

# STRATIFICATION FOR MODULE CATEGORIES OF FINITE GROUP SCHEMES

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ABSTRACT. The tensor ideal localising subcategories of the stable module category of all, including infinite dimensional, representations of a finite group scheme over a field of positive characteristic are classified. Various applications concerning the structure of the stable module category and the behavior of support and cosupport under restriction and induction are presented.

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## INTRODUCTION

This paper is about the representation theory of finite group schemes over a field  $k$  of positive characteristic. A finite group scheme  $G$  is an affine group scheme whose coordinate algebra is a finite dimensional vector space over  $k$ . In that case, the linear dual of the coordinate algebra, called the *group algebra* of  $G$ , is a finite dimensional cocommutative Hopf algebra, whose representation theory is equivalent to that of  $G$ . This means that all our results can be restated for finite dimensional cocommutative Hopf  $k$ -algebras, but we adhere to the geometric language.

Examples of finite group schemes include finite groups, restricted enveloping algebras of finite dimensional  $p$ -Lie algebras, and Frobenius kernels of algebraic groups. The representation theory of finite group schemes over  $k$  is often wild, even in such small cases as the finite group  $\mathbb{Z}/3 \times \mathbb{Z}/3$  over a field of characteristic three, or the 3-dimensional Heisenberg Lie algebra. In constructive terms, this means that it is not possible to classify the finite dimensional indecomposable modules. One thus has to find better ways to organise our understanding of the structure of the module category of  $G$ . Developments in stable homotopy theory and algebraic geometry suggest a natural extension of the process of building modules up to direct sums. Namely, in addition to (possibly infinite) direct sums and summands, one allows taking syzygies (both positive and negative), extensions, and tensoring (over  $k$ ) with simple  $G$ -modules. We say  $M$  is *built out* of  $N$  if  $M$  can be constructed out of  $N$  using these operations. What follows is one of the main results of this work.

**Theorem** (Corollary 10.2). *Let  $M$  and  $N$  be non-zero  $G$ -modules. One can build  $M$  out of  $N$  if (and only if) there is an inclusion  $\pi\text{-supp}_G(M) \subseteq \pi\text{-supp}_G(N)$ .*

Here  $\pi\text{-supp}_G(M)$  denotes the  $\pi$ -support of  $M$  introduced by Friedlander and Pevtsova [21], and recalled in Section 1. It is a subset of the space of  $\pi$ -points of  $G$  and the latter is homeomorphic to  $\text{Proj } H^*(G, k)$ . Recall that  $H^*(G, k)$ , the cohomology algebra of  $G$ , is a finitely generated graded-commutative  $k$ -algebra, by a result of Friedlander and Suslin [22]. Thus,  $\pi\text{-supp}_G(M)$  may be seen as an algebro-geometric portrait of  $M$  and the gist of the theorem is that this is fine enough to capture at least homological aspects of the  $G$ -module  $M$ .

The proper context for the result above is  $\text{StMod } G$ , the stable module category of  $G$ , and  $\mathbf{K}(\text{Inj } G)$ , the homotopy category of complexes of injective  $G$ -modules. These are compactly generated triangulated categories that inherit the tensor product of  $G$ -modules. We deduce Corollary 10.2 from an essentially equivalent statement, Theorem 10.1, that gives a classification of the tensor ideal localising subcategories of  $\text{StMod } G$ . See Corollary 10.6 for a version dealing with  $\mathbf{K}(\text{Inj } G)$ .

When  $M$  and  $N$  are finite dimensional,  $M$  is built out of  $N$  only if it is *finitely* built out of  $N$ , meaning that one needs only finite direct sums in the building process. Consequently, the results mentioned in the preceding paragraph yield a classification of the tensor ideal thick subcategories of  $\text{stmod } G$  and  $\mathbf{D}^b(\text{mod } G)$ , the stable module category and the bounded derived category, respectively, of finite dimensional modules. This is because these categories are equivalent to the subcategories of compact objects of  $\text{StMod } G$  and  $\mathbf{K}(\text{Inj } G)$ , respectively.

**A brief history.** The genesis of such results is a classification theorem of Devinatz, Hopkins, and Smith for the stable homotopy category of finite spectra [19].

Classification theorems for other “small” categories followed: see Hopkins [24] and Neeman [28] for perfect complexes over a commutative noetherian ring; Thomason [36] for perfect complexes of sheaves over a quasi-compact, quasi-separated scheme; Benson, Carlson, and Rickard [4] for finite dimensional modules of a finite group, as well as many more recent developments. Our results cover not only finite dimensional modules, but also the “big” category of all  $G$ -modules, so the closest precursor is Neeman’s classification [28] for all complexes over a commutative noetherian ring. An analogous statement for group schemes arising from finite groups is proved in [7]. Theorem 10.1 is new for all other classes of finite groups schemes.

**Structure of the proof.** Arbitrary finite group schemes lack many of the structural properties of finite groups, as we explain further below. Consequently the methods we use in this work are fundamentally different from the ones that lead to the successful resolution of the classification problem for finite groups in [7]. In fact, our proof of Theorem 10.1 provides another proof for the case of finite groups. The two new ideas developed and exploited in this work are that of  $\pi$ -cosupport of a  $G$ -module introduced in [9], and a technique of reduction to closed points that enhances a local to global principle from [6, 7].

For a finite group  $G$ , the proof of the classification theorem given in [7] proceeds by a reduction to the case of elementary abelian groups. This hinges on Chouinard’s theorem that a  $G$ -module is projective if and only if its restriction to all elementary abelian subgroups of  $G$  is projective. Such a reduction is an essential step also in a second proof of the classification theorem for finite groups described in [9]. For general finite group schemes there is no straightforward replacement for a detecting family of subgroups akin to elementary abelian subgroups of finite groups: for any such family one needs to allow scalar extensions. See Example 3.8 and the discussion around Corollary 5.5.

The first crucial step in the proof of the classification theorem is to verify that  $\pi$ -support detects projectivity:

**Theorem** (Theorem 5.3). *Any  $G$ -module  $M$  with  $\pi\text{-supp}_G(M) = \emptyset$  is projective.*

There is a flaw in the proof of this statement given in [21, Theorem 5.3], as we explain in Remark 5.7. For this reason, Part II of this paper is devoted to a proof of this result. Much of the argument is already available in the literature but is spread across various places. The new idea that allowed us to repair the proof appears in a “subgroup reduction principle”, Theorem 3.7, which also led to some simplifications of the known arguments.

As a consequence of the  $\pi$ -support detection theorem we prove:

**Theorem** (Theorem 6.1). *For any  $G$ -module  $M$  there is an equality*

$$\pi\text{-supp}_G(M) = \text{supp}_G(M).$$

Here  $\text{supp}_G(M)$  is the support of  $M$  defined in [5] using the action of  $H^*(G, k)$  on  $\text{StMod } G$ , recalled in Section 2. This allows us to apply the machinery developed in [5, 6, 7]. The first advantage that we reap from this is the following *local to global principle*: for the desired classification it suffices to verify that for each point  $\mathfrak{p}$  in  $\text{Proj } H^*(G, k)$  the subcategory of  $\text{StMod } G$  consisting of modules with support in  $\mathfrak{p}$  is *minimal*, in that it has no proper tensor ideal localising subcategories. This is

tantamount to proving that when  $M, N$  are  $G$ -modules whose support equals  $\{\mathfrak{p}\}$ , the  $G$ -module of homomorphisms  $\mathrm{Hom}_k(M, N)$  is not projective.

When  $\mathfrak{p}$  is a closed point in  $\mathrm{Proj} H^*(G, k)$ , we verify this by using a new invariant of  $G$ -modules called  $\pi$ -cosupport introduced in [9], and recalled in Section 1. Its relevance to the problem on hand stems from the equality below; see Theorem 1.9.

$$\pi\text{-cosupp}_G(\mathrm{Hom}_k(M, N)) = \pi\text{-supp}_G(M) \cap \pi\text{-cosupp}_G(N)$$

The minimality for a general point  $\mathfrak{p}$  in  $\mathrm{Proj} H^*(G, k)$  is established by a reduction to the case of closed points. To this end, in Section 8 we develop a technique that mimics the construction of generic points for irreducible algebraic varieties in the classical theory of Zariski and Weil. The results from commutative algebra that are required for this last step are established in Section 7. The ideas in these sections may be seen as an elaboration of the local to global principle eluded to above. They in fact suggest a richer version of the local to global principle and a passage to closed points that is likely to have further applications.

**Applications.** One of the many known consequences of Theorem 10.1 is a classification of the tensor-ideal thick subcategories of  $\mathbf{stmod} G$ , anticipated in [25], and of  $\mathbf{D}^b(\mathbf{mod} G)$ . This was mentioned earlier and is the content of Theorem 10.3 and Corollary 10.6. A few others are described in Section 10. Further applications specific to the context of finite group schemes are treated in Section 9. These include a proof that, akin to supports, the  $\pi$ -cosupport of a  $G$ -module coincides with its cosupport in the sense of [8]:

**Theorem** (Theorem 9.3). *For any  $G$ -module  $M$  there is an equality*

$$\pi\text{-cosupp}_G(M) = \mathrm{cosupp}_G(M).$$

This in turn is used to track support and cosupport under restriction and induction for subgroup schemes; see Proposition 9.5. That the result above is a consequence of the classification theorem also illustrates a key difference between the approach developed in this paper and the one in [9] where we give a new proof of the classification theorem for finite groups. There we prove that  $\pi$ -cosupport coincides with cosupport, using linear algebra methods and special properties of finite groups, and deduce the classification theorem from it. In this paper our route is the opposite: we have to develop a new method to prove classification and then deduce the equality of cosupports from it. See also Remark 5.6.

The methods developed in this work have led to other new results concerning the structure of the stable module category of a finite group scheme, including a classification of its Hom closed *colocalising* subcategories [10].

## Part I. Recollections

There have been two, rather different, approaches to studying representations of finite groups and finite groups schemes using geometric methods: via the theory of  $\pi$ -points and via the action of the cohomology ring on the stable category. Both are critical for our work. In this part we summarise basic constructions and results in the two approaches.

1.  $\pi$ -SUPPORT AND  $\pi$ -COSUPPORT

In this section we recall the notion of  $\pi$ -points for finite group schemes. The primary references are the papers of Friedlander and Pevtsova [20, 21]. We begin by summarising basic properties of modules over affine group schemes; for details we refer the reader to Jantzen [27] and Waterhouse [37].

**Affine group schemes and their representations.** Let  $k$  be a field and  $G$  a group scheme over  $k$ ; this work concerns only *affine* group schemes. The coordinate ring of  $G$  is denoted  $k[G]$ ; it is a commutative Hopf algebra over  $k$ . One says that  $G$  is *finite* if  $k[G]$  is finite dimensional as a  $k$ -vector space. The  $k$ -linear dual of  $k[G]$  is then a cocommutative Hopf algebra, called the *group algebra* of  $G$ , and denoted  $kG$ . A finite group scheme  $G$  over a field  $k$  is *connected* (or *infinitesimal*) if its coordinate ring  $k[G]$  is local; it is *unipotent* if its group algebra  $kG$  is local.

**Example 1.1** (Finite groups). A finite group  $G$  defines a finite group scheme over any field  $k$ : The group algebra  $kG$  is a finite dimensional cocommutative Hopf algebra, hence its dual is a commutative Hopf algebra and so defines a group scheme over  $k$ ; it is also denoted  $G$ . A finite group  $E$  is *elementary abelian* if it is isomorphic to  $(\mathbb{Z}/p)^r$ , for some prime number  $p$ . The integer  $r$  is then the *rank* of  $E$ . Over a field  $k$  of characteristic  $p$ , there are isomorphisms of  $k$ -algebras

$$k[E] \cong k^{\times p^r} \quad \text{and} \quad kE \cong k[z_1, \dots, z_r]/(z_1^p, \dots, z_r^p).$$

**Example 1.2** (Frobenius kernels). Let  $k$  be a field of positive characteristic  $p$  and  $f: k \rightarrow k$  its Frobenius endomorphism; thus  $f(\lambda) = \lambda^p$ . The *Frobenius twist* of a commutative  $k$ -algebra  $A$  is the base change  $A^{(1)} := k \otimes_f A$  over the Frobenius map. There is a  $k$ -linear algebra map  $F_A: A^{(1)} \rightarrow A$  given by  $F_A(\lambda \otimes a) = \lambda a^p$ .

If  $G$  is a group scheme over  $k$ , then the Frobenius twist  $k[G]^{(1)}$  is again a Hopf algebra over  $k$  and therefore defines another group scheme  $G^{(1)}$  called the *Frobenius twist* of  $G$ . The algebra map  $F_{k[G]}: k[G]^{(1)} = k[G]^{(1)} \rightarrow k[G]$  induces the Frobenius map of group schemes  $F: G \rightarrow G^{(1)}$ . The *r*th *Frobenius kernel* of  $G$  is the group scheme theoretic kernel of the  $r$ -fold iteration of the Frobenius map:

$$G_{(r)} = \text{Ker}(F^r: G \rightarrow G^{(r)}).$$

The Frobenius kernel of  $G$  is connected if the  $k$ -algebra  $k[G]$  is finitely generated.

Let  $\mathbb{G}_a$  denote the additive group over  $k$ . For the  $r$ -th Frobenius kernel  $\mathbb{G}_{a(r)}$  there are isomorphism of  $k$ -algebras

$$k[\mathbb{G}_{a(r)}] \cong k[t]/(t^{p^r}) \quad \text{and} \quad k\mathbb{G}_{a(r)} \cong k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p).$$

**Example 1.3** (Quasi-elementary group schemes). Following Bendel [1], a group scheme is said to be *quasi-elementary* if it is isomorphic to  $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ . Thus a quasi-elementary group scheme is unipotent abelian and its group algebra structure is the same as that of an elementary abelian  $p$ -group of rank  $r + s$ .

A module over an affine group scheme  $G$  over  $k$  is called  *$G$ -module*; it is equivalent to a comodule over the Hopf algebra  $k[G]$ . The term “module” will mean “left module”, unless stated otherwise. We write  $\text{Mod } G$  for the category of  $G$ -modules and  $\text{mod } G$  for its full subcategory consisting of finite dimensional  $G$ -modules. When  $G$  is finite, we identify  $G$ -modules with modules over the group algebra  $kG$ ; this is justified by [27, I.8.6].

**Induction.** For each subgroup scheme  $H$  of  $G$  restriction is a functor

$$\text{res}_H^G: \text{Mod } G \longrightarrow \text{Mod } H.$$

We often write  $(-)\downarrow_H$  instead of  $\text{res}_H^G$ . This has a right adjoint called induction<sup>1</sup>

$$\text{ind}_H^G: \text{Mod } H \longrightarrow \text{Mod } G$$

as described in [27, I.3.3]. If the quotient  $G/H$  is affine then  $\text{ind}_H^G$  is exact. This holds, for example, when  $H$  is finite; see [27, I.5.13].

**Extending the base field.** Let  $G$  be a group scheme over  $k$ . If  $K$  is a field extension of  $k$ , we write  $K[G]$  for  $K \otimes_k k[G]$ , which is a commutative Hopf algebra over  $K$ . This defines a group scheme over  $K$  denoted  $G_K$ . If  $G$  is a finite group scheme, then there is a natural isomorphism  $KG_K \cong K \otimes_k kG$  and we simplify notation by writing  $KG$ . For a  $G$ -module  $M$ , we set

$$M_K := K \otimes_k M.$$

The induction functor commutes with the extension of scalars (see [27, I.3.5]), that is, there is a canonical isomorphism:

$$\text{ind}_{H_K}^{G_K}(M_K) \cong (\text{ind}_H^G M)_K.$$

The assignment  $M \mapsto M_K$  defines a functor from  $\text{Mod } G$  to  $\text{Mod } G_K$  which is left adjoint to the restriction functor  $\text{Mod } G_K \rightarrow \text{Mod } G$ .

For  $G$  a *finite* group scheme and a  $G$ -module  $M$  we set

$$M^K := \text{Hom}_k(K, M),$$

again viewed as a  $G_K$ -module. This is right adjoint to restriction. It is essential for the group scheme to be finite to make sense of this definition; see Remark 5.6.

