

# MONODROMY EIGENVALUES AND POLES OF ZETA FUNCTIONS

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ABSTRACT. We study the poles of the topological and related zeta functions associated to a polynomial  $f$  and a suitable differential form, and establish a strong link between these poles and monodromy eigenvalues of  $f$ .

## INTRODUCTION

**Archimedean zeta functions.** Let  $f : X \rightarrow \mathbb{C}$  be a non-constant analytic function on an open part  $X$  of  $\mathbb{C}^n$ . We consider  $C^\infty$  functions  $\varphi$  with compact support on  $X$  and the corresponding differential forms  $\omega = \varphi dx \wedge d\bar{x}$ . Here and further  $x = (x_1, \dots, x_n)$  and  $dx = dx_1 \wedge \dots \wedge dx_n$ . For such  $f$  the integral

$$Z(f, \omega; s) := \int_X |f(x)|^{2s} \omega,$$

where  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 0$ , has been the object of intensive study. One verifies that  $Z(f, \omega; s)$  is holomorphic in  $s$ . Either by resolution of singularities [Ati70], [BG69], or by the theory of Bernstein polynomials [Ber72], one can show that it admits a meromorphic continuation to  $\mathbb{C}$ , and that all its poles are among the translates by  $\mathbb{Z}_{<0}$  of a finite number of rational numbers. Combining results of Barlet [Bar84b], Kashiwara [Kas83] and Malgrange [Mal83], the poles of (the extended)  $Z(f, \omega; s)$  are strongly linked to the eigenvalues of (local) monodromy at points of  $f^{-1}\{0\}$ ; see §1 for the concept of monodromy.

**Theorem 0.1.** (1) *If  $s_0$  is a pole of  $Z(f, \omega; s)$  for some differential form  $\omega$ , then  $\exp(2\pi\sqrt{-1}s_0)$  is a monodromy eigenvalue of  $f$  at some point of  $f^{-1}\{0\}$ .*

(2) *If  $\lambda$  is a monodromy eigenvalue of  $f$  at a point of  $f^{-1}\{0\}$ , then there exists a differential form  $\omega$  and a pole  $s_0$  of  $Z(f, \omega; s)$  such that  $\lambda = \exp(2\pi\sqrt{-1}s_0)$ .*

There are also more precise local versions in a neighbourhood of a point of  $f^{-1}\{0\}$ . Similar results hold for a real analytic function  $f : X(\subset \mathbb{R}^n) \rightarrow \mathbb{R}$  and integrals  $\int_{X \cap \{f > 0\}} f^s \varphi dx$ ; we refer to e.g. [Bar84a], [Bar85], [Bar04], [BM93], [JM98]. Nowadays such (complex or real) integrals are also called Archimedean zeta functions.

**Topological and related zeta functions.** There are similar integrals over  $p$ -adic fields, called ( $p$ -adic) Igusa zeta functions or non-Archimedean zeta functions, whose meromorphic continuation was proved by Igusa using resolution of singularities [Igu75]. Strongly related

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to these  $p$ -adic zeta functions, there are various ‘algebraic-geometric’ zeta functions: the motivic, Hodge and topological zeta functions. (We have for instance that the motivic zeta function specializes to the various  $p$ -adic Igusa zeta functions, for all but finitely many  $p$ ). In this algebraic-geometric context one rather considers polynomial functions  $f$  and algebraic differential forms  $\omega$ .

Let now  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  be a non-constant polynomial function and  $\omega$  a regular (algebraic) differential  $n$ -form on  $\mathbb{C}^n$ . The topological zeta function associated to  $f$  and  $\omega$  is described in terms of an embedded resolution of  $f^{-1}\{0\} \cup \text{div } \omega$ . Fix such an embedded resolution  $\pi : X \rightarrow \mathbb{C}^n$ , where we assume that  $\pi$  is an isomorphism outside the inverse image of  $f^{-1}\{0\} \cup \text{div } \omega$ . We denote by  $E_i, i \in I$ , the irreducible components of the inverse image  $\pi^{-1}(f^{-1}\{0\} \cup \text{div } \omega)$ , and by  $N_i$  and  $\nu_i - 1$  the multiplicities of  $E_i$  in the divisor of  $\pi^*f$  and  $\pi^*\omega$ , respectively. We put  $E_J^\circ := (\cap_{j \in J} E_j) \setminus (\cup_{\ell \notin J} E_\ell)$  for  $J \subset I$ , in particular  $E_\emptyset^\circ = X \setminus (\cup_{i \in I} E_i)$ . Note that the  $E_J^\circ$  form a stratification of  $X$  in locally closed subsets. Finally, we denote by  $\chi(\cdot)$  the topological Euler characteristic.

