensional over \mathbf{k} . Also recall that an additive category is Krull–Schmidt if each object has a unique decomposition into a direct sum of indecomposable objects. Examples are bounded derived categories of Hom-finite abelian categories.

Definition 1.1. Let \mathcal{D} be a Hom-finite, Krull–Schmidt \mathbf{k} -linear triangulated category.

- (1) \mathcal{D} is called *cone finite* if for any two objects $D_1, D_2 \in \mathcal{D}$, the set of cones of morphisms $D_1 \rightarrow D_2$, i.e. the set $\{C \in \mathcal{D} \mid \exists D_1 \rightarrow D_2 \rightarrow C \rightarrow \Sigma D_1\}$, is finite.
- (2) \mathcal{D} is called *hom b -bounded* for some $b \in \mathbb{N}$ if $\dim \text{Hom}(D_1, D_2) \leq b$ for all $D_1, D_2 \in \text{ind}(\mathcal{D})$. The minimal such b is called the *hom bound* of \mathcal{D} , and \mathcal{D} is called *hom bounded* if it is hom b -bounded for some $b \in \mathbb{N}$.
- (3) \mathcal{D} is called *countable*, if the set of all objects up to isomorphism is countable. (Because of the Krull–Schmidt and Hom-finite assumptions this is equivalent to $\text{ind}(\mathcal{D})$ being a countable set.)

All of these definitions could be stated in greater generality: cone finiteness makes sense for all triangulated categories (no field needed); hom boundedness applies to arbitrary \mathbf{k} -linear categories (no triangulated structure required); countability of objects applies to arbitrary categories. The latter notion is crude, and depends strongly on the cardinality of the field \mathbf{k} ; see [Remark 1.3](#). We will not explore these properties beyond the setting of Hom-finite triangulated categories.

In this article, we also study the relationship with Vossieck’s notion of discreteness, see [Section 2](#), and we introduce and investigate its co- t -structure counterpart in [Section 4](#). For now, we only deal with the above three notions: they have the advantage of applying in a general setting, i.e. without additional data. Also note each condition (countable objects, hom bounded, cone finite) is automatically passed on to triangulated subcategories.

Theorem 1.2. (i) *A hom 1-bounded triangulated category is cone finite.*

(ii) *A cone finite triangulated category with a classical generator is countable.*

Proof. (i) Suppose that \mathcal{D} is hom bounded with bound 1, i.e. $\text{hom}(A, B) \leq 1$ for all $A, B \in \text{ind}(\mathcal{D})$. In particular, this implies that nonzero morphisms $A \rightarrow B$ with $A, B \in \text{ind}(\mathcal{D})$ have isomorphic cones. Consider a morphism of the form

$$A_1 \oplus \cdots \oplus A_n \xrightarrow{(a_1, \dots, a_n)^t} B$$

where A_1, \dots, A_n are indecomposable. If one $a_i = 0$, then a standard application of the octahedral axiom shows that the cone splits up as follows:

$$A_1 \oplus \cdots \oplus A_n \xrightarrow{(a_1, \dots, a_n)^t} B \longrightarrow C((a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)^t) \oplus \Sigma A_i.$$

Therefore, we can assume that all $a_i \neq 0$. Any other such morphism $A_1 \oplus \cdots \oplus A_n$ is of the form $(\lambda_1 a_1, \dots, \lambda_n a_n)^t$ for scalars $\lambda_1, \dots, \lambda_n \in \mathbf{k}$ and hence induces a commutative diagram of distinguished triangles

$$\begin{array}{ccccccc} A_1 \oplus \cdots \oplus A_n & \xrightarrow{(\lambda_1 a_1 \dots \lambda_n a_n)^t} & B & \longrightarrow & C & \longrightarrow & \Sigma(A_1 \oplus \cdots \oplus A_n) \\ \text{diag}(\lambda_1, \dots, \lambda_n) \downarrow & & \parallel & & \downarrow & & \downarrow \\ A_1 \oplus \cdots \oplus A_n & \xrightarrow{(a_1 \dots a_n)^t} & B & \longrightarrow & C' & \longrightarrow & \Sigma(A_1 \oplus \cdots \oplus A_n). \end{array}$$

As above, we may assume that all $\lambda_i \neq 0$. But then both vertical morphisms in the left hand square are isomorphisms, hence the dashed arrow is also an isomorphism. It follows that there are finitely many possible cones C for morphisms $A_1 \oplus \cdots \oplus A_n \rightarrow B$.

Now consider the cone of an arbitrary morphism $\bigoplus_{i=1}^n A_i \rightarrow \bigoplus_{j=1}^m B_j$. We proceed by induction on m . For $m = 1$, we are done above, so assume $m > 1$. We have the following diagram coming from the octahedral axiom:

$$\begin{array}{ccccc}
 & & B_1 & \xlongequal{\quad} & B_1 \\
 & & \downarrow & & \downarrow \\
 \bigoplus_{i=1}^n A_i & \longrightarrow & \bigoplus_{j=1}^m B_j & \longrightarrow & C \\
 \parallel & & \downarrow & & \downarrow \\
 \bigoplus_{i=1}^n A_i & \longrightarrow & \bigoplus_{j=2}^m B_j & \longrightarrow & C'.
 \end{array}$$

By induction, there are finitely many possible C' . By the case for $m = 1$ above, for each C' there are finitely many possible cones of $C' \rightarrow \Sigma B_1$, and in particular, finitely many possibilities for C . Hence \mathbf{D} is cone finite.

(ii) Recall that a *classical generator* of a triangulated category \mathbf{D} is an object G such that every object is obtained from G in finitely many steps by taking shifts, cones and summands. (There does not have to be a universal bound on the number of steps; if such bounds exist, their minimum is the Rouquier dimension of \mathbf{D} .) Bounded derived categories of finite-dimensional algebras and projective varieties have classical generators.

