LOCAL FUNCTIONAL EQUATIONS FOR SUBMODULE ZETA FUNCTIONS ASSOCIATED TO NILPOTENT ALGEBRAS OF ENDOMORPHISMS

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Abstract. We give a sufficient criterion for generic local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms defined over number fields. This allows us, in particular, to prove various conjectures on such functional equations for ideal zeta functions of nilpotent Lie lattices. Via the Mal'cev correspondence, these results have corollaries pertaining to zeta functions enumerating normal subgroups of finite index in finitely generated nilpotent groups, most notably free such groups.

1. Introduction

1.1. Submodule zeta functions for nilpotent associative algebras of endomorphisms.

Let R be the ring of integers $\mathcal{O}$ of a number field or the completion $\mathcal{O}_\mathfrak{p}$ of such a ring at a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$. Let $\mathcal{L}$ be a free R-module of finite rank n and $\mathcal{E}$ be a subalgebra of the associative algebra $\mathrm{End}_R(\mathcal{L})$. For $m \in \mathbb{N}$, let

$$a_m(\mathcal{E} \curvearrowright \mathcal{L}) = \#\left\{H \leq \mathcal{L} \mid H \text{ is } R\text{-submodule of } \mathcal{L},\ |\mathcal{L} : H| = m,\ \text{and } \mathcal{E}H \subseteq H\right\}.$$

We define the *submodule zeta function of* $\mathcal{E}$ *acting on* $\mathcal{L}$ as the formal Dirichlet generating function

$$\zeta_{\mathcal{E} \curvearrowright \mathcal{L}}(s) = \sum_{m=1}^{\infty} a_m(\mathcal{E} \curvearrowright \mathcal{L}) m^{-s},$$

where s is a complex variable; cf. [17, Definition 2.1]. The submodule zeta function $\zeta_{\mathcal{E} \curvearrowright \mathcal{L}}(s)$ may be viewed as a (non-unital) analogue of Solomon's zeta function; see [23]. Assume now that R is the ring of integers $\mathcal{O}$ of a number field. Then $\zeta_{\mathcal{E} \curvearrowright \mathcal{L}}(s)$ satisfies the Euler product

$$\zeta_{\mathcal{E} \curvearrowright \mathcal{L}}(s) = \prod_{\mathfrak{p}} \zeta_{\mathcal{E}(\mathcal{O}_\mathfrak{p}) \curvearrowright \mathcal{L}(\mathcal{O}_\mathfrak{p})}(s), \tag{1.1}$$

where the product ranges over the nonzero prime ideals of $\mathcal{O}$ and $\mathcal{E}(\mathcal{O}_\mathfrak{p}) := \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{p}$ and $\mathcal{L}(\mathcal{O}_\mathfrak{p}) := \mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{p}$, regarded as an $\mathcal{O}_\mathfrak{p}$-algebra and $\mathcal{O}_\mathfrak{p}$-module, respectively. It follows from well-known results expressing counting functions such as $\zeta_{\mathcal{E}(\mathcal{O}_\mathfrak{p}) \curvearrowright \mathcal{L}(\mathcal{O}_\mathfrak{p})}(s)$ in terms of $\mathfrak{p}$-adic integrals that each of the Euler factors is a rational function in $|\mathcal{O} : \mathfrak{p}|^{-s}$; see, for instance, [8] for the case $\mathcal{O} = \mathbb{Z}$.

Assume now that $\mathcal{E}$ is nilpotent. The main objective of this paper is to establish in this case, under suitable conditions, functional equations for $\mathfrak{p}$-adic submodule zeta functions occurring as generic factors in Euler products of the form (1.1); see Theorem 1.2.

