THE ARTINIAN CONJECTURE (FOLLOWING DJAMENT, PUTMAN, SAM, AND SNOWDEN)

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Abstract. This note provides a self-contained exposition of the proof of the artinian conjecture, following closely Djament's Bourbaki lecture. The original proof is due to Putman, Sam, and Snowden.

Contents

1. Introduction

This note provides a complete proof of the celebrated artinian conjecture. The proof is due to Putman, Sam, and Snowden [6, 7]. Here, we follow closely the elegant exposition of Djament in [3]. For the origin of the conjecture and its consequences, we refer to those papers and Djament's Bourbaki lecture [4]. In addition, the expository articles by Kuhn, Powell and Schwartz in [5] are recommended.

There are two main result. Fix a locally noetherian Grothendieck abelian category $\mathcal{A}$, for instance, the category of modules over a noetherian ring.

Theorem 1.1. *Let A be a ring whose underlying set is finite. For the category $\mathcal{P}(A)$ of free A-modules of finite rank, the functor category $\operatorname{Fun}(\mathcal{P}(A)^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

This result amounts to the assertion of the artinian conjecture when A is a finite field and $\mathcal{A}$ is the category of A-modules.

The first theorem is a direct consequence of the following.

Theorem 1.2. *For the category Γ of finite sets, the functor category $\operatorname{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

The paper is in final form and no version of it will be submitted for publication elsewhere.

The basic idea for the proof is to formulate finiteness conditions on an essentially small category $\mathcal{C}$ such that $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})$ is locally noetherian. This leads to the notion of a Gröbner category. Such finiteness conditions have a 'direction'. For that reason we consider contravariant functors $\mathcal{C}\to\mathcal{A}$, because then the direction is preserved (via Yoneda's lemma) when one passes from $\mathcal{C}$ to $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})$.

2. Noetherian posets

Let $\mathcal{C}$ be a poset. A subset $\mathcal{D}\subseteq\mathcal{C}$ is a *sieve* if the conditions $x\le y$ in $\mathcal{C}$ and $y\in\mathcal{D}$ imply $x\in\mathcal{D}$. The sieves in $\mathcal{C}$ are partially ordered by inclusion.

Definition 2.1. A poset $\mathcal{C}$ is called

(1) *noetherian* if every ascending chain of elements in $\mathcal{C}$ stabilises, and
(2) *strongly noetherian* if every ascending chain of sieves in $\mathcal{C}$ stabilises.

For a poset $\mathcal{C}$ and $x\in\mathcal{C}$, set $\mathcal{C}(x)=\{t\in\mathcal{C}\mid t\le x\}$. The assignment $x\mapsto\mathcal{C}(x)$ yields an embedding of $\mathcal{C}$ into the poset of sieves in $\mathcal{C}$.

Lemma 2.2. *For a poset $\mathcal{C}$ the following are equivalent:*

(1) *The poset $\mathcal{C}$ is strongly noetherian.*
(2) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there exists $i\in\mathbb{N}$ such that $x_j\le x_i$ for infinitely many $j\in\mathbb{N}$.*
(3) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there is a map $\alpha\colon\mathbb{N}\to\mathbb{N}$ such that $i<j$ implies $\alpha(i)<\alpha(j)$ and $x_{\alpha(j)}\le x_{\alpha(i)}$.*
(4) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there are $i<j$ in $\mathbb{N}$ such that $x_j\le x_i$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathcal{C}$ is strongly noetherian and let $(x_i)_{i\in\mathbb{N}}$ be elements in $\mathcal{C}$. For $n\in\mathbb{N}$ set $\mathcal{C}_n=\bigcup_{i\le n}\mathcal{C}(x_i)$. The chain $(\mathcal{C}_n)_{n\in\mathbb{N}}$ stabilises, say $\mathcal{C}_n=\mathcal{C}_N$ for all $n\ge N$. Thus there exists $i\le N$ such that $x_j\le x_i$ for infinitely many $i\in\mathbb{N}$.