In each step, there are finitely many possibilities for summands by Krull–Schmidt; countably many possibilities for sums; and finitely many possibilities for cones by the assumption. Therefore, with G generating \mathbf{D} in countably many steps, the cardinality of objects of \mathbf{D} is countable as well. \square

- Remark 1.3.**
- (1) The assumption of a classical generator in the theorem is necessary: if \mathbf{D} is any cone finite triangulated category, and I some uncountable set, then $\bigoplus_I \mathbf{D}$ is still cone finite but uncountable.
 - (2) On the other hand, a cone finite classically generating set does not imply cone finiteness. For example, for any two objects of the generating and suspension-stable set $\{\Sigma^i \mathcal{O}(j) \mid j \in \{0, 1\}, i \in \mathbb{Z}\} \subset \mathbf{D}^b(\mathbb{P}^1)$, there are only finitely many cones. Nonetheless, $\mathbf{D}^b(\mathbb{P}^1)$ is not cone finite.
 - (3) The same proof shows a bit more: if \mathbf{D} has a classical generator and for all $A, B \in \mathbf{D}$, there are only countably many cones of morphisms $A \rightarrow B$, then \mathbf{D} is countable. In particular, this applies if \mathbf{k} is a countable field.
 - (4) We remark that if \mathbf{k} is a finite field, then \mathbf{D} is trivially cone finite. It seems as if for fields of arbitrary cardinality, hom boundedness captures ‘smallness’ best.

Conjecture 1.4. *Hom bounded triangulated categories are cone finite.*

2. DISCRETENESS WITH RESPECT TO T-STRUCTURES

In this section we consider a slightly generalised version of Vossieck’s [21] original definition of derived-discrete algebras. Examining the derived categories of these algebras in [12] was our motivation to introduce the more abstract notions in this article.

Recall that a *torsion pair* in a triangulated category \mathbf{D} consists of a pair of full subcategories (\mathbf{X}, \mathbf{Y}) such that $\mathrm{Hom}(X, Y) = 0$ and

$$\mathbf{D} = \mathbf{X} * \mathbf{Y} := \{D \in \mathbf{D} \mid \exists X \rightarrow D \rightarrow Y \rightarrow \Sigma X \text{ with } X \in \mathbf{X}, Y \in \mathbf{Y}\}.$$

It is a *t-structure* if $\Sigma X \subseteq X$ and a *co-t-structure* if $\Sigma^{-1}X \subseteq X$. Any t-structure induces an abelian category, its *heart* $\mathbf{H} := X \cap \Sigma Y$. The analogous construction $\mathbf{C} = X \cap \Sigma^{-1}Y$ for a co-t-structure is called the *co-heart*; it is a silting subcategory, see [Section 4](#) for a definition and [\[1\]](#) for further details.

If a t-structure is bounded, then it can be reconstructed from its heart. As we only deal with bounded t-structures in this note, we will specify them by their hearts. For brevity, we will simply say ‘bounded heart’ to mean a full abelian subcategory which is the heart of a bounded t-structure. Recall that to any t-structure in a triangulated category, there are associated truncation (or cohomology) functors. Below, we denote these by $H^i: \mathbf{D} \rightarrow \mathbf{H}$ for a heart $\mathbf{H} \subset \mathbf{D}$.

Definition 2.1. Let \mathbf{D} be a Hom-finite, Krull–Schmidt \mathbf{k} -linear triangulated category admitting bounded t-structures.

- (1) Let \mathbf{H} be the heart of a bounded t-structure. Then \mathbf{D} is said to be *discrete with respect to \mathbf{H}* , or *\mathbf{H} -discrete*, if for every group valued function $v: \mathbb{Z} \rightarrow K_0(\mathbf{D})$, the set of objects $\{D \in \mathbf{D} \mid [H^i(D)] = v(i) \in K_0(\mathbf{D}) \forall i \in \mathbb{Z}\}$ is finite.
- (2) \mathbf{D} is said to have *finite hearts* if the heart of any bounded t-structure in \mathbf{D} has only finitely many indecomposable objects.

Let us establish the link between this definition of discreteness and Vossieck’s original notion: in [\[21\]](#), he exclusively considers categories of the form $\mathbf{D} = \mathbf{D}^b(\Lambda)$ for finite-dimensional \mathbf{k} -algebras over an algebraically closed field \mathbf{k} . He calls the derived category $\mathbf{D}^b(\Lambda)$ of the algebra Λ *discrete* if for any sequence $v: \mathbb{Z} \rightarrow K_0(\Lambda)$ with only finitely many non-zero terms, the set of isomorphism classes of indecomposable complexes $A \in \mathbf{D}^b(\Lambda)$ with dimension vector $\underline{\dim}(A) := (\underline{\dim}(H^i(A)))_{i \in \mathbb{Z}} = v$ is finite. Note that there are standard, canonical isomorphisms $K_0(\mathbf{D}^b(\Lambda)) \cong K_0(\Lambda) \cong \mathbb{Z}^N$, where N is the number of simple modules of Λ : the first isomorphism sends the class of a complex to the alternating sum of the classes of its cohomology, and the second isomorphism maps the class of a module to its dimension vector. It is immediate that $\mathbf{D}^b(\Lambda)$ is discrete in Vossieck’s sense if and only if $\mathbf{D}^b(\Lambda)$ is discrete with respect to $\text{mod}(\Lambda)$ in the above sense.

It seems to be hard to make use of the finite hearts property, but it does occur in examples, and is a curious feature of a triangulated category.

Because we have to work with hearts in triangulated categories, we now also introduce some notions that capture ‘smallness’ of abelian categories. Let \mathbf{H} be a Hom-finite, \mathbf{k} -linear abelian category, then \mathbf{H} is Krull–Schmidt [\[5\]](#). We denote $\pi: \text{Ob}(\mathbf{H}) \rightarrow K_0(\mathbf{H})$. Recall that we identify objects up to isomorphism. For an object $H \in \mathbf{H}$, we denote by $\text{Sub}(H)$ the set of subobjects $H' \hookrightarrow H$, and by $\text{Fac}(H)$ the set of factors $H \twoheadrightarrow H'$.

- \mathbf{H} is *finite* if the set $\text{ind}(\mathbf{H})$ is finite.
- \mathbf{H} is (*abelian*) *discrete* if π has finite fibres, i.e. for any $c \in K_0(\mathbf{H})$, the set of objects $\{A \in \mathbf{H} \mid [A] = c\}$ is finite.
- \mathbf{H} is *modular* if $\pi(\text{Sub}(H))$ is a finite set for all $H \in \mathbf{H}$, i.e. $\{[H'] \mid \exists H' \hookrightarrow H\} \subseteq K_0(\mathbf{H})$ is finite.

