THE KRULL–GABRIEL DIMENSION OF DISCRETE DERIVED CATEGORIES

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Abstract. We compute the Krull–Gabriel dimension of the category of perfect complexes for finite dimensional algebras which are derived discrete.

Introduction

Let k be an algebraically closed field and Λ a finite dimensional k-algebra. We denote by $\operatorname{mod}\Lambda$ the category of finitely presented Λ-modules and by $\operatorname{proj}\Lambda$ the full subcategory of finitely generated projective Λ-modules.

The Krull–Gabriel dimension of the representation theory of Λ is an invariant first studied by Geigle [11]. For this invariant one considers the abelian category $\mathcal{C} = \operatorname{Ab}(\operatorname{mod}\Lambda)$ of finitely presented functors $\operatorname{mod}\Lambda \to \mathrm{Ab}$ into the category of abelian groups. The Krull–Gabriel dimension $\operatorname{KGdim}\mathcal{C}$ of $\mathcal{C}$ is by definition the smallest integer n such that $\mathcal{C}$ admits a filtration by Serre subcategories

$$0 = \mathcal{C}_{-1} \subseteq \mathcal{C}_0 \subseteq \ldots \subseteq \mathcal{C}_n = \mathcal{C},$$

where $\mathcal{C}_i/\mathcal{C}_{i-1}$ is the full subcategory of all objects of finite length in $\mathcal{C}/\mathcal{C}_{i-1}$.

We have $\operatorname{KGdim}\mathcal{C} = 0$ if and only if Λ is of finite representation type by a classical result of Auslander [1], and $\operatorname{KGdim}\mathcal{C} \neq 1$ by a result of Herzog [14] and Krause [16]. In his thesis [11], Geigle proved that $\operatorname{KGdim}\mathcal{C} = 2$, when Λ is tame hereditary.

In this work we investigate the category of perfect complexes which is by definition the bounded derived category $\mathcal{D}^b(\operatorname{proj}\Lambda)$. We compute the Krull–Gabriel dimension of the ablian category $\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda))$, when Λ is derived discrete in the sense of Vossieck [19]. The main result is the following.

Main Theorem. *Let Λ be a finite dimensional k-algebra.*

(1) *If Λ is derived discrete and piecewise hereditary, then*

$$\operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda)) = 0.$$

(2) *If Λ is derived discrete and not piecewise hereditary, then*

$$\operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda)) = \begin{cases} 1 & \text{if } \operatorname{gl.dim}\Lambda = \infty, \\ 2 & \text{if } \operatorname{gl.dim}\Lambda < \infty. \end{cases}$$

(3) *If Λ is not derived discrete, then*

$$\operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda)) \geq 2.$$

The rest of this note is devoted to proving this theorem. For an elementary description of the Krull–Gabriel dimension, see Proposition 2.2.

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Conventions. By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$, we denote the sets of integers, nonnegative integers, and positive integers, respectively. For $i, j \in \mathbb{Z}$, set

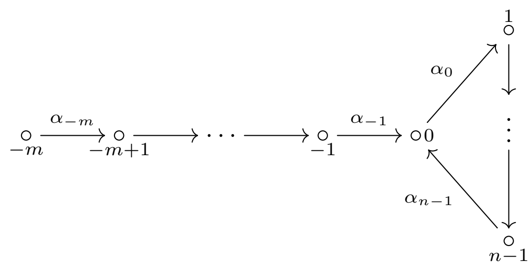
$$[i, j] := \{l \in \mathbb{Z} \mid i \leq l \leq j\}.$$

Furthermore, $[i, \infty) := \{l \in \mathbb{Z} \mid i \leq l\}$ and $(-\infty, j] := \{l \in \mathbb{Z} \mid l \leq j\}$.

1. Derived discrete algebras

Let Λ be a finite dimensional k-algebra. The algebra Λ is called *derived discrete* if for each sequence $(h_n)_{n\in\mathbb{Z}}$ of nonnegative integers there are only finitely many isomorphism classes of indecomposable objects X in $\mathcal{D}^b(\operatorname{proj}\Lambda)$ such that $\dim_k H^n(X) = h_n$ for each $n \in \mathbb{Z}$. Note that Vossieck's original definition [19] uses the category $\mathcal{D}^b(\operatorname{mod}\Lambda)$, but he has shown that both versions are equivalent. The one we use is more suitable for our setup.

In [19], it is shown that an algebra Λ is derived discrete if and only if either Λ is piecewise hereditary of Dynkin type or Λ is a one-cycle gentle algebra not satisfying the clock condition. Recall from [13] that Λ is *piecewise hereditary* if it is derived equivalent to a finite dimensional hereditary algebra. The class of one-cycle gentle algebras not satisfying the clock condition has been further studied in [4]. There it is shown that if Λ is a derived discrete algebra and not piecewise hereditary of Dynkin type, then Λ is derived equivalent to an algebra of the form $\Lambda(r, n, m)$, for some triple $(r, n, m) \in \Omega$. Here, Ω denotes the set of all triples (r, n, m) of nonnegative integers such that $1 \leq r \leq n$, and $\Lambda(r, n, m)$ is the path algebra of the quiver



bound by the relations

$$\alpha_{n-r+1}\alpha_{n-r}, \ldots, \alpha_{n-1}\alpha_{n-2}, \alpha_0\alpha_{n-1}.$$

Prototypical examples to have in mind are the algebra $\Lambda(1, 1, 0)$ which equals the algebra $k[\varepsilon]$ of dual numbers ($\varepsilon^2 = 0$), and its Auslander algebra $\Lambda(1, 2, 0)$. Note that $\operatorname{gl.dim}\Lambda(1, 1, 0) = \infty$ while $\operatorname{gl.dim}\Lambda(1, 2, 0) = 2$.