(2) $\Rightarrow$ (3): Define $\alpha\colon\mathbb{N}\to\mathbb{N}$ recursively by taking for $\alpha(0)$ the smallest $i\in\mathbb{N}$ such that $x_j\le x_i$ for infinitely many $j\in\mathbb{N}$. For $n>0$ set

$$\alpha(n)=\min\{i>\alpha(n-1)\mid x_j\le x_i\le x_{\alpha(n-1)}\text{ for infinitely many }j\in\mathbb{N}\}.$$

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (1): Suppose there is a properly ascending chain $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of sieves in $\mathcal{C}$. Choose $x_n\in\mathcal{C}_{n+1}\setminus\mathcal{C}_n$ for each $n\in\mathbb{N}$. There are $i<j$ in $\mathbb{N}$ such that $x_j\le x_i$. This implies $x_j\in\mathcal{C}_{i+1}\subseteq\mathcal{C}_j$ which is a contradiction. □

3. Functor categories

Let $\mathcal{C}$ be an essentially small category and $\mathcal{A}$ a Grothendieck abelian category. We denote by $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})$ the category of functors $\mathcal{C}^{\mathrm{op}}\to\mathcal{A}$. The morphisms between two functors are the natural transformations. Note that $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})$ is a Grothendieck abelian category.

Given an object $x\in\mathcal{C}$, the evaluation functor

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})\longrightarrow\mathcal{A},\quad F\mapsto F(x)$$

admits a left adjoint

$$\mathcal{A}\longrightarrow\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A}),\quad M\mapsto M[\mathcal{C}(-,x)]$$

where for any set X we denote by $M[X]$ a coproduct of copies of M indexed by the elements of X. Thus we have a natural isomorphism

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})(M[\mathcal{C}(-,x)],F)\cong\mathcal{A}(M,F(x)).\tag{3.1}$$

Lemma 3.1. *If $(M_i)_{i\in I}$ is a set of generators of $\mathcal{A}$, then the functors $M_i[\mathcal{C}(-,x)]$ with $i\in I$ and $x\in\mathcal{C}$ generate* $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})$.

Proof. Use the adjointness isomorphism (3.1). □

A Grothendieck abelian category $\mathcal{A}$ is *locally noetherian* if $\mathcal{A}$ has a generating set of noetherian objects. In that case an object $M\in\mathcal{A}$ is noetherian iff M is *finitely presented* (that is, the representable functor $\mathcal{A}(M,-)$ preserves filtered colimits); see [8, Chap. V] for details.

Lemma 3.2. *Let $\mathcal{A}$ be locally noetherian. Then* $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{A})$ *is locally noetherian iff $M[\mathcal{C}(-,x)]$ is noetherian for every noetherian $M\in\mathcal{A}$ and $x\in\mathcal{C}$.*

Proof. First observe that $M[\mathcal{C}(-,x)]$ is finitely presented if M is finitely presented. This follows from the isomorphism (3.1) since evaluation at $x\in\mathcal{C}$ preserves colimits. Now the assertion of the lemma is an immediate consequence of Lemma 3.1. □

4. Noetherian functors

Let $\mathcal{C}$ be an essentially small category and fix an object $x\in\mathcal{C}$. Set

$$\mathcal{C}(x)=\coprod_{t\in\mathcal{C}}\mathcal{C}(t,x).$$

Given $f,g\in\mathcal{C}(x)$, let $\langle f\rangle$ denote the set of morphisms in $\mathcal{C}(x)$ that factor through f, and set $f\le_x g$ if $\langle f\rangle\subseteq\langle g\rangle$. We identify f and g when $\langle f\rangle=\langle g\rangle$. This yields a poset which we denote by $\bar{\mathcal{C}}(x)$.

A functor is *noetherian* if every ascending chain of subfunctors stabilises.

Lemma 4.1. *The functor $\mathcal{C}(-,x)\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}$ is noetherian iff the poset $\bar{\mathcal{C}}(x)$ is strongly noetherian.*

Proof. Sending $F\subseteq\mathcal{C}(-,x)$ to $\bigcup_{t\in\mathcal{C}}F(t)$ induces an inclusion preserving bijection between the subfunctors of $\mathcal{C}(-,x)$ and the sieves in $\bar{\mathcal{C}}(x)$. □

For a poset $\mathcal{T}$ let $\mathrm{Set}\wr\mathcal{T}$ denote the category consisting of pairs (X,ξ) such that X is a set and $\xi\colon X\to\mathcal{T}$ is a map. A morphism $(X,\xi)\to(X',\xi')$ is a map $f\colon X\to X'$ such that $\xi(a)\le\xi' f(a)$ for all $a\in X$.

