

# MORPHISMS DETERMINED BY OBJECTS AND FLAT COVERS

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ABSTRACT. We describe a procedure for constructing morphisms in additive categories, combining Auslander's concept of a morphism determined by an object with the existence of flat covers. Also, we show how flat covers are turned into projective covers, and we interpret these constructions in terms of adjoint functors.

## INTRODUCTION

Functors and morphisms determined by objects were introduced by Auslander in his Philadelphia notes [2]. These concepts provide a method to construct and organise morphisms in additive categories, generalising previous work of Auslander and Reiten on almost split sequences [5]. More recently, Ringel presented a survey of these results, rearranged them as lattice isomorphisms (the Auslander bijections), and added a host of interesting examples [15].

The starting point for our work is the following natural question: Is there a procedure for constructing morphisms ending at a fixed object in an additive category? More precisely, we are looking for

- *invariants* of morphisms ending at some fixed object, and
- *constructions* for universal morphisms with respect to these invariants.

An answer to this question is presented in Theorem 1.1. This combines Auslander's concept of a morphism determined by an object with a deep result about functor categories, which says that every additive functor admits a flat cover [6].

The second part of this note is inspired by the first. We show in Theorem 2.2 that every flat cover is in fact a projective cover, when viewed in an appropriate abelian category. Building on another of Auslander's paradigms [1], we use the Yoneda embedding

$$\mathbf{A} \longrightarrow \mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab}), \quad X \mapsto \text{Hom}_{\mathbf{A}}(-, X)$$

of an additive category  $\mathbf{A}$  into the category of finitely presented functors  $\mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$ . Take for instance the category  $\mathbf{A} = \text{Mod } \Lambda$  of modules over a ring  $\Lambda$ . It is somewhat surprising that any functor  $\mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$  preserving filtered colimits in  $\mathbf{A}$  belongs to  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  and admits a projective cover, even a minimal projective presentation. This is precisely what we exploit in Theorem 1.1 and explain in Theorem 2.2.

The final section discusses the connection with almost split sequences, and we complement Theorem 1.1 by a non-existence result for an almost split sequence ending at a module which is not finitely presented (Proposition 3.1).

## 1. MORPHISMS DETERMINED BY OBJECTS

**Invariants of morphisms.** We fix an additive category  $\mathbf{A}$  and let  $\mathbf{C}$  be a set of objects. We shall view  $\mathbf{C}$  as a full subcategory of  $\mathbf{A}$ . A  *$\mathbf{C}$ -module* is by definition an additive functor  $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Ab}$  into the category  $\mathbf{Ab}$  of abelian groups, and a morphism

between two  $\mathbf{C}$ -modules is a natural transformation. The  $\mathbf{C}$ -modules form an abelian category which we denote by  $(\mathbf{C}^{\text{op}}, \mathbf{Ab})$ . For example, if  $\mathbf{C}$  consists of one object  $C$ , then  $(\mathbf{C}^{\text{op}}, \mathbf{Ab})$  is the category of modules over the endomorphism ring of  $C$ . Note that (co)kernels and (co)products in  $(\mathbf{C}^{\text{op}}, \mathbf{Ab})$  are computed pointwise: for instance, a sequence  $X \rightarrow Y \rightarrow Z$  of morphisms between  $\mathbf{C}$ -modules is exact if and only if the sequence  $X(C) \rightarrow Y(C) \rightarrow Z(C)$  is exact in  $\mathbf{Ab}$  for all  $C$  in  $\mathbf{C}$ .

Every object  $X$  in  $\mathbf{A}$  gives rise to a  $\mathbf{C}$ -module

$$\text{Hom}_{\mathbf{A}}(\mathbf{C}, X) = \text{Hom}_{\mathbf{A}}(-, X)|_{\mathbf{C}}: \mathbf{C}^{\text{op}} \longrightarrow \mathbf{Ab}$$

and every morphism  $\alpha: X \rightarrow Y$  in  $\mathbf{A}$  yields a morphism

$$\text{Hom}_{\mathbf{A}}(\mathbf{C}, \alpha): \text{Hom}_{\mathbf{A}}(\mathbf{C}, X) \longrightarrow \text{Hom}_{\mathbf{A}}(\mathbf{C}, Y).$$

The image  $\text{Im Hom}_{\mathbf{A}}(\mathbf{C}, \alpha)$  of  $\text{Hom}_{\mathbf{A}}(\mathbf{C}, \alpha)$  is the invariant we shall use. In fact, we construct a ‘right adjoint’ which takes a submodule of  $\text{Hom}_{\mathbf{A}}(\mathbf{C}, Y)$  to a morphism ending at  $Y$ .

**Constructing morphisms.** We work in an additive category  $\mathbf{A}$  which is *locally finitely presented* [7]. This means the full subcategory  $\text{fp } \mathbf{A}$  of finitely presented objects is essentially small and each object in  $\mathbf{A}$  is a filtered colimit of objects in  $\text{fp } \mathbf{A}$ . Recall that an object  $X$  is *finitely presented* if the functor  $\text{Hom}_{\mathbf{A}}(X, -): \mathbf{A} \rightarrow \mathbf{Ab}$  preserves filtered colimits.

For example, the category  $\text{Mod } \Lambda$  of modules over a ring  $\Lambda$  is locally finitely presented. Then  $\text{mod } \Lambda$  denotes the full subcategory of finitely presented  $\Lambda$ -modules.

A morphism  $\alpha: X \rightarrow Y$  is called *right minimal* if every endomorphism  $\phi: X \rightarrow X$  with  $\alpha\phi = \alpha$  is invertible.

The following theorem is the main result of this work.

**Theorem 1.1.** *Let  $\mathbf{A}$  be a locally finitely presented additive category and  $\mathbf{C}$  a set of finitely presented objects. For an object  $Y$  in  $\mathbf{A}$  and a submodule  $H \subseteq \text{Hom}_{\mathbf{A}}(\mathbf{C}, Y)$ , there exists a morphism  $\alpha: X \rightarrow Y$  in  $\mathbf{A}$  (unique up to non-unique isomorphism) such that the following holds:*

- (1)  $\text{Im Hom}_{\mathbf{A}}(\mathbf{C}, \alpha) = H$  and any morphism  $\alpha': X' \rightarrow Y$  with  $\text{Im Hom}_{\mathbf{A}}(\mathbf{C}, \alpha') \subseteq H$  factors through  $\alpha$ .
- (2)  $\alpha$  is right minimal.