2. Krull–Gabriel dimension

Let $\mathcal{C}$ be an abelian category. A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called a *Serre subcategory* if it is closed under subobjects, quotients and extensions. If $\mathcal{C}' \subseteq \mathcal{C}$ is a Serre subcategory, then one defines the *quotient category* $\mathcal{C}/\mathcal{C}'$ as follows. The objects of $\mathcal{C}/\mathcal{C}'$ coincide with the objects of $\mathcal{C}$, and if X and Y are objects of $\mathcal{C}$, then

$$\operatorname{Hom}_{\mathcal{C}/\mathcal{C}'}(X, Y) := \varinjlim \operatorname{Hom}_{\mathcal{C}}(X', Y/Y'),$$

where X' and Y' run through all subobjects of X and Y, respectively, such that X/X' and Y' belong to $\mathcal{C}'$.

Following Gabriel [10, IV.1] and Geigle [11, §2], the *Krull–Gabriel dimension* $\operatorname{KGdim}\mathcal{C}$ of $\mathcal{C}$ is defined as follows. Let $\mathcal{C}_{-1} := 0$, and for each $n \in \mathbb{N}$ denote by $\mathcal{C}_n$ the full subcategory of all objects X in $\mathcal{C}$ which are of finite length, when viewed as objects of $\mathcal{C}/\mathcal{C}_{n-1}$. Then $\operatorname{KGdim}\mathcal{C}$ equals the smallest n such that $\mathcal{C}_n = \mathcal{C}$ (and ∞ when such n does not exist).

Let $\mathcal{T}$ be a triangulated category. Following Freyd [9, §3] and Verdier [18, II.3], we consider the *abelianisation* $\mathrm{Ab}(\mathcal{T})$ of $\mathcal{T}$ which is the abelian category of finitely presented functors $F\colon \mathcal{T} \to \mathrm{Ab}$ into the category Ab of abelian groups. Recall that a functor $F\colon \mathcal{T} \to \mathrm{Ab}$ is *finitely presented* (*finitely generated*, respectively) if there exists an exact sequence of the form $H_Y \to H_X \to F \to 0$ ($H_X \to F \to 0$, respectively). Here, for an object X in $\mathcal{T}$, we denote by H_X the representable functor $\mathrm{Hom}(X, -)\colon \mathcal{T} \to \mathrm{Ab}$. Similarly, if $f\colon X \to Y$ is a morphism in $\mathcal{T}$, then we denote by H_f the induced morphism $H_Y \to H_X$. The cohomological functor $\iota\colon \mathcal{T} \to \mathrm{Ab}(\mathcal{T})$ sending $X \in \mathcal{T}$ to H_X is universal in the following sense. If $\varphi\colon \mathcal{T} \to \mathcal{A}$ is a contravariant cohomological functor, then there exists a unique exact functor $\varphi'\colon \mathrm{Ab}(\mathcal{T}) \to \mathcal{A}$, such that $\varphi = \varphi' \circ \iota$.

Now let Λ be any ring. We wish to compute the Krull–Gabriel dimension of $\mathrm{Ab}(\mathcal{D}^b(\mathrm{proj}\,\Lambda))$ and begin with an elementary observation. To this end fix a modular lattice L. Denote by L' the quotient which is obtained by collapsing all finite length intervals in L. Set $L_{-1} = L$ and $L_n = (L_{n-1})'$ for $n \in \mathbb{N}$. The *dimension* of L is the smallest n such that $L_n = 0$.

Lemma 2.1 ([16, Lemma 1.1])**.** *Let $\mathcal{C}$ be an abelian category and X an object. For the lattice $L_{\mathcal{C}}(X)$ of subobjects we have $L_{\mathcal{C}}(X)_n \cong L_{\mathcal{C}/\mathcal{C}_n}(X)$ for all $n \in \mathbb{N}$.* □

This lemma suggests an alternative description of the Krull–Gabriel dimension which avoids the formation of quotient categories.

Proposition 2.2. *Let Λ be any ring.*

1. *The finitely generated subfunctors of the forgetful functor* $\mathrm{mod}\,\Lambda \to \mathrm{Ab}$ *form a modular lattice and its dimension equals* $\mathrm{KGdim}\,\mathrm{Ab}(\mathrm{mod}\,\Lambda)$.
2. *The finitely generated subfunctors of* $H^0\colon \mathcal{D}^b(\mathrm{proj}\,\Lambda) \to \mathrm{Ab}$ *form a modular lattice and its dimension equals* $\mathrm{KGdim}\,\mathrm{Ab}(\mathcal{D}^b(\mathrm{proj}\,\Lambda))$.

Proof. (1) The lattice of finitely generated subfunctors of $F = \mathrm{Hom}_\Lambda(\Lambda, -)$ equals the lattice of subobjects of F in $\mathrm{Ab}(\mathrm{mod}\,\Lambda)$. Given a Serre subcategory $\mathcal{C} \subseteq \mathrm{Ab}(\mathrm{mod}\,\Lambda)$, we have $F \in \mathcal{C}$ iff $\mathcal{C} = \mathrm{Ab}(\mathrm{mod}\,\Lambda)$. Now apply Lemma 2.1.

(2) The lattice of finitely generated subfunctors of H^0 equals the lattice of subobjects of H_Λ in $\mathrm{Ab}(\mathcal{D}^b(\mathrm{proj}\,\Lambda))$, where Λ is viewed as a complex concentrated in degree zero. Now apply Lemma 2.1, keeping in mind that Λ generates $\mathcal{D}^b(\mathrm{proj}\,\Lambda))$ as a triangulated category. □

From now on suppose that Λ is a finite dimensional k-algebra and set $\mathcal{C} := \mathrm{Ab}(\mathcal{D}^b(\mathrm{proj}\,\Lambda))$. For the description of $\mathcal{C}_0$ one uses the well-known fact that the simple objects in $\mathcal{C}$ correspond to the Auslander–Reiten triangles in $\mathcal{D}^b(\mathrm{proj}\,\Lambda)$. Namely, if $X \xrightarrow{f} Y \to Z \to \Sigma X$ is an Auslander–Reiten triangle in $\mathcal{D}^b(\mathrm{proj}\,\Lambda)$, then $H_X/\operatorname{Im} H_f$ is a simple object in $\mathcal{C}$, and every simple object in $\mathcal{C}$ is of this form. This follows directly from the definition of an Auslander–Reiten triangle. Consequently, if $F \in \mathcal{C}$, then $F \in \mathcal{C}_0$ if and only if

$$\sum_{X \in \mathrm{ind}\,\mathcal{D}^b(\mathrm{proj}\,\Lambda)} \dim_k F(X) < \infty,$$

where $\mathrm{ind}\,\mathcal{D}^b(\mathrm{proj}\,\Lambda)$ denotes a fixed set of representatives of the indecomposable objects in $\mathcal{D}^b(\mathrm{proj}\,\Lambda)$; see also [1, §2] for the above description of simple and finite length objects. This condition immediately implies the first part of the Main Theorem.

