# LIMIT THEOREMS FOR NUMBER OF EDGES IN GENERALIZED RANDOM GRAPHS WITH RANDOM VERTEX WEIGHTS

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Abstract. We get central limit type theorems for the total number of edges in the generalized random graphs with random vertex weights under different moment conditions on the distributions of the weights.

Complex networks are attracting increased attention of researchers in various fields of science. In last years numerous network models have been proposed. In view of the uncertainty and the lack of regularity in real-world networks, these models are usually random graphs. Random graphs were first defined by Paul Erdős and Alfréd Rényi in their 1959 paper "On Random Graphs", see [5], and independently by Gilbert in [7]. The suggested models are closely related: there are $n$ isolated vertices and every possible edge occurs independently with probability $p:\ 0 < p < 1$. It is assumed that there are no self-loops. Later on these models have been generalized considerably. A natural generalization of the Erdős and Rényi random graph is that the equal edge probabilities are replaced by probabilities depending on the vertex weights. Vertices with higher weights are more likely to have more neighbors than vertices with small weights. Vertices with extremely high weights could act as hubs which are observed in many real-world networks.

The following generalized random graph model was first introduced by Britton et al., see [3]. Let $\{1, 2, ..., n\}$ be the set of vertices, and $W_i > 0$ be the weight of vertex $i, 1 \le i \le n$. The edge probability of the edge between any two vertices $i$ and $j$ is equal to

$$p_{ij} = \frac{W_i W_j}{L_n + W_i W_j},$$

where $L_n = \sum_{i=1}^n W_i$ denotes the total weight of all vertices, and the weights $W_i, i = 1, 2, \ldots, n$ can be taken to be deterministic or random. If we take all $W_i$-s as the same constant: $W_i \equiv n\lambda/(n - \lambda)$ for some $0 < \lambda < n$, it is easy to see that $p_{ij} = \lambda/n$ holds for all $1 \le i < j \le n$. That is, the Erdős–Rényi random graph with $p = \lambda/n$ is a special case of the generalized random graph. There are many versions of

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the generalized random graphs, such as the Poissonian random graphs (introduced by Norros and Reittu in [11] and studied by Bhamidi et al.[1]), rank-1 inhomogeneous random graphs (see [2]), random graphs with given prescribed degrees (see [4]), etc. Under unifying conditions (see [9]), all of the above mentioned random graph models are asymptotically equivalent, meaning that all events have asymptotically equal probabilities. For an updated review on results about such inhomogeneous random graphs see the Chapters 6 and 9 in [12].

In the present paper we assume that $W_i, i = 1, 2, \dots, n$, are independent identically distributed random variables distributed as $W$. Let $E_n$ be the total number of edges in a generalized random graph with vertex weights $W_1, W_2, \dots, W_n$. In [8], under the conditions that $W$ has a finite or infinite mean, several weak laws of large numbers for $E_n$ are established, see also Ch.6, [12]. For instance, in [8] and Ch.6, [12], it is proved that $E_n/n$ tends in probability to $\mathbb{E}W/2$, provided $\mathbb{E}W$ is finite.

Note that

$$E_n = \frac{1}{2}\sum_{i=1}^{n} D_i,$$

where $D_i, i = 1, 2, \dots, n$ is a degree of vertex $i$, i.e. the number of edges coming out from vertex $i$. It is clear, the random variables $D_i, i = 1, 2, \dots, n$ are dependent. The aim of the present paper is to refine the law of large numbers type results for $E_n$ and to get the central limit type theorems under different moment conditions for $W$. In Theorem 1 we assume that $\mathbb{E}W^2 < \infty$. It implies the standard normal limit distribution for $\{E_n\}$ after proper normalization. In Theorem 2 we assume that the distribution of $W$ belongs to the domain of attraction of a stable law $F$ with characteristic exponent $\alpha : 1 < \alpha < 2$. Here we prove that the limit distribution for normalized $E_n$ is $F$.