**$\pi$ -points.** Let  $G$  be a finite group scheme over  $k$ . A  $\pi$ -point of  $G$ , defined over a field extension  $K$  of  $k$ , is a morphism of  $K$ -algebras

$$\alpha: K[t]/(t^p) \longrightarrow KG$$

that factors through the group algebra of a unipotent abelian subgroup scheme  $C$  of  $G_K$ , and such that  $KG$  is flat when viewed as a left (equivalently, as a right) module over  $K[t]/(t^p)$  via  $\alpha$ . It should be emphasised that  $C$  need not be defined over  $k$ ; see Example 3.8. Restriction along  $\alpha$  defines a functor

$$\alpha^*: \text{Mod } G_K \longrightarrow \text{Mod } K[t]/(t^p),$$

and the functor  $KG \otimes_{K[t]/(t^p)} -$  provides a left adjoint

$$\alpha_*: \text{Mod } K[t]/(t^p) \longrightarrow \text{Mod } G_K.$$

**Definition 1.4.** A pair of  $\pi$ -points  $\alpha: K[t]/(t^p) \rightarrow KG$  and  $\beta: L[t]/(t^p) \rightarrow LG$  are *equivalent*, denoted  $\alpha \sim \beta$ , if they satisfy the following condition: for any finite dimensional  $kG$ -module  $M$ , the module  $\alpha^*(M_K)$  is projective if and only if  $\beta^*(M_L)$  is projective (see [21, Section 2] for a discussion of the equivalence relation).

<sup>1</sup>Warning: in representation theory of finite groups, *induction* is commonly used for the *left adjoint*. We stick with the convention in [27] pointing out that for group schemes the left adjoint does not always exist and when it does, it is not necessarily isomorphic to the right adjoint.

*Remark 1.5.* For ease of reference, we list some basic properties of  $\pi$ -points.

(1) Let  $\alpha: K[t]/(t^p) \rightarrow KG$  be a  $\pi$ -point and  $L$  a field extension of  $K$ . Then  $L \otimes_K \alpha: L[t]/(t^p) \rightarrow LG$  is a  $\pi$ -point and it is easy to verify that  $\alpha \sim L \otimes_K \alpha$ .

(2) Every  $\pi$ -point is equivalent to one that factors through some quasi-elementary subgroup scheme. This is proved in [20, Proposition 4.2].

(3) Every  $\pi$ -point of a subgroup scheme  $H$  of  $G$  is naturally a  $\pi$ -point of  $G$ .

(4) A  $\pi$ -point  $\alpha$  of  $G$  defined over  $L$  naturally gives rise to a  $\pi$ -point of  $G_K$  defined over a field containing  $K$  and  $L$ .

**$\pi$ -points and cohomology.** Let  $G$  be a finite group scheme over  $k$ . The cohomology of  $G$  with coefficients in a  $G$ -module  $M$  is denoted  $H^*(G, M)$ . It can be identified with  $\text{Ext}_G^*(k, M)$ , with the trivial action of  $G$  on  $k$ . Recall that  $H^*(G, k)$  is a  $k$ -algebra that is graded-commutative (because  $kG$  is a Hopf algebra) and finitely generated, by a theorem of Friedlander–Suslin [22, Theorem 1.1].

Let  $\text{Proj } H^*(G, k)$  denote the set of homogeneous prime ideals  $H^*(G, k)$  that are properly contained in the maximal ideal of positive degree elements.

Given a  $\pi$ -point  $\alpha: K[t]/(t^p) \rightarrow KG$  we write  $H^*(\alpha)$  for the composition of homomorphisms of  $k$ -algebras.

$$H^*(G, k) = \text{Ext}_G^*(k, k) \xrightarrow{K \otimes_k -} \text{Ext}_{G_K}^*(K, K) \longrightarrow \text{Ext}_{K[t]/(t^p)}^*(K, K),$$

where the one on the right is induced by restriction. Evidently, the radical of the ideal  $\text{Ker } H^*(\alpha)$  is a prime ideal in  $H^*(G, k)$  different from  $H^{\geq 1}(G, k)$  and so defines a point in  $\text{Proj } H^*(G, k)$ .

*Remark 1.6.* Fix a point  $\mathfrak{p}$  in  $\text{Proj } H^*(G, k)$ . There exists a field  $K$  and a  $\pi$ -point

$$\alpha_{\mathfrak{p}}: K[t]/(t^p) \longrightarrow KG$$

such that  $\sqrt{\text{Ker } H^*(\alpha_{\mathfrak{p}})} = \mathfrak{p}$ . In fact, there is such a  $K$  that is a finite extension of the degree zero part of the homogenous residue field at  $\mathfrak{p}$ ; see [21, Theorem 4.2]. It should be emphasised that  $\alpha_{\mathfrak{p}}$  is not uniquely defined.

In this way, one gets a bijection between the set of equivalence classes of  $\pi$ -points of  $G$  and  $\text{Proj } H^*(G, k)$ ; see [21, Theorem 3.6]. In the sequel, we identify a prime in  $\text{Proj } H^*(G, k)$  and the corresponding equivalence class of  $\pi$ -points.

The following definitions of  $\pi$ -support and  $\pi$ -cosupport of a  $G$ -module  $M$  are from [21] and [9] respectively.

**Definition 1.7.** Let  $G$  be a finite group scheme and  $M$  be a  $G$ -module. The  $\pi$ -support of  $M$  is the subset of  $\text{Proj } H^*(G, k)$  defined by

$$\pi\text{-supp}_G(M) := \{\mathfrak{p} \in \text{Proj } H^*(G, k) \mid \alpha_{\mathfrak{p}}^*(K \otimes_k M) \text{ is not projective}\}.$$

The  $\pi$ -cosupport of  $M$  is the subset of  $\text{Proj } H^*(G, k)$  defined by

$$\pi\text{-cosupp}_G(M) := \{\mathfrak{p} \in \text{Proj } H^*(G, k) \mid \alpha_{\mathfrak{p}}^*(\text{Hom}_k(K, M)) \text{ is not projective}\}.$$

Here  $\alpha_{\mathfrak{p}}: K[t]/(t^p) \rightarrow KG$  denotes a representative of the equivalence class of  $\pi$ -points corresponding to  $\mathfrak{p}$ ; see Remark 1.6. Both  $\pi$ -supp and  $\pi$ -cosupp are well defined on the equivalence classes of  $\pi$ -points; see [9, Theorem 2.1].

The following observation will be useful; see Corollary 9.4 for a better statement.

**Lemma 1.8.** *Let  $\mathfrak{m}$  be a closed point of  $\text{Proj } H^*(G, k)$  and  $M$  a  $G$ -module. Then  $\mathfrak{m}$  is in  $\pi\text{-supp}_G(M)$  if and only if it is in  $\pi\text{-cosupp}_G(M)$ .*

*Proof.* The key observation is that as  $\mathfrak{m}$  is a closed point there is a corresponding  $\pi$ -point  $\alpha_{\mathfrak{m}}: K[t]/(t^p) \rightarrow KG$  with  $K$  a finite extension of  $k$ ; see Remark 1.6. It then remains to note that the natural evaluation map is an isomorphism:

$$\text{Hom}_k(K, k) \otimes_k M \xrightarrow{\cong} \text{Hom}_k(K, M)$$

so that  $K \otimes_k M$  and  $\text{Hom}_k(K, M)$  are isomorphic as  $G_K$ -modules.  $\square$

In the next result, the formula for the support of tensor products is [21, Proposition 5.2] while the one for the cosupport of function objects is [9, Theorem 3.4]. These play a key role in what follows.

**Theorem 1.9.** *Let  $M$  and  $N$  be  $G$ -modules. Then there are equalities*

- (i)  $\pi\text{-supp}_G(M \otimes_k N) = \pi\text{-supp}_G(M) \cap \pi\text{-supp}_G(N)$ ,
- (ii)  $\pi\text{-cosupp}_G(\text{Hom}_k(M, N)) = \pi\text{-supp}_G(M) \cap \pi\text{-cosupp}_G(N)$ .  $\square$

## 2. SUPPORT AND COSUPPORT VIA COHOMOLOGY

This section provides a quick summary of the techniques developed in [5, 8], with a focus on modules over finite group schemes. Throughout  $G$  will be a finite group scheme over a field  $k$ .

**The stable module category.** The stable module category of  $G$  is denoted  $\text{StMod } G$ . Its objects are all (possibly infinite dimensional)  $G$ -modules. The set of morphisms between  $G$ -modules  $M$  and  $N$  is by definition

$$\underline{\text{Hom}}_G(M, N) := \frac{\text{Hom}_G(M, N)}{\text{P}\text{Hom}_G(M, N)}$$

where  $\text{P}\text{Hom}_G(M, N)$  consists of all  $G$ -maps between  $M$  and  $N$  which factor through a projective  $G$ -module. Since  $G$ -modules are precisely modules over the group algebra  $kG$  and the latter is a Frobenius algebra [27, Lemma I.8.7], the stable module category is triangulated, with suspension equal to  $\Omega^{-1}(-)$ , the inverse of the syzygy functor. The tensor product  $M \otimes_k N$  of  $G$ -modules, with the usual diagonal  $G$ -action, is inherited by  $\text{StMod } G$  making it a tensor triangulated category. This category is compactly generated and the subcategory of compact objects is equivalent to  $\text{stmod } G$ , the stable module category of finite dimensional  $G$ -modules. For details, readers might consult Carlson [17, §5] and Happel [23, Chapter 1].

A subcategory of  $\text{StMod } G$  is said to be *thick* if it is a triangulated subcategory that is closed under direct summands. Note that any triangulated subcategory is closed under finite direct sums. A triangulated subcategory that is closed under all set-indexed direct sums is said to be *localising*. A localising subcategory is also thick. We say that a subcategory is a *tensor ideal* if it is closed under  $M \otimes_k -$  for any  $G$ -module  $M$ .

We write  $\underline{\text{Hom}}_G^*(M, N)$  for the graded abelian group with  $\underline{\text{Hom}}_G^*(M, \Omega^{-i}N)$  the component in degree  $i$ . Composition of morphisms endows  $\underline{\text{Hom}}_G^*(M, M)$  with a structure of a graded ring and  $\underline{\text{Hom}}_G^*(M, N)$  with the structure of a graded left- $\underline{\text{Hom}}_G^*(N, N)$  and right- $\underline{\text{Hom}}_G^*(M, M)$  bimodule. There is a natural map

$$\text{Ext}_G^*(M, N) \longrightarrow \underline{\text{Hom}}_G^*(M, N)$$

of graded abelian groups; it is a homomorphism of graded rings for  $M = N$ . Composing this with the homomorphism

$$- \otimes_k M: H^*(G, k) \longrightarrow \text{Ext}_G^*(M, M)$$

yields a homomorphism of graded rings

$$\phi_M: H^*(G, k) \longrightarrow \underline{\text{Hom}}_G^*(M, M).$$

It is clear from the construction that  $\underline{\text{Hom}}_G^*(M, N)$  is a graded bimodule over  $H^*(G, k)$  with left action induced by  $\phi_N$  and right action induced by  $\phi_M$ , and that the actions differ by the usual sign. Said otherwise,  $H^*(G, k)$  acts on  $\text{StMod } G$ , in the sense of [8, Section 3].

**The spectrum of the cohomology ring.** We write  $\text{Spec } H^*(G, k)$  for the set of homogenous prime ideals in  $H^*(G, k)$ . It has one more point than  $\text{Proj } H^*(G, k)$ , namely, the maximal ideal consisting of elements of positive degree. A subset  $\mathcal{V}$  of  $\text{Spec } H^*(G, k)$  is *specialisation closed* if whenever  $\mathfrak{p}$  is in  $\mathcal{V}$  so is any prime  $\mathfrak{q}$  containing  $\mathfrak{p}$ . Thus  $\mathcal{V}$  is specialisation closed if and only if it is a union of Zariski closed subsets, where a subset is *Zariski closed* if it is of the form

$$\mathcal{V}(I) := \{\mathfrak{p} \in \text{Spec } H^*(G, k) \mid \mathfrak{p} \subseteq I\}$$

for some ideal  $I$  in  $H^*(G, k)$ .

**Local cohomology.** Let  $\mathcal{V}$  be a specialisation closed subset of  $\text{Spec } H^*(G, k)$ . A  $G$ -module  $M$  is called  $\mathcal{V}$ -torsion if  $\underline{\text{Hom}}_G^*(C, M)_{\mathfrak{q}} = 0$  for any finite dimensional  $G$ -module  $C$  and  $\mathfrak{q} \notin \mathcal{V}$ . We write  $(\text{StMod } G)_{\mathcal{V}}$  for the full subcategory of  $\mathcal{V}$ -torsion modules. This is a localising subcategory and the inclusion  $(\text{StMod } G)_{\mathcal{V}} \subseteq \text{StMod } G$  admits a right adjoint, denoted  $\Gamma_{\mathcal{V}}$ . Thus, for each  $M$  in  $\text{StMod } G$  one gets a functorial exact triangle

$$\Gamma_{\mathcal{V}}M \longrightarrow M \longrightarrow L_{\mathcal{V}}M \longrightarrow$$

and this provides a localisation functor  $L_{\mathcal{V}}$  that annihilates precisely the  $\mathcal{V}$ -torsion modules. For details, see [5, Section 4].

A noteworthy special case pertains to a point  $\mathfrak{p}$  in  $\text{Proj } H^*(G, k)$  and the subset

$$\mathcal{Z}(\mathfrak{p}) := \{\mathfrak{q} \in \text{Spec } H^*(G, k) \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

This is evidently a specialisation closed subset. The corresponding localisation functor  $L_{\mathcal{Z}(\mathfrak{p})}$  models localisation at  $\mathfrak{p}$ , that is to say, for any  $G$ -module  $M$  and finite dimensional  $G$ -module  $C$ , the morphism  $M \rightarrow L_{\mathcal{Z}(\mathfrak{p})}M$  induces an isomorphism

$$\underline{\text{Hom}}_G^*(C, M)_{\mathfrak{p}} \xrightarrow{\cong} \underline{\text{Hom}}_G^*(C, L_{\mathcal{Z}(\mathfrak{p})}M)$$

of graded  $H^*(G, k)$ -modules; see [5, Theorem 4.7]. For this reason, we usually write  $M_{\mathfrak{p}}$  in lieu of  $L_{\mathcal{Z}(\mathfrak{p})}M$ . When the natural map  $M \rightarrow M_{\mathfrak{p}}$  is an isomorphism we say  $M$  is  $\mathfrak{p}$ -local, and that  $M$  is  $\mathfrak{p}$ -torsion if it is  $\mathcal{V}(\mathfrak{p})$ -torsion.

We write  $\Gamma_{\mathfrak{p}}$  for the exact functor on  $\text{StMod } G$  defined on objects by

$$\Gamma_{\mathfrak{p}}M := \Gamma_{\mathcal{V}(\mathfrak{p})}(M_{\mathfrak{p}}) = (\Gamma_{\mathcal{V}(\mathfrak{p})}M)_{\mathfrak{p}}.$$

The equality is a special case of a general phenomenon: the functors  $\Gamma_{\mathcal{V}}$  and  $L_{\mathcal{W}}$  commute for any specialisation closed subsets  $\mathcal{V}$  and  $\mathcal{W}$ ; see [5, Proposition 6.1]. This property will be used often and without further comment.

**Support and cosupport.** We introduce the *support* of a  $G$ -module  $M$  to be the following subset of  $\text{Proj } H^*(G, k)$ .

$$\text{supp}_G(M) := \{\mathfrak{p} \in \text{Proj } H^*(G, k) \mid \Gamma_{\mathfrak{p}}M \text{ is not projective}\}$$

As in [8, Section 4], the *cosupport* of  $M$  is the set

$$\text{cosupp}_G(M) := \{\mathfrak{p} \in \text{Proj } H^*(G, k) \mid \text{Hom}_k(\Gamma_{\mathfrak{p}}k, M) \text{ is not projective}\}.$$

Note that we are ignoring the closed point of  $\text{Spec } H^*(G, k)$ . It turns that the support and the cosupport of  $M$  coincide with its  $\pi$ -support and  $\pi$ -cosupport introduced in Section 1; see Theorems 6.1 and 9.3.

**Stratification.** Let  $(\mathbb{T}, \otimes, \mathbf{1})$  be a compactly generated tensor triangulated category and  $R$  a graded-commutative noetherian ring acting on  $\mathbb{T}$  via homomorphisms of rings  $R \rightarrow \text{End}^*(X)$ , for each  $X$  in  $\mathbb{T}$ ; see [5, Section 8] for details. For each  $\mathfrak{p}$  in  $\text{Spec } R$ , one can construct a functor  $\Gamma_{\mathfrak{p}}: \mathbb{T} \rightarrow \mathbb{T}$  as above and use it to define support and cosupport for objects in  $\mathbb{T}$ . The subcategory

$$\Gamma_{\mathfrak{p}}\mathbb{T} := \{X \in \mathbb{T} \mid X \cong \Gamma_{\mathfrak{p}}X\}$$

consists of all objects  $X$  in  $\mathbb{T}$  such that  $\text{Hom}_{\mathbb{T}}^*(C, X)$  is  $\mathfrak{p}$ -local and  $\mathfrak{p}$ -torsion for each compact object  $C$ , and has the following alternative description:

$$\Gamma_{\mathfrak{p}}\mathbb{T} = \{X \in \mathbb{T} \mid \text{supp}_R(X) \subseteq \{\mathfrak{p}\}\};$$

see [5, Corollary 5.9]. The subcategory  $\Gamma_{\mathfrak{p}}\mathbb{T}$  of  $\mathbb{T}$  is tensor ideal and localising.