Proposition 2.3. *Let Λ be a derived discrete algebra which is piecewise hereditary. Then*

$$\mathrm{KGdim}\,\mathrm{Ab}(\mathcal{D}^b(\mathrm{proj}\,\Lambda)) = 0.$$

Proof. If Λ is a derived discrete algebra, which is piecewise hereditary, then Λ is piecewise hereditary of Dynkin type. The well-known description of $\mathcal{D}^b(\operatorname{proj}\Lambda)$ in this case (see for example [13]), immediately implies that

$$\sum_{X \in \operatorname{ind} \mathcal{D}^b(\operatorname{proj}\Lambda)} \dim_k H_M(X) < \infty$$

for each complex M in $\mathcal{D}^b(\operatorname{proj}\Lambda)$. Consequently,

$$\sum_{X \in \operatorname{ind} \mathcal{D}^b(\operatorname{proj}\Lambda)} \dim_k F(X) < \infty$$

for each $F \in \operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda))$, hence the claim follows. □

3. Krull–Gabriel dimension and generic objects

Let Λ be a finite dimensional k-algebra. Generic modules were introduced by Crawley-Boevey in order to describe the representation type of an algebra [7, 8]. Following [12], an indecomposable object X of the unbounded derived category $\mathcal{D}(\operatorname{Mod}\Lambda)$ of all Λ-modules is called *generic*, if $H^i(X)$ is a finite length $\operatorname{End}(X)$-module, for each $i \in \mathbb{Z}$, but X is not in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Derived discrete algebras can be characterised in terms of generic complexes. This follows from work of Bautista [2] and we recall the following result.

Proposition 3.1 ([2, Theorem 1.1])**.** *Let Λ be a finite dimensional k-algebra which is not derived discrete. Then there exists a generic object X in $\mathcal{D}(\operatorname{Mod}\Lambda)$ such that the division ring $\operatorname{End}(X)/\operatorname{rad}\operatorname{End}(X)$ contains an element which is transcendental over k.* □

Note that the description of the endomorphism ring in [2, Theorem 1.1] is a consequence of the proof which uses [8, Theorem 9.5].

Next we combine Bautista's result with an argument due to Herzog [14]. To be precise, Herzog proves a result about the abelianisation $\operatorname{Ab}(\operatorname{mod}\Lambda)$, but the same argument works for $\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda))$ and yields the following.

Proposition 3.2 ([14, Theorem 3.6])**.** *Let Λ be a finite dimensional k-algebra. If there exists a generic object X in $\mathcal{D}(\operatorname{Mod}\Lambda)$ such that $\operatorname{End}(X)/\operatorname{rad}\operatorname{End}(X)$ contains an element which is transcendental over k, then*

$$\operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda)) \geq 2.$$ □

Our claim about the Krull–Gabriel dimension of an algebra which is not derived discrete is an immediate consequence.

Corollary 3.3. *Let Λ be a finite dimensional k-algebra which is not derived discrete. Then*

$$\operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda)) \geq 2.$$ □

4. The category of perfect complexes

Throughout this section we fix $(r, n, m) \in \Omega$ such that $r < n$ and we put $\Lambda := \Lambda(r, n, m)$. Note that the condition $r < n$ is equivalent to $\operatorname{gl.dim}\Lambda < \infty$. In this section we follow [3] and describe a quiver Γ together with a set $\mathcal{R}$ of relations such that the category $\mathcal{D}^b(\operatorname{proj}\Lambda)$ is equivalent to the path category $k\Gamma$ modulo the given relations (see for example [17, §2.1] for the definition of $k\Gamma$). We refer to [5] for a detailed study of morphisms in $\mathcal{D}^b(\operatorname{proj}\Lambda)$; the diagrams in there might help to understand our calculations.

For $i \in [0, r-1]$ we set

$$I_i := \mathbb{Z}^2,$$
$$I'_i := \{(a,b) \in \mathbb{Z}^2 \mid a \leq b + \delta_{i,0} \cdot m\},$$
$$I''_i := \{(a,b) \in \mathbb{Z}^2 \mid a + \delta_{i,0} \cdot n \leq b\},$$

where $\delta_{x,y}$ is the Kronecker delta. The set of vertices of Γ is

$$\Gamma_0 := \{X_v^{(i)} \mid i \in [0,r-1],\, v \in I'_i\} \cup \{Y_v^{(i)} \mid i \in [0,r-1],\, v \in I''_i\} \cup \{Z_v^{(i)} \mid i \in [0,r-1],\, v \in I_i\}.$$

Now we describe the arrows in Γ and associate to each arrow a *degree*. There are three cases.