**Theorem 1.** *If* $\mathbb{E}W^2 < \infty$, *then*

$$\frac{2E_n - n\mathbb{E}W}{\sqrt{n\,(2\mathbb{E}W + Var(W))}} \xrightarrow{d} N(0,1).$$

*Proof.* Put for all integer $n \geq 1$

$$b_n = \frac{1}{2}n\,\mathbb{E}W,\ c_n = \frac{1}{2}\sqrt{n\,\mathrm{Var}(W)}. \tag{1}$$

For any $t \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\exp\Big\{it\frac{E_n - b_n}{c_n}\Big\} &= \mathbb{E}\exp\Big\{\frac{it}{c_n}\Big(\sum_{1\le i<j\le n} I_{ij} - b_n\Big)\Big\} \\
&= \mathbb{E}\Big(\mathbb{E}\Big(\exp\Big\{\frac{it}{c_n}\Big(\sum_{1\le i<j\le n} I_{ij} - b_n\Big)\Big\}\Big|W_1,\cdots,W_n\Big)\Big) \\
&= \mathbb{E}\Big(e^{-itb_n/c_n}\prod_{1\le i<j\le n}\frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j}\Big)
\end{aligned}$$

$$:= \mathbb{E}e^{Y_n},$$

where

$$Y_n = \sum_{1\le i<j\le n} \log\frac{L_n + e^{it/c_n}W_iW_j}{L_n+W_iW_j} - \frac{itb_n}{c_n}$$

$$(2) \quad = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \log\frac{L_n + e^{it/c_n}W_iW_j}{L_n+W_iW_j} - \frac{itb_n}{c_n} - \sum_{i=1}^{n}\log\frac{L_n+e^{it/c_n}W_i^2}{L_n+W_i^2}$$

and $\log(\cdot)$ is the principal value of the complex logarithm function.

By using the Maclaurin series expansion of $\log(1+x)$ for complex $x$ with $|x|<1$, we have that

$$\frac{|\log(1+x)|}{|x|} \longrightarrow 1, \quad \frac{|\log(1+x)-x|}{|x|^2} \longrightarrow \frac{1}{2} \text{ as } |x|\to 0.$$

Hence there exists some constant $c_0>0$ such that $|\log(1+x)|\le 2|x|$ and $|\log(1+x)-x|\le |x|^2$ hold for any $|x|\le c_0$.

Clearly, for any fixed $t$, there exists $n_0=n_0(t)\in\mathbb{N}$ such that for all $n\ge n_0$ and any $1\le i,j\le n$ one has

$$\Big|\frac{(e^{it/c_n}-1)W_iW_j}{L_n+W_iW_j}\Big| \le |e^{it/c_n}-1| \le |t|/c_n \le c_0.$$

Thus, since

$$(3) \qquad \frac{L_n}{n} \to \mathbb{E}W \ a.s. \text{ and } \frac{\sum_{i=1}^n W_i^2}{n} \to \mathbb{E}W^2 \ a.s.,$$

we have for any $n\ge n_0$

$$\begin{aligned}\Big|\sum_{i=1}^{n}\log\frac{L_n+e^{it/c_n}W_i^2}{L_n+W_i^2}\Big| &\le \sum_{i=1}^{n}\Big|\log\Big(1+\frac{(e^{it/c_n}-1)W_i^2}{L_n+W_i^2}\Big)\Big| \\ &\le 2|e^{it/c_n}-1|\sum_{i=1}^{n}\frac{W_i^2}{L_n+W_i^2} \\ (4) \qquad &\le 2\frac{|t|}{c_n}\frac{\sum_{i=1}^n W_i^2}{L_n} \to 0 \ a.s.\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log\frac{L_n+e^{it/c_n}W_iW_j}{L_n+W_iW_j} - \frac{itb_n}{c_n} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log\Big(1+\frac{(e^{it/c_n}-1)W_iW_j}{L_n+W_iW_j}\Big) - \frac{itb_n}{c_n} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)W_iW_j}{L_n+W_iW_j} - \frac{itb_n}{c_n} + O_1\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(e^{it/c_n}-1)^2W_i^2W_j^2}{(L_n+W_iW_j)^2}\end{aligned}$$

$$= \frac{1}{2}\Big(e^{it/c_n} - 1 - \frac{it}{c_n} + \frac{t^2}{2c_n^2}\Big)\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j}{L_n + W_iW_j}$$
$$+\frac{1}{2}\Big(\frac{it}{c_n} - \frac{t^2}{2c_n^2}\Big)\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n}$$
$$+O_1\sum_{i=1}^n\sum_{j=1}^n \frac{(e^{it/c_n}-1)^2W_i^2W_j^2}{(L_n+W_iW_j)^2}$$
$$= I_1 + I_2 + I_3, \tag{5}$$

where $|O_1| \le 1/2$. By (3) and the inequality $|e^{ix} - 1 - ix + x^2/2| \le |x|^3/6$ for any $x \in \mathbb{R}$, we have

$$|I_1| \le \frac{|t|^3}{12c_n^3}\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j}{L_n} = \frac{|t|^3L_n}{12c_n^3} \longrightarrow 0 \ a.s. \tag{6}$$