We say that  $\Gamma_{\mathfrak{p}}\mathbb{T}$  is *minimal* if it is non-zero and contains no proper non-zero tensor ideal localising subcategories. Following [6, Section 7] we say  $\mathbb{T}$  is *stratified* by  $R$  if for each  $\mathfrak{p}$  the subcategory  $\Gamma_{\mathfrak{p}}\mathbb{T}$  is either zero or minimal. When this property holds, the tensor ideal localising subcategories of  $\mathbb{T}$  are parameterised by subsets of  $\text{Spec } R$ ; see [7, Theorem 3.8]. The import of this statement is that the classification problem we have set out to solve can be tackled one prime at a time.

Lastly, we recall from [8, Section 7] the behaviour of support under change of rings and categories.

**Change of rings and categories.** In this paragraph,  $(\mathbb{T}, R)$  denotes a pair consisting of a compactly generated triangulated category  $\mathbb{T}$  endowed with an action of a graded-commutative noetherian ring  $R$ . A functor  $(F, \phi): (\mathbb{T}, R) \rightarrow (\mathbb{U}, S)$  between such pairs consists of an exact functor  $F: \mathbb{T} \rightarrow \mathbb{U}$  that preserves set-indexed products and coproducts, and a homomorphism  $f: R \rightarrow S$  of rings such that, for each  $X \in \mathbb{T}$ , the following diagram is commutative.

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ \text{End}_{\mathbb{T}}^*(X) & \xrightarrow{F} & \text{End}_{\mathbb{U}}^*(FX) \end{array}$$

The result below is extracted from [8, Corollary 7.8].

**Proposition 2.1.** *Let  $(F, f): (\mathbb{T}, R) \rightarrow (\mathbb{U}, S)$  be a functor between compactly generated triangulated categories with ring actions. Let  $E$  be a left adjoint of  $F$ , let  $G$  be a right adjoint of  $F$ , and  $\phi: \text{Spec } S \rightarrow \text{Spec } R$  the map that assigns  $f^{-1}(\mathfrak{p})$  to  $\mathfrak{p}$ . Then for  $X$  in  $\mathbb{T}$  and  $Y$  in  $\mathbb{U}$  there are inclusions:*

- (1)  $\phi(\text{supp}_S(FX)) \subseteq \text{supp}_R(X)$  and  $\text{supp}_R(EY) \subseteq \phi(\text{supp}_S(Y))$ ,  
 (2)  $\phi(\text{cosupp}_S(FX)) \subseteq \text{cosupp}_R(X)$  and  $\text{cosupp}_R(GY) \subseteq \phi(\text{cosupp}_S(Y))$ .

Each inclusion is an equality if the corresponding functor is faithful on objects.  $\square$

## Part II. Detecting projectivity with $\pi$ -support

Let  $G$  be a finite group scheme over a field of positive characteristic. This part is dedicated to a proof of Theorem 5.3 that asserts that  $\pi$ -support detects projectivity of  $G$ -modules, by which we mean that a  $G$ -module  $M$  is projective if (and only if)  $\pi\text{-supp}_G(M) = \emptyset$ . This result was claimed in [21], but the argument there has a flaw (see Remark 5.4) which we repair here. Most of the different pieces of our proof are already available in the literature; we collect them here for the sake of both completeness and comprehensibility.

The essential new ingredient is a “subgroup reduction principle”, Theorem 3.7 which allows us to extend the detection theorem from the case of a connected finite group scheme to an arbitrary one. Theorem 3.7 relies on a remarkable result of Suslin (see also [1] for the special case of a unipotent finite group scheme) on detection of nilpotents in the cohomology ring  $H^*(G, \Lambda)$  for a  $G$ -algebra  $\Lambda$ , generalising work of Quillen and Venkov for finite groups.

The first step in our proof of the detection theorem is to settle the case of a *connected unipotent* finite group scheme. This is achieved in Section 4. The argument essentially follows the one of Bendel [1] but is simpler, for two reasons: because of the connectedness assumption and because we employ the subgroup reduction principle that allows one to apply induction on  $\dim_k k[G]$  in certain cases.

The subgroup reduction principle cannot be used for general connected finite groups schemes; see Example 3.8. To tackle that case, we import a result from [31] which readily implies that  $\pi$ -support detects projectivity for Frobenius kernels of connected reductive groups; in fact it would suffice to treat  $\text{GL}_{n(r)}$ , but the proof is no different in general. A connected group scheme can be realised as a subgroup of a Frobenius kernel and so we deduce the desired property for the former from that for the latter using a descent theorem. This is done in Section 5 and essentially repeats the argument in [32]. This also takes care of group schemes that are a direct product of their identity component with an elementary abelian  $p$ -group. After all, the statement of the theorem does not mention the coalgebra part of the structure, and in this case the algebra structure is identical to that of a suitably chosen connected finite group scheme.

Armed with these results, we tackle the general case, also in Section 5, but not without yet another invocation of the subgroup reduction principle, Theorem 3.7.

### 3. A SUBGROUP REDUCTION PRINCIPLE

In this section we establish basic results, including the general subgroup reduction principle alluded to above, Theorem 3.7, that will be used repeatedly in proving that  $\pi$ -support detects projectivity. Throughout,  $G$  will be a finite group scheme over a field  $k$  of positive characteristic.

**Lemma 3.1.** *Let  $G$  be a finite group scheme with the property that for any  $G$ -module  $M$  with  $\pi\text{-supp}_G(M) = \emptyset$  one has  $H^i(G, M) = 0$  for  $i \gg 0$ . Then  $\pi$ -support detects projectivity of  $G$ -modules.*

*Proof.* Let  $M$  be a  $G$ -module with  $\pi\text{-supp}_G(M) = \emptyset$ . Then, for any simple  $G$ -module  $S$ , Theorem 1.9 yields

$$\pi\text{-supp}_G(\text{Hom}_k(S, k) \otimes_k M) = \pi\text{-supp}_G(\text{Hom}_k(S, k)) \cap \pi\text{-supp}_G(M) = \emptyset.$$

Thus, for  $i \gg 0$  the hypothesis on  $G$  gives the second equality below:

$$\text{Ext}_G^i(S, M) \cong H^i(G, \text{Hom}_k(S, k) \otimes_k M) = 0,$$

where the isomorphism holds since all simple  $G$ -modules are finite dimensional. It follows that  $M$  is projective, as desired.  $\square$

The following observation will be of some use in what follows.

*Remark 3.2.* If  $G$  and  $G'$  are unipotent abelian group schemes such that their group algebras are isomorphic, then  $\pi$ -support detects projectivity of  $G$ -modules if and only if it detects projectivity of  $G'$ -modules.

Indeed, this is because projectivity of a  $G$ -module  $M$  does not involve the comultiplication on  $kG$ , and when  $G$  is unipotent abelian  $\pi$ -points are just flat homomorphism of  $K$ -algebras  $K[t]/(t^p) \rightarrow KG$ , for some field extension  $K/k$ , and again have nothing to do with the comultiplication on  $KG$ .

To establish that  $\pi$ -support detects projectivity we need a version of Dade's lemma proved by Benson, Carlson, and Rickard in [3, Lemma 4.1]. For our purposes we restate the result in terms of  $\pi$ -support as can be found in [9, Theorem 4.4].

**Theorem 3.3.** *If  $\mathcal{E}$  is a quasi-elementary group scheme, then  $\pi$ -support detects projectivity of  $\mathcal{E}$ -modules.*

*Proof.* The group algebra  $k\mathcal{E}$  of a quasi-elementary group scheme is isomorphic to the group algebra of an elementary abelian finite group as seen in Example 1.3. In view of Remark 3.2, the result follows from [9, Theorem 4.4].  $\square$

The next result, which is a corollary of Suslin's theorem on detection of nilpotence in cohomology [34, Theorem 5.1], is critical to what follows.

**Theorem 3.4.** *Let  $G$  be a finite group scheme over a field  $k$  and  $\Lambda$  a unital associative  $G$ -algebra. If  $\pi\text{-supp}_G(\Lambda) = \emptyset$ , then any element in  $H^{\geq 1}(G, \Lambda)$  is nilpotent.*

*Proof.* For any extension field  $K$  of  $k$  and any quasi-elementary subgroup scheme  $\mathcal{E}$  of  $G_K$ , the hypothesis of the theorem yields

$$\pi\text{-supp}_{\mathcal{E}}(\Lambda_K)\downarrow_{\mathcal{E}} = \emptyset.$$

Theorem 3.3 then yields that  $(\Lambda_K)\downarrow_{\mathcal{E}}$  is projective, so  $H^{\geq 1}(\mathcal{E}, \Lambda_K) = 0$ . This implies that for any element  $z \in H^{\geq 1}(G, \Lambda)$  the restriction of  $z_K$  to  $H^*(\mathcal{E}, (\Lambda_K)\downarrow_{\mathcal{E}})$  is trivial. Therefore,  $z$  is nilpotent, by [34, Theorem 5.1].  $\square$

The next result has been proved in a larger context by Burke [15, Theorem]. For finite group schemes a simpler argument is available and is given below.

**Lemma 3.5.** *Let  $G$  be a finite group scheme and  $M$  a  $G$ -module. If each element in  $\text{Ext}_G^{\geq 1}(M, M)$  is nilpotent, then  $M$  is projective.*

*Proof.* The  $k$ -algebra  $H^*(G, k)$  is finitely generated so Noether normalisation provides homogeneous algebraically independent elements  $z_1, \dots, z_r$  in  $H^{\geq 1}(G, k)$  such that the extension  $k[z_1, \dots, z_r] \subseteq H^*(G, k)$  is finite; see [14, Theorem 1.5.17]. By assumption, the image of any  $z_i$  under the composition

$$k[z_1, \dots, z_r] \longrightarrow H^*(G, k) \longrightarrow \text{Ext}_G^*(M, M)$$

is nilpotent. By taking powers of the generators  $z_1, \dots, z_r$ , if necessary, one may assume that these images are zero.

Represent each  $z_i$  by a homomorphism  $\Omega^{|z_i|}(k) \rightarrow k$  of  $G$ -modules and let  $L_{z_i}$  denote its kernel; this is the Carlson module [16] associated to  $z_i$ . Vanishing of  $z_1$  in  $\text{Ext}_G^*(M, M)$  implies that for some integer  $n$ , the  $G$ -modules  $L_{z_1} \otimes_k M$  and  $\Omega M \oplus \Omega^n M$  are isomorphic up to projective summands; this is proved in [2, 5.9] for finite groups and the argument carries over to finite group schemes. Setting  $L_{\mathbf{z}} := L_{z_r} \otimes_k \cdots \otimes_k L_{z_1}$ , an iteration yields that the  $G$ -modules

$$L_{\mathbf{z}} \otimes_k M \quad \text{and} \quad \bigoplus_{i=1}^{2^r} \Omega^{n_i} M$$

are isomorphic up to projective summands. However, since  $H^*(G, k)$  is finite as a module over  $k[z_1, \dots, z_r]$ , one gets the second equality below

$$\pi\text{-supp}_G(L_{\mathbf{z}}) = \bigcap_{i=1}^r \pi\text{-supp}_G(L_{z_i}) = \emptyset.$$

The first one is by Theorem 1.9. As the  $G$ -module  $L_{\mathbf{z}}$  is finitely generated, by construction, it follows that  $L_{\mathbf{z}}$ , and hence also  $L_{\mathbf{z}} \otimes_k M$ , is projective; see, for example, [20, Theorem 5.6]. Thus  $\bigoplus_{i=1}^{2^r} \Omega^{n_i} M$  is projective, and hence so is  $M$ .  $\square$

The next result is well-known; for a proof see, for example, [9, Lemma 3.2].

**Lemma 3.6.** *Let  $M$  and  $N$  be  $G$ -modules.*

- (i) *If  $M$  or  $N$  is projective, then so is  $\text{Hom}_k(M, N)$ .*
- (ii)  *$M$  is projective if and only if  $\text{End}_k(M)$  is projective.*  $\square$

We can now establish the following subgroup reduction principle.

**Theorem 3.7.** *Let  $G$  be a finite group scheme over  $k$  with the property that every  $\pi$ -point for  $G$  is equivalent to a  $\pi$ -point that factors through an embedding  $H_K \hookrightarrow G_K$  where  $H$  is a proper subgroup scheme of  $G$  and  $K/k$  is a field extension. If  $\pi$ -support detects projectivity for all proper subgroup schemes of  $G$ , then it detects projectivity for  $G$ .*

We emphasise that the subgroup scheme  $H$  of  $G$  is already defined over  $k$ .

*Proof.* Let  $M$  be a  $G$ -module with  $\pi\text{-supp}_G(M) = \emptyset$ . Let  $H$  be a proper subgroup scheme of  $G$ . Any  $\pi$ -point of  $H$  is a  $\pi$ -point of  $G$  so  $\pi\text{-supp}_H(M \downarrow_H) = \emptyset$  and hence  $M \downarrow_H$  is projective, by hypothesis. Therefore  $\text{End}_k(M) \downarrow_H$  is also projective, by Lemma 3.6. Since any  $\pi$ -point of  $G$  factors through a proper subgroup scheme, again by hypothesis, one gets that  $\pi\text{-supp}_G(\text{End}_k(M)) = \emptyset$ . By Theorem 3.4, any element in  $H^*(G, \text{End}_k(M)) = \text{Ext}_G^*(M, M)$  of positive degree is nilpotent. Lemma 3.5 then implies that  $M$  is projective, as desired.  $\square$

The hypothesis of Theorem 3.7 is quite restrictive, as the next example shows.

**Example 3.8.** Let  $k$  be a field of characteristic at least 3 and  $\mathfrak{g}$  the three dimensional Heisenberg Lie algebra over  $k$ , that is to say, the Lie algebra of  $3 \times 3$  strictly upper triangular matrices. It has generators  $\langle x_1, x_2, x_{12} \rangle$  subject to relations

$$[x_1, x_{12}] = 0 = [x_2, x_{12}] \quad \text{and} \quad [x_1, x_2] = x_{12}.$$

Then  $u(\mathfrak{g})$ , the restricted enveloping algebra of  $\mathfrak{g}$ , is a cocommutative Hopf algebra and hence its dual defines a group scheme over  $k$ . Its support variety is  $\mathbb{P}^2$  with coordinate algebra  $k[y_1, y_2, y_{12}]$ . Let  $K = k(y_1, y_2, y_{12})$  be the field of fractions, and let  $\alpha : K[t]/(t^p) \rightarrow K \otimes_k u(\mathfrak{g})$  be a “generic”  $\pi$ -point given by

$$\alpha_K(t) = y_1 x_1 + y_2 x_2 + y_{12} x_{12}.$$

Specialising  $\alpha$  to points  $[a_1, a_2, a_{12}] \in \mathbb{P}^2$  we get all  $\pi$ -points of  $u(\mathfrak{g})$  defined over  $k$ . Therefore  $\alpha$  cannot factor through any proper Lie subalgebra of  $\mathfrak{g}$  defined over  $k$ .

For contexts where Theorem 3.7 does apply see Theorems 4.1 and 5.3.

#### 4. CONNECTED UNIPOTENT GROUP SCHEMES

In this section we prove that  $\pi$ -support detects projectivity for modules over connected unipotent finite group schemes. Our strategy mimics the one used in [1], with one difference: it does not use the analogue of Serre’s cohomological criterion for a quasi-elementary group scheme as developed in [35]. This is because Theorem 3.7 allows us to invoke [35, Theorem 1.6] in the step where Bendel’s proof uses Serre’s criterion, significantly simplifying the argument.

**Theorem 4.1.** *If  $G$  is a connected unipotent finite group scheme over a field  $k$ , then  $\pi$ -support detects projectivity.*

*Proof.* If  $G \cong \mathbb{G}_{a(r)}$ , then a  $\pi$ -point for  $G$  is nothing more than a flat map of  $K$ -algebras  $K[t]/(t^p) \rightarrow KG$ , with  $K$  a field extension of  $k$ . The desired result follows from Theorem 3.3, given the description of its group algebra in Example 1.2.

In the remainder of the proof we may thus assume  $G$  is not isomorphic to  $\mathbb{G}_{a(r)}$ . The proof proceeds by induction on  $\dim_k k[G]$ . The base case, where this dimension is one, is trivial. Assume that the theorem holds for all proper subgroup schemes of  $G$ . We consider two cases.

*Case 1.* Suppose  $\dim_k \text{Hom}_{\text{Gr}/k}(G, \mathbb{G}_{a(1)}) = 1$ . Let  $\phi : G \rightarrow \mathbb{G}_{a(1)}$  be a generator of  $\text{Hom}_{\text{Gr}/k}(G, \mathbb{G}_{a(1)})$ , and  $x$  a generator of  $H^2(\mathbb{G}_{a(1)}, k)$ . By [35, Theorem 1.6], either  $G \cong \mathbb{G}_{a(r)}$  or  $\phi^*(x) \in H^*(G, k)$  is nilpotent. Since we have ruled out the former case, we may assume  $\phi^*(x)$  is nilpotent.