(1) Fix $i \in [0, r-1]$ and $v := (a,b) \in I'_i$. We put

$$\mathcal{I}'^{(i)}_v := [a, b + \delta_{i,0} \cdot m] \times [b, \infty),$$
$$\mathcal{X}^{(i)}_v := [a, b + \delta_{i,0} \cdot m] \times \mathbb{Z},$$
$$\mathcal{X}'^{(i)}_v := (-\infty, a + \delta_{i,r-1} \cdot m] \times [a, b + \delta_{i,0} \cdot m].$$

For $u \in \mathcal{I}'^{(i)}_v$, $u \neq v$, there is an arrow $f'^{(i)}_{v,u}\colon X^{(i)}_v \to X^{(i)}_u$ of degree 0. Next, for $u \in \mathcal{X}^{(i)}_v$ there is an arrow $g'^{(i)}_{v,u}\colon X^{(i)}_v \to Z^{(i)}_u$ of degree 1. Finally, for $u \in \mathcal{X}'^{(i)}_v$ there is an arrow $e'^{(i)}_{v,u}\colon X^{(i)}_v \to X^{(i+1)}_u$ of degree 2, where we always change the upper index modulo r.

(2) Fix $i \in [0, r-1]$ and $v := (a,b) \in I''_i$. We put

$$\mathcal{I}''^{(i)}_v := [a, b - \delta_{i,0} \cdot n] \times [b, \infty),$$
$$\mathcal{Y}^{(i)}_v := \mathbb{Z} \times [a, b - \delta_{i,0} \cdot n],$$
$$\mathcal{Y}'^{(i)}_v := (-\infty, a - \delta_{i,r-1} \cdot n] \times [a, b - \delta_{i,0} \cdot n].$$

For $u \in \mathcal{I}''^{(i)}_v$, $u \neq v$, there is an arrow $f''^{(i)}_{v,u}\colon Y^{(i)}_v \to Y^{(i)}_u$ of degree 0. Next, for $u \in \mathcal{Y}^{(i)}_v$ there is an arrow $g''^{(i)}_{v,u}\colon Y^{(i)}_v \to Z^{(i)}_u$ of degree 1. Finally, for $u \in \mathcal{Y}'^{(i)}_v$ there is an arrow $e''^{(i)}_{v,u}\colon Y^{(i)}_v \to Y^{(i+1)}_u$ of degree 2.

(3) Fix $i \in [0, r-1]$ and $v := (a,b) \in I_i$. We put

$$\mathcal{I}^{(i)}_v := [a, \infty) \times [b, \infty),$$
$$\mathcal{Z}'^{(i)}_v := (-\infty, a + \delta_{i,r-1} \cdot m] \times [a, \infty),$$
$$\mathcal{Z}''^{(i)}_v := (-\infty, b - \delta_{i,r-1} \cdot n] \times [b, \infty),$$
$$\mathcal{Z}^{(i)}_v := (-\infty, a + \delta_{i,r-1} \cdot m] \times (\infty, b - \delta_{i,r-1} \cdot n].$$

For $u \in \mathcal{I}^{(i)}_v$, $u \neq v$, there is an arrow $f^{(i)}_{v,u}\colon Z^{(i)}_v \to Z^{(i)}_u$ of degree 0. Next, for $u \in \mathcal{Z}'^{(i)}_v$ there is an arrow $h'^{(i)}_{v,u}\colon Z^{(i)}_v \to X^{(i+1)}_u$ of degree 1. Similarly, for $u \in \mathcal{Z}''^{(i)}_v$ there is an arrow $h''^{(i)}_{v,u}\colon Z^{(i)}_v \to Y^{(i+1)}_u$ of degree 1. Finally, for $u \in \mathcal{Z}^{(i)}_v$ there is an arrow $e^{(i)}_{v,u}\colon Z^{(i)}_v \to Z^{(i+1)}_u$ of degree 2.

Now we describe the set $\mathcal{R}$ of relations. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be arrows of degree p and q, respectively. If there is an arrow $h\colon X \to Z$ of degree $p+q$, then we have the relation $gf = h$, otherwise we have the relation $gf = 0$.

We summarise our construction.

Proposition 4.1. *There exists a k-linear equivalence $k\Gamma/\mathcal{R} \xrightarrow{\sim} \mathcal{D}^b(\operatorname{proj}\Lambda)$.*

Proof. The Auslander–Reiten quiver of $\mathcal{D}^b(\operatorname{proj}\Lambda)$ has been described in [4]. It consists of $2r$ components of type $\mathbb{Z}\mathbb{A}_\infty$ (they correspond to X- and Y-vertices of Γ) and r components of type $\mathbb{Z}\mathbb{A}^\infty_\infty$ (they correspond to Z-vertices of Γ). Moreover, under the action of the shift Σ the components fall into three orbits, consisting of r components each. The objects lying on the border of $\mathbb{Z}\mathbb{A}_\infty$ components have been also identified. Using string combinatorics [6, 15] it is straightforward to verify the description of H_X for X lying on the border of a component of type $\mathbb{Z}\mathbb{A}_\infty$. By induction, using Auslander–Reiten triangles, the description of H_X follows for each X in the components of type $\mathbb{Z}\mathbb{A}_\infty$. Then we verify this description for two (cleverly) chosen neighbouring objects in a component of type $\mathbb{Z}\mathbb{A}_\infty^\infty$ and proceed again by induction (using Auslander–Reiten triangles) to finish the proof. □

For future use we introduce the following notation. For $i \in [0, r-1]$ set

$$f'^{(i)}_{v,v} := \mathrm{id}_{X^{(i)}_v},\ v \in I'_i, \quad f''^{(i)}_{v,v} := \mathrm{id}_{Y^{(i)}_v},\ v \in I''_i, \quad f^{(i)}_{v,v} := \mathrm{id}_{Z^{(i)}_v},\ v \in I_i.$$

Also, we let $f'^{(i)}_{v,u}$ denote the zero morphism $X^{(i)}_v \to 0$, if $v \in I'_i$, and $u \notin I'_i$. The same convention applies to $f''^{(i)}_{v,u}$, $h'^{(i)}_{v,u}$ and $h''^{(i)}_{v,u}$.