The proof will be given later in this section. A second and more elementary proof for  $\mathbf{A} = \text{Mod } \Lambda$  can be found in an appendix.

**Morphisms determined by objects.** Following [2], a morphism  $\alpha: X \rightarrow Y$  in  $\mathbf{A}$  is called *right determined by  $\mathbf{C}$*  (or simply *right  $\mathbf{C}$ -determined*) if any morphism  $\alpha': X' \rightarrow Y$  satisfying

$$\text{Im Hom}_{\mathbf{A}}(\mathbf{C}, \alpha') \subseteq \text{Im Hom}_{\mathbf{A}}(\mathbf{C}, \alpha)$$

factors through  $\alpha$ . In fact, Auslander established Theorem 1.1 for module categories with  $\mathbf{C}$  consisting of a single object [2, Theorem I.3.19], generalising previous work of Auslander and Reiten on almost split sequences [5].

An obvious question to ask is when a morphism is right determined by some set of finitely presented objects. We give a general answer in Proposition 1.13. For an Artin algebra, every morphism between finitely presented modules is right determined by some finitely presented module [3, Theorem 2.6].

**Functoriality.** The assignment  $(C, H, Y) \mapsto \alpha$  in Theorem 1.1 is functorial. This has been pointed out by Ringel in his survey of Auslander's results [15].

Let us formulate the functoriality. We fix an object  $Y \in \mathbf{A}$ . The morphisms ending at  $Y$  are *preordered*. This means for morphisms  $\alpha: X \rightarrow Y$  and  $\alpha': X' \rightarrow Y$  that  $\alpha' \leq \alpha$  when  $\alpha'$  factors through  $\alpha$ . We obtain a *poset* by identifying  $\alpha$  and  $\alpha'$  when  $\alpha' \leq \alpha$  and  $\alpha \leq \alpha'$ . Let us denote this poset by  $[A/Y]$  because it is derived from the slice category  $\mathbf{A}/Y$ .<sup>1</sup>

For a pair  $C \subseteq \text{fp } \mathbf{A}$  and  $H \subseteq \text{Hom}_{\mathbf{A}}(C, Y)$  let  $\alpha_{C,H}: X_{C,H} \rightarrow Y$  denote the right minimal and right  $C$ -determined morphism such that  $\text{Im Hom}_{\mathbf{A}}(C, \alpha_{C,H}) = H$ ; it exists and is well-defined up to a non-unique isomorphism by Theorem 1.1. Note that  $\alpha_{C,H}$  is unique when viewed as an element of  $[A/Y]$ .

**Lemma 1.2.** *Let  $\phi: Y' \rightarrow Y$  be a morphism in  $\mathbf{A}$ ,  $C \subseteq C' \subseteq \text{fp } \mathbf{A}$ ,  $H \subseteq \text{Hom}_{\mathbf{A}}(C, Y)$ , and  $H' \subseteq \text{Hom}_{\mathbf{A}}(C', Y')$ . Then  $\text{Hom}_{\mathbf{A}}(C, \phi)(H') \subseteq H$  implies  $\phi\alpha_{C',H'} \leq \alpha_{C,H}$ .*

*Proof.* The assumptions imply

$$\text{Im Hom}_{\mathbf{A}}(C, \phi\alpha_{C',H'}) = \text{Hom}_{\mathbf{A}}(C, \phi)(\text{Im Hom}_{\mathbf{A}}(C, \alpha_{C',H'})) \subseteq H.$$

Thus  $\phi\alpha_{C',H'}$  factors through  $\alpha_{C,H}$ . □

For an object  $X$  in an abelian category let  $\text{sub}(X)$  denote its lattice of subobjects.

*Remark 1.3.* Viewing a poset as a category, the map

$$\text{sub}(\text{Hom}_{\mathbf{A}}(C, Y)) \longrightarrow [A/Y], \quad H \mapsto \alpha_{C,H}$$

is right adjoint to

$$[A/Y] \longrightarrow \text{sub}(\text{Hom}_{\mathbf{A}}(C, Y)), \quad \alpha \mapsto \text{Im Hom}_{\mathbf{A}}(C, \alpha).$$

There are some natural choices of triples  $(C, H, Y)$ . For example, right almost split morphisms arise from triples  $(Y, \text{rad End}_{\mathbf{A}}(Y), Y)$  for a finitely presented object  $Y$  with local endomorphism ring [2, §II.2]. Another obvious choice is  $H = 0$ ; not much seems to be known in this case.

**Problem 1.4** (Auslander [4]). Describe the morphism  $\alpha_{C,0}$  for  $Y \in \mathbf{A}$  and  $C \subseteq \text{fp } \mathbf{A}$ .

Note the extremes:  $\alpha_{\emptyset,0} = \text{id}_Y$  is the identity morphism and  $\alpha_{C,0} = 0$  when  $C$  contains a generator of  $\text{fp } \mathbf{A}$ .

The assignment  $(C, H) \mapsto \alpha_{C,H}$  has been studied in some detail for modules over Artin algebras. We include Ringel's formulation of the Auslander bijection as an example.

**Example 1.5** (Ringel [15]). Let  $\Lambda$  be an Artin algebra. For  $Y \in \text{mod } \Lambda$  the assignment  $(C, H) \mapsto \alpha_{C,H}$  induces an isomorphism<sup>2</sup>

$$\text{colim}_{C \in \text{mod } \Lambda} \text{sub}(\text{Hom}_{\Lambda}(C, Y)) \xrightarrow{\sim} [\text{mod } \Lambda/Y].$$

<sup>1</sup>In [15], the poset  $[A/Y]$  is called *right factorisation lattice* for  $Y$  and is denoted by  $[\rightarrow Y]$ .

<sup>2</sup>The colimit is taken over the collection of finite subsets  $C \subseteq \text{mod } \Lambda$ , identifying  $C = \{C_1, \dots, C_n\}$  with  $C = C_1 \oplus \dots \oplus C_n$ .