**Definition 0.2.** The (global) *topological zeta function* of  $(f, \omega)$  and its local version at  $b \in \mathbb{C}^n$  are

$$Z_{\text{top}}(f, \omega; s) := \sum_{J \subset I} \chi(E_J^\circ) \prod_{j \in J} \frac{1}{\nu_j + sN_j},$$

and

$$Z_{\text{top},b}(f, \omega; s) := \sum_{J \subset I} \chi(E_J^\circ \cap \pi^{-1}\{b\}) \prod_{j \in J} \frac{1}{\nu_j + sN_j},$$

respectively, where  $s$  is a variable.

They were introduced by Denef and Loeser in [DL92] for the ‘standard’  $\omega$ , i.e., for  $\omega = dx$ . Their original proof that these expressions do not depend on the chosen resolution proceeds by describing them as a kind of limit of  $p$ -adic Igusa zeta functions. Later they obtained them as a specialization of the intrinsically defined motivic zeta functions [DL98]. Another technique is applying the Weak Factorization Theorem [AKMW02], [Wło03] to compare two different resolutions.

Note that the numbers  $-\nu_i/N_i, i \in I$ , form a complete list of candidate poles for the topological zeta function associated to  $f$  and  $\omega$ . (For the Archimedean zeta functions, there exist similar complete lists of candidate poles in terms of embedded resolutions.)

The finer variants, Hodge and motivic zeta functions, involve, instead of Euler characteristics, Hodge polynomials and classes in the Grothendieck ring of varieties, respectively. Concerning Hodge and motivic zeta functions, we refer to e.g. [DL98], [Rod05], [Vey06a] for versions with  $\omega = dx$ , and to [ACLM02], [ACLM05], [Vey01], [Cau14] involving more general  $\omega$ . We mention that, in contrast with topological zeta functions, Hodge and motivic zeta functions can be defined intrinsically as formal power series with coefficients determined by the behaviour of the arcs on  $\mathbb{C}^n$  with respect to their intersection with  $f^{-1}\{0\}$  and with  $\text{div } \omega$ . Then one shows that they are rational functions by proving explicit formulae as above in terms of an embedded resolution.

**Towards a generalized monodromy conjecture.** It is very natural to investigate the validity of the analogous statements of Theorem 0.1 in the context of the non-Archimedean and related zeta functions. Here we focus on the topological zeta function, because this is the ‘most difficult’ one in the context of our main result, see Remark 2.2. First of all, there is the famous *Monodromy Conjecture* (originally stated for the  $p$ -adic Igusa zeta function), being an analog of Theorem 0.1(1) in the special case where  $\omega = dx$ . Without loss of generality, we assume from now on that  $f(0) = 0$ .

**Conjecture 0.3.** [DL92] If  $s_0$  is a pole of  $Z_{top,0}(f, dx; s)$ , then  $\exp(2\pi\sqrt{-1}s_0)$  is a monodromy eigenvalue of  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  at a point of  $f^{-1}\{0\}$ , close to 0.

This conjecture was proved for  $n = 2$  by Loeser [Loe88]. There are by now various other partial results as [ACLM02], [ACLM05], [BV13], [BMT11], [Cau14], [LVP11], [LV09], [Vey93], [Vey06b]. In contrast with the Archimedean case, such a statement is certainly *not* true for arbitrary  $\omega$ . There are plenty of examples involving a pole  $s_0$  of  $Z_{top,0}(f, \omega; s)$ , such that  $\exp(2\pi\sqrt{-1}s_0)$  is not a monodromy eigenvalue of  $f$ . Concerning Theorem 0.1(2), the second author investigated analogous statements in [Vey07]. He showed for instance the following.

**Theorem 0.4.** *Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be a nonzero polynomial function (germ). Let  $\lambda$  be a monodromy eigenvalue of  $f$  at 0. Then there exists a differential  $n$ -form  $\omega$  and a point  $b \in f^{-1}\{0\}$ , close to 0, such that  $Z_{top,b}(f, \omega; s)$  has a pole  $s_0$  satisfying  $\exp(2\pi\sqrt{-1}s_0) = \lambda$ .*

*If  $f^{-1}\{0\}$  has an isolated singularity at 0, then we can take 0 itself as point  $b$ .*

The zeta functions associated to  $f$  and the constructed  $\omega$  in the theorem above can have *other* poles that do not induce monodromy eigenvalues of  $f$ . So for those zeta functions the analog of Theorem 0.1(1) is (unfortunately) not true. Then, in [Vey07], the following questions were raised, aiming at a deeper analog of Theorem 0.1.