5. The Krull–Gabriel dimension for algebras of finite global dimension

Let $(r, n, m) \in \Omega$ be such that $r < n$. We consider $\mathcal{C} := \mathrm{Ab}(\mathcal{D}^b(\mathrm{proj}\,\Lambda))$ for $\Lambda := \Lambda(r, n, m)$. Our aim is to prove that $\mathcal{C}_1 \neq \mathcal{C}$, but $\mathcal{C}_2 = \mathcal{C}$. In order to show the latter claim, it is enough to prove that $H_U \in \mathcal{C}_2$ for each indecomposable $U \in \mathcal{D}^b(\mathrm{proj}\,\Lambda)$. In fact, we will prove that H_U is either zero or simple in $\mathcal{C}/\mathcal{C}_1$. Note that in order to prove that H_U (more generally, $H_U / \operatorname{Im} H_g$, where $g\colon U \to M$ is a morphism) is either zero or simple in $\mathcal{C}/\mathcal{C}_{n-1}$ for some $n \in \mathbb{N}$, it is enough to prove that, for every non-zero map $f\colon U \to V$ with V indecomposable, either $\operatorname{Im} H_f \underset{\mathcal{C}/\mathcal{C}_{n-1}}{=} 0$ or $\operatorname{Im} H_f \underset{\mathcal{C}/\mathcal{C}_{n-1}}{=} H_U$ (either $\operatorname{Im} H_f/(\operatorname{Im} H_f \cap \operatorname{Im} H_g) \underset{\mathcal{C}/\mathcal{C}_{n-1}}{=} 0$ or $\operatorname{Im} H_f/(\operatorname{Im} H_f \cap \operatorname{Im} H_g) \underset{\mathcal{C}/\mathcal{C}_{n-1}}{=} H_U / \operatorname{Im} H_g$), where the subscript means that the equalities hold in $\mathcal{C}/\mathcal{C}_{n-1}$.

We begin with the description of the simple objects in $\mathcal{C}$.

Lemma 5.1. *The simple object in $\mathcal{C}$ are*

(1) $A'^{(i)}_v := H_{X^{(i)}_v} / \operatorname{Im} H_{(f'^{(i)}_{v,v+(1,0)}, f'^{(i)}_{v,v+(0,1)})^{\mathrm{tr}}}$, $i \in [0, r-1]$, $v \in I'_i$,

(2) $A''^{(i)}_v := H_{Y^{(i)}_v} / \operatorname{Im} H_{(f''^{(i)}_{v,v+(1,0)}, f''^{(i)}_{v,v+(0,1)})^{\mathrm{tr}}}$, $i \in [0, r-1]$, $v \in I''_i$,

(3) $A^{(i)}_v := H_{Z^{(i)}_v} / \operatorname{Im} H_{(f^{(i)}_{v,v+(1,0)}, f^{(i)}_{v,v+(0,1)})^{\mathrm{tr}}}$, $i \in [0, r-1]$, $v \in I_i$.

Proof. This follows from the well-known description of the Auslander–Reiten triangles in $\mathcal{D}^b(\mathrm{proj}\,\Lambda)$; see Section 2. □

Now we move to the category $\mathcal{C}/\mathcal{C}_0$. We first describe the simple objects.

Lemma 5.2. *The objects*

(1) $B'^{(i)}_{a,b,b'} := H_{X^{(i)}_{(a,b)}} / \operatorname{Im} H_{(f'^{(i)}_{(a,b),(a+1,b)}, g'^{(i)}_{(a,b),(a,b')})^{\mathrm{tr}}}$ *for* $i \in [0, r-1]$, $(a, b) \in I'_i$, $b' \in \mathbb{Z}$,

(2) $B''^{(i)}_{a,b,b'} := H_{Y^{(i)}_{(a,b)}} / \operatorname{Im} H_{(f''^{(i)}_{(a,b),(a+1,b)}, g''^{(i)}_{(a,b),(b',a)})^{\mathrm{tr}}}$ *for* $i \in [0, r-1]$, $(a, b) \in I''_i$, $b' \in \mathbb{Z}$,

(3) $C'^{(i)}_{a,b,b'} := H_{Z^{(i)}_{(a,b)}} / \operatorname{Im} H_{(f^{(i)}_{(a,b),(a+1,b)}, h'^{(i)}_{(a,b),(b',a)})^{\mathrm{tr}}}$ *for* $i \in [0, r-1]$, $(a, b) \in I_i$, $b' \in (-\infty, a + \delta_{i,r-1} \cdot m + 1]$,

(4) $C''^{(i)}_{a,b,a'} := H_{Z^{(i)}_{(a,b)}} / \operatorname{Im} H_{(f^{(i)}_{(a,b),(a,b+1)}, h''^{(i)}_{(a,b),(a',b)})^{\mathrm{tr}}}$ *for* $i \in [0, r-1]$, $(a, b) \in I_i$, $a' \in (-\infty, b - \delta_{i,r-1} \cdot n + 1]$,

are simple in $\mathcal{C}/\mathcal{C}_0$.

Remark 5.3. One may easily show that every simple object in $\mathcal{C}/\mathcal{C}_0$ is (up to isomorphism) of the above form. It is also not difficult to describe the isomorphism classes of the above objects.

Proof. We only prove the first claim; the remaining ones are proved similarly. Let $i \in [0, r-1]$, $(a, b) \in I'_i$ and $b' \in \mathbb{Z}$. It is clear that $B'^{(i)}_{a,b,b'} \underset{\mathcal{C}/\mathcal{C}_0}{\neq} 0$. This follows, since for each $n \in \mathbb{N}$ we have in $\mathcal{C}$ a short exact sequence

$$0 \to B'^{(i)}_{a,b+n+1,b'} \to B'^{(i)}_{a,b+n,b'} \to A'^{(i)}_{(a,b+n)} \to 0.$$

Indeed, first observe that

$$\dim_k B'^{(i)}_{a,b+n+1,b'}(X) \leq 1, \qquad \dim_k B'^{(i)}_{a,b+n,b'}(X) \leq 1$$

and

$$\dim_k A'^{(i)}_{(a,b+n)}(X) \leq 1$$

for each indecomposable object X of $\mathcal{D}^b(\operatorname{proj} \Lambda)$. Next, $B'^{(i)}_{a,b+n,b'}(X) \neq 0$ if and only if either $X \simeq X^{(i)}_{(a,d)}$ for $d \geq b+n$ or $X \simeq Z^{(i)}_{(a,d)}$ for $d > b'$. Similarly, $B'^{(i)}_{a,b+n+1,b'}(X) \neq 0$ if and only if either $X \simeq X^{(i)}_{(a,d)}$ for $d > b+n$ or $X \simeq Z^{(i)}_{(a,d)}$ for $d > b'$. Finally, $A'^{(i)}_{(a,b+n)}(X) \neq 0$ if and only if $X \simeq X^{(i)}_{a,b+n}$.