Of these, length and modular are mild restrictions. For example, they hold for $\text{mod}(\Lambda)$ with Λ a finite-dimensional algebra. The other two conditions (discrete and finite) are severe restrictions.

Remark 2.2. For derived-discrete algebras, all hearts are module categories of finite representation type. Therefore, they satisfy a stronger property than abelian discrete: the class of an object determines the object uniquely!

In general, preprojective and preinjective indecomposable modules of an algebra are uniquely determined by their dimension vectors; see, for example [\[4, Ch. IX\]](#).

We will justify the terminology ‘modular’ below in [Remark 2.4](#). For now, just observe that the condition is equivalent to the finiteness of $\pi(\text{Fac}(H))$, since $[H''] = [H] - [H']$ for any short exact sequence $0 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 0$.

Lemma 2.3. *Let H', H'' be objects of a modular abelian category \mathbf{H} . Then the K_0 -classes of objects H with exact sequences $H' \rightarrow H \rightarrow H''$ are finitely determined by the classes $[H']$ and $[H'']$, i.e. the set $\{[H] \mid \exists H' \rightarrow H \rightarrow H'' \text{ exact}\} \subseteq K_0(\mathbf{H})$ is finite.*

Proof. The exact sequence $H' \xrightarrow{f} H \xrightarrow{g} H''$ leads to short exact sequences

$$0 \rightarrow \ker(f) \rightarrow H' \rightarrow \text{im}(f) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{im}(g) \rightarrow H'' \rightarrow \text{coim}(g) \rightarrow 0.$$

As \mathbf{H} is modular, the class $[\text{im}(f)]$ is finitely determined by $[H']$, and $[\text{im}(g)]$ is finitely determined by $[H'']$. Moreover, from $0 \rightarrow \ker(g) \rightarrow H \rightarrow \text{im}(g) \rightarrow 0$ and exactness we get $[H] = [\ker(g)] + [\text{im}(g)] = [\text{im}(f)] + [\text{im}(g)]$, so $[H]$ is finitely determined by $[H']$ and $[H'']$, as claimed. \square

Remark 2.4. The property of the lemma captures the positivity of dimension vectors for modules over a finite-dimensional algebra Λ . If Λ has N simple modules, then $K_0(\Lambda) = \mathbb{Z}^N$, and the class of a module M is encoded in its dimension vector $\underline{\dim}(M) \in \mathbb{N}^N$.

All submodules have smaller dimension vectors, hence $\mathbf{H} := \text{mod}(\Lambda)$ is a modular abelian category. The lemma is a generalisation of the inequality $\underline{\dim}(M) \leq \underline{\dim}(M') + \underline{\dim}(M'')$ which holds for any exact sequence $M' \rightarrow M \rightarrow M''$.

Moreover, over an algebraically closed field \mathbf{k} the equivalences

$$\text{mod}(\Lambda) \text{ discrete} \iff \Lambda \text{ has finite representation type} \iff \text{mod}(\Lambda) \text{ finite}$$

hold by the validity of the second Brauer–Thrall conjecture; see for example [4, Ch. IV.5] and the references therein. These equivalences fail in general, for example, the Hom-finite hereditary abelian category that is a tube of rank r is discrete but not finite.

Theorem 2.5. *Let (\mathbf{D}, \mathbf{H}) be a triangulated category together with the heart of a bounded t -structure. Then*

- (i) \mathbf{D} cone finite and \mathbf{H} abelian discrete $\implies \mathbf{D}$ is discrete with respect to \mathbf{H}
- (ii) \mathbf{D} is discrete with respect to $\mathbf{H} \implies \mathbf{H}$ abelian discrete.
- (iii) \mathbf{D} is discrete with respect to \mathbf{H} and \mathbf{H} modular $\implies \mathbf{D}$ cone finite.

Corollary 2.6. *Let (\mathbf{D}, \mathbf{H}) be a triangulated category with a modular heart of a bounded t -structure. Then*

$$\mathbf{D} \text{ is discrete with respect to } \mathbf{H} \iff \mathbf{D} \text{ cone finite and } \mathbf{H} \text{ abelian discrete.}$$

Proof. For an object $D \in \mathbf{D}$, we define the function $v_D: \mathbb{Z} \rightarrow K_0(\mathbf{D})$ by $v_D(i) := [H^i(D)]$.

For a function $v: \mathbb{Z} \rightarrow K_0(\mathbf{D})$, we define:

- $\mathbf{D}_v := \{A \in \mathbf{D} \mid [H^i(A)] = v(i) \forall i \in \mathbb{Z}\}$, a full subcategory;
- $\text{supp}(v) := \{i \in \mathbb{Z} \mid v(i) \neq 0\}$, the support of v ;
- $\text{length}(v) := \max_{i \in \mathbb{Z}} \{v(i) \neq 0\} - \min_{i \in \mathbb{Z}} \{v(i) \neq 0\}$, the length of v .

(i) Given v , we do induction on the length of v . If $\text{length}(v) = 0$, then we can assume that $v(0) \neq 0$, by suspending if necessary. Then all objects of \mathbf{D}_v have a single cohomology in degree 0, hence are in the heart \mathbf{H} . Thus, $\mathbf{D}_v = \{A \in \mathbf{H} \mid [A] = v(0)\}$, and this set is finite by our assumption that \mathbf{H} is abelian discrete.

Let now $\text{length}(v) = n > 0$. Again, without loss of generality, we can assume that $\text{supp}(v) \subseteq \{0, \dots, n\}$. Define $v', v'': \mathbb{Z} \rightarrow K_0(\mathbf{D})$ by $v''(i) = v(i)$ for $1 \leq i \leq n$ and zero otherwise; and $v'(0) = v(0)$ and zero otherwise. By induction, the subcategories $\mathbf{D}_{v'}$ and

$\mathcal{D}_{v''}$ are finite. Now any object $A \in \mathcal{D}_v$ has a unique decomposition $A' \rightarrow A \rightarrow A'' \rightarrow \Sigma A'$ with $A' \in \mathcal{D}_{v'}$ and $A'' \in \mathcal{D}_{v''}$. (This is the truncation triangle for A with respect to H^0 .) Hence, $\mathcal{D}_v \subseteq \mathcal{D}_{v'} * \mathcal{D}_{v''}$. However, as \mathcal{D} is cone finite, there are only finitely many cones out of the finitely many objects from the two subcategories. Hence, \mathcal{D}_v is also finite.