A functor $\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}\wr\mathcal{T}$ is given by a pair (F,ϕ) consisting of a functor $F\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}$ and a map $\phi\colon\coprod_{t\in\mathcal{C}}F(t)\to\mathcal{T}$ such that $\phi(a)\le\phi(F(f)(a))$ for every $a\in F(t)$ and $f\colon t'\to t$ in $\mathcal{C}$.

Lemma 4.2. *Let $\mathcal{T}$ be a noetherian poset. If $\mathcal{C}(-,x)$ is noetherian, then any functor $(\mathcal{C}(-,x),\phi)\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Set}\wr\mathcal{T}$ is noetherian.*

Proof. Let $(F_n,\phi_n)_{n\in\mathbb{N}}$ be a strictly ascending chain of subfunctors of (F,ϕ). The chain $(F_n)_{n\in\mathbb{N}}$ stabilises since $\mathcal{C}(-,x)$ is noetherian. Thus we may assume that $F_n=F$ for all $n\in\mathbb{N}$, and we find $f_n\in\coprod_{t\in\mathcal{C}}F(t)$ such that $\phi_n(f_n)<\phi_{n+1}(f_n)$. The poset $\bar{\mathcal{C}}(x)$ is strongly noetherian by Lemma 4.1. It follows from Lemma 2.2 that there is a map $\alpha\colon\mathbb{N}\to\mathbb{N}$ such that $i<j$ implies $\alpha(i)<\alpha(j)$ and $f_{\alpha(j)}\le_x f_{\alpha(i)}$. Thus

$$\phi_{\alpha(n)}(f_{\alpha(n)})<\phi_{\alpha(n)+1}(f_{\alpha(n)})\le\phi_{\alpha(n+1)}(f_{\alpha(n)})\le\phi_{\alpha(n+1)}(f_{\alpha(n+1)}).$$

This yields a strictly ascending chain in $\mathcal{T}$, contradicting the assumption on $\mathcal{T}$. □

Definition 4.3. A partial order $\le$ on $\mathcal{C}(x)$ is *admissible* if the following holds:

1. The order $\le$ restricted to $\mathcal{C}(t,x)$ is total and noetherian for every $t\in\mathcal{C}$.
2. For $f,f'\in\mathcal{C}(t,x)$ and $e\in\mathcal{C}(s,t)$, the condition $f<f'$ implies $fe<f'e$.

Fix an admissible partial order $\leq$ on $\mathcal{C}(x)$ and an object M in a Grothendieck abelian category $\mathcal{A}$. Let $\operatorname{Sub}(M)$ denote the poset of subobjects of M and consider the functor

$$\mathcal{C}(-,x) \wr M \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \operatorname{Sub}(M), \quad t \mapsto \big(\mathcal{C}(t,x), (M)_{f\in\mathcal{C}(t,x)}\big).$$

For a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ define a subfunctor $\tilde{F} \subseteq \mathcal{C}(-,x) \wr M$ as follows:

$$\tilde{F} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \operatorname{Sub}(M), \quad t \mapsto \Big(\mathcal{C}(t,x), \big(\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t))\big)_{f\in\mathcal{C}(t,x)}\Big)$$

where $\mathcal{C}(t,x)_f = \{g \in \mathcal{C}(t,x) \mid f \leq g\}$ and $\pi_f \colon M[\mathcal{C}(t,x)_f] \to M$ is the projection onto the factor corresponding to f. For a morphism $e \colon t' \to t$ in $\mathcal{C}$, the morphism $\tilde{F}(e)$ is induced by precomposition with e. Note that

$$\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t)) \subseteq \pi_{fe}(M[\mathcal{C}(t',x)_{fe}] \cap F(t'))$$

since $\leq$ is compatible with the composition in $\mathcal{C}$.

Lemma 4.4. *Suppose there is an admissible partial order on $\mathcal{C}(x)$. Then the assignment which sends a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ to $\tilde{F}$ preserves proper inclusions. Therefore $M[\mathcal{C}(-,x)]$ is noetherian provided that $\mathcal{C}(-,x) \wr M$ is noetherian.*