For a non-zero morphism $f\colon X^{(i)}_{(a,b)} \to V$ with V indecomposable we put

$$B'_f := \operatorname{Im} H_f / (\operatorname{Im} H_f \cap \operatorname{Im} H_{(f'^{(i)}_{(a,b),(a+1,b)}, g'^{(i)}_{(a,b),(a,b')})^{\mathrm{tr}}}).$$

We have to show that either $B'_f \underset{\mathcal{C}/\mathcal{C}_0}{=} 0$ or $B'_f \underset{\mathcal{C}/\mathcal{C}_0}{=} B'^{(i)}_{a,b,b'}$ for every f as above. We may assume that we are in one of the following cases:

(1) $V = X^{(i)}_{(c,d)}$ and $f = f'^{(i)}_{(a,b),(c,d)}$ for some $(c, d) \in \mathcal{I}'^{(i)}_{(a,b)}$,
(2) $V = Z^{(i)}_{(c,d)}$ and $f = g'^{(i)}_{(a,b),(c,d)}$ for some $(c, d) \in \mathcal{X}^{(i)}_{(a,b)}$,
(3) $V = X^{(i+1)}_{(c,d)}$ and $f = e'^{(i)}_{(a,b),(c,d)}$ for some $(c, d) \in \mathcal{X}'^{(i)}_{(a,b)}$.

Case (1). If $c > a$, then f factors through $f'^{(i)}_{(a,b),(a+1,b)}$, hence B'_f is the zero subobject of $B'^{(i)}_{a,b,b'}$. If $c = a$, then we prove by induction on d that $B'_f \underset{\mathcal{C}/\mathcal{C}_0}{=} B'^{(i)}_{a,b,b'}$. Indeed, if $d = b$, then the claim is obvious. If $d > b$, then we have a short exact sequence

$$0 \to B'_f \to B'_{f'^{(i)}_{(a,b),(a,d-1)}} \to A'^{(i)}_{(a,d-1)} \to 0,$$

hence $B'_f \underset{\mathcal{C}/\mathcal{C}_0}{=} B'_{f'^{(i)}_{(a,b),(a,d-1)}}$ by Lemma 5.1. Moreover, $B'_{f'^{(i)}_{(a,b),(a,d-1)}} \underset{\mathcal{C}/\mathcal{C}_0}{=} B'_{a,b,b'}$ by induction.

Case (2). We prove that $B'_f \underset{\mathcal{C}/\mathcal{C}_0}{=} 0$ in this case. Again, we may assume that $c = a$. If $d \geq b'$, then f factors through $g'^{(i)}_{(a,b),(a,b')}$, hence B'_f is again zero. On the other hand, if $d < b'$, then we have a short exact sequence

$$0 \to B'_{g'^{(i)}_{(a,b),(c,d+1)}} \to B'_f \to A_{(a,d)} \to 0,$$

hence the claim follows by an obvious induction.

Case (3). In this case f factors through $(f'^{(i)}_{(a,b),(a+1,b)}, g'^{(i)}_{(a,b),(a,b')})^{\mathrm{tr}}$, hence B'_f is zero. □

Now we show that some representable functors have finite length in $\mathcal{C}/\mathcal{C}_0$.

Lemma 5.4. *Let* $i \in [0, r-1]$.

(1) *If* $v \in I'_i$, *then* $H_{X_v^{(i)}} \in \mathcal{C}_1$.
(2) *If* $v \in I''_i$, *then* $H_{Y_v^{(i)}} \in \mathcal{C}_1$.

Proof. Again, we only prove the first claim. Let $v = (a, b)$. The claim is shown by induction on $b - a$. If $b - a = \delta_{i,0} \cdot m$, then we have a short exact sequence

$$0 \to C'^{(i)}_{a,0,a+\delta_{i,r-1}\cdot m+1} \to H_{X_v^{(i)}} \to B'^{(i)}_{a,b,0} \to 0,$$

hence the claim follows from Lemma 5.2. If $b - a > \delta_{i,0} \cdot m$, then we have exact sequences

$$H_{X^{(i)}_{(a+1,b)}} \to H_{X_v^{(i)}} \to H_{X_v'^{(i)}} / \operatorname{Im} H_{f'^{(i)}_{(a,b),(a+1,b)}} \to 0$$

and

$$0 \to C'^{(i)}_{a,0,a+\delta_{i,r-1}\cdot m+1} \to H_{X_v^{(i)}} / \operatorname{Im} H_{f'^{(i)}_{(a,b),(a+1,b)}} \to B'^{(i)}_{a,b,0} \to 0,$$

and the claim follows by induction and Lemma 5.2. □

Next we show that the remaining representable functors corresponding to the indecomposable objects in $\mathcal{D}^b(\operatorname{proj} \Lambda)$ are not of finite length in $\mathcal{C}/\mathcal{C}_0$.

Lemma 5.5. *If* $i \in [0, r-1]$ *and* $v \in I_i$, *then* $H_{Z_v^{(i)}} \notin \mathcal{C}_1$.

Proof. Let $v = (a, b)$. For each $n \in \mathbb{N}$ we have the following exact sequence

$$0 \to \operatorname{Im} H_{f^{(i)}_{(a,b),(a+n+1,b)}} \to \operatorname{Im} H_{f^{(i)}_{(a,b),(a+n,b)}} \to C'^{(i)}_{a+n,b,a+\delta_{i,r-1}\cdot m+1} \to 0,$$

which implies the claim (note that $\operatorname{Im} H_{f^{(i)}_{(a,b),(a,b)}} = H_{Z_v^{(i)}}$). □

Finally, we show that $\mathcal{C}_2 = \mathcal{C}$. For this we only need to prove the following.