(ii) This is immediate: given $c \in K_0(\mathbf{H})$, let $v: \mathbb{Z} \rightarrow K_0(\mathcal{D}) = K_0(\mathbf{H})$ be defined by $v(0) := c$ and $v(i) = 0$ for $i \neq 0$. By the assumption that \mathcal{D} is discrete with respect to \mathbf{H} , the set of objects $D \in \mathcal{D}$ with $[H^i(D)] = v(i)$ for all $i \in \mathbb{Z}$ is finite. This set includes the objects $D \in \mathbf{H}$ (i.e. all cohomology outside 0 vanishes) and the cohomology object of degree zero has class $c = v(0)$. Hence \mathbf{H} is abelian discrete.

(iii) For $A, B \in \mathcal{D}$, we want to show that there are only finitely many cones $A \xrightarrow{f} B \rightarrow C_f$, where $f \in \text{Hom}(A, B)$ is arbitrary. Any such triangle gives rise to a long exact cohomology sequence in \mathbf{H}

$$\cdots \rightarrow H^i(A) \rightarrow H^i(B) \rightarrow H^i(C_f) \rightarrow H^{i+1}(A) \rightarrow H^{i+1}(B) \rightarrow \cdots .$$

By [Lemma 2.3](#), $[H^i(C_f)] \in K_0(\mathbf{H})$ is determined up to finite ambiguity by $[H^i(A)]$ and $[H^{i+1}(B)]$. As the long exact sequence is finite (the t-structure is bounded), we see that all $[H^i(C_f)]$ are determined by A and B , up to finite ambiguity (even more, they are determined by the functions v_A, v_B , but we do not need this). Hence, for fixed A and B , there are only finitely many possibilities for v_{C_f} . Finally, since \mathcal{D} is \mathbf{H} -discrete it follows that for each such choice of v_{C_f} , there are only finitely many objects C_f realising this function. Altogether, the number of cones of morphisms $A \rightarrow B$ is finite. \square

We expect that the following statements hold in general. In the next section, we show that they do hold for derived-discrete algebras. Note that a triangulated category \mathcal{D} can be discrete with respect to a bounded heart \mathbf{H} which is hom unbounded; see the tube category \mathcal{T}_1 in the table on [page 12](#). This example also yields a bounded heart with infinitely many indecomposable objects.

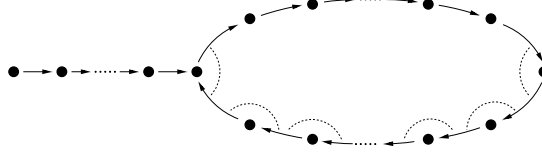
Conjecture 2.7. *Let \mathcal{D} be a Hom-finite Krull–Schmidt triangulated category and \mathbf{H} the heart of a bounded t-structure.*

- (i) *If \mathcal{D} is \mathbf{H} -discrete, then all bounded hearts in \mathcal{D} are discrete.*
- (ii) *If \mathcal{D} is \mathbf{H} -discrete, then so is $(\mathcal{D}, \mathbf{H}')$ for any bounded heart \mathbf{H}' .*
- (iii) *If \mathcal{D} is \mathbf{H} -discrete and \mathbf{H} is finite, then all bounded hearts are finite.*
- (iv) *\mathcal{D} is \mathbf{H} -discrete $\iff \mathcal{D}$ is cone finite.*

3. DERIVED-DISCRETE ALGEBRAS $\Lambda(r, n, m)$

Recall that in [\[21\]](#), a finite-dimensional algebra was defined to have a *discrete derived category* if $\mathcal{D}^b(\Lambda)$ is discrete with respect to $\text{mod}(\Lambda)$ in our sense, i.e. with respect to the standard heart. Following standard usage, we call such an algebra *derived-discrete*.

By the classification of G. Bociński, C. Geiß and A. Skowroński [\[9\]](#), such an algebra is derived equivalent to either a representation-finite hereditary algebra or to the path algebra $\Lambda(r, n, m)$ given by a cycle of length n to which a linearly oriented A_m -chain is attached; bound by r consecutive zero relations in the cycle, ending at the trivalent vertex. Here, $n \neq 1, m \geq 0, 1 \leq r \leq n$. In the following, we assume that $r < n$, which is equivalent to finite global dimension of $\Lambda(r, n, m)$.



We mention two basic facts about these categories from our previous work:

Proposition 3.1 ([12, Theorem 5.1 and Proposition 6.1]).

Let $\mathcal{D} = \mathcal{D}^b(\Lambda(r, n, m))$ be the bounded derived category of a derived-discrete algebra. Then

- (i) If $r = 1$, then \mathcal{D} is hom 2-bounded. If $r > 1$, then \mathcal{D} is hom 1-bounded.
- (ii) Hearts of bounded t -structures are finite.

We use this together with the results of Section 2 to show that discreteness for $\Lambda(r, n, m)$ is actually independent of the bounded t -structure.

Proposition 3.2. Let Λ be a derived-discrete algebra. Then $\mathcal{D}^b(\Lambda)$ is discrete with respect to any bounded heart \mathcal{H} .

Proof. As $\mathbf{mod}(\Lambda)$ is the module category of a finite-dimensional algebra (in particular modular) and $\mathcal{D}^b(\Lambda)$ is $\mathbf{mod}(\Lambda)$ -discrete, the triangulated category $\mathcal{D}^b(\Lambda)$ is cone finite by Theorem 2.5(iii). Moreover, any bounded heart \mathcal{H} is finite by Proposition 3.1. Therefore, $\mathcal{D}^b(\Lambda)$ is discrete with respect to \mathcal{H} by Theorem 2.5(i). \square

Propositions 3.1 and 3.2 show that Conjecture 2.7 holds for $\mathcal{D}^b(\Lambda(r, n, m))$. It turns out that these categories even enjoy a property slightly stronger than \mathcal{H} -discreteness:

Proposition 3.3. If $\mathcal{D} = \mathcal{D}^b(\Lambda(r, n, m))$ is a discrete derived category, then indecomposable objects D are uniquely determined by the sequences $[H^i(D)] \in K_0(\mathcal{D})$.