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Prominent examples of submodule zeta functions of nilpotent associative algebras of endomorphisms are ideal zeta functions of nilpotent Lie lattices, which we now recall. Let $\mathcal{L}$ be an *$\mathcal{O}$-Lie lattice*, i.e. a free and finitely generated $\mathcal{O}$-module of finite rank n equipped with an antisymmetric bi-additive form (or "Lie bracket") satisfying the Jacobi identity. By a *Lie ring* we mean a $\mathbb{Z}$-Lie lattice. For $m \in \mathbb{N}$, we write $a_m(\mathcal{L}) = \#\{H \triangleleft_{\mathcal{O}} \mathcal{L} \mid |\mathcal{L} : H| = m\}$ for the number of $\mathcal{O}$-ideals of $\mathcal{L}$ of index m in $\mathcal{L}$. The *ideal zeta function of* $\mathcal{L}$ is the Dirichlet generating series

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \sum_{m=1}^{\infty} a_m(\mathcal{L}) m^{-s};$$

cf. [13, Section 3]. It fits into the setup from above by considering the associative subalgebra $\mathcal{E} \subseteq \mathrm{End}_{\mathcal{O}}(\mathcal{L})$ generated by $\mathrm{ad}(\mathcal{L})$; clearly $a_m(\mathcal{L}) = a_m(\mathcal{E} \curvearrowright \mathcal{L})$. The Euler product (1.1) takes the form

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \prod_{\mathfrak{p}} \zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}_{\mathfrak{p}})}(s),$$

where, for each prime ideal $\mathfrak{p}$, the Euler factor $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}_{\mathfrak{p}})}(s)$ enumerates the $\mathcal{O}_{\mathfrak{p}}$-ideals of $\mathcal{L}(\mathcal{O}_{\mathfrak{p}})$ of finite additive index in $\mathcal{L}(\mathcal{O}_{\mathfrak{p}})$.

Returning to general nilpotent associative algebras of endomorphisms $\mathcal{E} \subseteq \mathrm{End}_{\mathcal{O}}(\mathcal{L})$ we define $\mathcal{O}$-submodules Z_i of $\mathcal{L}$, for $i \in \mathbb{N}_0$, by setting $Z_0 = \{0\}$ and

$$Z_{i+1}/Z_i = \mathrm{Cent}_{\mathcal{E}}(\mathcal{L}/Z_i) := \{x + Z_i \in \mathcal{L}/Z_i \mid \mathcal{E}(x) \subseteq Z_i\}$$

for $i > 0$. As $\mathcal{E}$ is nilpotent there exists $i \in \mathbb{N}$ such that $Z_i = \mathcal{L}$; cf. [14, Chapter 2, Section II]. We set

$$c = c(\mathcal{L}, \mathcal{E}) = \min\{i \in \mathbb{N}_0 \mid Z_i = \mathcal{L}\}.$$

(If $\mathcal{L}$ is a nilpotent $\mathcal{O}$-Lie lattice and $\mathcal{E}$ is the associative subalgebra generated by $\mathrm{ad}(\mathcal{L})$, then $(Z_i)_{i=0}^{c}$ is just the upper central series of $\mathcal{L}$ and c is the nilpotency class of $\mathcal{L}$.)

In this paper, we consider pairs $(\mathcal{L}, \mathcal{E})$ satisfying the following assumption.

Assumption 1.1. There exist free $\mathcal{O}$-submodules $\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_{c+1}$ of $\mathcal{L}$ such that

$$Z_i = \bigoplus_{j > c-i} \mathcal{L}_j$$

for $i = 0, 1, \dots, c+1$. Note that $\mathcal{L} = Z_c$, $\mathcal{L}_{c+1} = Z_0 = \{0\}$, and $\mathcal{L}_0 \cong Z_{c+1}/Z_c = \{0\}$. In particular,

$$\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_c$$

(direct sum of $\mathcal{O}$-modules).