Lemma 5.6. *If* $i \in [0, r-1]$ *and* $v \in I_i$, *then* $H_{Z_v^{(i)}}$ *is simple in* $\mathcal{C}/\mathcal{C}_1$.

Proof. Let $v = (a, b)$. We know from Lemma 5.5 that $H_{Z_v^{(i)}}$ is a non-zero object in $\mathcal{C}/\mathcal{C}_1$. In order to prove it is simple we fix a non-zero morphism $f \colon Z_v^{(i)} \to V$. We may assume that we are in one of the following cases:

(1) $V = Z^{(i)}_{(c,d)}$ and $f = f^{(i)}_{(a,b),(c,d)}$ for some $(c, d) \in \mathcal{I}^{(i)}_{(a,b)}$,
(2) $V = X^{(i+1)}_{(c,d)}$ and $f = h'^{(i)}_{(a,b),(c,d)}$ for some $(c, d) \in \mathcal{Z}'^{(i)}_{(a,b)}$,
(3) $V = Y^{(i+1)}_{(c,d)}$ and $f = h''^{(i)}_{(a,b),(c,d)}$ for some $(c, d) \in \mathcal{Z}''^{(i)}_{(a,b)}$,
(4) $V = Z^{(i+1)}_{(c,d)}$ and $f = e^{(i)}_{(a,b),(c,d)}$ for some $(c, d) \in \mathcal{Z}^{(i)}_{(a,b)}$.

Case (1). We prove by induction on $c+d$ that $\operatorname{Im} H_f \underset{\mathcal{C}/\mathcal{C}_1}{=} H_{Z_v^{(i)}}$. If $c + d = a + b$ (i.e., $c = a$ and $d = b$), the claim is obvious. Assume $c > a$. Then we have an exact sequence

$$0 \to \operatorname{Im} H_f \to \operatorname{Im} H_{f^{(i)}_{(a,b),(c-1,d)}} \to C'^{(i)}_{c-1,d,a+\delta_{i,r-1}\cdot m+1} \to 0,$$

hence $\operatorname{Im} H_f \underset{\mathcal{C}/\mathcal{C}_1}{=} \operatorname{Im} H_{f^{(i)}_{(a,b),(c-1,d)}}$ by Lemma 5.2. Moreover, we have by induction $\operatorname{Im} H_{f^{(i)}_{(a,b),(c-1,d)}} \underset{\mathcal{C}/\mathcal{C}_1}{=} H_{Z_v^{(i)}}$. We proceed similarly if $d > b$.

Case (2). We have an epimorphism $H_{X^{(i+1)}_{(c,d)}} \to \operatorname{Im} H_f$, hence $\operatorname{Im} H_f \underset{\mathcal{C}/\mathcal{C}_1}{=} 0$ by Lemma 5.4.

Case (3). Analogous to Case (2).

Case (4). Again, we have an epimorphism $H_{X^{(i+1)}_{(c,a)}} \to \operatorname{Im} H_f$, hence we get $\operatorname{Im} H_f \underset{\mathcal{C}/\mathcal{C}_1}{=} 0$ (in fact one may even prove that $\operatorname{Im} H_f \in \mathcal{C}_0$ in this case). □

6. The algebras of infinite global dimension

Throughout this section fix $n \in \mathbb{N}_+$ and $m \in \mathbb{N}$. We put $\Lambda := \Lambda(n,n,m)$. The aim of this section is to prove that $\operatorname{KGdim} \mathcal{C} = 1$, where $\mathcal{C} := \operatorname{Ab}(\mathcal{D}^b(\operatorname{proj} \Lambda))$. The basic idea is to extract the X-part of the arguments in the finite global dimension case. We explain this in more detail.

First we describe the category $\mathcal{D}^b(\operatorname{proj} \Lambda)$. Let Γ be the quiver with the vertices $X_v^{(i)}$ for $i \in [0, n-1]$ and $v \in I_i$, where

$$I_i := \{(a,b) \in \mathbb{Z}^2 \mid a \le b + \delta_{i,0} \cdot m\}.$$

For $i \in [0, n-1]$ and $v = (a,b) \in I_i$ we define

$$\mathcal{I}_v^{(i)} := [a, b + \delta_{i,0} \cdot m] \times [b, \infty)$$
$$\mathcal{X}_v^{(i)} := (-\infty, a + \delta_{i,n-1} \cdot m] \times [a, b + \delta_{i,0} \cdot m].$$

Then for each $i \in [0, n-1]$, $v \in I_i^{(i)}$, and $u \in \mathcal{I}_v^{(i)}$, $u \neq v$, we have an arrow $f_{v,u}^{(i)} : X_v^{(i)} \to X_u^{(i)}$ of degree 0, and for each $i \in [0, n-1]$, $v \in I_i^{(i)}$, and $u \in \mathcal{X}_v^{(i)}$, we have an arrow $e_{v,u}^{(i)} : X_v^{(i)} \to X_u^{(i+1)}$ of degree 1. Finally, by $\mathcal{R}$ we denote the set of the following relations. Let $f\colon X \to X'$ and $g\colon X' \to X''$ be arrows of degree p and q, respectively. If there is an arrow $h\colon X \to X''$ of degree $p+q$, then we have the relation $gf = h$, otherwise we have the relation $gf = 0$ (an explicit list of relations can be found in [3, §5]).