Proof. We first show that for any indecomposable complex $D \in \mathbf{ind}(\mathcal{D}^b(\Lambda(r, n, m)))$, each cohomology module $H^i(D)$ is indecomposable or zero in $\mathbf{mod}(\Lambda(r, n, m))$. To see this, recall that since $\Lambda(r, n, m)$ is gentle, by [6] the indecomposable complexes in $\mathbf{K}^{b,-}(\Lambda(r, n, m))$ are given by so-called *homotopy string complexes*; see [8] for the terminology, see also [2, Section 2] for an overview. The homotopy string complexes for $\Lambda(r, n, m)$ are listed in [2, Lemma 7.1]. Proving that $H^i(D)$ is either indecomposable or zero now follows from a straightforward computation. The only place where this is not completely trivial is in the following part of a homotopy complex:

$$P(0) \xrightarrow{(\bar{a}_j \ c_{n-1})} P(j) \oplus P(n-1) \xrightarrow{c_{n-2}} P(n-2) \longrightarrow \dots,$$

where we employ the notation of [2, Section 7], also setting $\bar{a}_j := a_j \cdots a_{-1}$ and assuming $-m \leq j \leq -1$ and $r > 1$. One then computes that the cohomology contributed in the same degree as $P(j) \oplus P(n-1)$ is the string module given by the string $\bar{a}_j c_{n-1}$, which is thus indecomposable. A similar computation holds when $r = 1$. \square

Example 3.4. The statement of Proposition 3.3 does not hold for all \mathcal{H} -discrete categories. For instance, the derived category $\mathcal{D}^b(\mathbb{P}_{\mathbf{k}}^1)$ for a finite field \mathbf{k} is \mathcal{H} -discrete by Remark 1.3 but the finitely many skyscraper sheaves have the same class in the Grothendieck group.

Example 3.5. A similar strengthening of the cone finite property akin to the proposition does not hold for discrete derived categories, however — they are not ‘cone unique’: there are indecomposable objects A, B of $\mathcal{D}^b(\Lambda(1, 2, 1))$ and non-zero maps $f, g: A \rightarrow B$ having non-isomorphic cones. Obviously such behaviour is impossible if $r > 1$, as then \mathbf{Hom} spaces between indecomposables are 1-dimensional and lead to unique cones.

Consider $\Lambda(1, 2, 1)$, i.e. $-1 \xrightarrow{a} 0 \xrightleftharpoons[c]{b} 1$ with the zero relation bc at the vertex 0.

$$\begin{array}{ccc}
P(0) & \xrightarrow{(cb \ a)} & P(0) \oplus P(-1) \\
\downarrow 1 & & \downarrow \begin{pmatrix} 0 \\ a \end{pmatrix} \\
P(0) & \xrightarrow{cba} & P(-1) \\
\downarrow a & & \\
P(-1) & &
\end{array}
\qquad
\begin{array}{ccc}
P(0) & \xrightarrow{(cb \ a)} & P(0) \oplus P(-1) \\
\downarrow cb & & \downarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
P(0) & \xrightarrow{cba} & P(-1) \\
\downarrow & & \downarrow \\
P(0) & \xrightarrow{(cb \ a)} & P(0) \oplus P(-1) \xrightarrow{\begin{pmatrix} cba \\ 0 \end{pmatrix}} & P(-1)
\end{array}$$

Then the cones are $P(-1)$ and $P(0) \rightarrow P(0) \oplus P(-1) \rightarrow P(-1)$. Note that above all the matrices are transposed because, to match up with string combinatorics for these examples, we read compositions of maps from left to right.

Remark 3.6. We expect that all results of this section are also true for the derived-discrete algebras $\Lambda(n, n, m)$ of infinite global dimension. In fact, the hom bound was established in [2, Theorem 7.4].

Our results suggest the following question.

Question 3.7. Are derived-discrete algebras Λ characterised (among finite-dimensional algebras) by $D^b(\Lambda)$ having only finite bounded hearts?

Relation to compactly generated triangulated categories. We briefly discuss ‘big’ triangulated categories, i.e. assuming the existence of all set-indexed coproducts. Note that such categories are not Hom-finite, and hence outside the scope of the rest of this article.

Recall that a (compactly generated) triangulated category \mathcal{D} is called *pure semisimple* if each object $D \in \mathcal{D}$ is pure-injective; we refer the reader to [13, Section 2] for the definition of pure-injectivity in the setting of triangulated categories.

Let Λ be a finite-dimensional algebra. By [7, Theorem 12.20], the category $\mathcal{D} = D(\text{Mod}(\Lambda))$ is pure semisimple if and only if Λ is derived equivalent to a representation-finite hereditary algebra. However, by [3], each indecomposable object in the homotopy category $K(\text{Proj}(\Lambda(r, n, m)))$ is pure injective but $K(\text{Proj}(\Lambda(r, n, m)))$ is not pure semisimple; similarly also for $D(\text{Mod}(\Lambda(r, n, m)))$. This raises the following question:

Question 3.8. Does the property that each indecomposable object of a (big) compactly generated triangulated category is pure injective, characterise the property of discreteness in the setting of big (= with set-indexed coproducts) triangulated categories?

4. DISCRETENESS WITH RESPECT TO CO-T-STRUCTURES

Let $(\mathcal{D}, \mathcal{C})$ be a Krull–Schmidt triangulated category together with the co-heart \mathcal{C} of a bounded co-t-structure. Then \mathcal{C} is a silting subcategory, i.e. $\text{Hom}^{>0}(C, C') = 0$ for all $C, C' \in \mathcal{C}$ and the thick subcategory generated by \mathcal{C} is \mathcal{D} ; see [1] for more details. If \mathcal{C} has an additive generator, i.e. $\mathcal{C} = \text{add } C$ for some object $C \in \mathcal{D}$, then C is a *silting object* and we say \mathcal{D} has a silting object.