**Question 1.** Given  $f$ , does there exist some set of distinguished differential forms  $\omega$  such that

- (1) for every monodromy eigenvalue  $\lambda$  of  $f$ , there exists a distinguished differential  $n$ -form  $\omega$  and a point  $b \in f^{-1}\{0\}$ , close to 0, such that  $Z_{top,b}(f, \omega; s)$  has a pole  $s_0$  satisfying  $\exp(2\pi\sqrt{-1}s_0) = \lambda$ ;
- (2) for all distinguished  $\omega$  and for *all* poles  $\tilde{s}$  of  $Z_{top,b}(f, \omega; s)$ , we have that  $\exp(2\pi\sqrt{-1}\tilde{s})$  is a monodromy eigenvalue of  $f$ ?

**Question 2.** Does there exist a set as in Question 1, such that moreover the standard form  $\omega = dx$  is distinguished?

Of course, Question 2 is quite ambitious because a positive answer would be a lot stronger than the (in arbitrary dimension) still wide open monodromy conjecture. For  $n = 2$  this was realized by Némethi and the second author, first for analytically irreducible germs [NV10] and later for arbitrary curve germs [NV12]. The first author then showed in [Cau14] that the set proposed by the second author and Némethi also works for the motivic zeta function.

The aim of this paper is to give an affirmative answer to Question 1, in full generality for arbitrary  $n$ . Statements and proofs are only given for the topological zeta function, but everything is also valid for the motivic and Hodge zeta function, see Remark 2.2 for more details. More precisely, we show the following.

**Theorem 0.5.** *Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be a non-zero polynomial function (germ).*

- (1) *Let  $\lambda$  be a monodromy eigenvalue of  $f$  at 0. Then there exists a differential  $n$ -form  $\omega$  and a point  $b \in f^{-1}\{0\}$ , close to 0, such that  $Z_{top,b}(f, \omega; s)$  has a pole  $s_0$  satisfying  $\exp(2\pi\sqrt{-1}s_0) = \lambda$ , and such that  $\tilde{s} \in \mathbb{Z}$  for all poles  $\tilde{s}$  of  $Z_{top,b}(f, \omega; s)$  different from  $s_0$ .*
- (2) *If  $f^{-1}\{0\}$  has an isolated singularity at 0, then we can take 0 itself as point  $b$  in (1).*

Since  $\exp(2\pi\sqrt{-1}\mathbb{Z}) = \{1\}$ , and 1 is always a monodromy eigenvalue of  $f$ , the set of differential forms in Theorem 0.5(1), for varying  $\lambda$ , satisfies the requirements of Question 1.

## 1. PREREQUISITES

**1.1. Monodromy.** Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  be a non-constant polynomial function satisfying  $f(b) = 0$ . Let  $B \subset \mathbb{C}^n$  be a small enough ball with centre  $b$ ; the restriction  $f|_B$  is a topological fibration over a small enough pointed disc  $D \subset \mathbb{C} \setminus \{0\}$  with centre 0. The fibre  $F_b$  of this fibration is called the (local) Milnor fibre of  $f$  at  $b$ ; see e.g. [Mil68]. The counterclockwise generator of the fundamental group of  $D$  induces an automorphism of the cohomologies  $H^q(F_b, \mathbb{C})$ , which is called the (local) monodromy of  $f$  at  $b$ . By a monodromy eigenvalue of  $f$  at  $b$  we mean an eigenvalue of the monodromy action on a least one of the  $H^q(F_b, \mathbb{C})$ . It is well known that  $H^q(F_b, \mathbb{C}) = 0$  for  $q \geq n$ , and that all monodromy eigenvalues are roots of unity.

Let  $P_q(t)$  denote the characteristic polynomial of the monodromy action on  $H^q(F_b, \mathbb{C})$ . If  $f = \prod_j f_j^{N_j}$  is the decomposition of  $f$  in irreducible components and  $d := \gcd_j N_j$ , then  $P_0(t) = t^d - 1$ . When  $b$  is an isolated singularity of  $f^{-1}\{0\}$ , then  $H^q(F_b, \mathbb{C}) = 0$  for  $q \neq 0, n-1$ ; and  $P_0(t) = t - 1$ .

**Definition 1.1.** The *monodromy zeta function*  $\zeta_{f,b}(t)$  of  $f$  at  $b$  is the alternating product of all characteristic polynomials  $P_q(t)$ :

$$\zeta_{f,b}(t) := \prod_{q=0}^{n-1} P_q(t)^{(-1)^q}.$$

(Note that there are also other conventions, see for example [A'C75], [AGV88].) In particular, for an isolated singularity the knowledge of  $\zeta_{f,b}(t)$  and of  $P_{n-1}(t)$  are equivalent. In general, the following result says that ‘knowing all eigenvalues is equivalent to knowing all zeroes and poles of all monodromy zeta functions’.