**Functors determined by objects.** Let  $\mathbf{A}$  be an additive category. We consider additive functors  $\mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$  into the category of abelian groups; they form an abelian category which we denote by  $(\mathbf{A}^{\text{op}}, \mathbf{Ab})$ .

Fix an additive functor  $F: \mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$  and a set  $\mathbf{C}$  of objects in  $\mathbf{A}$  (viewed as a full subcategory). We write  $F|_{\mathbf{C}}$  for the restriction  $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Ab}$ . Following Auslander [2], a subfunctor  $F' \subseteq F$  is called *C-determined* if for any subfunctor  $F'' \subseteq F$

$$F'' \subseteq F' \iff F''|_{\mathbf{C}} \subseteq F'|_{\mathbf{C}}.$$

For a subfunctor  $H \subseteq F|_{\mathbf{C}}$  define a subfunctor  $F_H \subseteq F$  by

$$(1.1) \quad F_H(X) = \bigcap_{\substack{\alpha: C \rightarrow X \\ C \in \mathbf{C}}} F(\alpha)^{-1}(H(C)) \quad \text{for } X \in \mathbf{A}.$$

**Lemma 1.6** (Auslander [2, Proposition I.1.2]). *The subfunctor  $F_H \subseteq F$  is C-determined and  $F_H|_{\mathbf{C}} = H$ .*

*Proof.* Let  $F' \subseteq F$  be a subfunctor and  $F'|_{\mathbf{C}} \subseteq H$ . This implies for each morphism  $\alpha: C \rightarrow X$  with  $C \in \mathbf{C}$  that  $F'(X) \subseteq F(\alpha)^{-1}(H(C))$ . Thus  $F' \subseteq F_H$ .  $\square$

*Remark 1.7.* Let  $F \in (\mathbf{A}^{\text{op}}, \mathbf{Ab})$ . Viewing a poset as a category, the map

$$\text{sub}(F|_{\mathbf{C}}) \longrightarrow \text{sub}(F), \quad H \mapsto F_H$$

is right adjoint to

$$\text{sub}(F) \longrightarrow \text{sub}(F|_{\mathbf{C}}), \quad G \mapsto G|_{\mathbf{C}}.$$

We include an explicit example.

**Example 1.8.** Let  $\Lambda$  be a ring and consider the forgetful functor  $F: \text{mod } \Lambda \rightarrow \mathbf{Ab}$  which takes a module to its underlying abelian group.

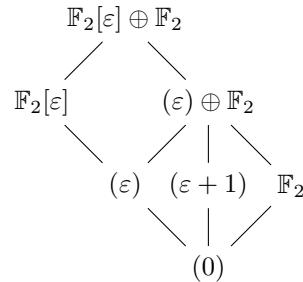
(1) For  $C \in \text{mod } \Lambda$  and an  $\text{End}_{\Lambda}(C)$ -submodule  $H \subseteq C$ , the  $C$ -determined subfunctor  $F_H$  is given by

$$F_H(X) = \bigcap_{\alpha: X \rightarrow C} \{x \in X \mid \alpha(x) \in H\} \quad \text{for } X \in \text{mod } \Lambda.$$

(2) For an Artin algebra  $\Lambda$ , let  $\text{sub}_{\text{fg}}(F)$  denote the poset of finitely generated subfunctors of  $F$ . Then the assignment  $H \mapsto F_H$  induces an isomorphism

$$\text{colim}_{C \in \text{mod } \Lambda} \text{sub}(C) \xrightarrow{\sim} \text{sub}_{\text{fg}}(F).$$

(3) Consider the algebra  $\Lambda = \mathbb{F}_2[\varepsilon]$  of dual numbers ( $\varepsilon^2 = 0$ ) and the  $\Lambda$ -module  $C = \mathbb{F}_2[\varepsilon] \oplus \mathbb{F}_2$ . Then the poset  $\text{sub}(C)$  is isomorphic to  $\text{sub}_{\text{fg}}(F)$  and its Hasse diagram is the following.



**Flat functors and flat covers.** Let  $\mathcal{C}$  be an essentially small additive category. We consider the category  $(\mathcal{C}^{\text{op}}, \mathbf{Ab})$  of additive functors  $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Ab}$ . Recall that  $F$  is *flat* if it can be written as a filtered colimit of representable functors.

The following result establishes a connection between locally finitely presented additive categories and categories of flat functors.

**Theorem 1.9** (Crawley-Boevey [7, §1.4]). *Let  $\mathbf{A}$  be a locally finitely presented category. Then the Yoneda functor*

$$h: \mathbf{A} \longrightarrow ((\mathbf{fp} \mathbf{A})^{\text{op}}, \mathbf{Ab}), \quad X \mapsto \text{Hom}_{\mathbf{A}}(-, X)|_{\mathbf{fp} \mathbf{A}}$$

*identifies  $\mathbf{A}$  with the full subcategory of flat functors  $(\mathbf{fp} \mathbf{A})^{\text{op}} \rightarrow \mathbf{Ab}$ .*  $\square$

A morphism  $\pi: F \rightarrow G$  in  $(\mathcal{C}^{\text{op}}, \mathbf{Ab})$  is a *flat cover* of  $G$  if the following holds:

- (1)  $F$  is flat and every morphism  $F' \rightarrow G$  with  $F'$  flat factors through  $\pi$ .
- (2)  $\pi$  is right minimal.

A *minimal flat presentation* of  $G$  is an exact sequence

$$F_1 \longrightarrow F_0 \xrightarrow{\pi} G \longrightarrow 0$$

such that  $F_0 \rightarrow G$  and  $F_1 \rightarrow \text{Ker } \pi$  are flat covers. A *projective cover* and a *minimal projective presentation* are defined analogously, replacing the term flat by projective.<sup>3</sup>

**Theorem 1.10** (Bican–El Bashir–Enochs [6]). *Every additive functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Ab}$  admits a flat cover.*  $\square$

**Proof of the main theorem.** We are ready to prove Theorem 1.1. The basic idea is to identify  $\mathbf{A}$  with the category of flat functors  $(\mathbf{fp} \mathbf{A})^{\text{op}} \rightarrow \mathbf{Ab}$  and to employ the existence of flat covers.