By [20], see also [1], the co-t-structure $(X_{\mathcal{C}}, Y_{\mathcal{C}})$ can be recovered from the co-heart \mathcal{C} using the formulas

$$X_{\mathcal{C}} = \bigcup_{k>0} \Sigma^{-k} \mathcal{C} * \Sigma^{-k+1} \mathcal{C} * \dots * \Sigma^{-1} \mathcal{C} \quad \text{and} \quad Y_{\mathcal{C}} = \bigcup_{k \geq 0} \mathcal{C} * \Sigma \mathcal{C} * \dots * \Sigma^k \mathcal{C}.$$

For integers $p \leq q$ we set $C^{p,q} := \Sigma^p \mathcal{C} * \Sigma^{p+1} \mathcal{C} * \dots * \Sigma^{q-1} \mathcal{C} * \Sigma^q \mathcal{C}$.

Let $D \in \mathcal{D}$. If $D \in \Sigma^m \mathcal{C}^{p,q}$ for some $m, p, q \in \mathbb{Z}$ we shall say that D is $(q - p)$ -term with respect to \mathcal{C} .

Note that the formulas above are none other than the observation that given a co-t-structure with co-heart \mathcal{C} each object $0 \neq D \in \mathcal{D}$ admits a Postnikov tower

$$(1) \quad \begin{array}{ccccccccccc} 0 = D_0 & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & \cdots & \longrightarrow & D_{n-1} & \longrightarrow & D_n = D \\ & & \downarrow & \nearrow & \downarrow & & & & \downarrow & \nearrow & \downarrow \\ & & \Sigma^{i_1} C_1 & & \Sigma^{i_2} C_2 & & & & \Sigma^{i_{n-1}} C_{n-1} & & \Sigma^{i_n} C_n \end{array}$$

with $i_1 < i_2 < \cdots < i_n$ and $C_i \in \mathcal{C}$; see [11, Proposition 1.5.6].

Definition 4.1. Let $(\mathcal{D}, \mathcal{C})$ be a triangulated category with a bounded co-t-structure with co-heart \mathcal{C} . We call $(\mathcal{D}, \mathcal{C})$ *discrete with respect to \mathcal{C}* , or \mathcal{C} -*discrete*, if for each group valued function $v: \mathbb{Z} \rightarrow K_0(\mathcal{D})$ the set of objects

$$\{D \in \mathcal{D} \mid D \text{ admits a filtration (1) such that } \text{supp}(v) = \{i_1, \dots, i_n\} \text{ and } [C_j] = v(i_j)\}$$

is finite.

We have the following co-t-structure analogue of Theorem 2.5.

Theorem 4.2. *Let $(\mathcal{D}, \mathcal{C})$ be a triangulated category together with the co-heart \mathcal{C} of a bounded co-t-structure, and assume that \mathcal{C} has an additive generator. Then*

$$\mathcal{D} \text{ is cone-finite} \iff \mathcal{D} \text{ is discrete with respect to } \mathcal{C}.$$

Proof. \implies The proof is essentially the same as the proof of Theorem 2.5(i), where we instead write for a function $v: \mathbb{Z} \rightarrow K_0(\mathcal{D})$,

$$\mathcal{D}_v := \{A \in \mathcal{D} \mid A \text{ admits a filtration (1) with } \text{supp}(v) = \{i_1, \dots, i_n\} \text{ and } [C_j] = v(i_j)\}.$$

The only part where the proof differs is the base step of the induction, i.e. $\text{length}(v) = 0$. Again, without loss of generality we may assume $v(0) \neq 0$. Since $\mathcal{C} = \text{add } \mathcal{C}$, where $\mathcal{C} = C_1 \oplus \cdots \oplus C_n$ say, is a silting object, each object $C' \in \mathcal{C}$ decomposes uniquely as $C' = C_1^{m_1} \oplus C_2^{m_2} \oplus \cdots \oplus C_n^{m_n}$, whence $[C'] = m_1[C_1] + \cdots + m_n[C_n]$. Therefore, the class of $[C'] \in K_0(\mathcal{C})$ is uniquely determined by its Krull–Schmidt decomposition. This says that, in particular, that \mathcal{D}_v is a singleton when $\text{length}(v) = 0$. The remainder of the proof proceeds as in Theorem 2.5(i), noting that the uniqueness of the decomposition triangle $A' \rightarrow A \rightarrow A'' \rightarrow \Sigma A'$ is not required for the proof to work.

\impliedby Let $A, B \in \mathcal{D}$. We want to show that the set $\mathcal{Z} := \{Z \mid \exists A \xrightarrow{f} B \rightarrow Z \rightarrow \Sigma A\}$ is finite. We proceed in two steps.

Step 1: A is 1-term with respect to \mathcal{C} and B is n -term with respect to \mathcal{C} , for some $n \geq 1$.

Without loss of generality we may assume that $A = \Sigma^m C$ for some $C \in \mathcal{C}$ and some $m \in \mathbb{Z}$ and $B \in \mathcal{C}^{0,n}$. In particular, this means that B admits a filtration,

$$\begin{array}{ccccccccccc} 0 = B_{-1} & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_{n-1} & \longrightarrow & B_n = B \\ & & \downarrow & \nearrow & \downarrow & & & & \downarrow & \nearrow & \downarrow \\ & & C_0 & & \Sigma C_1 & & & & \Sigma^{n-1} C_{n-1} & & \Sigma^n C_n \end{array}$$

with the $C_i \in \mathcal{C}$, some of them possibly zero. We consider various possibilities for m .

If $m < 0$ then $\text{Hom}(A, B) = 0$ and \mathcal{Z} is trivially finite.

If $m \geq n$, then we get the following filtration for any $Z \in \mathbf{Z}$,

$$\begin{array}{ccccccccccc}
0 = B_{-1} & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_{n-1} & \longrightarrow & B_n = B & \longrightarrow & Z, \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
& & C_0 & & \Sigma C_1 & & & & \Sigma^{n-1} C_{n-1} & & \Sigma^n C_n & & \Sigma^{m+1} C
\end{array}$$

whence by \mathbf{C} -discreteness there are only finitely many Z admitting a filtration with these filtrands, making \mathbf{Z} finite.