Remark 1.1. Assumption 1.1 is only made for notational convenience. It is automatically satisfied if $\mathcal{O}$ is a unique factorization domain (e.g. $\mathbb{Z}$). In general, the "centralizers" Z_j will be isolated in $\mathcal{L}$ (viz. the quotients $\mathcal{L}/Z_j$ will be torsion-free), but may not allow complements. This may be mitigated by localizing $\mathcal{L}$ at a finite set of prime ideals of $\mathcal{O}$ or – by the general theory of finitely generated modules over Dedekind domains – by passing to a suitable finite index $\mathcal{O}$-submodule of $\mathcal{L}$; cf., for instance, the discussion in [26, Section 2.3]. In any case, only finitely many of the Euler factors in (1.1) are affected. As we are only looking to prove results for all but finitely many of these, making Assumption 1.1 means no loss of generality.

Our main results concern local submodule zeta functions associated to general nilpotent algebras $\mathcal{E}$ which satisfy the following condition.

Condition 1.1. *The nilpotent associative algebra* $\mathcal{E} \subseteq \mathrm{End}_{\mathcal{O}}(\mathcal{L})$ *is generated by elements* $c_1, \dots, c_d$ *such that, for all* $k = 1, \dots, d$ *and* $j = 1, \dots, c$,

$$c_k \mathcal{L}_j \subseteq \mathcal{L}_{j+1}. \tag{1.2}$$

For a matrix version of this condition, see Condition 2.1.

Given a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ we write q for the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$. The paper's main result establishes, in particular, functional equations upon inversion of q for all but finitely many of the Euler factors in (1.1) in case that $(\mathcal{L}, \mathcal{E})$ satisfies Condition 1.1. For $i \in \{0, 1, \dots, c\}$ we write

$$N_i = \mathrm{rk}_{\mathcal{O}} \bigoplus_{j \leq c-i} \mathcal{L}_j = \mathrm{rk}_{\mathcal{O}}(\mathcal{L}/Z_i),$$

noting that $N_0 = n = \mathrm{rk}_{\mathcal{O}}\mathcal{L}$ and $N_c = 0$. Throughout this paper, by a finite extension of a local ring of the form $\mathcal{O}_{\mathfrak{p}}$ we mean the ring of integers of a finite extension of the local field of fractions of $\mathcal{O}_{\mathfrak{p}}$.

Theorem 1.2. *Assume that* $\mathcal{E} \subseteq \mathrm{End}_{\mathcal{O}}(\mathcal{L})$ *satisfies Condition 1.1. Then, for almost all prime ideals* $\mathfrak{p}$ *of* $\mathcal{O}$ *and all finite extensions* $\mathfrak{O}$ *of* $\mathcal{O}_{\mathfrak{p}}$, *with residue field cardinality* q^f, *say, the following functional equation holds:*

$$\zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s)\big|_{q \to q^{-1}} = (-1)^n q^{f\left(\binom{n}{2} - s\left(\sum_{i=0}^{c-1} N_i\right)\right)} \zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s). \tag{1.3}$$

Example 1.3. For $c = 1$, Condition 1.1 is trivially satisfied as $\mathcal{E} = \{0\}$. Theorem 1.2 follows, in this case, by inspection of the classical formula, valid for any compact discrete valuation ring $\mathfrak{o}$ with residue field cardinality q, say,

$$\zeta_{\{0\} \curvearrowright \mathfrak{o}^n}(s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s}) \dots (1 - q^{n-1-s})} \tag{1.4}$$

for the Dirichlet generating series enumerating all finite index submodules of $\mathfrak{o}^n$; see, for instance, [13, Proposition 1.1].

Remark 1.4. In general, the operation $q \to q^{-1}$ means the following. Given a finite extension $\mathfrak{O}$ of $\mathcal{O}_{\mathfrak{p}}$, write $\mathfrak{P}$ for the maximal ideal of $\mathfrak{O}$. Thus $|\mathfrak{O}/\mathfrak{P}| = q^f$. Our proof of Theorem 1.2 will show that, excluding finitely many $\mathfrak{p}$, the zeta function $\zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s)$ may be written as a finite sum of the form