*Proof of Theorem 1.1.* We apply Theorems 1.9 and 1.10. Consider the subfunctor  $H \subseteq h(Y)|_{\mathcal{C}}$  and choose a flat cover  $\pi: h(X) \rightarrow h(Y)_H$  in  $((\mathbf{fp} \mathbf{A})^{\text{op}}, \mathbf{Ab})$ . The composite  $h(X) \rightarrow h(Y)_H \rightarrow h(Y)$  is of the form  $h(\alpha)$  for some morphism  $\alpha: X \rightarrow Y$  in  $\mathbf{A}$ . We check the properties of  $\alpha$  and apply Lemma 1.6. Thus  $\text{Im Hom}_{\mathbf{A}}(\mathcal{C}, \alpha) = h(Y)_H|_{\mathcal{C}} = H$ . For a morphism  $\alpha': X' \rightarrow Y$  in  $\mathbf{A}$  with  $\text{Im Hom}_{\mathbf{A}}(\mathcal{C}, \alpha') \subseteq H$ , we have  $\text{Im } h(\alpha')|_{\mathcal{C}} \subseteq H$  and therefore  $\text{Im } h(\alpha') \subseteq h(Y)_H = \text{Im } h(\alpha)$ . Using that  $\pi$  is a flat cover, we obtain a morphism  $\phi: X' \rightarrow X$  in  $\mathbf{A}$  satisfying  $\alpha\phi = \alpha'$ . The fact that  $\pi$  is right minimal implies that  $\alpha$  is right minimal.  $\square$

**Corollary 1.11.** *Let  $\alpha: X \rightarrow Y$  be a morphism which is right determined by a set of finitely presented objects. Then there is an essentially unique decomposition  $X = X' \oplus X''$  such that  $\alpha|_{X'}$  is right minimal and  $\alpha|_{X''} = 0$ .*

*Proof.* Suppose that  $\alpha$  is right  $\mathcal{C}$ -determined and let  $H = \text{Im Hom}_{\mathbf{A}}(\mathcal{C}, \alpha)$ . Then there exists a right minimal and right  $\mathcal{C}$ -determined morphism  $\alpha': X' \rightarrow Y$  with  $\text{Im Hom}_{\mathbf{A}}(\mathcal{C}, \alpha') = H$  by Theorem 1.1. The minimality of  $\alpha'$  yields a decomposition  $X = X' \oplus X''$  such that  $\alpha|_{X'} = \alpha'$  and  $\alpha|_{X''} = 0$ .  $\square$

<sup>3</sup>This definition of a projective cover is equivalent to the usual one which requires the kernel to be superfluous.

**Weak kernels.** Let  $\mathbf{A}$  be an additive category. A morphism  $X \rightarrow Y$  is a *weak kernel* of a morphism  $Y \rightarrow Z$  in  $\mathbf{A}$  if the induced sequence

$$\mathrm{Hom}_{\mathbf{A}}(-, X) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(-, Y) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(-, Z)$$

is exact. A weak kernel is called *minimal* if it is a right minimal morphism. Note that a minimal weak kernel is unique up to a non-unique isomorphism; it is a kernel if a kernel exists.

**Proposition 1.12.** *In a locally finitely presented additive category every morphism admits a minimal weak kernel.*

*Proof.* We apply Theorems 1.9 and 1.10. Fix a morphism  $\beta: Y \rightarrow Z$  and choose a flat cover  $h(X) \rightarrow \mathrm{Ker} h(\beta)$  in  $((\mathrm{fp} \mathbf{A})^{\mathrm{op}}, \mathbf{Ab})$ . The composite  $h(X) \rightarrow \mathrm{Ker} h(\beta) \rightarrow h(Y)$  is of the form  $h(\alpha)$  for some morphism  $\alpha: X \rightarrow Y$  in  $\mathbf{A}$ , which is a minimal weak kernel for  $\beta$ .  $\square$

**Morphisms determined by finitely presented objects.** Let  $\mathbf{A}$  be a locally finitely presented additive category. Recall that a sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  of morphisms in  $\mathbf{A}$  is *pure-exact* if for each  $C \in \mathrm{fp} \mathbf{A}$  the induced sequence

$$0 \rightarrow \mathrm{Hom}_{\mathbf{A}}(C, X) \rightarrow \mathrm{Hom}_{\mathbf{A}}(C, Y) \rightarrow \mathrm{Hom}_{\mathbf{A}}(C, Z) \rightarrow 0$$

is exact. An object  $X$  in  $\mathbf{A}$  is *pure-injective* if every pure-exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is split exact.

The following proposition characterises the morphisms which are right determined by a set of finitely presented objects. The relevance of pure-injectives was noticed by Auslander in some special cases [2, §I.10].

**Proposition 1.13.** *Let  $\mathbf{A}$  be a locally finitely presented additive category.*

- (1) *If a morphism in  $\mathbf{A}$  is right minimal and right determined by a set of finitely presented objects in  $\mathbf{A}$ , then its minimal weak kernel is pure-injective.*
- (2) *If a morphism in  $\mathbf{A}$  has a pure-injective kernel, then it is right determined by a set of finitely presented objects in  $\mathbf{A}$ .*

*Proof.* We set  $\mathbf{C} = \mathrm{fp} \mathbf{A}$  and recall that a functor  $F \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Ab})$  is *cotorsion* if  $\mathrm{Ext}^1(E, F) = 0$  for each flat  $E \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Ab})$ . Clearly,  $X \in \mathbf{A}$  is pure-injective if and only if  $h(X)$  is cotorsion.

Observe that the kernel of a flat cover in  $(\mathbf{C}^{\mathrm{op}}, \mathbf{Ab})$  is cotorsion. Also, the flat cover of a cotorsion functor is cotorsion; see [6, §2] for details.