Let  $\alpha_K : K[t]/(t^p) \rightarrow KG$  be a  $\pi$ -point. Consider the composition

$$K[t]/(t^p) \xrightarrow{\alpha_K} KG \xrightarrow{\phi_K} K\mathbb{G}_{a(1)}$$

and the induced map in cohomology:

$$H^*(\mathbb{G}_{a(1)}, K) \xrightarrow{\phi_K^*} H^*(G, K) \xrightarrow{\alpha_K^*} \text{Ext}_{K[t]/(t^p)}^*(K, K)$$

Since  $\phi^*(x) \in H^2(G, k)$  is nilpotent,  $(\phi_K \circ \alpha_K)^*(x_K) = 0$ . This implies that  $\phi_K \circ \alpha_K$  factors through the augmentation map  $K[t]/(t^p) \rightarrow K$ , and hence that  $\alpha_K$  factors through  $(\text{Ker } \phi)_K$ . The statement now follows from Theorem 3.7.

*Case 2.* Suppose  $\dim_k \mathrm{Hom}_{\mathbb{G}_r/k}(G, \mathbb{G}_{a(1)}) \geq 2$ . Let  $\phi, \psi: G \rightarrow \mathbb{G}_{a(1)}$  be linearly independent morphisms. Fix an algebraically closed non-trivial field extension  $K$  of  $k$ . Note that  $\phi_K, \psi_K: G_K \rightarrow \mathbb{G}_{a(1),K}$  remain linearly independent, and hence for any pair of elements  $\lambda, \mu \in K$  not both zero,  $\lambda^{1/p}\phi_K + \mu^{1/p}\psi_K \neq 0$ . This implies that for any non-zero element  $x$  in  $H^2(\mathbb{G}_{a(1)}, K)$ , the element

$$(\lambda^{1/p}\phi_K + \mu^{1/p}\psi_K)^*(x) = \lambda\phi_K^*(x) + \mu\psi_K^*(x)$$

in  $H^2(G, K)$  is non-zero; this follows by the semilinearity of the Bockstein map, which also explains  $1/p$  in the exponents (see the proof of [9, Theorem 4.3] for more details on this formula). The induction hypothesis implies that  $M_K$  is projective when restricted to the kernel of  $\lambda^{1/p}\phi_K + \mu^{1/p}\psi_K$ . Thus  $\lambda\phi_K^*(x) + \mu\psi_K^*(x)$  induces a periodicity isomorphism

$$H^1(G, M_K) = H^1(G, M)_K \longrightarrow H^3(G, M)_K = H^3(G, M_K).$$

As this is so for any pair  $\lambda, \mu$  not both zero, the analogue of the Kronecker quiver lemma [3, Lemma 4.1] implies that

$$H^1(G, M)_K = H^1(G, M_K) = 0$$

Since  $G$  is unipotent, this implies that  $M$  is projective, as desired.  $\square$

## 5. FINITE GROUP SCHEMES

In this section we prove that  $\pi$ -support detects projectivity for any finite group scheme. It uses the following result that can be essentially found in [32]. However the pivotal identity (5.1) in the proof was only justified later in [34].

**Theorem 5.1.** *Let  $G \hookrightarrow G'$  be an embedding of connected finite group schemes over  $k$ . If  $\pi$ -support detects projectivity for  $G'$ , then it detects projectivity for  $G$ .*

*Proof.* Let  $M$  be a  $G$ -module such that  $\pi\text{-supp}_G(M) = \emptyset$ . By Lemma 3.1 and Frobenius reciprocity, it suffices to show that  $\pi\text{-supp}_{G'}(\mathrm{ind}_G^{G'} M) = \emptyset$ .

Consequently, we need to show that for any  $\pi$ -point  $\alpha: K[t]/(t^p) \rightarrow KG'$ , the restriction  $\alpha^*((\mathrm{ind}_G^{G'} M)_K)$  is free. By Remark 1.5, we may assume that  $\alpha$  factors through some quasi-elementary subgroup scheme  $\mathcal{E}' \leq G'_K$  defined over  $K$ . Since induction commutes with extension of scalars and we are only going to work with one  $\pi$ -point at a time, we may extend scalars and assume that  $k = K$ .

Let  $\mathcal{E} = \mathcal{E}' \cap G \leq G'$  (this can be the trivial group scheme). Let  $\Lambda = \mathrm{End}_k(M)$ . Since  $\mathcal{E}$  is quasi-elementary, the assumption on  $M$  together with Theorem 3.3 imply that  $M \downarrow_{\mathcal{E}}$  is free. Hence, Lemma 3.6(ii) implies that  $\Lambda \downarrow_{\mathcal{E}}$  is free.

Consider the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(\mathrm{ind}_G^{G'} \Lambda, \Lambda) \cong \mathrm{Hom}_{\mathcal{E}'}(\mathrm{ind}_G^{G'} \Lambda, \mathrm{ind}_{\mathcal{E}'}^{\mathcal{E}'} \Lambda),$$

and let

$$\theta: \mathrm{ind}_G^{G'} \Lambda \longrightarrow \mathrm{ind}_{\mathcal{E}'}^{\mathcal{E}'} \Lambda$$

be the homomorphism of  $\mathcal{E}'$ -modules which corresponds to the standard evaluation map  $\epsilon_{\Lambda}: \mathrm{ind}_G^{G'} \Lambda \longrightarrow \Lambda$  (see [27, 3.4]) considered as a map of  $\mathcal{E}$ -modules. By [34, pp. 216–217], the map  $\theta$  is surjective and the ideal  $I = \mathrm{Ker} \theta$  is nilpotent.

Indeed, it is shown in [34] that

$$(5.1) \quad \text{ind}_{\mathcal{E}}^{\mathcal{E}'} \Lambda \cong k[\mathcal{E}'/\mathcal{E}] \otimes_{k[G'/G]} \text{ind}_G^{G'} \Lambda$$

with the map  $\theta$  induced by the extension of scalars from  $k[G'/G]$  to  $k[\mathcal{E}'/\mathcal{E}]$ . Hence, the surjectivity follows from the fact that  $\mathcal{E}'/\mathcal{E} \rightarrow G'/G$  is a closed embedding, see [34, Thm. 5.3], and the nilpotency of  $I$  follows from the fact that  $k[G'/G]$  is a local artinian ring.

Consequently, we have an exact sequence of  $\mathcal{E}'$ -modules

$$0 \rightarrow I \rightarrow \text{ind}_G^{G'} \Lambda \xrightarrow{\theta} \text{ind}_{\mathcal{E}}^{\mathcal{E}'} \Lambda \rightarrow 0$$

where  $I$  is a nilpotent ideal and  $\text{ind}_{\mathcal{E}}^{\mathcal{E}'} \Lambda$  is projective since  $\Lambda$  is projective as an  $\mathcal{E}$ -module. The exact sequence in cohomology now implies that any positive degree element in  $H^*(k[t]/(t^p), \text{ind}_G^{G'} \Lambda)$  is nilpotent, where the action of  $t$  is via the  $\pi$ -point  $\alpha: k[t]/(t^p) \rightarrow k\mathcal{E}' \rightarrow kG'$ .

Note that the linear action of  $k$  on  $\text{ind}_G^{G'} M$  factors as follows:

$$k \otimes_k \text{ind}_G^{G'} M \longrightarrow \text{ind}_G^{G'} \Lambda \otimes_k \text{ind}_G^{G'} M \longrightarrow \text{ind}_G^{G'} (\Lambda \otimes_k M) \longrightarrow \text{ind}_G^{G'} M.$$

Thus the Yoneda action of  $H^*(k[t]/(t^p), k)$  on  $H^*(k[t]/(t^p), \text{ind}_G^{G'} M)$  factors through the action of  $H^*(k[t]/(t^p), \text{ind}_G^{G'} \Lambda)$ . We conclude that  $H^{*>0}(k[t]/(t^p), k)$  acts nilpotently. On the other hand, the action of a generator in degree 2 for  $p > 2$  (or degree 1 for  $p = 2$ ) induces the periodicity isomorphism on  $H^*(k[t]/(t^p), \text{ind}_G^{G'} M)$ . Hence,  $H^{*>0}(k[t]/(t^p), \text{ind}_G^{G'} M) = 0$ , and therefore the equivalence class of  $\alpha$  is not in  $\pi\text{-supp}_{G'}(\text{ind}_G^{G'} M)$ . Since  $\alpha$  was any  $\pi$ -point, the statement follows.  $\square$

We also require the following detection criterion; see [31, Theorem 1.6].

**Theorem 5.2.** *Let  $\mathcal{G}$  be a connected reductive algebraic group over  $k$  and let  $\mathcal{G}_{(r)}$  be its  $r$ -th Frobenius kernel. If  $M$  is a  $\mathcal{G}_{(r)}$ -module such that for any field extension  $K/k$  and any embedding of group schemes  $\mathbb{G}_{a(r),K} \hookrightarrow \mathcal{G}_{(r),K}$ , the restriction of  $M_K$  to  $\mathbb{G}_{a,K}$  is projective, then  $M$  is projective as a  $\mathcal{G}_{(r)}$ -module.  $\square$*

We come now to the central result of the first part of this article.

**Theorem 5.3.** *Let  $G$  be a finite group scheme over  $k$ . A  $G$ -module  $M$  is projective if and only if  $\pi\text{-supp}_G(M) = \emptyset$ .*

*Proof.* Assume  $M$  is projective and let  $\alpha: K[t]/(t^p) \rightarrow KG$  be a  $\pi$ -point of  $G$ . The  $G_K$ -module  $M_K$  is then projective, and hence so is the  $K[t]/(t^p)$ -module  $\alpha^*(M_K)$ , for  $\alpha$  is flat when viewed as a map of algebras. Thus  $\pi\text{-supp}_G(M) = \emptyset$ .

The proof of the converse builds up in a number of steps.

**Frobenius kernels.** Suppose  $G := \mathcal{G}_{(r)}$ , the  $r$ th Frobenius kernel of a connected reductive group  $\mathcal{G}$  over  $k$ . Let  $M$  be a  $G$ -module with  $\pi\text{-supp}_G(M) = \emptyset$ . For any field extension  $K/k$  and embedding  $\phi: \mathbb{G}_{a(r),K} \hookrightarrow \mathcal{G}_{(r),K}$  of group schemes over  $K$ , one then has  $\pi\text{-supp}_{\mathcal{G}_{(r),K}}(M_K) = \emptyset$ , and hence it follows that

$$\pi\text{-supp}_{\mathbb{G}_{a(r),K}}(\phi^*(M_K)) = \emptyset.$$

Theorem 3.3 then implies that  $\phi^*(M_K)$  is projective as a  $\mathbb{G}_{a(r),K}$ -module. It remains to apply Theorem 5.2.

**Connected finite group schemes.** This case is immediate from the preceding one and Theorem 5.1 since any connected finite group scheme can be embedded into  $\mathrm{GL}_{n(r)}$  for some positive integers  $n, r$ ; see [37, 3.4].

$G \cong G^\circ \times (\mathbb{Z}/p)^r$  **where  $G^\circ$  is the connected component at the identity.** Let  $M$  be a  $G$ -module with  $\pi\text{-supp}_G(M) = \emptyset$ . Let  $\mathcal{E} = (\mathbb{G}_{a(1)})^{\times r}$  and observe that the  $k$ -algebras  $kG$  and  $k(G^\circ \times (\mathbb{Z}/p)^r)$  are isomorphic, and hence so are  $H^*(G, M)$  and  $H^*(G^\circ \times \mathcal{E}, M)$ . Moreover,  $(\mathbb{Z}/p)^r$  and  $\mathcal{E}$  are both unipotent abelian group schemes, so the maximal unipotent abelian subgroup schemes of  $G$  and  $G^\circ \times \mathcal{E}$ , and hence also their  $\pi$ -points, are in bijection. In summary:  $\pi\text{-supp}_{G^\circ \times \mathcal{E}}(M) = \pi\text{-supp}_G(M) = \emptyset$ . Since we have verified already that  $\pi$ -support detects projectivity for connected finite group schemes, and  $G^\circ \times \mathcal{E}$  is one such, one gets the equality below

$$H^i(G, M) \cong H^*(G^\circ \times \mathcal{E}, M) = 0 \quad \text{for } i \geq 1.$$

It remains to recall Lemma 3.1 to deduce that  $M$  is projective as a  $G$ -module.

**General finite group schemes.** Extending scalars, if needed, we may assume that  $k$  is algebraically closed. The proof is by induction on  $\dim_k k[G]$ . The base case is trivial. Suppose the theorem holds for all proper subgroup schemes of  $G$ . Let  $G = G^\circ \rtimes \pi_0(G)$  where  $G^\circ$  is the connected component at the identity and  $\pi_0(G)$  is the (finite) group of connected components. If the product is direct and  $\pi_0(G)$  is elementary abelian, then we have already verified that the desired result holds for  $G$ . We may thus assume that this is not the case; this implies that for any elementary abelian subgroup  $E < \pi_0(G)$ , the subgroup scheme  $(G^\circ)^E \times E$  is a proper subgroup scheme of  $G$ .

It follows from the Quillen stratification for the space of equivalence classes of  $\pi$ -points [21, 4.12] that any  $\pi$ -point for  $G$  is equivalent to one of the form  $\alpha: K[t]/t^p \rightarrow KG$  that factors through  $((G^\circ)^E \times E)_K < (G^\circ \rtimes \pi_0(G))_K$ , where  $E < \pi_0(G)$  is an elementary abelian subgroup. Thus, the hypotheses of Theorem 3.7 holds, and we can conclude that  $M$  is projective, as needed.  $\square$

*Remark 5.4.* The implication (ii)  $\implies$  (i) in Theorem 5.3 is the content of [21, Theorem 5.3]. However, the proof given in [21] is incorrect. The problem occurs in the third paragraph of the proof where what is asserted translates to: the  $\pi$ -support of  $\mathrm{End}_k(M)$  is contained in the  $\pi$ -support of  $M$ . This is not so; see [9, Example 5.4]. What is true is that the  $\pi$ -cosupport of  $\mathrm{End}_k(M)$  is contained in the  $\pi$ -support of  $M$ , by Theorem 1.9. This is why it is useful to consider cosupports even if one is interested only in supports.

Chouinard [18, Corollary 1.1] proved that a module  $M$  over a finite group  $G$  is projective if its restriction to every elementary abelian subgroup of  $G$  is projective. This result is fundamental to the development of the theory of support varieties for finite groups. For finite group schemes Theorem 5.3 yields the following analogue of Chouinard's theorem. There are two critical differences: one has to allow for field extensions and there are infinitely many subgroup schemes involved.

**Corollary 5.5.** *Let  $G$  be a finite group scheme over  $k$ . A  $G$ -module  $M$  is projective if for every field extension  $K/k$  and quasi-elementary subgroup scheme  $\mathcal{E}$  of  $G_K$ , the  $\mathcal{E}$ -module  $(M_K)_{\downarrow \mathcal{E}}$  is projective.*

*Proof.* As noted in Remark 1.5, every  $\pi$ -point factors through some  $\mathcal{E}$  as above, so if  $(M_K)_{\downarrow \mathcal{E}}$  is projective for each such  $\mathcal{E}$ , it follows that  $\pi\text{-supp}_G(M) = \emptyset$ , and hence that  $M$  is projective, by Theorem 5.3.  $\square$

*Remark 5.6.* All the steps in the proof of Theorem 5.3 except for the one dealing with Frobenius kernels, Theorem 5.2, work equally well for  $\pi$ -cosupport: namely, they can be used with little change to show that if  $\pi\text{-cosupp } M = \emptyset$  then  $M$  is projective. Explicitly, the following changes need to be made:

- (1) Theorem 3.7: Simply replace  $\pi$ -support with  $\pi$ -cosupport.
- (2) Theorem 4.1: In the proof of Case 2, use [9, 4.1] which is an analogue for cosupports of the Kronecker quiver lemma.
- (3) Theorem 5.1: The proof carries over almost verbatim. One replaces the extension  $M_K$  with coextension  $M^K$  and uses repeatedly that coextension commutes with induction for finite group schemes [9, Lemma 1.2].

The trouble with establishing the analogue of Theorem 5.2 for cosupports can be pinpointed to the fact that the induction functor  $\text{ind}: \text{Mod } H \rightarrow \text{Mod } G$  does not commute with coextension of scalars for general affine group schemes. Even worse, when  $G$  is not finite and  $K/k$  is of infinite degree, given a  $G$ -module  $M$  there is no natural action of  $G_K$  on  $M^K$ .

In Part IV we prove that  $\pi$ -cosupport detects projectivity, taking an entirely different approach. This uses the support detection theorem in an essential way.