If $0 \leq m < n$, we consider the diagram coming from the octahedral axiom.

$$\begin{array}{ccccc}
& & A = \Sigma^m C & & \\
& & \downarrow & & \downarrow 0 \\
B_{n-1} & \longrightarrow & B & \longrightarrow & \Sigma^n C_n \\
\parallel & & \downarrow & & \downarrow \\
B_{n-1} & \longrightarrow & Z & \longrightarrow & X
\end{array}$$

Thus, $X = \Sigma^n C_n \oplus \Sigma^{m+1} C$. If $m = n - 1$ then we get the filtration:

$$\begin{array}{ccccccccccc}
0 = B_{-1} & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_{n-1} & \longrightarrow & Z. \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
& & C_0 & & \Sigma C_1 & & & & \Sigma^{n-1} C_{n-1} & & \Sigma^n (C_n \oplus C)
\end{array}$$

Otherwise, using [17, Lemmas 7.1 and 7.2] in sequence gives the filtration

$$\begin{array}{ccccccccccc}
0 = B_{-1} & \longrightarrow & B_0 & \longrightarrow & \cdots & \longrightarrow & B'_{m+1} & \longrightarrow & \cdots & \longrightarrow & B'_{n-1} & \longrightarrow & Z. \\
& & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow \\
& & C_0 & & & & \Sigma^{m+1} (C_{m+1} \oplus C) & & & & \Sigma^{n-1} C_{n-1} & & \Sigma^n C_n
\end{array}$$

In either case, \mathbf{C} -discreteness affirms the finiteness of \mathbf{Z} .

Step 2: Both A and B are of arbitrary length with respect to \mathbf{C} .

We proceed by induction on the length of A with respect to \mathbf{C} ; cf. proof of Theorem 1.2. Suppose A is n -term with respect to \mathbf{C} . Then A admits a decomposition $A' \rightarrow A \rightarrow A'' \rightarrow \Sigma A'$ in which A' is $(n-1)$ -term and A'' is 1-term with respect to \mathbf{C} . Now consider the diagram coming from the octahedral axiom:

$$\begin{array}{ccccc}
& & \Sigma A' = \Sigma A' & & \\
& & \downarrow & & \downarrow \\
Z & \longrightarrow & \Sigma A & \longrightarrow & \Sigma B \\
\parallel & & \downarrow & & \downarrow \\
Z & \longrightarrow & \Sigma A'' & \longrightarrow & \Sigma X.
\end{array}$$

By induction, there are finitely many possible ΣX , whence by Step 1, and the fact that A'' is 1-term with respect to \mathbf{C} there are finitely many possible Z . \square

Corollary 4.3. *If \mathbf{D} is discrete with respect to a silting subcategory \mathbf{C} of \mathbf{D} , then \mathbf{D} is discrete with respect to any other silting subcategory \mathbf{C}' .*

For the next corollary, we remark that $D^b(\mathbf{mod}(\Lambda))$ has a natural bounded t-structure, with heart $\mathbf{mod}(\Lambda)$, and $K^b(\mathbf{proj}(\Lambda))$ has a natural co-t-structure, with co-heart $\mathbf{proj}(\Lambda)$. By the previous corollary, the actual choice of (co-)t-structure does not matter, however.

Corollary 4.4. *Let Λ be a finite-dimensional \mathbf{k} -algebra. If $D^b(\mathbf{mod}(\Lambda))$ is discrete with respect to $\mathbf{mod}(\Lambda)$ then $K^b(\mathbf{proj}(\Lambda))$ is discrete with respect to $\mathbf{proj}(\Lambda)$.*

Proof. Since $\mathbf{mod}(\Lambda)$ is modular, by [Theorem 2.5](#), $D^b(\mathbf{mod}(\Lambda))$ is cone finite. Since $K^b(\mathbf{proj}(\Lambda))$ embeds into $D^b(\mathbf{mod}(\Lambda))$ as a full subcategory, we have that $K^b(\mathbf{proj}(\Lambda))$ is also cone finite, whence [Theorem 4.2](#) implies $K^b(\mathbf{proj}(\Lambda))$ is discrete with respect to $\mathbf{proj}(\Lambda)$. \square

Remark 4.5. Vossieck's main result [[21](#), §2, Theorem] asserts that $D^b(\mathbf{mod}(\Lambda))$ is discrete if and only if $K^b(\mathbf{proj}(\Lambda))$ is discrete (taking homology with respect to $\mathbf{mod}(\Lambda)$).

However, in general, there is no intrinsic definition of discreteness in $K^b(\mathbf{proj}(\Lambda))$. For example, $D^b(\mathbf{mod}(\mathbf{k}[x]/(x^2)))$ is discrete with respect to $\mathbf{mod}(\Lambda)$, but $K^b(\mathbf{proj}(\mathbf{k}[x]/(x^2)))$ has no bounded t-structure [[16](#)] so that the discreteness notion of [[21](#)] does not apply. Nevertheless, $K^b(\mathbf{proj}(\mathbf{k}[x]/(x^2))) = K^b(\mathbf{proj}(\Lambda(1, 1, 0)))$ is discrete with respect to $\mathbf{proj}(\Lambda)$ by [Corollary 4.4](#), and more generally, the same holds for all $K^b(\mathbf{proj}(\Lambda(n, n, m)))$.

5. EXAMPLES

In the following table, we present some triangulated categories exhibiting interesting behaviour with regards to the various smallness notions studied in this article. For the convenience of the reader, we briefly summarise these notions:

H-discrete: There is a bounded heart \mathbf{H} such that for any $v: \mathbb{Z} \rightarrow K_0(\mathbf{H})$, the set of objects $D \in \mathbf{D}$ with $[H^i(D)] = v(i)$ for all i is finite. In all example classes below, if this property holds for one bounded heart, it holds for all.

C-discrete: There is a silting subcategory \mathbf{C} such that for any $v: \mathbb{Z} \rightarrow K_0(\mathbf{H})$, the set of objects admitting a Postnikov tower having filtrands $\Sigma^{i_j} C_j$ such that $i_1 < \dots < i_n$ and $C_i \in \mathbf{C}$ is finite. In the examples below, this property is independent of \mathbf{C} .