Now fix a sequence  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$  of morphisms such that  $\alpha$  is the minimal weak kernel of  $\beta$ . If  $\beta$  is right minimal and right determined by any set of finitely presented objects, then  $h(Y) \rightarrow \mathrm{Im} h(\beta)$  is a flat cover; this follows from the proof of Theorem 1.1. Thus  $\mathrm{Im} h(\alpha) = \mathrm{Ker} h(\beta)$  is cotorsion. The proof of Proposition 1.12 shows that  $h(X) \rightarrow \mathrm{Im} h(\alpha)$  is a flat cover. Thus  $h(X)$  is cotorsion and therefore  $X$  is pure-injective. Conversely, if  $\alpha$  is a kernel of  $\beta$  and  $X$  is pure-injective, then every morphism  $h(Y') \rightarrow \mathrm{Im} h(\beta)$  factors through  $h(Y) \rightarrow \mathrm{Im} h(\beta)$ . Thus every morphism  $\beta': Y' \rightarrow Z$  with  $\mathrm{Im} \mathrm{Hom}_{\mathbf{A}}(\mathbf{C}, \beta') \subseteq \mathrm{Im} \mathrm{Hom}_{\mathbf{A}}(\mathbf{C}, \beta)$  factors through  $\beta$ . This means that  $\beta$  is right  $\mathbf{C}$ -determined.  $\square$

In view of Corollary 1.11 we obtain the following consequence.

**Corollary 1.14.** *Let  $\alpha: X \rightarrow Y$  be a morphism which admits a kernel. Then  $\alpha$  is right determined by a set of finitely presented objects if and only if there is a decomposition  $X = X' \oplus X''$  such that  $\mathrm{Ker} \alpha|_{X'}$  is pure-injective and  $\alpha|_{X''} = 0$ .  $\square$*

## 2. FLAT VERSUS PROJECTIVE COVERS

In this section we show how flat covers are turned into projective covers. In fact, we prove a general result about locally finitely presented additive categories which is tantamount to the the existence of flat covers in functor categories.

**Finitely presented functors.** Let  $\mathbf{A}$  be an additive category. We denote by  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  the category of finitely presented functors  $F: \mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$ . Recall that  $F$  is *finitely presented* (or *coherent*) if it fits into an exact sequence

$$\text{Hom}_{\mathbf{A}}(-, X) \longrightarrow \text{Hom}_{\mathbf{A}}(-, Y) \longrightarrow F \longrightarrow 0.$$

We call such a presentation *minimal* if the morphisms  $\text{Hom}_{\mathbf{A}}(-, Y) \rightarrow F$  and  $\text{Hom}_{\mathbf{A}}(-, X) \rightarrow \text{Hom}_{\mathbf{A}}(-, Y)$  are right minimal.

The following lemma is well-known and easily proved (using Yoneda's lemma).

**Lemma 2.1.** *The category  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  is abelian if and only if  $\mathbf{A}$  admits weak kernels. In this case the Yoneda embedding  $X \mapsto \text{Hom}_{\mathbf{A}}(-, X)$  identifies the idempotent completion of  $\mathbf{A}$  with the category of projective objects in  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$ .  $\square$*

If  $\mathbf{A}$  is locally finitely presented then we consider the *evaluation functor*

$$(2.1) \quad \mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab}) \longrightarrow ((\mathbf{fp} \mathbf{A})^{\text{op}}, \mathbf{Ab}), \quad F \mapsto F|_{\mathbf{fp} \mathbf{A}}.$$

The following theorem establishes a right adjoint which identifies flat covers with projective covers.

**Theorem 2.2.** *Let  $\mathbf{A}$  be a locally finitely presented category.*

- (1) *The category  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  of finitely presented functors  $\mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$  is abelian.*
- (2) *For an additive functor  $F: (\mathbf{fp} \mathbf{A})^{\text{op}} \rightarrow \mathbf{Ab}$ , the unique functor  $\tilde{F}: \mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$  extending  $F$  and preserving filtered colimits in  $\mathbf{A}$  is finitely presented and admits a minimal projective presentation in  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$ .*
- (3) *The assignment  $F \mapsto \tilde{F}$  provides a fully faithful right adjoint to the evaluation functor (2.1).*

We postpone the proof and illustrate the theorem by a couple of examples. The first one shows how flat covers of modules over a ring are derived from this result.

**Example 2.3.** Let  $\Lambda$  be a ring and denote by  $\mathbf{A}$  the category of flat  $\Lambda$ -modules. Then  $\mathbf{fp} \mathbf{A}$  equals the category of finitely generated projective  $\Lambda$ -modules and evaluation at  $\Lambda$  yields an equivalence  $((\mathbf{fp} \mathbf{A})^{\text{op}}, \mathbf{Ab}) \xrightarrow{\sim} \text{Mod } \Lambda$  (which we view as identification). For a  $\Lambda$ -module  $Y$ , the theorem yields a projective cover  $\text{Hom}_{\mathbf{A}}(-, X) \rightarrow \tilde{Y}$ . Evaluation at  $\Lambda$  then gives a flat cover  $X \rightarrow Y$ .

**Example 2.4.** Not all functors in  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  admit a projective cover. For instance, take  $\mathbf{A} = \text{Mod } \mathbb{Z}$  and consider the canonical morphism  $\mathbb{Z} \rightarrow \mathbb{Z}/p$  for any prime  $p$ . Then the image of the induced morphism  $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/p)$  admits no projective cover, because a projective cover  $\text{Hom}_{\mathbb{Z}}(-, X) \rightarrow F$  would give a projective cover  $X \rightarrow \mathbb{Z}/p$  in  $\mathbf{A}$  (which is known not to exist).

*Proof of Theorem 2.2.* The category  $\mathbf{A}$  has weak kernels by Proposition 1.12. Thus  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  is abelian by Lemma 2.1.

Now fix  $F \in ((\mathbf{fp} \mathbf{A})^{\text{op}}, \mathbf{Ab})$  and observe that  $\tilde{F} \cong \text{Hom}(h-, F)$  since  $h$  preserves filtered colimits. This yields for  $X \in \mathbf{A}$  a functorial isomorphism

$$(2.2) \quad \text{Hom}(\text{Hom}_{\mathbf{A}}(-, X), \tilde{F}) \cong \tilde{F}(X) \cong \text{Hom}(\text{Hom}_{\mathbf{A}}(-, X)|_{\mathbf{fp} \mathbf{A}}, F)$$

which extends to an isomorphism

$$\text{Hom}(E, \tilde{F}) \cong \text{Hom}(E|_{\mathbf{fp} \mathbf{A}}, F)$$

for all  $E \in \mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$ . The adjointness property of the assignment  $F \mapsto \tilde{F}$  then follows. Also, the functor is fully faithful since  $\tilde{F}|_{\mathbf{fp}\mathbf{C}} = F$ , and it identifies the flat functors in  $((\mathbf{fp}\mathbf{A})^{\text{op}}, \mathbf{Ab})$  with the projective objects in  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$ .