### Part III. Minimal localising subcategories

Let  $G$  be a finite group scheme over a field  $k$ . From now on we consider the stable module category  $\text{StMod } G$  whose construction and basic properties were recalled in Section 2. For each  $\mathfrak{p}$  in  $\text{Proj } H^*(G, k)$ , we focus on the subcategory  $\Gamma_{\mathfrak{p}}(\text{StMod } G)$  consisting of modules with support in  $\{\mathfrak{p}\}$ . These are precisely the modules whose cohomology is  $\mathfrak{p}$ -local and  $\mathfrak{p}$ -torsion.

This part of the paper is dedicated to proving that  $\Gamma_{\mathfrak{p}}(\text{StMod } G)$  is *minimal*, meaning that it contains no proper non-zero tensor ideal localising subcategories. As noted in Section 2, this is the crux of the classification of the tensor ideal localising subcategories of  $\text{StMod } G$ .

For closed points in  $\text{Proj } H^*(G, k)$  the desired minimality is verified in Section 6. The general case is settled in Section 8, by reduction to a closed point. The key idea here is to construct good generic points for projective varieties. The necessary commutative algebra is developed in Section 7.

## 6. SUPPORT EQUALS $\pi$ -SUPPORT

Henceforth it becomes necessary to have at our disposal the methods developed in [5, 6] and recalled in Section 2. We begin by establishing that the  $\pi$ -support of a  $G$ -module coincides with its support. Using this, we track the behaviour of supports under extensions of scalars and verify that for a closed point  $\mathfrak{p}$  the tensor ideal localising subcategory  $\Gamma_{\mathfrak{p}}(\text{StMod } G)$  is minimal.

**Theorem 6.1.** *Let  $G$  be a finite group scheme defined over  $k$ . Viewed as subsets of  $\text{Proj } H^*(G, k)$  one has  $\pi\text{-supp}_G(M) = \text{supp}_G(M)$  for any  $G$ -module  $M$ .*

*Proof.* From [21, Proposition 6.6] one gets that  $\pi\text{-supp}_G(\Gamma_{\mathfrak{p}}k) = \{\mathfrak{p}\}$ . Given this, the tensor product formula (Theorem 1.9) and the detection of projectivity (Theorem 5.3), the calculation is purely formal; see the proof of [9, Theorem 5.1].  $\square$

The preceding result reconciles two rather different points of view of support and so makes available a panoply of new tools for studying representation of finite group schemes. The next result, required in Section 8, well illustrates this point.

**Proposition 6.2.** *Let  $G$  be a finite group scheme over  $k$ , let  $K/k$  be an extension of fields, and  $\rho: \text{Proj } H^*(G_K, K) \rightarrow \text{Proj } H^*(G, k)$  the induced map.*

- (1)  $\text{supp}_{G_K}(M_K) = \rho^{-1}(\text{supp}_G(M))$  for any  $G$ -module  $M$ .
- (2)  $\text{supp}_G(N\downarrow_G) = \rho(\text{supp}_{G_K}(N))$  for any  $G_K$ -module  $N$ .

*Proof.* The equality in (1) is clear for  $\pi$ -supports; now recall Theorem 6.1.

We deduce the equality in (2) by applying twice Proposition 2.1. The action of  $H^*(G_K, K)$  on  $\text{StMod } G_K$  induces an action also of  $H^*(G, k)$  via restriction of scalars along the homomorphism  $K \otimes_k -: H^*(G, k) \rightarrow H^*(G_K, K)$ . Applying Proposition 2.1 to the functor  $(\text{id}, K \otimes_k -)$  on  $\text{StMod } G_K$  yields an equality

$$\text{supp}_{H^*(G, k)}(N) = \rho(\text{supp}_{G_K}(N)).$$

Next observe that the restriction functor  $(-)\downarrow_G$  is compatible with the actions of  $H^*(G, k)$ . Also,  $(-)\downarrow_G$  is exact, preserves set-indexed coproducts and products, and is faithful on objects. Everything is obvious, except the last property. So suppose  $N$  is a  $G_K$ -module such that  $N\downarrow_G$  is projective. Then, for any simple  $G$ -module  $S$  and integer  $i \geq 1$  one has

$$\text{Ext}_{G_K}^i(K \otimes_k S, N) \cong \text{Ext}_G^i(S, N\downarrow_G) = 0$$

Since any simple  $G_K$ -module is a direct summand of  $K \otimes_k S$ , for some choice of  $S$ , it follows that  $N$  is projective, as desired.

Now we apply Proposition 2.1 to the functor

$$((-)\downarrow_G, \text{id}_{H^*(G, k)}): \text{StMod } G_K \longrightarrow \text{StMod } G$$

and obtain the equality

$$\text{supp}_G(N\downarrow_G) = \text{supp}_{H^*(G, k)}(N).$$

In conjunction with the one above, this gives (2).  $\square$

We can now begin to address the main task of this part of the paper.

**Proposition 6.3.** *When  $\mathfrak{m}$  is a closed point of  $\text{Proj } H^*(G, k)$ , the tensor ideal localising subcategory  $\Gamma_{\mathfrak{m}}(\text{StMod } G)$  of  $\text{StMod } G$  is minimal.*

*Proof.* It suffices to verify that the  $G$ -module  $\text{Hom}_k(M, N)$  is not projective for any non-zero modules  $M, N$  in  $\Gamma_{\mathfrak{m}}(\text{StMod } G)$ ; see [7, Lemma 3.9].

A crucial observation is that since  $\mathfrak{m}$  is a closed point, it is in  $\pi\text{-supp}_G(M)$  if and only if it is in  $\pi\text{-cosupp}_G(M)$  for any  $G$ -module  $M$ ; see Lemma 1.8. This will be used (twice) without comment in what follows. For any non-zero modules  $M, N$  in  $\Gamma_{\mathfrak{m}}(\text{StMod } G)$  Theorem 1.9 yields

$$\pi\text{-cosupp}_G(\text{Hom}_k(M, N)) = \pi\text{-supp}_G(M) \cap \pi\text{-cosupp}_G(N) = \{\mathfrak{m}\}.$$

Thus,  $\mathfrak{m}$  is also in the support of  $\text{Hom}_k(M, N)$ . It remains to recall Theorem 5.3.  $\square$

## 7. GENERIC POINTS IN GRADED-COMMUTATIVE ALGEBRAS

A standard technique in classical algebraic geometry is to ensure that irreducible varieties have generic points by enlarging their field of definition. For affine varieties this amounts to the following: Given a prime ideal  $\mathfrak{p}$  in an algebra  $A$  finitely generated over a field  $k$ , there is an extension of fields  $K/k$  such that in the ring  $B := A \otimes_k K$  there is a maximal ideal  $\mathfrak{m}$  lying over  $\mathfrak{p}$ , that is to say,  $\mathfrak{m} \cap A = \mathfrak{p}$ . In Section 8 we need a more precise version of this result, namely that there is such a  $K$  where  $\mathfrak{m}$  is cut out from  $B/\mathfrak{p}B$ , the fiber over  $\mathfrak{p}$ , by a complete intersection in  $B$ ; also, we have to deal with projective varieties. This is what is achieved in this section; see Theorem 7.7. The statement and its proof require some care, for in our context the desired property holds only outside a hypersurface.

Let  $B$  be a graded-commutative ring; a graded abelian group  $B = \{B^i\}_{i \in \mathbb{Z}}$  with an associative product satisfying  $a \cdot b = (-1)^{|a||b|} b \cdot a$  for all elements  $a, b$  in  $B$ , where  $||$  denotes degree. We consider only homogenous elements of graded objects.

**Definition 7.1.** Let  $N$  be a graded  $B$ -module. Mimicking [14, Definition 1.1.1], we say that a sequence  $\mathbf{b} := b_1, \dots, b_n$  of elements in  $B$  is a *weak  $N$ -sequence* if  $b_i$  is not a zerodivisor on  $N/(b_1, \dots, b_{i-1})N$ , for  $i = 1, \dots, n$ . We drop the adjective “weak” if, in addition,  $\mathbf{b}N \neq N$  holds.

**Lemma 7.2.** *Let  $A \rightarrow B$  be a homomorphism of graded-commutative rings and  $\mathbf{b} := b_1, \dots, b_n$  a weak  $B$ -sequence. If the  $A$ -module  $B/(b_1, \dots, b_i)B$  is flat for each  $i = 1, \dots, n$ , then  $\mathbf{b}$  is a weak  $(M \otimes_A B)$ -sequence for each graded  $A$ -module  $M$ .*

*Proof.* Set  $\mathbf{b}_{\leq i} := b_1, \dots, b_i$  for  $i = 0, \dots, n$ . For  $i = 1, \dots, n$ , since  $b_i$  is not zero divisor on  $B/(\mathbf{b}_{\leq i-1})$ , one gets the following exact sequence of graded  $B$ -modules.

$$0 \longrightarrow \frac{B}{(\mathbf{b}_{\leq i-1})B} \xrightarrow{b_i} \frac{B}{(\mathbf{b}_{\leq i-1})B} \longrightarrow \frac{B}{(\mathbf{b}_{\leq i})B} \longrightarrow 0$$

Since  $B/(\mathbf{b}_{\leq i})$  is flat as an  $A$ -module, applying  $M \otimes_A -$  to the exact sequence above and noting that  $M \otimes_A B/(\mathbf{b}_{\leq i-1})$  is naturally isomorphic to  $(M \otimes_A B)/(\mathbf{b}_{\leq i-1})$ , one gets the following exact sequence.

$$0 \longrightarrow \frac{M \otimes_A B}{(\mathbf{b}_{\leq i-1})(M \otimes_A B)} \xrightarrow{b_i} \frac{M \otimes_A B}{(\mathbf{b}_{\leq i-1})(M \otimes_A B)} \longrightarrow \frac{M \otimes_A B}{(\mathbf{b}_{\leq i})(M \otimes_A B)} \longrightarrow 0$$

This is the desired conclusion.  $\square$

**A model for localisation.** To prepare for the next step, we recall some basic properties of the kernel of a diagonal map. Let  $k$  be a field and  $k[\mathbf{x}]$  a polynomial ring over  $k$  in indeterminates  $\mathbf{x} := x_0, \dots, x_n$  of the same degree. Let  $\mathbf{t} := t_1, \dots, t_n$  be indeterminates over  $k$  and  $k(\mathbf{t})$  the corresponding field of rational functions, and consider the homomorphism of  $k$ -algebra

$$\mu: k(\mathbf{t})[\mathbf{x}] \rightarrow k\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)[x_0] \quad \text{where } \mu(t_i) = \frac{x_i}{x_0} \text{ for each } i.$$

The range of  $\mu$  is viewed as subring of the field of rational functions in  $\mathbf{x}$ .

**Lemma 7.3.** *The ideal  $\text{Ker}(\mu)$  is generated by  $x_1 - x_0 t_1, \dots, x_n - x_0 t_n$ , and the latter is a  $k(\mathbf{t})[\mathbf{x}]$ -sequence*

*Proof.* It is clear that the kernel of  $\mu$  is generated by the given elements. That these elements form a  $k(\mathbf{t})[\mathbf{x}]$ -sequence can be readily verified by, for example, an induction on  $n$ . Another way is to note that they are  $n$  elements in a polynomial ring and the Krull dimension of  $k(\mathbf{t})[\mathbf{x}]/\text{Ker}(\mu)$ , is one; see [14, Theorem 2.1.2(c)].  $\square$

Let now  $A$  be a graded-commutative  $k$ -algebra, and  $\mathbf{a} := a_0, \dots, a_n$  an algebraically independent set over  $k$ , with each  $a_i$  of the same degree. Observe that the following subset of  $A$  is multiplicatively closed.

$$(7.1) \quad U_{\mathbf{a}} := \{f(a_0, \dots, a_n) \mid f \text{ a non-zero homogeneous polynomial}\}$$

The algebraic independence of  $\mathbf{a}$  is equivalent to the condition that 0 is not in  $U_{\mathbf{a}}$ . For example,  $U_{\mathbf{a}}$  is the multiplicatively closed subset  $\cup_{i \geq 0} ka^i$ . For any  $A$ -module  $M$  one has the localisation at  $U_{\mathbf{a}}$ , namely equivalence classes of fractions

$$U_{\mathbf{a}}^{-1}M := \left\{ \left[ \frac{m}{f} \right] \mid m \in M \text{ and } f \in U_{\mathbf{a}} \right\}$$

The result below provides a concrete realisation of this localisation.

**Lemma 7.4.** *Let  $\mathbf{t} := t_1, \dots, t_n$  be indeterminates over  $k$  and  $k(\mathbf{t})$  the corresponding field of rational functions. Set  $B := A \otimes_k k(\mathbf{t})$  and  $b_i := a_i - a_0 t_i$ , for  $i = 1, \dots, n$ . The following statements hold.*

- (1) *The canonical map  $A \rightarrow B/(\mathbf{b})$  of  $k$ -algebras induces an isomorphism*

$$U_{\mathbf{a}}^{-1}A \xrightarrow{\cong} U_{a_0}^{-1}(B/(\mathbf{b})).$$

- (2)  *$\mathbf{b}$  is a weak  $U_{a_0}^{-1}(M \otimes_k k(\mathbf{t}))$ -sequence for any graded  $A$ -module  $M$ .*

*Proof.* We first verify the statements when  $A = k[\mathbf{x}]$ , a polynomial ring over  $k$  in indeterminates  $\mathbf{x} := x_0, \dots, x_n$  of the same degree, and  $a_i = x_i$  for each  $i$ . Then  $B = k(\mathbf{t})[\mathbf{x}]$ , the polynomial ring over the same indeterminates, but over the field  $k(\mathbf{t})$ , and  $U_{x_0}^{-1}k(\mathbf{t})[\mathbf{x}]$  can be naturally identified with  $k(\mathbf{t})[\mathbf{x}, x_0^{-1}]$ .

Consider the commutative diagram of morphisms of graded  $k$ -algebras

$$\begin{array}{ccc} k[\mathbf{x}] & \longrightarrow & k(\mathbf{t})[\mathbf{x}] & \longrightarrow & k(\mathbf{t})[\mathbf{x}, x_0^{-1}] \\ \downarrow & & & & \downarrow U_{x_0}^{-1}\mu \\ U_{\mathbf{x}}^{-1}k[\mathbf{x}] & \xrightarrow{\cong} & k(x_1, \dots, x_n)[x_0^{\pm 1}] & & \end{array} \quad \text{where } \mu(t_i) = \frac{x_i}{x_0}.$$

The unlabeled arrows are all canonical inclusions and the isomorphism is obvious. It follows from Lemma 7.3 that  $\text{Ker}(U_{x_0}^{-1}\mu)$  is the ideal generated by  $\{x_i - x_0 t_i\}_{i=1}^n$ . This justifies the assertion in (1).

As to (2), since  $x_1 - x_0 t_1, \dots, x_n - x_0 t_n$  is a  $k(\mathbf{t})[\mathbf{x}]$ -sequence, by Lemma 7.3, it is also a weak  $k(\mathbf{t})[\mathbf{x}, x_0^{-1}]$ -sequence. Moreover, arguing as above one gets that there is an isomorphism of graded rings

$$\frac{k(\mathbf{t})[\mathbf{x}, x_0^{-1}]}{(x_1 - x_0 t_1, \dots, x_i - x_0 t_i)} \cong k(x_1, \dots, x_i, t_{i+1}, \dots, t_n)[\mathbf{x}, x_0^{-1}]$$

for each  $1 \leq i \leq n$ . In particular, these are all flat as modules over  $k[\mathbf{x}, x_0^{-1}]$ , for they are obtained by localisation followed by an extension of scalars. Thus Lemma 7.2 applied to the morphism  $k[\mathbf{x}, x_0^{-1}] \rightarrow k(\mathbf{t})[\mathbf{x}, x_0^{-1}]$ , yields (2).

This completes the proof of the result when  $A = k[\mathbf{x}]$ .

The desired statements for a general  $A$  follow readily by base change. Indeed, consider the morphism of graded  $k$ -algebras  $k[\mathbf{x}] \rightarrow A$  given by the assignment  $x_i \mapsto a_i$ , for each  $i$ . It is easy to see then that  $B \cong k(\mathbf{t})[\mathbf{x}] \otimes_{k[\mathbf{x}]} A$ , so that applying  $- \otimes_{k[\mathbf{x}]} A$  to the isomorphism

$$U_{\mathbf{x}}^{-1}k[\mathbf{x}] \xrightarrow{\cong} k(x_1, \dots, x_n)[x_0^{\pm 1}]$$

gives the isomorphism in (1). As to (2), viewing a graded  $A$ -module  $M$  as an module over  $k[\mathbf{x}]$  via restriction of scalars, and applying the already established result for  $k[\mathbf{x}]$  gives the desired conclusion.  $\square$

Let  $k$  be a field and  $A = \{A^i\}_{\geq 0}$  a finitely generated graded-commutative  $k$ -algebra with  $A^0 = k$ . As usual  $\text{Proj } A$  denotes the collection of homogeneous prime ideals in  $A$  not containing  $A^{\geq 1}$ . Given a point  $\mathfrak{p}$  in  $\text{Proj } A$ , we write  $k(\mathfrak{p})$  for the localisation of  $A/\mathfrak{p}$  at the set of non-zero homogenous elements of  $A/\mathfrak{p}$ . Note that  $k(\mathfrak{p})$  is a graded field and its component in degree zero is the field of functions at  $\mathfrak{p}$ .