**Lemma 1.2.** [Den93, Lemma 4.6] *Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  be a non-constant polynomial function. If  $\lambda$  is a monodromy eigenvalue of  $f$  at  $b \in f^{-1}\{0\}$ , then there exists  $P \in f^{-1}\{0\}$  (arbitrarily close to  $b$ ) such that  $\lambda$  is a zero or pole of the monodromy zeta function of  $f$  at  $P$ .*

Also the monodromy zeta function can be described in terms of an embedded resolution.

**Theorem 1.3** (A’Campo’s formula [A’C75]). *Take an embedded resolution  $\pi : X \rightarrow \mathbb{C}^n$  of  $f^{-1}\{0\}$  (that is an isomorphism outside the inverse image of  $f^{-1}\{0\}$ ). Denote by  $E_i, i \in I$ ,*

the irreducible components of the inverse image  $\pi^{-1}(f^{-1}\{0\})$ , and by  $N_i$  the multiplicity of  $E_i$  in the divisor of  $\pi^*f$ . We put  $E_i^\circ := E_i \setminus \cup_{\ell \neq i} E_\ell$  for  $i \in I$ . Then we have

$$\zeta_{f,b}(t) = \prod_{i \in I} (t^{N_i} - 1)^{\chi(E_i^\circ \cap \pi^{-1}\{b\})}.$$

**1.2. ‘Curvettes’ in higher dimension.** We briefly recall a construction in [Vey07], generalizing the notion of curvette in dimension two.

Let  $X_0$  be a smooth quasi-projective (complex) variety of dimension  $n$  and let

$$(1) \quad X_0 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_m} X_m$$

be a composition  $\pi$  of  $m$  blowing-ups  $\pi_i$  at a smooth irreducible centre  $Z_{i-1}(\subset X_{i-1})$ , having normal crossings with the exceptional locus of  $\pi_1 \circ \cdots \circ \pi_{i-1}$ . Denote the exceptional locus of  $\pi_i$ , as well as its consecutive strict transforms, by  $E_i$ .

**Proposition 1.4.** [Vey07, Proposition 3.2 and 3.3] *One can construct consecutively for  $i = 1, \dots, m$  a smooth hypersurface  $C_j$  on  $X_j$  such that*

- (1)  $C_j$  has normal crossing with  $E_1 \cup \dots \cup E_j$ , with (the strict transform of) previously created  $C_1, \dots, C_{j-1}$ , and with the next centre of blowing-up  $Z_j$  (and such that  $Z_j \not\subset C_j$ ).
- (2) Given another hypersurface  $H$  on  $X_m$ , having normal crossings with  $E_1 \cup \dots \cup E_m$ , we can choose  $C_1, \dots, C_m$  such that furthermore  $H$  and all  $E_i$  and  $C_i$  form a normal crossings divisor on  $X_m$ .
- (3) Denote  $\overline{C}_j = \pi(C_j) \subset X_0$  and  $\pi^*\overline{C}_j = \sum_{i=1}^m a_{ji}E_i + C_j$  for  $j = 1, \dots, m$ . Then the determinant of the matrix  $(a_{ij}) \in \mathbb{Z}^{m \times m}$  is equal to 1.

## 2. RESULTS

Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be a non-zero polynomial function (germ). We will use an appropriate embedded resolution of  $f^{-1}\{0\}$ , for which we first fix notation.

**2.1. Notation.** Let  $f : X_0(\subset \mathbb{C}^n) \rightarrow \mathbb{C}$  be a relevant representative of  $f$ , in the sense that some embedded resolution of  $f^{-1}\{0\} \subset X_0$  only has exceptional components that intersect the inverse image of 0. Take such an embedded resolution  $\pi : X_m \rightarrow X_0$ , which is a composition of  $m$  blowing-ups as in (1).

- Let  $B_i, i \in I^s$ , be the irreducible components of the strict transform of  $f^{-1}\{0\}$ . We may and will assume that these are pairwise disjoint. Let also  $E_i, i \in I^e = \{1, \dots, m\}$ , be the exceptional components of  $\pi$ .
- Define  $N_i, i \in I^e$ , and  $M_i, i \in I^s$ , such that

$$\operatorname{div}(\pi^*f) = \sum_{i \in I^e} N_i E_i + \sum_{i \in I^s} M_i B_i.$$

Define  $\Delta_i, i \in I^e$ , such that

$$K_\pi = \operatorname{div}(\pi^*dx) = \sum_{i \in I^e} \Delta_i E_i.$$

- Introduce  $C_i, i \in I^e = \{1, \dots, m\}$ , by taking them as in Proposition 1.4. (The extra hypersurface  $H$  considered in Proposition 1.4(2) is  $f^{-1}\{0\}$ .) Let  $g_i$  be a defining polynomial for  $\overline{C}_i = \pi(C_i)$ . Recall that

$$\pi^*\overline{C}_j = \sum_{i \in I^e} a_{ji} E_i + C_j$$

with  $\det(a_{ij}) = 1$ .