Next we show that  $\tilde{F}$  is finitely presented. In fact, we obtain a minimal projective presentation of  $\tilde{F}$  in  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  by choosing a minimal flat presentation

$$h(X) \longrightarrow h(Y) \xrightarrow{\pi} F \longrightarrow 0$$

in  $((\mathbf{fp}\mathbf{A})^{\text{op}}, \mathbf{Ab})$ ; see Theorem 1.10. From (2.2) it follows that the corresponding sequence

$$\text{Hom}_{\mathbf{A}}(-, X) \longrightarrow \text{Hom}_{\mathbf{A}}(-, Y) \xrightarrow{\tilde{\pi}} \tilde{F} \longrightarrow 0$$

is a minimal projective presentation.  $\square$

**Corollary 2.5.** *The evaluation functor (2.1) is exact and admits both adjoints; it induces an equivalence*

$$\frac{\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})}{\{F \mid F|_{\mathbf{fp}\mathbf{A}} = 0\}} \xrightarrow{\sim} ((\mathbf{fp}\mathbf{A})^{\text{op}}, \mathbf{Ab}).$$

*Proof.* The right adjoint exists by Theorem 2.2, and the left adjoint is the unique colimit preserving functor sending  $\text{Hom}_{\mathbf{A}}(-, X)$  to  $\text{Hom}_{\mathbf{A}}(-, X)$  for all  $X \in \mathbf{fp}\mathbf{A}$ .

Let  $\mathbf{S}$  denote the kernel of the evaluation functor. A quasi-inverse for the functor  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})/\mathbf{S} \rightarrow ((\mathbf{fp}\mathbf{A})^{\text{op}}, \mathbf{Ab})$  is obtained by composing the left (or right) adjoint of the evaluation functor with the quotient functor  $\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab}) \rightarrow \mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})/\mathbf{S}$ .  $\square$

**Auslander's formula.** Let  $\mathbf{A}$  be an abelian category. A somewhat hidden result in Auslander's account on coherent functors [1, p. 205] shows that the Yoneda functor  $\mathbf{A} \rightarrow \mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})$  admits an exact left adjoint which sends a representable functor  $\text{Hom}_{\mathbf{A}}(-, X)$  to  $X$  and yields *Auslander's formula* [13]

$$\frac{\mathbf{Fp}(\mathbf{A}^{\text{op}}, \mathbf{Ab})}{\{F \mid F \text{ is exactly presented}\}} \xrightarrow{\sim} \mathbf{A}.$$

Here, a functor  $F: \mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$  is *exactly presented* if it fits into an exact sequence

$$(2.3) \quad 0 \rightarrow \text{Hom}_{\mathbf{A}}(-, X) \rightarrow \text{Hom}_{\mathbf{A}}(-, Y) \rightarrow \text{Hom}_{\mathbf{A}}(-, Z) \rightarrow F \rightarrow 0$$

such that the corresponding sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\mathbf{A}$  is exact.

One may think of Auslander's formula as a prototype for Corollary 2.5. In particular, we see that the kernel of the evaluation functor is given by the functors  $F: \mathbf{A}^{\text{op}} \rightarrow \mathbf{Ab}$  with a presentation (2.3) such that the corresponding sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\mathbf{A}$  is pure-exact.

### 3. ALMOST SPLIT SEQUENCES

The concept of a morphism determined by an object generalises that of an almost split morphism. In fact, a morphism  $\alpha: X \rightarrow Y$  in an additive category  $\mathbf{A}$  is *right almost split* (that is,  $\alpha$  is not a retraction and every morphism  $X' \rightarrow Y$  that is not a retraction factors through  $\alpha$ ) if and only if  $\Gamma = \text{End}_{\mathbf{A}}(Y)$  is a local ring,  $\alpha$  is right determined by  $Y$ , and  $\text{Im Hom}_{\mathbf{A}}(Y, \alpha) = \text{rad } \Gamma$  [2, §II.2].

Now let  $\mathbf{A}$  be an abelian category. Recall that an exact sequence  $0 \rightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Y \rightarrow 0$  is *almost split* if the morphism  $\alpha$  is left almost split and  $\beta$  is right almost split. This is equivalent to  $\beta$  being a right minimal and right almost split morphism [2, §II.4].

The construction of morphisms in Theorem 1.1 requires the determining objects to be finitely presented. The following proposition shows that this assumption is

necessary. A similar non-existence result for finite dimensional representations of infinite quivers is due to Paquette [14].

**Proposition 3.1.** *The category of modules over  $\mathbb{Z}$  does not admit an almost split sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  for  $Z = \mathbb{Q}$ .*

Thus the category  $\text{Mod } \mathbb{Z}$  does not admit a right minimal and right determined morphism for the triple  $(C, H, Y) = (\mathbb{Q}, 0, \mathbb{Q})$ .

The end terms of an almost split sequence determine each other, and this correspondence enjoys some weak functoriality. The proof of Proposition 3.1 uses this argument, and I am grateful to Helmut Lenzing for suggesting it.

**Lemma 3.2** ([12, Lemma A.10]). *Let  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an almost split sequence in any abelian category  $\mathbf{A}$ . Then there is an isomorphism*

$$\text{End}_{\mathbf{A}}(X)/\text{rad } \text{End}_{\mathbf{A}}(X) \cong \text{End}_{\mathbf{A}}(Z)/\text{rad } \text{End}_{\mathbf{A}}(Z). \quad \square$$

*Proof of Proposition 3.1.* Suppose there exists an almost split sequence  $0 \rightarrow X \rightarrow Y \rightarrow \mathbb{Q} \rightarrow 0$  in  $\text{Mod } \mathbb{Z}$ . Then the lemma implies

$$\text{End}_{\mathbb{Z}}(X)/\text{rad } \text{End}_{\mathbb{Z}}(X) \cong \mathbb{Q}.$$

It follows that  $X$  is divisible since multiplication with any non-zero integer induces an endomorphism of  $X$  which is invertible. Thus the sequence splits which is impossible.  $\square$

Very little seems to be known about the non-existence of almost split sequences. We conjecture the following.