**Definition 7.5.** Let  $A$  be a domain and set  $Q := k((0))$ , the graded field of fractions of  $A$ . We say that elements  $\mathbf{a} := a_0, \dots, a_n$  in  $A$  give a *Noether normalisation* of  $A$  if the  $a_i$  all have the same positive degree, are algebraically independent over  $k$ , and  $A$  is a finitely generated module over the subalgebra  $k[\mathbf{a}]$ . Noether normalisations exist; see, for example, [14, Theorem 1.5.17], noting that, in the language of *op. cit.*, a sequence  $a_0, \dots, a_n$  is a system of parameters for  $A$  if and only so is the sequence  $a_0^{e_0}, \dots, a_n^{e_n}$ , for any positive integers  $e_0, \dots, e_n$ .

Observe that if  $\mathbf{a}$  is a Noether normalisation of  $A$ , then the set  $\{a_1/a_0, \dots, a_n/a_0\}$  is a transcendence basis for the extension of fields  $k \subseteq Q_0$ .

The result below, though not needed in the sequel, serves to explain why in constructing generic points it suffices to enlarge the field of definition to function fields of Noether normalisations.

**Lemma 7.6.** *The inclusion  $A \rightarrow Q$  induces an isomorphism  $U_{\mathbf{a}}^{-1}A \xrightarrow{\cong} Q$ .*

*Proof.* By the universal property of localisations, it suffices to verify that  $U_{\mathbf{a}}^{-1}A$  is a graded field. Recall that  $U_{\mathbf{a}}$  is the set  $k[\mathbf{a}] \setminus \{0\}$ . By definition,  $A$  is finitely generated as a module over  $k[\mathbf{a}]$ , so  $U_{\mathbf{a}}^{-1}A$  is finitely generated as a module over  $U_{\mathbf{a}}^{-1}k[\mathbf{a}]$ . The latter is a graded field, hence so is the former, as it is a domain.  $\square$

Fix a point  $\mathfrak{p}$  in  $\text{Proj } A$  and elements  $\mathbf{a} := a_0, \dots, a_n$  in  $A$  whose residue classes modulo  $\mathfrak{p}$  give a Noether normalisation of  $A/\mathfrak{p}$ ; see Definition 7.5. Let  $K := k(\mathbf{t})$ , the field of rational functions in indeterminates  $\mathbf{t} := t_1, \dots, t_n$  over  $k$ . Set

$$B := A \otimes_k K \quad \text{and} \quad b_i := a_i - a_0 t_i \quad \text{for } i = 1, \dots, n.$$

Thus  $B$  is a  $K$ -algebra. The next result is probably well-known but we were unable to find an adequate reference. Recall that a point  $\mathfrak{m}$  in  $\text{Proj } B$  is *closed* if it is maximal with respect to inclusion; equivalently, the Krull dimension of  $B/\mathfrak{m}$  is one.

**Theorem 7.7.** *For  $\mathbf{b} := b_1, \dots, b_n$  and  $\mathfrak{m} := (\mathfrak{p}, \mathbf{b})B$ , the following statements hold.*

- (1)  $\mathfrak{m}$  is prime ideal in  $B$  and defines a closed point in  $\text{Proj } B$ .
- (2)  $\mathfrak{m} \cap A = \mathfrak{p}$ .
- (3)  $\mathbf{b}$  is a weak  $U_{a_0}^{-1}B$ -sequence.

*Proof.* Note that the set  $\mathbf{a}$  is algebraically independent over  $k$ , since it has that property modulo  $\mathfrak{p}$ . Thus (3) is a special case of Lemma 7.4(2).

As to (1) and (2), replacing  $A$  by  $A/\mathfrak{p}$ , we can assume  $A$  is a domain with Noether normalisation  $\mathbf{a} := a_0, \dots, a_n$  and  $\mathfrak{p} = 0$ . We have to verify that the ideal  $\mathfrak{m}$  is prime,  $\mathfrak{m} \cap A = (0)$ , and that the Krull dimension of  $B/\mathfrak{m}$  is one.

From Lemma 7.4 one gets the isomorphism in the following commutative diagram of homomorphism of graded rings.

$$(7.2) \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & B/\mathfrak{m} \\ \downarrow & & \downarrow \beta \\ U_{\mathbf{a}}^{-1}A & \xrightarrow{\cong} & U_{a_0}^{-1}(B/\mathfrak{m}) \end{array}$$

The map  $\alpha$  is the composition  $A \rightarrow B \rightarrow B/\mathfrak{m}$  while  $\beta$  is localisation at  $U_{a_0}$ . Since  $A$  is a domain, the vertical map on the left is one-to-one, and hence so is the map  $\alpha$ . This proves that  $\mathfrak{m} \cap A = (0)$ .

Recall that  $\text{Spec } B$  denote the collection of homogeneous prime ideals of  $B$ .

*Claim.*  $\text{ht}(\mathfrak{q}) = n$  for any  $\mathfrak{q} \in \text{Spec } B$  minimal over  $\mathfrak{m}$ ; hence  $\dim(B/\mathfrak{m}) = 1$ .

Indeed,  $\mathfrak{m}$  is generated by  $n$  elements, so the Krull Height Theorem [14, Theorem A.1.] yields that  $\text{ht}(\mathfrak{q}) \leq n$  for each  $\mathfrak{q}$  minimal over  $\mathfrak{m}$ . On the other hand, by construction,  $B$  is finitely generated as a module over its subalgebra  $K[\mathbf{a}]$ . Notice that  $\mathfrak{b}$  is contained in  $K[\mathbf{a}]$ , so it follows that  $B/(\mathfrak{b})B$ , that is to say,  $B/\mathfrak{m}B$ , is a finitely generated module over  $K[\mathbf{a}]/(\mathfrak{b})$ . Since  $\mathbf{a}$  is algebraically independent,  $K[\mathbf{a}]/(\mathfrak{b})$  is isomorphic to  $k(a_1/a_0, \dots, a_n/a_0)[a_0]$ , see Lemma 7.3, and hence of Krull dimension one. It follows that  $\dim(B/\mathfrak{m}B) \leq 1$ , and therefore that  $\text{ht}(\mathfrak{q}) \geq n$ , because  $B$  is a catenary ring. This completes the proof of the claim.

It now remains to verify that  $\mathfrak{m}$  is a prime ideal.

*Claim.*  $a_0 \notin \mathfrak{q}$  for any  $\mathfrak{q} \in \text{Spec } B$  minimal over  $\mathfrak{m}$ .

Indeed, suppose  $a_0$  is in some  $\mathfrak{q} \in \text{Spec } B$  minimal over  $\mathfrak{m}$ . Then  $\mathfrak{q}$  contains the ideal  $(a_0, \dots, a_n)$ , because  $\mathfrak{m} \subseteq \mathfrak{q}$ . Recall that  $B$  is finitely generated as a module over its subalgebra  $K[\mathbf{a}]$ . Thus,  $B/(\mathfrak{a})$  is finitely generated as module over  $K[\mathbf{a}]/(\mathfrak{a}) \cong K$  and hence the Krull dimension of  $B/(\mathfrak{a})$  is zero. Said otherwise, the radical of  $\mathfrak{a}$  equals  $B^{\geq 1}$ , the (unique) homogeneous maximal ideal of  $B$ . This justifies the first equality below.

$$\text{ht}(\mathfrak{q}) \geq \text{ht}(\mathfrak{a}) = \dim B = n + 1$$

The inequality holds because  $\mathfrak{q} \supseteq \mathfrak{a}$ . As to the second equality:  $B$  is a domain that is a finitely generated module over  $K[\mathbf{a}]$ , which is of Krull dimension  $n + 1$ . The resulting inequality  $\text{ht}(\mathfrak{q}) \geq n + 1$  contradicts the conclusion of the previous claim.

Observe that the elements in  $\text{Spec}(U_{a_0}^{-1}B)$  minimal over  $\mathfrak{m}U_{a_0}^{-1}B$  are in bijection with the elements of  $\text{Spec } B$  minimal over  $\mathfrak{m}$  and not containing  $a_0$ . Since  $U_{a_0}^{-1}(B/\mathfrak{m})$  is a domain, by (7.2), it follows from preceding claim that  $\mathfrak{m}$  has only one prime ideal minimal over it, and hence that it has no embedded associated primes. Given this the preceding claim implies that  $a_0$  is not a zero divisor on  $B/\mathfrak{m}$  and hence that the map  $\beta$  in (7.2) is also one-to-one. Recalling once again that  $U_{a_0}^{-1}(B/\mathfrak{m})$

is a domain, we deduce that so is  $B/\mathfrak{m}$ ; in other words,  $\mathfrak{m}$  is a prime ideal. This completes the proof of the result.  $\square$

## 8. PASSAGE TO CLOSED POINTS

As usual let  $G$  be a finite group scheme over a field  $k$  of positive characteristic. In this section we prove that for any point  $\mathfrak{p}$  in  $\text{Proj } H^*(G, k)$  the category  $\Gamma_{\mathfrak{p}}(\text{StMod } G)$  consisting of the  $\mathfrak{p}$ -local and  $\mathfrak{p}$ -torsion  $G$ -modules is minimal. The main step in this proof is a concrete model for localisation at multiplicatively closed subsets of the form  $U_{\mathbf{a}}$ ; see (7.1). With an eye towards future applications, we establish a more general statement than needed for the present purpose.

We begin by recalling the construction of Koszul objects from [5, Definition 5.10].

**Koszul objects.** Each element  $a$  in  $H^d(G, k)$  defines a morphism  $k \rightarrow \Omega^{-d}k$  in  $\text{StMod } G$ ; we write  $k//a$  for its mapping cone. This is nothing but a shift of the Carlson module,  $L_a$ , that came up in Lemma 3.5. We have opted to stick to  $k//a$  for this is what is used in [5, 6] which are the main references for this section.

It follows from the construction that, in  $\text{StMod } G$ , there is an exact triangle

$$\Omega^d k \xrightarrow{a} k \xrightarrow{q_a} \Omega^d(k//a) \longrightarrow$$

Given a sequence of elements  $\mathbf{a} := a_1, \dots, a_n$  in  $H^*(G, k)$ , consider the  $G$ -module

$$k//\mathbf{a} := (k//a_1) \otimes_k \cdots \otimes_k (k//a_n).$$

It comes equipped with a morphism in  $\text{StMod } G$

$$(8.1) \quad q_{\mathbf{a}} := q_{a_0} \otimes_k \cdots \otimes_k q_{a_0} : k \longrightarrow \Omega^d(k//\mathbf{a})$$

where  $d = |a_1| + \cdots + |a_n|$ . For any  $G$ -module  $M$ , set

$$M//\mathbf{a} := M \otimes_k (k//\mathbf{a}).$$

In the sequel, we need the following computation:

$$(8.2) \quad \text{supp}_G(M//\mathbf{a}) = \text{supp}_G(M) \cap \mathcal{V}(\mathbf{a}).$$

This is a special case of [6, Lemma 2.6].

*Remark 8.1.* We say that an element  $a$  in  $H^d(G, k)$  is *invertible* on a  $G$ -module  $M$  if the canonical map  $M \xrightarrow{a} \Omega^{-d}M$  in  $\text{StMod } G$  is an isomorphism. This is equivalent to the condition that  $M//a = 0$ . A subset  $U$  of  $H^*(G, k)$  is said to be invertible on  $M$  if each element in it has that property.

Fix a multiplicatively closed subset  $U$  of  $H^*(G, k)$  and set

$$\mathcal{Z}(U) := \{\mathfrak{p} \in \text{Spec } H^*(G, k) \mid \mathfrak{p} \cap U \neq \emptyset\}.$$

This subset is specialisation closed. The associated localisation functor  $L_{\mathcal{Z}(U)}$ , whose construction was recalled in Section 2, is characterised by the property that for any  $G$ -modules  $M$  and  $N$ , with  $M$  finite dimensional, the induced morphism

$$\underline{\text{Hom}}_G^*(M, N) \longrightarrow \underline{\text{Hom}}_G^*(M, L_{\mathcal{Z}(U)}N)$$

of graded  $H^*(G, k)$ -modules is localisation at  $U$ ; see, for example, [26, Theorem 3.3.7]. In particular, the set  $U$  is invertible on  $L_{\mathcal{Z}(U)}N$ . For this reason, in what follows we use the more suggestive notation  $U^{-1}N$  instead of  $L_{\mathcal{Z}(U)}N$ .

**Notation 8.2.** Let  $\mathbf{a} := a_0, \dots, a_n$  be elements in  $H^*(G, k)$ , of the same positive degree, that are algebraically independent over  $k$ . Let  $K$  be the field of rational functions in indeterminates  $\mathbf{t} := t_1, \dots, t_n$ . Since there is a canonical isomorphism

$$H^*(G_K, K) \cong H^*(G, k) \otimes_k K$$

as  $K$ -algebras, we view  $H^*(G, k)$  as a subring of  $H^*(G_K, K)$ , and consider elements

$$b_i := a_i - a_0 t_i \quad \text{for } i = 1, \dots, n$$

in  $H^*(G_K, K)$ . Set  $d = n|a_0|$ . Composing the canonical map  $k \rightarrow K \downarrow_G$  with restriction to  $G$  of  $K \rightarrow \Omega^d(K//\mathbf{b})$  in  $\text{StMod } G_K$  from (8.1), one gets a morphism

$$(8.3) \quad f: k \longrightarrow \Omega^d(K//\mathbf{b}) \downarrow_G$$

in  $\text{StMod } G$ . Let  $U_{\mathbf{a}}$  be the multiplicatively closed set defined in (7.1).

**Theorem 8.3.** *The set  $U_{\mathbf{a}}$  is invertible on  $U_{a_0}^{-1} \Omega^d(K//\mathbf{b}) \downarrow_G$ , and the morphism  $U_{\mathbf{a}}^{-1} k \rightarrow U_{a_0}^{-1} \Omega^d(K//\mathbf{b}) \downarrow_G$  induced by (8.3) is an isomorphism. Thus there is a commutative diagram*

$$\begin{array}{ccc} k & \xrightarrow{f} & \Omega^d(K//\mathbf{b}) \downarrow_G \\ \downarrow & & \downarrow \\ U_{\mathbf{a}}^{-1} k & \xrightarrow{\cong} & U_{a_0}^{-1} \Omega^d(K//\mathbf{b}) \downarrow_G \end{array}$$

in  $\text{StMod } G$  where the vertical maps are the canonical localisations.

The proof takes a little preparation. Given a  $G$ -module  $M$ , we write  $\text{Loc}_G(M)$  for the smallest localising subcategory of  $\text{StMod } G$  that contains  $M$ , and  $\text{Loc}_G^{\otimes}(M)$  for the smallest tensor ideal localising subcategory of  $\text{StMod } G$  containing  $M$ .

**Lemma 8.4.** *Let  $g: M \rightarrow N$  be a morphism in  $\text{StMod } G$ . If  $M, N$  are in  $\text{Loc}_G(k)$  and  $\underline{\text{Hom}}_G^*(k, g)$  is an isomorphism, then so is  $g$ .*

*Proof.* Let  $C$  be the cone of  $g$  in  $\text{StMod } G$ ; the hypotheses is that  $\underline{\text{Hom}}_G^*(k, C) = 0$ . Since  $C$  is also in  $\text{Loc}_G(k)$ , it follows that it is zero, in  $\text{StMod } G$ , and hence that  $g$  is an isomorphism.  $\square$

The result below is well-known, and is recalled here for convenience.

**Lemma 8.5.** *For any element  $a$  in  $H^*(G, k)$  of positive degree and  $G$ -module  $M$ , the natural map  $\text{Ext}_G^*(k, M) \rightarrow \underline{\text{Hom}}_G^*(k, M)$  induces an isomorphism*

$$U_a^{-1} \text{Ext}_G^*(k, M) \xrightarrow{\cong} U_a^{-1} \underline{\text{Hom}}_G^*(k, M)$$

*Proof.* The main point is that there is an exact sequence

$$0 \longrightarrow \text{PHom}_G(k, M) \longrightarrow \text{Ext}_G^*(k, M) \longrightarrow \underline{\text{Hom}}_G^*(k, M) \longrightarrow C \longrightarrow 0$$

of graded  $H^*(G, k)$ -modules, where  $C$  is concentrated in negative degrees; see, for example, [11, Section 2]. For degree reasons, it is clear that  $\text{PHom}_G(k, M)$  and  $C$  are torsion with respect to  $H^{\geq 1}(G, k)$ , and so are annihilated when  $a$  is inverted.  $\square$

The next result concerns weak sequences; see Definition 7.1.