Similarly, by (3) and the inequality $|e^x - 1| \le |x|$, we get

$$|I_3| \le \frac{t^2}{2c_n^2}\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2}{L_n^2} = \frac{t^2}{c_n^2}\Big(\frac{1}{n}\sum_{i=1}^n W_i^2\Big)^2\Big(\frac{n}{L_n}\Big)^2 \longrightarrow 0 \ a.s. \tag{7}$$

Recalling the definition (1) for $b_n$ and $c_n$, we have

$$I_2 = \frac{1}{2}\Big(\frac{it}{c_n} - \frac{t^2}{2c_n^2}\Big)\Big(\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j}{L_n} - \sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2}{L_n(L_n+W_iW_j)}\Big) - \frac{itb_n}{c_n}$$
$$= it\frac{L_n - n\mathbb{E}W}{\sqrt{n\mathrm{Var}(W)}} - \frac{t^2L_n}{n\mathrm{Var}(W)} - \frac{1}{2}\Big(\frac{it}{c_n} - \frac{t^2}{2c_n^2}\Big)\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2}{L_n(L_n+W_iW_j)}.$$

Moreover, by (3) we get

$$\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2}{L_n(L_n+W_iW_j)} \le \sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2}{L_n^2} = \frac{\big(\sum_{i=1}^n W_i^2\big)^2}{L_n^2} \to \Big(\frac{\mathbb{E}W^2}{\mathbb{E}W}\Big)^2 \ a.s.$$

The central limit theorem yields

$$I_2 \xrightarrow{d} it\mathbf{N} - t^2\mathbb{E}W/\mathrm{Var}(W), \tag{8}$$

where $\mathbf{N}$ is a standard normal random variable. Now, it follows from (2)–(8) that

$$Y_n \xrightarrow{d} it\mathbf{N} - t^2\mathbb{E}W/\mathrm{Var}(W).$$

Hence, by noting that $|e^{Y_n}| \le 1$ and applying the Lebesgue dominated convergence theorem, we get that, for any $t \in \mathbb{R}$,

$$\mathbb{E}\exp\Big\{it\frac{E_n - b_n}{c_n}\Big\} = \mathbb{E}e^{Y_n} \to \mathbb{E}\exp\{it\mathbf{N} - t^2\mathbb{E}W/\mathrm{Var}(W)\}$$

$$= \exp\{-(1/2)t^2(1 + 2\mathbb{E}W/\mathrm{Var}(W))\}.$$

Thus, Theorem 1 is proved. □

In the following theorem we get convergence of the sequence $\{E_n\}$ under weaker moment conditions on $W_i$'s.

**Theorem 2.** *Let $W, W_1, W_2, \cdots$ be a sequence of i.i.d. nonnegative random variables and*

$$\frac{W_1 + \cdots + W_n - n\mathbb{E}W}{a_n} \xrightarrow{d} F, \tag{9}$$

*where $F$ is a stable distribution with characteristic exponent $\alpha : 1 < \alpha < 2$, then*

$$\frac{2E_n - n\mathbb{E}W}{a_n} \xrightarrow{d} F.$$

Before we start to prove the theorem, let us state some properties of the distribution of $W$.

If (9) holds true, then $a_n$ (see e.g. [6], ch.XVII, §5) is a regularly varying function with exponent $1/\alpha$ satisfying

$$n\mathbb{E}W^2 I(W \le a_n) \sim a_n^2, \tag{10}$$

and there exists some constant $c > 0$ and $h(x)$, a slowly varying function at $\infty$, such that

$$P(W > x) \sim cx^{-\alpha}h(x). \tag{11}$$

We shall use the following lemma.

**Lemma 1.** *If (11) holds with $\alpha : 1 < \alpha < 2$, then we have*

$$\mathbb{E}W^2 I(W \le x) \sim \frac{c\alpha}{2-\alpha} x^{2-\alpha}h(x),$$
$$\mathbb{E}W I(W \ge x) \sim c\,\frac{2-\alpha}{\alpha - 1} x^{1-\alpha}h(x).$$

The proof of the lemma see e.g. [6], ch.XVII, §5.

Now we are ready to prove Theorem 2.