Proof. Let $F \subseteq G \subseteq M[\mathcal{C}(-,x)]$. Then $\tilde{F} \subseteq \tilde{G}$. Now suppose that $F \neq G$. Thus there exists $t \in \mathcal{C}$ such that $F(t) \neq G(t)$. We have $\mathcal{C}(t,x) = \bigcup_{f\in\mathcal{C}(t,x)} \mathcal{C}(t,x)_f$, and this union is directed since $\leq$ is total. Thus

$$F(t) = \sum_{f\in\mathcal{C}(t,x)} \big(M[\mathcal{C}(t,x)_f] \cap F(t)\big)$$

since filtered colimits in $\mathcal{A}$ are exact. This yields f such that

$$M[\mathcal{C}(t,x)_f] \cap F(t) \neq M[\mathcal{C}(t,x)_f] \cap G(t).$$

Choose $f \in \mathcal{C}(t,x)$ maximal with respect to this property, using that $\leq$ is noetherian. Now observe that the projection π_f induces an exact sequence

$$0 \longrightarrow \sum_{f<g} \big(M[\mathcal{C}(t,x)_g] \cap F(t)\big) \longrightarrow F(t) \longrightarrow \pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \longrightarrow 0$$

since the kernel of π_f equals the directed union $\sum_{f<g} M[\mathcal{C}(t,x)_g]$. For the directedness one uses again that $\leq$ is total. Thus

$$\pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \neq \pi_f\big(M[\mathcal{C}(t,x)_f] \cap G(t)\big)$$

and therefore $\tilde{F} \neq \tilde{G}$. □

Proposition 4.5. *Let $x \in \mathcal{C}$. Suppose that $\mathcal{C}(-,x)$ is noetherian and that $\mathcal{C}(x)$ has an admissible partial order. If $M \in \mathcal{A}$ is noetherian, then $M[\mathcal{C}(-,x)]$ is noetherian.*

Proof. Combine Lemmas 4.2 and 4.4. □

5. Gröbner categories

Definition 5.1. An essentially small category $\mathcal{C}$ is a *Gröbner category* if the following holds:

(1) The functor $\mathcal{C}(-,x)$ is noetherian for every $x \in \mathcal{C}$.
(2) There is an admissible partial order on $\mathcal{C}(x)$ for every $x \in \mathcal{C}$.

Theorem 5.2. *Let $\mathcal{C}$ be a Gröbner category and $\mathcal{A}$ a Grothendieck abelian category. If $\mathcal{A}$ is locally noetherian, then* $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof. Combine Lemma 3.1 and Proposition 4.5. □

Example 5.3. A strongly noetherian poset (viewed as a category) is a Gröbner category. For instance, let $\mathcal{C} = (\mathbb{N}, \leq)^{\mathrm{op}}$ and $\mathcal{A}$ be the module category of a noetherian ring A. Then $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ equals the module category of the polynomial ring in one variable over A. Thus Theorem 5.2 generalises Hilbert's Basis Theorem.

6. Base change

Given functors $F, G \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, we write $F \rightsquigarrow G$ if there is a finite chain

$$F = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = G$$

of epimorphisms and monomorphisms of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

Definition 6.1. A functor $\phi \colon \mathcal{C} \to \mathcal{D}$ is *contravariantly finite*[1] if the following holds:

(1) Every object $y \in \mathcal{D}$ is isomorphic to $\phi(x)$ for some $x \in \mathcal{C}$.
(2) For every object $y \in \mathcal{D}$ there are objects $x_1, \ldots, x_n$ in $\mathcal{C}$ such that

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) \rightsquigarrow \mathcal{D}(\phi -, y).$$

The functor ϕ is *covariantly finite* if $\phi^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is contravariantly finite.

Note that a composite of contravariantly finite functors is contravariantly finite.

Lemma 6.2. *Let $f \colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a Grothendieck abelian category. Fix $M \in \mathcal{A}$ and suppose that $M[\mathcal{C}(-, x)]$ is noetherian for all $x \in \mathcal{C}$. Then $M[\mathcal{D}(-, y)]$ is noetherian for all $y \in \mathcal{D}$.*

Proof. A finite chain

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = \mathcal{D}(\phi -, y)$$

of epimorphisms and monomorphisms induces a chain

$$\coprod_{i=1}^{n} M[\mathcal{C}(-, x_i)] = \bar{F}_0 \twoheadrightarrow \bar{F}_1 \leftarrowtail \bar{F}_2 \twoheadrightarrow \cdots \twoheadrightarrow \bar{F}_{n-1} \leftarrowtail \bar{F}_n = M[\mathcal{D}(\phi -, y)]$$

of epimorphisms and monomorphisms in $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. Thus $M[\mathcal{D}(\phi -, y)]$ is noetherian. It follows that $M[\mathcal{D}(-, y)]$ is noetherian, since precomposition with ϕ yields a faithful and exact functor $\operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A}) \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. □