**Lemma 8.6.** *When  $\mathbf{b} := b_1, \dots, b_n$  is a weak  $U_a^{-1}H^*(G, M)$ -sequence for some element  $a$  in  $H^*(G, k)$ , the natural map  $M \rightarrow \Omega^d M // \mathbf{b}$ , where  $d = \sum_{i=1}^n |b_i|$ , induces an isomorphism of graded  $H^*(G, k)$ -modules*

$$U_a^{-1} \frac{H^*(G, M)}{\mathbf{b}H^*(G, M)} \xrightarrow{\cong} U_a^{-1}H^*(G, \Omega^d M // \mathbf{b}).$$

*Proof.* It suffices to verify the claim for  $n = 1$ ; the general case follows by iteration. The exact triangle  $\Omega^d M \xrightarrow{b} M \rightarrow \Omega^d M // b$  induces an exact sequence

$$0 \longrightarrow \frac{H^*(G, M)}{\mathbf{b}H^*(G, M)} \longrightarrow H^*(G, \Omega^d M // b) \longrightarrow \Sigma^{d+1}(0 : b) \longrightarrow 0$$

of graded  $H^*(G, k)$ -modules. Here  $(0 : b)$  denotes the elements of  $H^*(G, M)$  annihilated by  $b$ . Localising the sequence above at  $a$  gives the desired isomorphism, since  $b$  is not a zerodivisor on  $U_a^{-1}H^*(G, M)$ .  $\square$

*Proof of Theorem 8.3.* Set  $W := \Omega^d(K // \mathbf{b})$ . Since  $K \downarrow_G$  is a direct sum of copies of  $k$ , it follows that  $W \downarrow_G$  is in  $\text{Loc}_G(k)$ . Thus, in view of Lemma 8.4, it suffices to prove that the morphism  $f : k \rightarrow W \downarrow_G$  induces an isomorphism

$$U_{\mathbf{a}}^{-1} \underline{\text{Hom}}_G^*(k, k) \longrightarrow U_{a_0}^{-1} \underline{\text{Hom}}_G^*(k, W \downarrow_G)$$

of graded  $H^*(G, k)$ -modules. Note that this map is isomorphic to

$$U_{\mathbf{a}}^{-1}H^*(G, k) \longrightarrow U_{a_0}^{-1}H^*(G, W \downarrow_G)$$

by Lemma 8.5, since the degree of elements in  $\mathbf{a}$  is positive.

As  $H^*(G_K, K) \cong H^*(G, k) \otimes_k K$  it follows from Lemma 7.4(2) that  $\mathbf{b}$  is a weak  $U_{a_0}^{-1}H^*(G_K, K)$ -sequence. Thus Lemma 8.6 gives the first isomorphism below

$$U_{a_0}^{-1} \frac{H^*(G_K, K)}{\mathbf{b}H^*(G_K, K)} \xrightarrow{\cong} U_{a_0}^{-1}H^*(G_K, W) \xrightarrow{\cong} U_{a_0}^{-1}H^*(G, W \downarrow_G).$$

The second isomorphism is a standard adjunction. It remains to compose this with the isomorphism in Lemma 7.4(1).  $\square$

**Notation 8.7.** Fix a point  $\mathfrak{p} \in \text{Proj } H^*(G, k)$ , and let  $a_0, \dots, a_n$  be elements in  $H^*(G, k)$  that give a Noether normalisation of  $H^*(G, k) / \mathfrak{p}$ ; see Definition 7.5.

Let  $K$  be the field of rational functions in indeterminates  $t_1, \dots, t_n$ . Consider the ideal in  $H^*(G_K, K)$  given by

$$\mathfrak{m} := (\mathfrak{p}, b_1, \dots, b_n) \quad \text{where } b_i = a_i - a_0 t_i.$$

By Theorem 7.7 this defines a closed point in  $\text{Proj } H^*(G_K, K)$  lying over  $\mathfrak{p}$ .

**Theorem 8.8.** *The  $G$ -module  $(K // \mathfrak{m}) \downarrow_G$  is  $\mathfrak{p}$ -local and  $\mathfrak{p}$ -torsion and  $f \otimes_k k // \mathfrak{p}$ , with  $f$  as in (8.3), induces an isomorphism  $(k // \mathfrak{p})_{\mathfrak{p}} \cong \Omega^d(K // \mathfrak{m}) \downarrow_G$ , where  $d = n|a_0|$ . Thus in  $\text{StMod } G$  there is a commutative diagram*

$$\begin{array}{ccc} & k // \mathfrak{p} & \\ & \swarrow & \searrow f \otimes_k k // \mathfrak{p} \\ (k // \mathfrak{p})_{\mathfrak{p}} & \xrightarrow{\cong} & \Omega^d(K // \mathfrak{m}) \downarrow_G \end{array}$$

where the map pointing left is localisation. In particular, there is an equality

$$\Gamma_{\mathfrak{p}}(\mathrm{StMod} G) = \mathrm{Loc}_G^{\otimes}((K//\mathfrak{m})\downarrow_G).$$

*Proof.* Since  $\mathrm{supp}_{G_K}(K//\mathfrak{m})$  equals  $\{\mathfrak{m}\}$ , by (8.2), it follows from the construction of  $\mathfrak{m}$  and Proposition 6.2 that  $\mathrm{supp}_G(K//\mathfrak{m})\downarrow_G$  equals  $\{\mathfrak{p}\}$ . Said otherwise,  $(K//\mathfrak{m})\downarrow_G$  is  $\mathfrak{p}$ -local and  $\mathfrak{p}$ -torsion, as claimed.

Set  $W := \Omega^d(K//\mathfrak{b})\downarrow_G$ . Observe that the restriction functor  $(-)\downarrow_G$  is compatible with construction of Koszul objects with respect to elements of  $H^*(G, k)$ . This gives a natural isomorphism

$$W//\mathfrak{p} \cong \Omega^d(K//\mathfrak{m})\downarrow_G.$$

Since we already know that the module on the right is  $\mathfrak{p}$ -local, so is the one on the left. This justifies the last isomorphism below.

$$(k//\mathfrak{p})_{\mathfrak{p}} \cong (k_{\mathfrak{p}})//\mathfrak{p} \cong (W_{\mathfrak{p}})//\mathfrak{p} \cong (W//\mathfrak{p})_{\mathfrak{p}} \cong W//\mathfrak{p}$$

The second is the one induced by the isomorphism in Theorem 8.3, since  $U_{\mathfrak{a}}$  is not contained in  $\mathfrak{p}$ . The other isomorphisms are standard. The concatenation of the isomorphisms is the one in the statement of the theorem.

By [5, Corollary 8.3] one has  $\Gamma_{\mathfrak{p}}k \otimes_k N \cong N$  for any  $\mathfrak{p}$ -local and  $\mathfrak{p}$ -torsion  $G$ -module  $N$ . This justifies the first equality below.

$$\Gamma_{\mathfrak{p}}(\mathrm{StMod} G) = \mathrm{Loc}_G^{\otimes}(\Gamma_{\mathfrak{p}}k) = \mathrm{Loc}_G^{\otimes}((k//\mathfrak{p})_{\mathfrak{p}}).$$

For the second one see, for example, [6, Lemma 3.8]. Thus, the already established part of the theorem gives the desired equality.  $\square$

**Lemma 8.9.** *Let  $K/k$  be an extension of fields and  $M$  a  $G$ -module. If a  $G_K$ -module  $N$  is in  $\mathrm{Loc}_{G_K}^{\otimes}(M_K)$ , then  $N\downarrow_G$  is in  $\mathrm{Loc}_G^{\otimes}(M)$ .*

*Proof.* Let  $S(G)$  denote a direct sum of a representative set of simple  $G$ -modules. Then  $\mathrm{Loc}_G^{\otimes}(M) = \mathrm{Loc}_G(S(G) \otimes_k M)$ . Note that  $S(G_K)$  is a direct summand of  $S(G)_K$ . Now suppose that

$$N \in \mathrm{Loc}_{G_K}^{\otimes}(M_K) = \mathrm{Loc}_{G_K}(S(G)_K \otimes_K M_K).$$

Since there is an isomorphism of  $G$ -modules

$$(S(G)_K \otimes_K M_K)\downarrow_G \cong S(G) \otimes_k (M_K)\downarrow_G,$$

one gets the following

$$N\downarrow_G \in \mathrm{Loc}_G(S(G) \otimes_k (M_K)\downarrow_G) = \mathrm{Loc}_G^{\otimes}((M_K)\downarrow_G) = \mathrm{Loc}_G^{\otimes}(M),$$

where the last equality uses that  $(M_K)\downarrow_G$  equals a direct sum of copies of  $M$ .  $\square$

**Theorem 8.10.** *Let  $G$  be a finite group scheme over  $k$ . The tensor triangulated category  $\Gamma_{\mathfrak{p}}(\mathrm{StMod} G)$  is minimal for each point  $\mathfrak{p}$  in  $\mathrm{Proj} H^*(G, k)$ .*

*Proof.* Given the description of  $\Gamma_{\mathfrak{p}}(\mathrm{StMod} G)$  in Theorem 8.8, it suffices to verify that if  $\mathfrak{p}$  is in the support of a  $G$ -module  $M$ , then  $(K//\mathfrak{m})\downarrow_G$  is in  $\mathrm{Loc}_G^{\otimes}(M)$ . Let  $K/k$  be the extension of fields and  $\mathfrak{m}$  the closed point of  $\mathrm{Proj} H^*(G_K, K)$  lying over  $\mathfrak{p}$  constructed in 8.7. Then  $\mathrm{supp}_{G_K}(M_K)$  contains  $\mathfrak{m}$ , by Proposition 6.2. By (8.2),  $\mathrm{supp}_{G_K}(K//\mathfrak{m}) = \{\mathfrak{m}\}$  so Proposition 6.3 implies  $K//\mathfrak{m}$  is in  $\mathrm{Loc}_{G_K}^{\otimes}(M_K)$ . It follows from Lemma 8.9 that  $(K//\mathfrak{m})\downarrow_G$  is in  $\mathrm{Loc}_G^{\otimes}(M)$ .  $\square$

## Part IV. Applications

The final part of this paper is devoted to applications of the results proved in the preceding part. We proceed in several steps and derive global results about the module category of a finite group scheme from local properties.

As before,  $G$  denotes a finite group scheme over a field  $k$  of positive characteristic.

### 9. COSUPPORT EQUALS $\pi$ -COSUPPORT

In this section we show that  $\pi$ -cosupport of any  $G$ -module coincides with its cosupport introduced in Section 2. The link between them is provided by a naturally defined  $G$ -module,  $\alpha_*(K)\downarrow_G$ , that is the subject of the result below. For its proof we recall [27, I.8.14] that given any subgroup scheme  $H$  of  $G$  there is a functor

$$\text{coind}_H^G: \text{Mod } H \longrightarrow \text{Mod } G.$$

that is *left* adjoint to the restriction functor  $(-)\downarrow_H$  from  $\text{Mod } G$  to  $\text{Mod } H$ .

**Lemma 9.1.** *Fix a point  $\mathfrak{p}$  in  $\text{Proj } H^*(G, k)$ . If  $\alpha: K[t]/(t^p) \rightarrow KG$  is a  $\pi$ -point corresponding to  $\mathfrak{p}$ , then  $\text{supp}_G(\alpha_*(K)\downarrow_G) = \{\mathfrak{p}\}$  holds.*

*Proof.* We proceed in several steps. Suppose first that  $K = k$  and that  $G$  is unipotent. Since  $\alpha_*(k)$  is a finite dimensional  $k$ -vector space  $\text{supp}_G(\alpha_*(k))$  coincides with the set of prime ideals in  $\text{Proj } H^*(G, k)$  containing the annihilator of the  $H^*(G, k)$ -module  $\text{Ext}_G^*(\alpha_*(k), \alpha_*(k))$ ; see [5, Theorem 5.5]. This annihilator coincides with that of  $\text{Ext}_G^*(\alpha_*(k), k)$ , since  $G$  is unipotent, where  $H^*(G, k)$  acts via the canonical map  $H^*(G, k) \rightarrow \text{Ext}_G^*(\alpha_*(k), \alpha_*(k))$ . Adjunction yields an isomorphism

$$\text{Ext}_G^*(\alpha_*(k), k) \cong \text{Ext}_{k[t]/(t^p)}^*(k, k)$$

and we see that the action of  $H^*(G, k)$  factors through the canonical map

$$H^*(\alpha): H^*(G, k) \longrightarrow H^*(k[t]/(t^p), k)$$

that is induced by restriction via  $\alpha$ . Thus the annihilator of  $\text{Ext}_G^*(\alpha_*(k), \alpha_*(k))$  has the same radical as  $\text{Ker } H^*(\alpha)$ , which is  $\mathfrak{p}$ . It follows that  $\text{supp}_G(\alpha_*(k)) = \{\mathfrak{p}\}$ .

Now let  $\alpha$  be arbitrary. We may assume that it factors as

$$K[t]/(t^p) \xrightarrow{\beta} KU \longrightarrow KG$$

where  $U$  is a quasi-elementary subgroup scheme of  $G_K$ ; see Remark 1.5(2). Note that  $\beta$  defines a  $\pi$ -point of  $U$ ; call it  $\mathfrak{m}$ . The first part of this proof yields an equality

$$\text{supp}_U(\beta_*(K)) = \{\mathfrak{m}\}.$$

Let  $f: H^*(G_K, K) \rightarrow H^*(U, K)$  be the restriction map and  $\phi$  the map it induces on  $\text{Proj}$ . Note that  $\phi(\mathfrak{m})$  is the  $\pi$ -point of  $G_K$  corresponding to  $\alpha$ . Therefore, applying Proposition 2.1 to the pair

$$((-)\downarrow_U, f): \text{StMod } G_K \rightarrow \text{StMod } U,$$

one gets the inclusion below

$$\text{supp}_{G_K}(\alpha_*(K)) = \text{supp}_{G_K}(\text{coind}_U^{G_K}(\beta_*(K))) \subseteq \phi(\text{supp}_U(\beta_*(K))) = \{\phi(\mathfrak{m})\}.$$

Since  $\text{Ext}_{G_K}^*(\alpha_*(K), K)$  is non-zero, by adjointness,  $\alpha_*(K)$  is not projective. Thus its support equals  $\{\phi(\mathfrak{m})\}$ . It remains to apply Proposition 6.2(2).  $\square$

**Lemma 9.2.** *Let  $\alpha: K[t]/(t^p) \rightarrow KG$  be a  $\pi$ -point corresponding to a point  $\mathfrak{p}$  in  $\text{Proj } H^*(G, k)$ , and  $M$  a  $G$ -module. The following conditions are equivalent.*

- (1)  $\mathfrak{p}$  is in  $\pi$ -cosupp $_G(M)$ ;
- (2)  $\text{Hom}_k(\alpha_*(K)\downarrow_G, M)$  is not projective;
- (3)  $\underline{\text{Hom}}_G(\alpha_*(K)\downarrow_G, M) \neq 0$ .

*Proof.* The equivalence of (1) and (3) follows from the definition of  $\pi$ -cosupport and the following standard adjunction isomorphisms

$$\underline{\text{Hom}}_{K[t]/(t^p)}(K, \alpha^*(M^K)) \cong \underline{\text{Hom}}_{GK}(\alpha_*(K), M^K) \cong \underline{\text{Hom}}_G(\alpha_*(K)\downarrow_G, M)$$

(1)  $\iff$  (2) Let  $S$  be the direct sum of a representative set of simple  $kG$ -modules. Since  $\pi$ -supp $_G(S)$  equals  $\text{Proj } H^*(G, k)$ , Theorem 1.9 yields an equality

$$\pi\text{-cosupp}_G(M) = \pi\text{-cosupp}_G(\text{Hom}_k(S, M)).$$

This justifies the first of the following equivalences.

$$\begin{aligned} \mathfrak{p} \in \pi\text{-cosupp}_G(M) &\iff \mathfrak{p} \in \pi\text{-cosupp}_G(\text{Hom}_k(S, M)) \\ &\iff \underline{\text{Hom}}_G(\alpha_*(K)\downarrow_G, \text{Hom}_k(S, M)) \neq 0 \\ &\iff \underline{\text{Hom}}_G(\alpha_*(K)\downarrow_G \otimes_k S, M) \neq 0 \\ &\iff \underline{\text{Hom}}_G(S, \text{Hom}_k(\alpha_*(K)\downarrow_G, M)) \neq 0 \\ &\iff \text{Hom}_k(\alpha_*(K)\downarrow_G, M) \text{ is not projective.} \end{aligned}$$

The second one is (1)  $\iff$  (3) applied to  $\text{Hom}_k(S, M)$ ; the third and the fourth are standard adjunctions, and the last one is clear.  $\square$

**Theorem 9.3.** *Let  $G$  be a finite group scheme over a field  $k$ . Viewed as subsets of  $\text{Proj } H^*(G, k)$  one has  $\pi\text{-cosupp}_G(M) = \text{cosupp}_G(M)$  for any  $G$ -module  $M$ .*

*Proof.* The first of the following equivalences is Lemma 9.2.

$$\begin{aligned} \mathfrak{p} \in \pi\text{-cosupp}_G(M) &\iff \text{Hom}_k(\alpha_*(K)\downarrow_G, M) \text{ is not projective} \\ &\iff \text{Hom}_k(\Gamma_{\mathfrak{p}}k, M) \text{ is not projective} \\ &\iff \mathfrak{p} \in \text{cosupp}_G(M). \end{aligned}$$

The second one holds because  $\alpha_*(K)\downarrow_G$  and  $\Gamma_{\mathfrak{p}}k$  generate the same tensor ideal localising subcategory of  $\text{StMod } G$ . This is a consequence of Theorem 8.10 because  $\text{supp}_G(\alpha_*(K)\downarrow_G) = \{\mathfrak{p}\}$  by Lemma 9.1. The final equivalence is simply the definition of cosupport.  $\square$

Here is a first consequence of this result. We have been unable to verify the statement about maximal elements directly, except for closed points in the  $\pi$ -support and  $\pi$ -cosupport; see Lemma 1.8.