- Let  $f_i, i \in I^s$ , be a defining polynomial for  $\overline{B}_i = \pi(B_i)$ , and denote

$$\pi^*\overline{B}_j = \sum_{i \in I^e} b_{ji} E_i + B_j.$$

**Theorem 2.1.** *Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be a non-zero polynomial function (germ). Let  $\lambda$  be a zero or pole of the monodromy zeta function  $\zeta_{f,0}(t)$  of  $f$  at 0. Then there exists a differential  $n$ -form  $\omega$  such that  $Z_{top,0}(f, \omega; s)$  has a pole  $s_0$  satisfying  $\lambda = \exp(2\pi\sqrt{-1}s_0)$  and such that  $\tilde{s} \in \mathbb{Z}$  for all poles  $\tilde{s}$  of  $Z_{top,0}(f, \omega; s)$  different from  $s_0$ .*

*Proof.* We first treat the obvious case where  $\pi$  is the identity. This means that  $f$  is of the form  $f_1^{M_1}$ , with  $f_1^{-1}\{0\}$  smooth at 0, since we assumed that the components of the strict transform are disjoint. Then  $\zeta_{f,0}(t) = P_0(t) = t^{M_1} - 1$ . Taking  $\lambda = \exp\left(-2\pi\sqrt{-1}\frac{a}{M_1}\right)$ , with  $a \in \{1, 2, \dots, M_1\}$ , the  $n$ -form  $\omega = f_1^{a-1} dx$  does the job, since  $Z_{top,0}(f, \omega; s) = 1/(a + sM_1)$ .

From now on, we assume that  $\pi$  is not the identity. Then every point in  $\pi^{-1}\{0\}$  belongs to some exceptional component. This implies that  $E_i^\circ \cap \pi^{-1}\{0\} = \emptyset$  for  $i \in I^s$ , and consequently the product in Theorem 1.3 is in fact over  $i \in I^e$ . Hence there exist by assumption some  $i_0 \in I^e$  and  $a \in \{1, \dots, N_{i_0}\}$  such that

$$\lambda = \exp\left(-2\pi\sqrt{-1}\frac{a}{N_{i_0}}\right) \quad \text{and} \quad \chi(E_{i_0}^\circ \cap \pi^{-1}\{0\}) \neq 0.$$

We will look for adequate  $m_i \in \mathbb{Z}_{\geq 0}, i \in I^e$ , and  $n_i \in \mathbb{Z}_{\geq 0}, i \in I^s$ , such that the differential form

$$\omega = \left(\prod_{i \in I^s} f_i^{n_i}\right) \left(\prod_{i \in I^e} g_i^{m_i}\right) dx$$

satisfies the desired properties. Define  $\nu_i \in \mathbb{Z}_{\geq 0}, i \in I^e$ , such that

$$\operatorname{div}(\pi^*\omega) = \sum_{i \in I^e} (\nu_i - 1) E_i + \sum_{j \in I^e} m_j C_j + \sum_{i \in I^s} n_i B_i.$$

We can also write

$$\operatorname{div}(\pi^*\omega) = \sum_{i \in I^e} \Delta_i E_i + \sum_{j \in I^e} m_j \pi^*\overline{C}_j + \sum_{j \in I^s} n_j \pi^*\overline{B}_j,$$

and thus

$$(2) \quad \nu_i - 1 = \Delta_i + \sum_{j \in I^e} m_j a_{ji} + \sum_{j \in I^s} n_j b_{ji} \quad \text{for } i \in I^e.$$

Then we obtain the following expression for the topological zeta function:

$$(3) \quad Z_{top,0}(f, \omega; s) = \sum_{\substack{J^e \subset I^e, J^s \subset I^s, L^e \subset I^e \\ J^e \cup J^s \cup L^e \neq \emptyset}} \chi(E_{J^e, J^s, L^e}^\circ \cap \pi^{-1}\{0\}) \\ \prod_{i \in J^e} \frac{1}{N_i s + \nu_i} \prod_{i \in J^s} \frac{1}{M_i s + n_i + 1} \prod_{i \in L^e} \frac{1}{1 + m_i},$$

where

$$E_{J^e, J^s, L^e}^\circ = \left( \bigcap_{i \in J^e} E_i \cap \bigcap_{i \in J^s} B_i \cap \bigcap_{i \in L^e} C_i \right) \setminus \left( \bigcup_{i \notin J^e} E_i \cup \bigcup_{i \notin J^s} B_i \cup \bigcup_{i \notin L^e} C_i \right).$$