Proposition 6.1. *There exists a k-linear equivalence $k\Gamma/\mathcal{R} \xrightarrow{\sim} \mathcal{D}^b(\operatorname{proj} \Lambda)$.*

Proof. Analogous to the proof of Proposition 4.1. □

It is obvious that $\mathcal{C}_0 \neq \mathcal{C}$. In order to prove $\mathcal{C}_1 = \mathcal{C}$, it suffices to show that $H_U \in \mathcal{C}_1$ for each indecomposable $U \in \mathcal{D}^b(\operatorname{proj} \Lambda)$. The arguments are similar to those used in Section 5 and we state the analogues of Lemmas 5.1, 5.2 and 5.4 without proofs. Again, we use the convention that $f_{v,u}^{(i)}$ ($e_{v,u}^{(i)}$) denotes the zero morphism $X_v^{(i)} \to 0$ if $i \in [0, n-1]$, $v \in I_i$, and $u \notin I_i$ ($u \notin I_{i+1}$, respectively).

Lemma 6.2. *The simple objects in $\mathcal{C}$ are*

$$A_v^{(i)} := H_{X_v^{(i)}} / \operatorname{Im} H_{(f_{v,v+(1,0)}^{(i)}, f_{v,v+(0,1)}^{(i)})^{\mathrm{tr}}}$$

for $i \in [0, n-1]$ and $v \in I_i$. □

Lemma 6.3. *The objects*

$$B_{a,b,b'}^{(i)} := H_{X_{(a,b)}^{(i)}} / \operatorname{Im} H_{(f_{(a,b),(a+1,b)}^{(i)}, e_{(a,b),(b',a)}^{(i)})^{\mathrm{tr}}}$$

for $i \in [0, n-1]$, $(a,b) \in I_i$, and $b' \in (-\infty, a + \delta_{i,n-1} \cdot m]$, are simple in $\mathcal{C}/\mathcal{C}_0$. □

Lemma 6.4. *If $i \in [0, n-1]$ and $v \in I_i$, then $H_{X_v^{(i)}} \in \mathcal{C}_1$.* □

7. Concluding remarks

There are two other important triangulated categories, which one often studies for a finite dimensional algebra Λ: the bounded derived category $\mathcal{D}^b(\operatorname{mod} \Lambda)$ and the stable category $\underline{\operatorname{mod}}\, \hat{\Lambda}$ of the repetitive algebra $\hat{\Lambda}$. Thus one may also ask about the Krull–Gabriel dimensions of the abelianisations of these two categories. We have the following result.

Theorem 7.1. *Let Λ be a finite dimensional k-algebra and denote by $\mathcal{C}$ either $\operatorname{Ab}(\mathcal{D}^b(\operatorname{mod} \Lambda))$ or $\operatorname{Ab}(\underline{\operatorname{mod}}\, \hat{\Lambda})$. Then $\operatorname{KGdim} \mathcal{C} \neq 1$, and $\operatorname{KGdim} \mathcal{C} = 0$ if and only if Λ is piecewise hereditary of Dynkin type.*

Proof. We apply the Main Theorem and use the chain of fully faithful exact functors

$$\mathcal{D}^b(\operatorname{proj}\Lambda) \to \mathcal{D}^b(\operatorname{mod}\Lambda) \to \underline{\operatorname{mod}}\,\hat{\Lambda}.$$

If Λ is piecewise hereditary of Dynkin type, then Λ is derived discrete and $\mathcal{D}^b(\operatorname{proj}\Lambda) = \mathcal{D}^b(\operatorname{mod}\Lambda) = \underline{\operatorname{mod}}\,\hat{\Lambda}$. Thus $\operatorname{KGdim}\mathcal{C} = 0$.

If Λ is not derived discrete, then Lemma 7.2 below yields

$$2 \leq \operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda)) \leq \operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{mod}\Lambda)) \leq \operatorname{KGdim}\operatorname{Ab}(\underline{\operatorname{mod}}\,\hat{\Lambda}).$$

Finally, assume Λ is derived discrete, but not piecewise hereditary. If $\operatorname{gl.dim}\Lambda < \infty$, then again $\mathcal{D}^b(\operatorname{proj}\Lambda) = \mathcal{D}^b(\operatorname{mod}\Lambda) = \underline{\operatorname{mod}}\,\hat{\Lambda}$. Thus assume $\operatorname{gl.dim}\Lambda = \infty$. In this case the description of $\underline{\operatorname{mod}}\,\hat{\Lambda}$ is the same as the description of $\mathcal{D}^b(\operatorname{proj}\Lambda)$ given in Section 4, hence the arguments from Section 5 apply. On the other hand, $\mathcal{D}^b(\operatorname{mod}\Lambda)$ lies strictly between $\mathcal{D}^b(\operatorname{proj}\Lambda)$ and $\underline{\operatorname{mod}}\,\hat{\Lambda}$, but some of the Z-modules from Section 4 survive (see [4] for details) and the corresponding representable functors are not of finite length in $\mathcal{C}/\mathcal{C}_0$ when $\mathcal{C} = \operatorname{Ab}(\mathcal{D}^b(\underline{\operatorname{mod}}\,\hat{\Lambda}))$. □

In the above proof the following lemma is used.

Lemma 7.2. *Let $\mathcal{S}$ be a thick subcategory of a triangulated category $\mathcal{T}$. Then* $\operatorname{KGdim}\operatorname{Ab}(\mathcal{S}) \leq \operatorname{KGdim}\operatorname{Ab}(\mathcal{T})$.

Proof. The universal property of the abelianisation yields an exact embedding $\operatorname{Ab}(\mathcal{S}) \to \operatorname{Ab}(\mathcal{T})$. An easy induction shows that $\operatorname{Ab}(\mathcal{T})_n \cap \operatorname{Ab}(\mathcal{S}) \subseteq \operatorname{Ab}(\mathcal{S})_n$ for each $n \in \mathbb{N}$. □

We observe that

$$\operatorname{KGdim}\operatorname{Ab}(\mathcal{D}^b(\operatorname{mod}\Lambda)) = \operatorname{KGdim}\operatorname{Ab}(\underline{\operatorname{mod}}\,\hat{\Lambda})$$

if Λ is derived discrete. We do not know whether this equality holds for an arbitray finite dimensional k-algebra Λ.

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