Proposition 6.3. *Let $f \colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a locally noetherian Grothendieck abelian category. If $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian, then $\operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

Proof. Combine Lemmas 3.2 and 6.2. □

7. Categories of finite sets

Let Γ denote the category of finite sets (a skeleton is given by the sets $\mathbf{n} = \{1, 2, \ldots, n\}$). The subcategory of finite sets with surjective morphisms is denoted by Γ_{sur}. A surjection $f \colon \mathbf{m} \to \mathbf{n}$ is *ordered* if $i < j$ implies $\min f^{-1}(i) < \min f^{-1}(j)$. We write Γ_{os} for the subcategory of finite sets whose morphisms are ordered surjections. Given a surjection $f \colon \mathbf{m} \to \mathbf{n}$, let $f^! \colon \mathbf{n} \to \mathbf{m}$ denote the map given by $f^!(i) = \min f^{-1}(i)$. Note that $ff^! = \mathrm{id}$, and $gf = f^! g^!$ provided that f and g are ordered surjections.

[1]The terminology follows that introduced by Auslander and Smalø [1] for an inclusion functor.

Lemma 7.1. (1) *The inclusion* $\Gamma_{\mathrm{sur}} \to \Gamma$ *is contravariantly finite.*
(2) *The inclusion* $\Gamma_{\mathrm{os}} \to \Gamma_{\mathrm{sur}}$ *is contravariantly finite.*

Proof. (1) For each integer $n \geq 0$ there is an isomorphism

$$\coprod_{\mathbf{m} \hookrightarrow \mathbf{n}} \Gamma_{\mathrm{sur}}(-, \mathbf{m}) \xrightarrow{\sim} \Gamma(-, \mathbf{n})$$

which is induced by the injective maps $\mathbf{m} \to \mathbf{n}$.

(2) For each integer $n \geq 0$ there is an isomorphism

$$\Gamma_{\mathrm{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \xrightarrow{\sim} \Gamma_{\mathrm{sur}}(-, \mathbf{n})$$

which sends a pair (f, σ) to σf. The inverse sends a surjective map $g \colon \mathbf{m} \to \mathbf{n}$ to $(\tau^{-1} g, \tau)$ where $\tau \in \mathfrak{S}_n$ is the unique permutation such that $g^! \tau$ is increasing. □

Fix an integer $n \geq 0$. Given $f, g \in \Gamma(\mathbf{n})$ we set $f \leq g$ if there exists an ordered surjection h such that $f = gh$.

Lemma 7.2. *The poset* $(\Gamma(\mathbf{n}), \leq)$ *is strongly noetherian.*

Proof. We fix some notation for each $f \in \Gamma(\mathbf{m}, \mathbf{n})$. Set $\lambda(f) = m$. If f is not injective, set

$$\mu(f) = m - \max\{i \in \mathbf{m} \mid \text{there exists } j < i \text{ such that } f(i) = f(j)\}$$

and $\pi(f) = f(m - \mu(f))$. Define $\tilde{f} \in \Gamma(\mathbf{m-1}, \mathbf{n})$ by setting $\tilde{f}(i) = f(i)$ for $i < m - \mu(f)$ and $\tilde{f}(i) = f(i+1)$ otherwise.

Note that $f \leq \tilde{f}$. Moreover, $\mu(f) = \mu(g)$, $\pi(f) = \pi(g)$, and $\tilde{f} \leq \tilde{g}$ imply $f \leq g$.

Suppose that $(\Gamma(\mathbf{n}), \leq)$ is not strongly noetherian. Then there exists an infinite sequence $(f_r)_{r \in \mathbb{N}}$ in $\Gamma(\mathbf{n})$ such that $i < j$ implies $f_j \not\leq f_i$; see Lemma 2.2. Call such a sequence *bad*. Choose the sequence *minimal* in the sense that $\lambda(f_i)$ is minimal for all bad sequences $(g_r)_{r \in \mathbb{N}}$ with $g_j = f_j$ for all $j < i$. There is an infinite subsequence $(f_{\alpha(r)})_{r \in \mathbb{N}}$ (given by some increasing map $\alpha \colon \mathbb{N} \to \mathbb{N}$) such that μ and π agree on all $f_{\alpha(r)}$, since the values of μ and π are bounded by n. Now consider the sequence $f_0, f_1, \ldots, f_{\alpha(0)-1}, \tilde{f}_{\alpha(0)}, \tilde{f}_{\alpha(1)}, \ldots$ and denote this by $(g_r)_{r \in \mathbb{N}}$. This sequence is not bad, since $(f_r)_{r \in \mathbb{N}}$ is minimal. Thus there are $i < j$ in $\mathbb{N}$ with $g_j \leq g_i$. Clearly, $j < \alpha(0)$ is impossible. If $i < \alpha(0)$, then