Note that

$$E_{i_0}^\circ = E_{i_0} \setminus \left( \bigcup_{i \in I^e \setminus \{i_0\}} E_i \cup \bigcup_{i \in I^s} B_i \right) = \cup_{L^e \subset I^e} E_{\{i_0\}, \emptyset, L^e}^\circ.$$

Define  $N = \text{lcm}(\{N_i, i \in I^e\} \cup \{M_i, i \in I^s\})$ , let  $p$  be coprime to  $N$  and let  $q \in \mathbb{Z}$  such that  $pq \equiv 1 \pmod{N}$ .

The remainder of the proof consists in putting conditions on the  $m_i$  and  $n_i$ , and arguing that they can be satisfied. First, we assume that  $n_i \equiv -1 \pmod{pN}$  for all  $i \in I^s$ . Next, we put also on the  $m_i, i \in I^e$ , congruence conditions mod  $pN$ . Using (2) and the fact that  $\det(a_{ij}) = 1$ , we do this indirectly by putting such conditions on the  $\nu_i$ . More precisely, we assume that  $\nu_i \equiv 0 \pmod{pN}$  if  $i \neq i_0$ , and that  $\nu_{i_0} \equiv 1 + pq(a-1) \pmod{pN}$ . This implies that  $\nu_{i_0} \equiv a \pmod{N_{i_0}}$  and that  $p$  does not divide  $\nu_{i_0}$ .

Hence  $s_0 = -\frac{\nu_{i_0}}{N_{i_0}}$  will never be an integer divisible by  $p$ . And on the other hand, *all other candidate poles*, i.e., elements of the set

$$\left\{ -\frac{\nu_i}{N_i} \mid i \in I^e \setminus \{i_0\} \right\} \cup \left\{ -\frac{n_i + 1}{M_i} \mid i \in I^s \right\},$$

are integers divisible by  $p$ . We conclude that  $s_0$  is a candidate pole of order at most one with the property  $\lambda = \exp\left(-2\pi\sqrt{-1}\frac{a}{N_{i_0}}\right) = \exp(2\pi\sqrt{-1}s_0)$ , and with residue

$$(4) \quad R = \frac{1}{N_{i_0}} \left( \sum_{J^e \subset I^e \setminus \{i_0\}, J^s \subset I^s, L^e \subset I^e} \chi(E_{J^e \cup \{i_0\}, J^s, L^e}^\circ \cap \pi^{-1}\{0\}) \right. \\ \left. \prod_{i \in J^e} \frac{1}{N_i s_0 + \nu_i} \prod_{i \in J^s} \frac{1}{M_i s_0 + n_i + 1} \prod_{i \in L^e} \frac{1}{1 + m_i} \right).$$

We will now prove that we can take  $m_i$  and  $n_i$  such that  $R$  is not zero, meaning that  $s_0$  is really a pole.

- Assume first that

$$\chi(E_{\{i_0\}, \emptyset, \emptyset}^\circ \cap \pi^{-1}\{0\}) = \chi((E_{i_0}^\circ \setminus \cup_{i \in I^e} C_i) \cap \pi^{-1}\{0\}) \neq 0.$$

Note that the denominators occurring in the other terms of  $R$  are polynomials in the  $n_i, i \in I^s$ , and  $m_i, i \in I^e$ , that are all not constant. Indeed, the expression  $N_i s_0 + \nu_i$  is equal to

$$\Delta_i + 1 - \frac{N_i}{N_{i_0}}(\Delta_{i_0} + 1) + \sum_{j \in I^e} (a_{ji} - \frac{N_i}{N_{i_0}} a_{ji_0}) m_j + \sum_{j \in I^s} (b_{ji} - \frac{N_i}{N_{i_0}} b_{ji_0}) n_j,$$

see (2), and at least one of the coefficients  $a_{ji} - \frac{N_i}{N_{i_0}} a_{ji_0}$  is non-zero because the rows of the matrix  $(a_{ij})$  are linearly independent. The other expressions  $M_i s_0 + n_i + 1$  and  $1 + m_i$  are clearly not constant.

We conclude that  $R$  is never identically zero as rational function in the  $m_i$  and  $n_i$ , and then there exist (positive)  $n_i, i \in I^s$ , and  $m_i, i \in I^e$ , such that  $R \neq 0$ .