**Conjecture 3.3.** *Given an almost split sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in any module category, the module  $Z$  is finitely presented.*

#### APPENDIX A. FUNCTORS ON MODULE CATEGORIES

The proof of Theorem 1.1 uses the existence of flat covers. In this appendix we provide an elementary argument for the case of a module category  $\mathbf{A} = \text{Mod } \Lambda$ . We proceed in three steps.

**Minimal presentations.** The existence of flat covers in  $((\text{mod } \Lambda)^{\text{op}}, \text{Ab})$  can be phrased as follows.

**Theorem A.1.** *Let  $\Lambda$  be a ring. An additive functor  $F: (\text{Mod } \Lambda)^{\text{op}} \rightarrow \text{Ab}$  preserving filtered colimits in  $\text{Mod } \Lambda$  admits a minimal presentation*

$$\text{Hom}_{\Lambda}(-, X) \longrightarrow \text{Hom}_{\Lambda}(-, Y) \longrightarrow F \longrightarrow 0.$$

*Remark A.2.* (1) The evaluation functor  $\text{Fp}((\text{Mod } \Lambda)^{\text{op}}, \text{Ab}) \rightarrow ((\text{mod } \Lambda)^{\text{op}}, \text{Ab})$  sends a minimal projective presentation to a minimal flat presentation.

(2) The theorem complements a result of Crawley-Boevey [8]: An additive functor  $F: \text{Mod } \Lambda \rightarrow \text{Ab}$  preserves filtered colimits and products if and only if it admits a presentation

$$\text{Hom}_{\Lambda}(Y, -) \longrightarrow \text{Hom}_{\Lambda}(X, -) \longrightarrow F \longrightarrow 0$$

with  $X, Y \in \text{mod } \Lambda$ .

*Proof.* Any  $\Lambda$ -module can be written as a filtered colimit of finitely presented modules. It follows that  $F$  is determined by its restriction  $F|_{\text{mod } \Lambda}$ . The functors

$$I_C = \text{Hom}_{\mathbb{Z}}(\text{Hom}_{\Lambda}(C, -), \mathbb{Q}/\mathbb{Z})$$

with  $C \in \mathbf{mod} \Lambda$  form a set of injective cogenerators for  $((\mathbf{mod} \Lambda)^{\mathrm{op}}, \mathbf{Ab})$  and this yields an injective copresentation

$$0 \longrightarrow F|_{\mathbf{mod} \Lambda} \longrightarrow \prod_i I_{C_i} \longrightarrow \prod_j I_{C_j}$$

with  $C_i, C_j \in \mathbf{mod} \Lambda$ . Thus we obtain an exact sequence

$$0 \longrightarrow F \longrightarrow \prod_i I_{C_i} \longrightarrow \prod_j I_{C_j}$$

in  $((\mathbf{Mod} \Lambda)^{\mathrm{op}}, \mathbf{Ab})$ . It suffices to show for each  $C \in \mathbf{mod} \Lambda$  that the functor  $I_C$  is finitely presented. Then one uses that the finitely presented functors are closed under taking products and kernels. Choose a presentation

$$P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0$$

such that each  $P_i$  is finitely generated projective. This yields a presentation

$$I_{P_1} \longrightarrow I_{P_0} \longrightarrow I_C \longrightarrow 0.$$

We have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\Lambda}(P_i, -), \mathbb{Q}/\mathbb{Z}) &\cong \mathrm{Hom}_{\mathbb{Z}}(- \otimes_{\Lambda} P_i^*, \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\Lambda}(-, \mathrm{Hom}_{\mathbb{Z}}(P_i^*, \mathbb{Q}/\mathbb{Z})) \end{aligned}$$

where  $P_i^* = \mathrm{Hom}_{\Lambda}(P_i, \Lambda)$ . Thus  $I_C$  is finitely presented.

The morphisms  $\mathrm{Hom}_{\Lambda}(-, Y) \rightarrow F$  and  $\mathrm{Hom}_{\Lambda}(-, X) \rightarrow \mathrm{Hom}_{\Lambda}(-, Y)$  can be chosen to be right minimal; this follows from a standard argument [9, §7].  $\square$

**Coinduction.** Let  $\mathbf{A}$  be an additive category. For a full subcategory  $\mathbf{C} \subseteq \mathbf{A}$  consider the *evaluation functor*

$$\mathrm{ev}_{\mathbf{C}}: (\mathbf{A}^{\mathrm{op}}, \mathbf{Ab}) \longrightarrow (\mathbf{C}^{\mathrm{op}}, \mathbf{Ab}), \quad F \mapsto F|_{\mathbf{C}}$$

and its right adjoint, the *coinduction functor*

$$\mathrm{coind}_{\mathbf{C}}: (\mathbf{C}^{\mathrm{op}}, \mathbf{Ab}) \longrightarrow (\mathbf{A}^{\mathrm{op}}, \mathbf{Ab})$$

given by

$$\mathrm{coind}_{\mathbf{C}} I(X) = \mathrm{Hom}(\mathrm{Hom}_{\mathbf{A}}(-, X)|_{\mathbf{C}}, I) \quad \text{for } I \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Ab}), X \in \mathbf{A}.$$

For  $F \in (\mathbf{A}^{\mathrm{op}}, \mathbf{Ab})$  and  $I \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Ab})$ , the isomorphism

$$(A.1) \quad \mathrm{Hom}(F, \mathrm{coind}_{\mathbf{C}} I) \xrightarrow{\sim} \mathrm{Hom}(F|_{\mathbf{C}}, I), \quad \eta \mapsto \bar{\eta}$$

is given by  $\bar{\eta} = \eta|_{\mathbf{C}}$ , where we identify  $(\mathrm{coind}_{\mathbf{C}} I)|_{\mathbf{C}} \xrightarrow{\sim} I$ .

The coinduction functor assists in understanding the assignment (1.1)

$$F|_{\mathbf{C}} \supseteq H \longmapsto F_H \subseteq F$$

for an additive functor  $F: \mathbf{A}^{\mathrm{op}} \rightarrow \mathbf{Ab}$ .