*Proof.* Let $b_n = (1/2)\, n\,\mathbb{E}W$ and $c_n = (1/2)\, a_n$ with $a_n$ from (10). As in the proof of Theorem 1, for any $t \in \mathbb{R}$, we also write

$$\mathbb{E}\exp\left\{it\frac{E_n - b_n}{c_n}\right\} = \mathbb{E}e^{Y_n}$$

with new definition for $c_n$ and

$$Y_n = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log\frac{L_n + e^{it/c_n}W_iW_j}{L_n + W_iW_j} - \frac{itb_n}{c_n} - \sum_{i=1}^{n}\log\frac{L_n + e^{it/c_n}W_i^2}{L_n + W_i^2}.$$

For the last sum for any $n \geq n_0$, where $n_0 = n_0(t)$ is defined in the proof of Theorem 1, we have (cp. (4))

$$\Big|\sum_{i=1}^{n} \log \frac{L_n + e^{it/c_n} W_i^2}{L_n + W_i^2}\Big| \leq 2\frac{|t|}{c_n} \frac{\sum_{i=1}^n W_i^2}{L_n}.$$

Similarly to (5), we get

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \log \frac{L_n + e^{it/c_n} W_i W_j}{L_n + W_i W_j} - \frac{itb_n}{c_n} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(e^{it/c_n}-1) W_i W_j}{L_n + W_i W_j} - \frac{itb_n}{c_n} + O_1 \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(e^{it/c_n}-1)^2 W_i^2 W_j^2}{(L_n + W_i W_j)^2} \\
&= \frac{1}{2}\Big(e^{it/c_n} - 1 - \frac{it}{c_n}\Big) \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{W_i W_j}{L_n + W_i W_j} \\
&+ \frac{1}{2}\frac{it(L_n - 2b_n)}{c_n} - \frac{1}{2}\frac{it}{c_n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{W_i^2 W_j^2}{L_n(L_n + W_i W_j)} \\
&+ O_1 \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(e^{it/c_n}-1)^2 W_i^2 W_j^2}{(L_n + W_i W_j)^2}
\end{aligned}$$

with $|O_1| \leq 1/2$. Due to Theorem's condition we have $(L_n - 2b_n)/(2c_n) \xrightarrow{d} F$. Since

$$|e^{ix} - 1| \leq |x|, \ |e^{ix} - 1 - ix| \leq |x|^2/2 \text{ for all } x \in \mathbb{R},$$

in order to prove Theorem 2, we only need to show that

$$\frac{1}{a_n}\frac{\sum_{i=1}^n W_i^2}{L_n} \xrightarrow{p} 0, \tag{12}$$

$$\frac{1}{a_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{W_i W_j}{L_n + W_i W_j} \xrightarrow{p} 0, \tag{13}$$

$$\frac{1}{a_n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{W_i^2 W_j^2}{L_n(L_n + W_i W_j)} \xrightarrow{p} 0, \tag{14}$$

$$\frac{1}{a_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{W_i^2 W_j^2}{(L_n + W_i W_j)^2} \xrightarrow{p} 0. \tag{15}$$

For any $\gamma : \alpha > \gamma > 0$, we have $\mathbb{E}(W^2)^{(\alpha-\gamma)/2} = \mathbb{E}W^{\alpha-\gamma} < \infty$. Then by Marcinkiewicz–Zygmund's strong law of large numbers (see e.g. Theorem 4.23 in [10]) we have

$$n^{-2/(\alpha-\gamma)} \sum_{i=1}^{n} W_i^2 \to 0 \ a.s.$$

Since $a_n$ is a regularly varying function with exponent $1/\alpha$, then we have $1/a_n = o(n^{-1/\alpha+\gamma})$. Now choose $\gamma > 0$ such that

$$2/(\alpha-\gamma) - 1 - 1/\alpha + \gamma < 0 \ \text{ and } \ -2/\alpha + 1 + 2\gamma < 0.$$

Then we have

$$\frac{1}{a_n}\frac{\sum_{i=1}^n W_i^2}{L_n} = \frac{n^{2/(\alpha-\gamma)-1}}{a_n}\frac{\sum_{i=1}^n W_i^2/n^{2/(\alpha-\gamma)}}{L_n/n} = o(n^{2/(\alpha-\gamma)-1-1/\alpha+\gamma}) \longrightarrow 0 \ a.s.$$

and

$$\frac{1}{a_n^2}\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j}{L_n+W_iW_j} \le \frac{1}{a_n^2}\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j}{L_n} = \frac{n}{a_n^2}\frac{L_n}{n} = o(n^{-2/\alpha+1+2\gamma})\longrightarrow 0 \ a.s.$$

Thus we get (12) and (13).