**Corollary 9.4.** *For any  $G$ -module  $M$  the maximal elements, with respect to inclusion, in  $\pi\text{-cosupp}_G(M)$  and  $\pi\text{-supp}_G(M)$  coincide. In particular,  $M$  is projective if and only if  $\pi\text{-cosupp}_G(M) = \emptyset$ .*

*Proof.* Given Theorems 6.1 and 9.3, this is a translation of [8, Theorem 4.5].  $\square$

The next result describes support and cosupport for a subgroup scheme  $H$  of  $G$ ; this complements Proposition 6.2.

Recall that induction and coinduction are related as follows

$$(9.1) \quad \mathrm{ind}_H^G(M) \cong \mathrm{coind}_H^G(M \otimes_k \mu),$$

with  $\mu$  the character of  $H$  dual to  $(\delta_G)\downarrow_H \delta_H^{-1}$ , where  $\delta_G$  is a linear character of  $G$  called the *modular function*; see [27, Proposition I.8.17].

**Proposition 9.5.** *Let  $H$  be subgroup scheme of a finite group scheme  $G$  over  $k$  and  $\rho: \mathrm{Proj} H^*(H, k) \rightarrow \mathrm{Proj} H^*(G, k)$  the map induced by restriction.*

(1) *For any  $G$ -module  $N$  the following equalities hold*

$$\mathrm{supp}_H(N\downarrow_H) = \rho^{-1}(\mathrm{supp}_G(N)) \quad \text{and} \quad \mathrm{cosupp}_H(N\downarrow_H) = \rho^{-1}(\mathrm{cosupp}_G(N))$$

(2) *For any  $H$ -module  $M$  the following inclusions hold.*

$$\mathrm{supp}_G(\mathrm{ind}_H^G M) \subseteq \rho(\mathrm{supp}_H(M)) \quad \text{and} \quad \mathrm{cosupp}_G(\mathrm{ind}_H^G M) \subseteq \rho(\mathrm{cosupp}_H(M)).$$

*These become equalities when  $G$  is a finite group or  $H$  is unipotent.*

*Proof.* (1) Since any  $\pi$ -point of  $H$  induces a  $\pi$ -point of  $G$ , the stated equalities are clear when one replaces support and cosupport by  $\pi$ -support and  $\pi$ -cosupport, respectively. It remains to recall Theorems 6.1 and 9.3.

(2) Since  $\mathrm{ind}$  is right adjoint to restriction, the inclusion of cosupports is a consequence of Proposition 2.1 applied to the functor

$$((-)\downarrow_H, f): (\mathrm{StMod} G, H^*(G, k)) \longrightarrow (\mathrm{StMod} H, H^*(H, k)).$$

By the same token, since coinduction is left adjoint to restriction one gets

$$\mathrm{supp}_G(\mathrm{coind}_H^G M) \subseteq \rho(\mathrm{supp}_H(M)).$$

As noted in (9.1), there is a one-dimensional representation  $\mu$  of  $H$  such that

$$\mathrm{supp}_G(\mathrm{ind}_H^G M) = \mathrm{supp}_G(\mathrm{coind}_H^G(M \otimes_k \mu)).$$

This yields the inclusion below.

$$\mathrm{supp}_G(\mathrm{ind}_H^G M) \subseteq \rho(\mathrm{supp}_H(M \otimes_k \mu)) = \rho(\mathrm{supp}_H(M) \cap \mathrm{supp}_H(\mu)) = \rho(\mathrm{supp}_H(M)).$$

The first equality is by Theorem 1.9 while the second one holds because the support of any one-dimensional representation  $\mu$  is  $\mathrm{Proj} H^*(G, k)$ , as follows, for example, because  $\mathrm{Hom}_k(\mu, \mu)$  is isomorphic to  $k$ .

Concerning the equalities, the key point is that under the additional hypotheses  $\mathrm{ind}_H^G(-)$ , which is right adjoint to  $(-)\downarrow_H$ , is faithful on objects.  $\square$

**Example 9.6.** One of many differences between finite groups and connected group schemes is that Proposition 9.5(2) may fail for the latter, because induction is not faithful on objects in general.

For example, let  $\mathcal{G} = \mathrm{SL}_n$  and  $\mathcal{B}$  be its standard Borel subgroup. Take  $G = \mathcal{G}_{(r)}$ ,  $H = \mathcal{B}_{(r)}$ , and  $\lambda = \rho(p^r - 1)$  where  $\rho$  is the half sum of all positive roots for the root system of  $\mathcal{G}$ . Let  $k_\lambda$  be the one-dimensional representation of  $H$  given by the character  $\lambda$ . Then  $\mathrm{ind}_H^G k_\lambda$  is the *Steinberg module* for  $G$ ; in particular, it is projective. Hence,  $\mathrm{ind}: \mathrm{stmod} H \rightarrow \mathrm{stmod} G$  is not faithful on objects, and both inclusions of Proposition 9.5(2) are strict for  $M = k_\lambda$ .

## 10. STRATIFICATION

In this section we establish for a finite group scheme the classification of tensor ideal localising subcategories of its stable module category and draw some consequences. The development follows closely the one in [7, Sections 10 and 11]. For this reason, in the remainder of the paper, we work exclusively with supports as defined in Section 2, secure in the knowledge afforded by Theorem 6.1 that the discussion could just as well be phrased in the language of  $\pi$ -points.

**Theorem 10.1.** *Let  $G$  be a finite group scheme over a field  $k$ . Then the stable module category  $\mathbf{StMod} G$  is stratified as a tensor triangulated category by the natural action of the cohomology ring  $H^*(G, k)$ . Therefore the assignment*

$$(10.1) \quad \mathcal{C} \longmapsto \bigcup_{M \in \mathcal{C}} \operatorname{supp}_G(M)$$

*induces a bijection between the tensor ideal localising subcategories of  $\mathbf{StMod} G$  and the subsets of  $\operatorname{Proj} H^*(G, k)$ .*

*Proof.* The first part of the assertion is precisely the statement of Theorem 8.10. The second part of the assertion is a formal consequence of the first; see [7, Theorem 3.8]. The inverse map sends a subset  $\mathcal{V}$  of  $\operatorname{Proj} H^*(G, k)$  to the tensor ideal localising subcategory consisting of all  $G$ -modules  $M$  such that  $\operatorname{supp}_G(M) \subseteq \mathcal{V}$ .  $\square$

The result below contains the first theorem from the introduction.

**Corollary 10.2.** *Let  $M$  and  $N$  be non-zero  $G$ -modules. One can build  $M$  out of  $N$  if (and only if) there is an inclusion  $\pi\text{-supp}_G(M) \subseteq \pi\text{-supp}_G(N)$ .*

*Proof.* The canonical functor  $\mathbf{Mod} G \rightarrow \mathbf{StMod} G$  that assigns a module to itself respects tensor products and takes short exact sequences to exact triangles. It follows that  $M$  is built out of  $N$  in  $\mathbf{Mod} G$  if and only if  $M$  is in the tensor ideal localising subcategory of  $\mathbf{StMod} G$  generated by  $N$ ; see also [7, Proposition 2.1]. The desired result is thus a direct consequence of Theorem 10.1.  $\square$

**Thick subcategories.** As a corollary of Theorem 10.1 we deduce a classification of the tensor ideal thick subcategories of  $\mathbf{stmod} G$ , stated already in [21, Theorem 6.3]. The crucial input in the proof in *op. cit.* is [21, Theorem 5.3], which is flawed (see Remark 5.4) but the argument can be salvaged by referring to Theorem 5.3 instead. We give an alternative proof, mimicking [7, Theorem 11.4].

**Theorem 10.3.** *Let  $G$  be a finite group scheme over a field  $k$ . The assignment (10.1) induces a bijection between tensor ideal thick subcategories of  $\mathbf{stmod} G$  and specialisation closed subsets of  $\operatorname{Proj} H^*(G, k)$ .*

*Proof.* To begin with, if  $M$  is a finite dimensional  $G$ -module, then  $\operatorname{supp}_G(M)$  is a Zariski closed subset of  $\operatorname{Proj} H^*(G, k)$ ; conversely, each Zariski closed subset of  $\operatorname{Proj} H^*(G, k)$  is of this form. Indeed, given the identification of  $\pi$ -support and cohomological support, the forward implication statement follows from [21, Proposition 3.4] while the converse is [21, Proposition 3.7]. Consequently, if  $\mathcal{C}$  is a tensor ideal thick subcategory of  $\mathbf{stmod} G$ , then  $\operatorname{supp}_G(\mathcal{C})$  is a specialisation closed subset of  $\operatorname{Proj} H^*(G, k)$ , and every specialisation closed subset of  $\operatorname{Proj} H^*(G, k)$  is of this form. It remains to verify that the assignment  $\mathcal{C} \mapsto \operatorname{supp}_G(\mathcal{C})$  is one-to-one.

This can be proved as follows:  $\mathbf{StMod} G$  is a compactly generated triangulated category and the full subcategory of its compact objects identifies with  $\mathbf{stmod} G$ . Thus, if  $\mathbf{C}$  is a tensor ideal thick subcategory of  $\mathbf{stmod} G$  and  $\mathbf{C}'$  the tensor ideal localising subcategory of  $\mathbf{StMod} G$  that it generates, then  $\mathbf{C}' \cap \mathbf{stmod} G = \mathbf{C}$ ; see [30, §5]. Since  $\mathrm{supp}_G(\mathbf{C}') = \mathrm{supp}_G(\mathbf{C})$ , Theorem 10.1 gives the desired result.  $\square$

**Localising subcategories closed under products.** The following result describes the localising subcategories of  $\mathbf{StMod} G$  that are closed under products.

**Theorem 10.4.** *A tensor ideal localising subcategory of  $\mathbf{StMod} G$  is closed under products if and only if the complement of its support in  $\mathrm{Proj} H^*(G, k)$  is specialisation closed.*

*Proof.* For the case that  $kG$  is the group algebra of a finite group, see [7, Theorem 11.8]. The argument applies verbatim to finite group schemes; the main ingredient is the stratification of  $\mathbf{StMod} G$ , Theorem 10.1.  $\square$

**The telescope conjecture.** A localising subcategory of a compactly generated triangulated category  $\mathbf{T}$  is *smashing* if it arises as the kernel of a localisation functor  $\mathbf{T} \rightarrow \mathbf{T}$  that preserves coproducts. The telescope conjecture, due to Bousfield and Ravenel [13, 33], in its general form is the assertion that every smashing localising subcategory of  $\mathbf{T}$  is generated by objects that are compact in  $\mathbf{T}$ ; see [29]. The following result confirms this conjecture for  $\mathbf{StMod} G$ , at least for all smashing subcategories that are tensor ideal. Note that when the trivial  $kG$ -module  $k$  generates  $\mathbf{stmod} G$  as a thick subcategory (for example, when  $G$  is unipotent) each localising subcategory is tensor ideal.

**Theorem 10.5.** *Let  $\mathbf{C}$  be a tensor ideal localising subcategory of  $\mathbf{StMod} G$ . The following conditions are equivalent:*

- (i) *The localising subcategory  $\mathbf{C}$  is smashing.*
- (ii) *The localising subcategory  $\mathbf{C}$  is generated by objects compact in  $\mathbf{StMod} G$ .*
- (iii) *The support of  $\mathbf{C}$  is a specialisation closed subset of  $\mathrm{Proj} H^*(G, k)$ .*

*Proof.* If  $G$  is a finite group this result is [7, Theorem 11.12] and is deduced from the stratification of  $\mathbf{StMod} G$ . The proof carries over verbatim to group schemes.  $\square$

**The homotopy category of injectives.** Let  $\mathbf{K}(\mathrm{Inj} G)$  denote the triangulated category whose objects are the complexes of injective  $G$ -modules and whose morphisms are the homotopy classes of degree preserving maps of complexes. As a triangulated category  $\mathbf{K}(\mathrm{Inj} G)$  is compactly generated, and the compact objects are equivalent to  $\mathbf{D}^b(\mathrm{mod} G)$ , via the functor  $\mathbf{K}(\mathrm{Inj} G) \rightarrow \mathbf{D}(\mathrm{Mod} G)$ . The tensor product of modules extends to complexes and defines a tensor product on  $\mathbf{K}(\mathrm{Inj} G)$ . This category was investigated in detail by Benson and Krause [12] in case  $G$  is a finite group; the more general case of a finite group scheme is analogous. Taking Tate resolutions gives an equivalence of triangulated categories from the stable module category  $\mathbf{StMod} G$  to the full subcategory  $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj} G)$  of  $\mathbf{K}(\mathrm{Inj} G)$  consisting of acyclic complexes. This equivalence preserves the tensor product. The Verdier quotient of  $\mathbf{K}(\mathrm{Inj} G)$  by  $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj} G)$  is equivalent, as a triangulated category, to the unbounded derived category  $\mathbf{D}(\mathrm{Mod} G)$ . There are left and right adjoints, forming

a recollement

$$\mathrm{StMod} G \xrightarrow{\sim} \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj} G) \begin{array}{c} \xleftarrow{\mathrm{Hom}_k(tk, -)} \\ \xrightarrow{\quad\quad\quad} \\ \xleftarrow{-\otimes_k tk} \end{array} \mathrm{K}(\mathrm{Inj} G) \begin{array}{c} \xleftarrow{\mathrm{Hom}_k(pk, -)} \\ \xrightarrow{\quad\quad\quad} \\ \xleftarrow{-\otimes_k pk} \end{array} \mathrm{D}(\mathrm{Mod} G)$$

where  $pk$  and  $tk$  are a projective resolution and a Tate resolution of  $k$  respectively.

The cohomology ring  $H^*(G, k)$  acts on  $\mathrm{K}(\mathrm{Inj} G)$  and, as in [5, 8], the theory of supports and cosupports for  $\mathrm{StMod} G$  extends in a natural way to  $\mathrm{K}(\mathrm{Inj} G)$ . It associates to each  $X$  in  $\mathrm{K}(\mathrm{Inj} G)$  subsets  $\mathrm{supp}_G(X)$  and  $\mathrm{cosupp}_G(X)$  of  $\mathrm{Spec} H^*(G, k)$ . The Tate resolution of a  $G$ -module  $M$  is  $tk \otimes_k M$ , so there are equalities

$$\mathrm{supp}_G(M) = \mathrm{supp}_G(tk \otimes_k M) \quad \text{and} \quad \mathrm{cosupp}_G(M) = \mathrm{cosupp}_G(tk \otimes_k M),$$

where one views  $\mathrm{Proj} H^*(G, k)$  as a subset of  $\mathrm{Spec} H^*(G, k)$ . Thus Theorem 10.1 has the following consequence.

**Corollary 10.6.** *The homotopy category  $\mathrm{K}(\mathrm{Inj} G)$  is stratified as a tensor triangulated category by the natural action of the cohomology ring  $H^*(G, k)$ . Therefore the assignment  $\mathcal{C} \mapsto \bigcup_{X \in \mathcal{C}} \mathrm{supp}_G(X)$  induces a bijection between the tensor ideal localising subcategories of  $\mathrm{K}(\mathrm{Inj} G)$  and the subsets of  $\mathrm{Spec} H^*(G, k)$ . It restricts to a bijection between the tensor ideal thick subcategories of  $\mathrm{D}^b(\mathrm{mod} G)$  and specialisation closed subsets of  $\mathrm{Spec} H^*(G, k)$ .  $\square$*

With this result on hand, one can readily establish analogues of Theorems 10.4 and 10.5 for  $\mathrm{K}(\mathrm{Inj} G)$ . We leave the formulation of the statements and the proofs to the interested reader; see also [7, Sections 10 and 11].

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