- If however  $\chi(E_{\{i_0\}, \emptyset, \emptyset}^\circ \cap \pi^{-1}\{0\}) = 0$ , this expression for  $R$  could be identically zero. To remedy the problem, we adapt our choice of  $\omega$ , as in the proof of [Vey07, Theorem 3.5]. Namely, we take some  $t \geq 2$  and introduce for each  $i \in I^e$  different hypersurfaces  $C_{i,j}, j \in \{1, \dots, t\}$ , as in Proposition 1.4, with defining polynomial  $g_{i,j}$  for  $\pi(C_{i,j})$ . Then we choose adequate  $n_i$  and  $m_{i,j}$  such that the form

$$\omega = \left( \prod_{i \in I^s} f_i^{n_i} \right) \left( \prod_{i \in I^e} \prod_{j=1}^t g_{i,j}^{m_{i,j}} \right) dx$$

satisfies the desired properties. In fact, proceeding analogously as before, we have to show that the residue of the candidate pole  $s_0$  is not identically zero as rational function in the  $n_i$  and  $m_{i,j}$ . This follows from the fact that the relevant Euler characteristic in this case,

$$\chi((E_{i_0}^\circ \setminus \cup_{i \in I^e, 1 \leq j \leq t} C_{i,j}) \cap \pi^{-1}\{0\}),$$

is nonzero for  $t$  large enough. We refer for details to the last part of the proof of [Vey07, Theorem 3.5].  $\square$

**2.2. Proof of Theorem 0.5.** Part (1) is immediate from Theorem 2.1 and Lemma 1.2. Part (2) is shown by the same argument as in the proof of [Vey07, Theorem 3.6].

**Remark 2.2.** The results of Theorem 0.5 are also true for the Hodge and motivic zeta function. The point is that, for a given  $f$  and  $\omega$ , the motivic zeta function specializes to the Hodge zeta function, which in turn specializes to the topological zeta function. In particular, a pole of the topological zeta function will induce a pole of the other two. The converse is not clear.

The main aspect of the problem that we treated here is, given a monodromy eigenvalue  $\lambda$  of  $f$ , find a form  $\omega$  such that the zeta function associated to  $f$  and  $\omega$  has a pole ‘inducing  $\lambda$ ’. We proved this result on the level of the topological zeta function, implying the analogous result for the ‘finer’ zeta functions.

A priori there could be a problem concerning the other poles of the finer zeta functions. Indeed, in general a given candidate pole  $-\nu_i/N_i$  for the topological/Hodge/motivic zeta function associated to  $f$  and  $\omega$  could be a pole for the Hodge/motivic zeta function, whilst it is not a pole for the topological zeta function. However, in the proof of Theorem 2.1 we saw that not only  $\tilde{s} \in \mathbb{Z}$  for all poles of  $Z_{top,0}(f, \omega; s)$  different from  $s_0$ , but also for *all candidate poles*  $-\nu_i/N_i$  different from  $s_0$ . Hence, all these candidate poles for the Hodge/motivic zeta

function induce the monodromy eigenvalue 1, and we do not need to worry whether they are poles or not.