To prove (14), we write

$$\begin{aligned}
&\frac{1}{a_n}\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2}{L_n(L_n+W_iW_j)} \\
&= \frac{1}{a_n}\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2 I(W_iW_j \le n)}{L_n(L_n+W_iW_j)} + \frac{1}{a_n}\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2 I(W_iW_j > n)}{L_n(L_n+W_iW_j)} \\
&\le \frac{1}{a_n}\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2 I(W_iW_j \le n)}{L_n^2} + \frac{1}{a_n}\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j I(W_iW_j > n)}{L_n} \\
&\le \frac{n^2}{a_nL_n^2}\sum_{i=1}^n\sum_{j=1}^n \frac{W_i^2W_j^2 I(W_iW_j \le n)}{n^2} + \frac{n}{a_nL_n}\sum_{i=1}^n\sum_{j=1}^n \frac{W_iW_j I(W_iW_j > n)}{n}.
\end{aligned}$$

Further, by (3) and by using the fact that $\mathbb{E}|X_n| \to 0$ implies $X_n \xrightarrow{p} 0$, in order to prove (14), it is sufficient to show that

$$\frac{1}{a_n}\mathbb{E}W_1^2W_2^2 I(W_1W_2 \le n)\longrightarrow 0, \tag{16}$$

$$\frac{n}{a_n}\mathbb{E}W_1W_2 I(W_1W_2 > n)\longrightarrow 0. \tag{17}$$

For any $\alpha \in (1,2)$, we can choose $\delta > 0$ satisfying $2-\alpha-1/\alpha+2\delta < 0$. By Lemma 1, there exists some constant $c_1 = c_1(\alpha,\delta) > 0$ such that

$$\mathbb{E}W^2I(W \le x) \le c_1x^{2-\alpha+\delta}, \ \mathbb{E}WI(W \ge x) \le c_1x^{1-\alpha+\delta}$$

hold for all $x > 1$. Hence

$$\begin{aligned}
&\frac{1}{a_n}\mathbb{E}W_1^2W_2^2I(W_1W_2 \le n) \\
&= \frac{1}{a_n}\mathbb{E}\Big(W_2^2I(W_2 \le n)\mathbb{E}(W_1^2I(W_1 \le n/W_2)|W_2)\Big) \\
&+\frac{1}{a_n}\mathbb{E}\Big(W_2^2I(W_2 > n)\mathbb{E}(W_1^2I(W_1 \le n/W_2)|W_2)\Big) \\
&\le \frac{c_1}{a_n}\mathbb{E}\Big(W_2^2(n/W_2)^{2-\alpha+\delta}\Big) + \frac{1}{a_n}\mathbb{E}\Big(W_2^2I(W_2 > n)(n/W_2)^2\Big)
\end{aligned}$$

$$= \frac{c_1 n^{2-\alpha+\delta}}{a_n} \mathbb{E}W^{\alpha-\delta} + \frac{n^2}{a_n} P(W > n).$$

Since by (11) we have $P(W > n) \sim cn^{-\alpha}h(n) = o(n^{-\alpha+\delta})$ and $1/a_n = o(n^{-1/\alpha+\delta})$, we get

$$\frac{1}{a_n} \mathbb{E}W_1^2 W_2^2 I(W_1 W_2 \le n) = o(n^{2-\alpha-1/\alpha+2\delta}) \to 0 \text{ as } x \to \infty.$$

Thus, we get (16).

Similarly, we have

$$\begin{aligned}
&\frac{n}{a_n} \mathbb{E}W_1 W_2 I(W_1 W_2 > n) \\
&= \frac{n}{a_n} \mathbb{E}\Big(W_2 I(W_2 \le n) \mathbb{E}(W_1 I(W_1 > n/W_2)|W_2)\Big) \\
&\quad + \frac{n}{a_n} \mathbb{E}\Big(W_2 I(W_2 > n) \mathbb{E}(W_1 I(W_1 > n/W_2)|W_2)\Big) \\
&\le \frac{c_1 n}{a_n} \mathbb{E}\Big(W_2 (n/W_2)^{1-\alpha+\delta}\Big) + \frac{n}{a_n} \mathbb{E}\Big(W_2 I(W_2 > n) \mathbb{E}W_1\Big) \\
&= \frac{c_1 n^{2-\alpha+\delta}}{a_n} \mathbb{E}W^{\alpha-\delta} + \frac{n}{a_n} \mathbb{E}W \mathbb{E}(W I(W > n)) \\
&\le \frac{c_1 n^{2-\alpha+\delta}}{a_n} \mathbb{E}W^{\alpha-\delta} + \frac{c_1 n^{2-\alpha+\delta}}{a_n} \mathbb{E}W = o(n^{2-\alpha-1/\alpha+2\delta}) \to 0.
\end{aligned}$$

Hence (17), and then (14), are proved.

And (15) follows from (14). The proof of Theorem 2 is complete. □

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