**Lemma A.3.** *Let  $F: \mathbf{A}^{\mathrm{op}} \rightarrow \mathbf{Ab}$  be an additive functor and  $H \subseteq F|_{\mathbf{C}}$  a subfunctor. For a morphism  $\bar{\eta}: F|_{\mathbf{C}} \rightarrow I$  with kernel  $H$  we have  $F_H = \mathrm{Ker} \eta$ , where  $\eta$  and  $\bar{\eta}$  are related via (A.1).*

*Proof.* Let  $F' \subseteq F$  be a subfunctor. The isomorphism (A.1) is functorial and the inclusion  $F' \rightarrow F$  gives  $\eta|_{F'} = 0$  iff  $\bar{\eta}|_{F'|_{\mathbf{C}}} = 0$ . Thus  $F' \subseteq \mathrm{Ker} \eta$  iff  $F'|_{\mathbf{C}} \subseteq H$ . This means that  $\mathrm{Ker} \eta$  is  $\mathbf{C}$ -determined, and  $\mathrm{Ker} \eta|_{\mathbf{C}} = H$  by construction.  $\square$

**Morphisms determined by objects.** The following lemma provides the connection between morphisms and functors determined by objects.

**Lemma A.4.** *Let  $\mathbf{A}$  be an additive category and  $\mathbf{C} \subseteq \mathbf{A}$ . A morphism  $\alpha: X \rightarrow Y$  in  $\mathbf{A}$  is right  $\mathbf{C}$ -determined if and only if the subfunctor*

$$\mathrm{Im} \mathrm{Hom}_{\mathbf{A}}(-, \alpha) \subseteq \mathrm{Hom}_{\mathbf{A}}(-, Y)$$

*is  $\mathbf{C}$ -determined.*

*Proof.* This is clear because a morphism  $\alpha': X' \rightarrow Y$  factors through  $\alpha$  iff

$$\mathrm{Im} \mathrm{Hom}_{\mathbf{A}}(-, \alpha') \subseteq \mathrm{Im} \mathrm{Hom}_{\mathbf{A}}(-, \alpha). \quad \square$$

We are now ready for a second proof of Theorem 1.1 for  $\mathbf{A} = \mathrm{Mod} \Lambda$ .

*Proof of Theorem 1.1.* Set  $F = \mathrm{Hom}_{\mathbf{A}}(-, Y)$ . The subfunctor  $F_H$  arises as a kernel of a morphism  $F \rightarrow \mathrm{coind}_{\mathbf{C}} I$  for some  $I \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Ab})$  by Lemma A.3. The functor  $\mathrm{coind}_{\mathbf{C}} I$  preserves filtered colimits in  $\mathbf{A}$  since  $\mathbf{C}$  consists of finitely presented objects. Also  $F$  preserves filtered colimits in  $\mathbf{A}$ , and therefore  $F_H$  has this property. Thus  $F_H$  admits a projective cover  $\pi: \mathrm{Hom}_{\mathbf{A}}(-, X) \rightarrow F_H$  by Theorem A.1. The composite  $\mathrm{Hom}_{\mathbf{A}}(-, X) \rightarrow F_H \rightarrow F$  is represented by a morphism  $\alpha: X \rightarrow Y$  which is right  $\mathbf{C}$ -determined by Lemma A.4, and right minimal since  $\pi$  is a projective cover.  $\square$

## APPENDIX B. AUSLANDER VARIETIES

In [15, §6], Ringel pointed out the geometric nature of the correspondence between submodules of  $\mathrm{Hom}_{\mathbf{A}}(\mathbf{C}, Y)$  and right  $\mathbf{C}$ -determined morphisms ending at an object  $Y$  (see Theorem 1.1). More precisely, he defines under suitable assumptions for a dimension vector  $\mathbf{d}$  the *Auslander variety*

$$\mathrm{Gr}_{\mathbf{d}}(\mathrm{Hom}_{\mathbf{A}}(\mathbf{C}, Y))$$

as an algebraic variety given by all submodules of  $\mathrm{Hom}_{\mathbf{A}}(\mathbf{C}, Y)$  with dimension vector  $\mathbf{d}$ ; it parametrizes right  $\mathbf{C}$ -determined morphisms ending at  $Y$ . We illustrate this by giving an example.

Let  $k$  be a field and fix a projective variety  $\mathbb{X} = V(f_1, \dots, f_r)$  given by homogeneous polynomials  $f_i \in k[x_0, \dots, x_n]$  of degree at most  $p$ . This variety can be realised as a *quiver Grassmannian* [10, §2], and we follow the exposition in [11]. Let  $\Lambda = \Lambda_{n,p}$  denote the *Beilinson algebra* given by the path algebra of the following quiver

$$p \begin{array}{c} \xrightarrow{x_0} \\ \cdots \\ \xrightarrow{x_n} \end{array} p-1 \quad \cdots \quad 2 \begin{array}{c} \xrightarrow{x_0} \\ \cdots \\ \xrightarrow{x_n} \end{array} 1 \begin{array}{c} \xrightarrow{x_0} \\ \cdots \\ \xrightarrow{x_n} \end{array} 0$$

modulo all relations of the form  $x_i x_j - x_j x_i$ . Each homogeneous polynomial  $f \in k[x_0, \dots, x_n]$  of degree  $d$  yields  $p - d + 1$  elements of  $\Lambda$  represented by paths ending at vertices  $0, 1, \dots, p - d$ , and we denote by  $\langle f_1, \dots, f_r \rangle$  the ideal of  $\Lambda$  generated by all occurrences of each  $f_i$ . Consider the indecomposable injective  $\Lambda/\langle f_1, \dots, f_r \rangle$ -module  $I(0)$  corresponding to the vertex 0, but viewed as  $\Lambda$ -module. Then  $\mathbb{X}$  is isomorphic to the variety of subrepresentations of  $I(0)$  with dimension vector  $\mathbf{d} = (1, \dots, 1)$ . It follows that

$$\mathbb{X} \cong \mathrm{Gr}_{\mathbf{d}}(\mathrm{Hom}_{\mathbf{A}}(\mathbf{C}, Y))$$

for  $\mathbf{A} = \mathrm{Mod} \Lambda$ ,  $\mathbf{C} = \Lambda$ , and  $Y = I(0)$ .

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