## REFERENCES

- [A'C75] N. A'Campo. La fonction zêta d'une monodromie. *Comment. Math. Helv.*, 50:233–248, 1975.
- [ACLM02] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo, and A. Melle Hernández. Monodromy conjecture for some surface singularities. *Ann. Sci. École Norm. Sup. (4)*, 35(4):605–640, 2002.
- [ACLM05] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo, and A. Melle Hernández. Quasi-ordinary power series and their zeta functions. *Mem. Amer. Math. Soc.*, 178(841):vi+85, 2005.
- [AGV88] V. I. Arnol'd, S. M. Guseïn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1988. Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous, Translation revised by the authors and James Montaldi.
- [AKMW02] D. Abramovich, K. Karu, K. Matsuki, and J. Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.*, 15(3):531–572 (electronic), 2002.
- [Ati70] M. F. Atiyah. Resolution of singularities and division of distributions. *Comm. Pure Appl. Math.*, 23:145–150, 1970.
- [Bar84a] D. Barlet. Contribution du cup-produit de la fibre de Milnor aux pôles de  $|f|^{2\lambda}$ . *Ann. Inst. Fourier (Grenoble)*, 34(4):75–107, 1984.
- [Bar84b] D. Barlet. Contribution effective de la monodromie aux développements asymptotiques. *Ann. Sci. École Norm. Sup. (4)*, 17(2):293–315, 1984.
- [Bar85] D. Barlet. Contribution effective dans le réel. *Compositio Math.*, 56(3):351–359, 1985.
- [Bar04] D. Barlet. Singularités réelles isolées et développements asymptotiques d'intégrales oscillantes. In *Actes des Journées Mathématiques à la Mémoire de Jean Leray*, volume 9 of *Sémin. Congr.*, pages 25–50. Soc. Math. France, Paris, 2004.
- [Ber72] I. N. Bernšteïn. Analytic continuation of generalized functions with respect to a parameter. *Funkt. Anal. Appl.*, 6(4):26–40, 1972.
- [BG69] I. N. Bernšteïn and S. I. Gel'fand. Meromorphy of the function  $P^\lambda$ . *Funkt. Anal. Appl.*, 3(1):84–85, 1969.
- [BM93] D. Barlet and A. Mardhy. Un critère topologique d'existence de pôles pour le prolongement méromorphe de  $\int_A f^\lambda \square$ . *Ann. Inst. Fourier (Grenoble)*, 43(3):743–750, 1993.
- [BMT11] N. Budur, M. Mustață, and Z. Teitler. The monodromy conjecture for hyperplane arrangements. *Geom. Dedicata*, 153:131–137, 2011.
- [BV13] B. Bories and W. Veys. Igusa's  $p$ -adic local zeta function and the monodromy conjecture for non-degenerated surface singularities. *Memoirs of the American Mathematical Society*, (to appear).
- [Cau14] T. Cauwbergs. Splicing for motivic zeta functions (preprint), (arXiv:1411.0499).
- [Den93] J. Denef. Degree of local zeta functions and monodromy. *Compositio Math.*, 89(2):207–216, 1993.
- [DL92] J. Denef and F. Loeser. Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques. *J. Amer. Math. Soc.*, 5(4):705–720, 1992.
- [DL98] J. Denef and F. Loeser. Motivic Igusa zeta functions. *J. Algebraic Geom.*, 7(3):505–537, 1998.
- [Igu75] J. Igusa. Complex powers and asymptotic expansions. II. Asymptotic expansions. *J. Reine Angew. Math.*, 278/279:307–321, 1975.
- [JM98] A. Jeddi and A. Mardhy. Pôles de  $\int_A f^s \bullet$  pour une singularité presque isolée. *Manuscripta Math.*, 97(4):435–452, 1998.
- [Kas83] M. Kashiwara. Vanishing cycle sheaves and holonomic systems of differential equations. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 134–142. Springer, Berlin, 1983.
- [Loe88] F. Loeser. Fonctions d'Igusa  $p$ -adiques et polynômes de Bernstein. *Amer. J. Math.*, 110(1):1–21, 1988.
- [LV09] A. Lemahieu and W. Veys. Zeta functions and monodromy for surfaces that are general for a toric idealistic cluster. *Int. Math. Res. Not. IMRN*, (1):11–62, 2009.
- [LVP11] A. Lemahieu and L. Van Proeyen. Monodromy conjecture for nondegenerate surface singularities. *Trans. Amer. Math. Soc.*, 363(9):4801–4829, 2011.

- [Mal83] B. Malgrange. Polynômes de Bernstein-Sato et cohomologie évanescence. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 243–267. Soc. Math. France, Paris, 1983.
- [Mil68] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.
- [NV10] A. Némethi and W. Veys. Monodromy eigenvalues are induced by poles of zeta functions: the irreducible curve case. *Bull. Lond. Math. Soc.*, 42(2):312–322, 2010.
- [NV12] A. Némethi and W. Veys. Generalized monodromy conjecture in dimension two. *Geom. Topol.*, 16(1):155–217, 2012.
- [Rod05] B. Rodrigues. On the geometric determination of the poles of Hodge and motivic zeta functions. *J. Reine Angew. Math.*, 578:129–146, 2005.
- [Vey93] W. Veys. Poles of Igusa’s local zeta function and monodromy. *Bull. Soc. Math. France*, 121(4):545–598, 1993.
- [Vey01] W. Veys. Zeta functions and “Kontsevich invariants” on singular varieties. *Canad. J. Math.*, 53(4):834–865, 2001.
- [Vey06a] W. Veys. Arc spaces, motivic integration and stringy invariants. In *Singularity theory and its applications*, volume 43 of *Adv. Stud. Pure Math.*, pages 529–572. Math. Soc. Japan, Tokyo, 2006.
- [Vey06b] W. Veys. Vanishing of principal value integrals on surfaces. *J. Reine Angew. Math.*, 598:139–158, 2006.
- [Vey07] W. Veys. Monodromy eigenvalues and zeta functions with differential forms. *Adv. Math.*, 213(1):341–357, 2007.
- [Wło03] J. Włodarczyk. Toroidal varieties and the weak factorization theorem. *Invent. Math.*, 154(2):223–331, 2003.

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