Hom bound: There is a universal bound on Hom dimensions among indecomposable objects; the subscript indicates the maximal bound occurring in the family.

cone finite: Any two objects admit only finitely many cones, up to isomorphism.

finite hearts: Hearts of bounded t-structures have finitely many indecomposables.

discrete hearts: Any object $H \in \mathbf{H}$ of any bounded heart is determined up to finite ambiguity by $[H] \in K_0(\mathbf{H})$.

countable: The category has only countably many objects, up to isomorphism.

Several of these properties make no sense for triangulated categories without bounded (co)-t-structures. This is indicated by $-$ in the table. The examples assume that \mathbf{k} is an uncountable field, apart from the last two columns.

| | $\mathbf{k}\Gamma_{\text{ADE}}$ | $\mathbf{k}A_\infty$ | DDC | DDC ^c | \mathcal{C}_{ADE} | \mathbf{T}_1 | $\mathbf{T}_{>1}$ | $\mathbf{T}_{<1}$ | $\mathbf{T}_{1,n}$ | $\mathbb{Q}\tilde{A}_1$ | $\mathbb{F}_q\tilde{A}_1$ |
|-------------------|---------------------------------|----------------------|----------------|------------------|----------------------------|----------------|-------------------|-------------------|--------------------|-------------------------|---------------------------|
| H-discrete | ✓ | ✓ | ✓ | — | — | ✓ | ✓ | — | — | × | ✓ |
| C-discrete | ✓ | ✓ | ✓ | ✓ | — | — | — | ✓ | — | × | ✓ |
| hom bounded | ✓ ₆ | ✓ ₁ | ✓ ₂ | ✓ ₂ | ✓ ₆ | × | ✓ ₁ | ✓ ₂ | ✓ _[n/2] | × | × |
| cone finite | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | × | ✓ |
| finite hearts | ✓ | × | ✓ | — | — | × | ✓ | — | — | × | × |
| discrete hearts | ✓ | ✓ | ✓ | — | — | ✓ | ✓ | — | — | × | ✓ |
| countable objects | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |

We proceed to explain the example classes.

$\mathbf{k}\Gamma_{\text{ADE}}, \mathbf{k}A_\infty$ — **quiver algebras:** By listing finite-dimensional algebras, we mean their bounded derived categories. Γ_{ADE} stands for an ADE quiver, so that the corresponding algebra is hereditary and representation-finite. The maximal hom bound of 6 achieved in type E_8 . Next, A_∞ stands for the (one-sided) infinite, zigzag oriented quiver of type A . This example is interesting because $\mathbf{D}^b(\mathbf{k}A_\infty)$ is hom bounded, but has infinite hearts; note it does not have a classical generator.

DDC, DDC^c — derived-discrete algebras: DDC is a shorthand for $\Lambda(r, n, m)$, the derived-discrete algebra with r consecutive relations in an n -cycle and a tail of length m ; see [Section 3](#). We assume $r < n$, so that AR triangles, or equivalently, a Serre functor, exist. Note that $\mathbf{D}^b(\Lambda(n, n, m))$ has no bounded co-t-structures.

DDC^c stands for $K^b(\mathbf{proj}(\Lambda(n, n, m)))$, the bounded homotopy category of projective modules over a derived-discrete algebra of infinite global dimension. This is the subcategory of compact objects of $\mathbf{D}^b(\Lambda(n, n, m))$. Note that DDC^c has AR triangles: these algebras are gentle, hence Gorenstein.

\mathcal{C}_{ADE} — **cluster categories:** $\mathcal{C}_{\text{ADE}} = \mathbf{D}^b(\mathbf{k}\Gamma_{\text{ADE}})/\Sigma^{-1}\tau$ stands for the cluster category of type ADE, where τ is the Auslander–Reiten translation. It is triangulated by [\[18\]](#) and has finitely many indecomposables.

$\mathbb{T}_1, \mathbb{T}_{>1}, \mathbb{T}_{<1}$ — **spherical generators:** For $w \in \mathbb{Z}$, let \mathbb{T}_w be the triangulated category generated by a w -spherical object, i.e. an object with derived endomorphism algebra $\mathbf{k} \oplus \Sigma^{-w}\mathbf{k}$. Hom bounds for $\mathbb{T}_{<0}$ are 1, and for \mathbb{T}_0 it is 2, due to the 0-spherical object. Note that \mathbb{T}_1 is the triangulated hereditary standard homogeneous tube. $\mathbb{T}_{<1}$ has no bounded t-structures [\[16\]](#), making it pointless to ask for H-discreteness or finite hearts. Likewise, $\mathbb{T}_{\geq 1}$ has no bounded co-t-structures.

$\mathbb{T}_{1,n}$ — **truncated tubes:** For $n > 1$, we let $\mathbb{T}_{1,n} = \underline{\mathbf{mod}}(\mathbf{k}[x]/(x^n))$ be the stable module category, e.g. $\mathbf{k}[x]/(x^2) = \Lambda(1, 1, 0)$. The AR quiver of $\mathbb{T}_{1,n}$ is the following truncated homogeneous tube:

$$X_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} X_2 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} X_3 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \cdots \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} X_{n-2} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} X_{n-1}.$$

Note that the unique projective module X_n does not occur because we have taken the stable category. The algebra is selfinjective, hence $\mathbb{T}_{1,n}$ is triangulated. As $\mathbb{T}_{1,n}$ has only $n - 1$ many indecomposable objects and is Krull–Schmidt, it is cone finite. However, from $\dim \text{Hom}(X_i, X_i) = \min\{i, n - i\}$, we see that arbitrary hom bounds can be attained. Note that $\mathbb{T}_{1,n}$ has no bounded t-structures.

Small fields: \tilde{A}_1 is the Kronecker quiver, and the last two columns are the derived categories of this quiver over \mathbb{Q} or \mathbb{F}_q , respectively. Instead of \mathbb{Q} , any infinite countable field works. By [Remark 1.3](#), we could actually replace \tilde{A}_1 by any finite-dimensional algebra. We have chosen the Kronecker quiver because it is manifestly non-discrete for uncountable fields.

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