$$f_{\alpha(j-\alpha(0))} \leq \tilde{f}_{\alpha(j-\alpha(0))} = g_j \leq g_i = f_i,$$

which is a contradiction, since $i < \alpha(0) \leq \alpha(j - \alpha(0))$. If $i \geq \alpha(0)$, then $f_{\alpha(j-\alpha(0))} \leq f_{\alpha(i-\alpha(0))}$; this is a contradiction again. Thus $(\Gamma(\mathbf{n}), \leq)$ is strongly noetherian. □

Proposition 7.3. *The category* Γ_{os} *is a Gröbner category.*

Proof. Fix an integer $n \geq 0$. The poset $\bar{\Gamma}_{\mathrm{os}}(\mathbf{n})$ is strongly noetherian by Lemma 7.2, and it follows from Lemma 4.1 that the functor $\Gamma_{\mathrm{os}}(-, \mathbf{n})$ is noetherian.

The admissible partial order on $\Gamma_{\mathrm{os}}(\mathbf{n})$ is given by the lexicographic order. Thus for $f, g \in \Gamma_{\mathrm{os}}(\mathbf{m}, \mathbf{n})$, we have $f < g$ if there exists $j \in \mathbf{m}$ with $f(j) < g(j)$ and $f(i) = g(i)$ for all $i < j$. □

Theorem 7.4. *Let* $\mathcal{A}$ *be a locally noetherian Grothendieck abelian category. Then the category* $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof. The category Γ_{os} is a Gröbner category by Proposition 7.3. It follows from Theorem 5.2 that $\mathrm{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian. The inclusion $\Gamma_{\mathrm{os}} \to \Gamma$ is contravariantly finite by Lemma 7.1. Thus $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 6.3. □

8. The artinian conjecture

Let A be a ring. We denote by $\mathcal{P}(A)$ the category of free A-modules of finite rank. If A is finite, then the functor $\Gamma \to \mathcal{P}(A)$ sending X to $A[X]$ is a left adjoint of the forgetful functor $\mathcal{P}(A) \to \Gamma$.

Lemma 8.1. *Let A be finite. Then the functor $\Gamma \to \mathcal{P}(A)$ is contravariantly finite.*

Proof. The assertion follows from the adjointness isomorphism
$$\mathcal{P}(A)(A[X], P) \cong \Gamma(X, P).$$ □

Theorem 8.2. *Let A be a finite ring and $\mathcal{A}$ a locally noetherian Grothendieck abelian category. Then the category $\operatorname{Fun}(\mathcal{P}(A)^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

Proof. Combine Theorem 7.4 with Lemma 8.1 and Proposition 6.3. □

9. FI-modules

The proof of the artinian conjecture yields an alternative proof of the following result due to Church, Ellenberg, Farb, and Nagpal.

Let Γ_{inj} denote the category whose objects are finite sets and whose morphisms are injective maps.

Theorem 9.1 ([2, Theorem A])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck abelian category. Then the category $\operatorname{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ is locally noetherian.*

Proof. The following argument has been suggested by Kai-Uwe Bux. Consider the functor $\phi\colon \Gamma_{\mathrm{os}} \to (\Gamma_{\mathrm{inj}})^{\mathrm{op}}$ which is the identity on objects and takes a map $f\colon \mathbf{m} \to \mathbf{n}$ to $f^!\colon \mathbf{n} \to \mathbf{m}$ given by $f^!(i) = \min f^{-1}(i)$. This functor is contravariantly finite, since for each integer $n \geq 0$ the morphism
$$\Gamma_{\mathrm{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \longrightarrow \Gamma_{\mathrm{inj}}(\mathbf{n}, \phi-)$$
which sends a pair (f, σ) to $f^!\sigma$ is an epimorphism.

It follows from Proposition 6.3 that $\operatorname{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ is locally noetherian, since $\operatorname{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 7.3 and Theorem 5.2. □

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