$$\sum_{i=1}^{M} |\overline{V_i}(\mathfrak{O}/\mathfrak{P})| W_i(q^f, q^{-fs}), \tag{1.5}$$

where the $|\overline{V_i}(\mathfrak{O}/\mathfrak{P})|$ denote the numbers of $\mathfrak{O}/\mathfrak{P}$-rational points of the reductions modulo $\mathfrak{p}$ of $\mathcal{O}$-defined smooth projective algebraic varieties V_i and rational functions $W_i(X, Y) \in \mathbb{Q}(X, Y)$. By the Weil conjectures, the numbers $|\overline{V_i}(\mathfrak{O}/\mathfrak{P})|$ may be written as alternating sums of Frobenius eigenvalues. By $q \to q^{-1}$ we mean the operation of inverting these eigenvalues and, of course, evaluating W_i at (q^{-f}, q^{fs}). If the reductions $\overline{V_i}$ are smooth (i.e. the V_i have good reduction modulo $\mathfrak{p}$), then Poincaré duality for étale cohomology entails symmetries among the Frobenius eigenvalues which imply that $|\overline{V_i}(\mathfrak{O}/\mathfrak{P})|_{q \to q^{-1}} = q^{-f \dim V_i} |\overline{V_i}(\mathfrak{O}/\mathfrak{P})|$. In the special case that $|\overline{V_i}(\mathfrak{O}/\mathfrak{P})|$ is given by a polynomial $F_{V_i} \in \mathbb{Z}[X]$ in the residue field cardinality q^f, this amounts to the palindromic symmetry $F_{V_i}(X^{-1}) = X^{-\dim V_i} F_{V_i}(X)$. The rational functions W_i admit a common denominator of the form $\prod_{j=1}^{r}(1 - X^{a_j} Y^{b_j})$ for $a_j \in \mathbb{N}_0$, $b_j \in \mathbb{N}$, $j \in \{1, \dots, r\}$. That the functional equation (1.3) does not depend on the chosen formula (1.5) is shown in [18, Section 5]. By [18, Corollary 5.3], it suffices to show (1.3) for $\mathfrak{O} = \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$.

Whilst we cannot make any quantitative statements on the finite set of prime ideals of $\mathcal{O}$ to be excluded in Theorem 1.2, there are examples illustrating that this set of "bad primes" is not empty in general.

Remark 1.5. Condition 1.1 is satisfied if $\mathcal{E}$ is cyclic (i.e. one may choose $d = 1$) or if $\mathcal{E}^2 = 0$ (i.e. $c \leq 2$). Moreover, it is stable under taking taking direct products and (certain) central quotients: If $\mathcal{E}_1 \subseteq \mathrm{End}_\mathcal{O}(\mathcal{L}_1)$ and $\mathcal{E}_2 \subseteq \mathrm{End}_\mathcal{O}(\mathcal{L}_2)$ both satisfy Condition 1.1, then so does $\mathcal{E}_1 \times \mathcal{E}_2 \subseteq \mathrm{End}_\mathcal{O}(\mathcal{L}_1 \oplus \mathcal{L}_2)$. If $\mathcal{E} \subseteq \mathrm{End}_\mathcal{O}(\mathcal{L})$ satisfies Condition 1.1 and $M \leq Z_1 = \mathcal{L}_c$ is an isolated central $\mathcal{O}$-submodule admitting a complement in $\mathcal{L}_c$, then the induced algebra of endomorphisms $\overline{\mathcal{E}} \subseteq \mathrm{End}_\mathcal{O}(\mathcal{L}/M)$ also satisfies Condition 1.1. As in Remark 1.1, the condition on M to be isolated and complementable may be dropped at the cost of disregarding finitely many (further) prime ideals $\mathfrak{p}$ of $\mathcal{O}$.

In contrast to Assumption 1.1, Condition 1.1 *does* delineate an interesting property. In Section 4 we discuss, along with several examples and applications of Theorem 1.2 in the context of ideal zeta functions of nilpotent Lie lattices, a number of known examples of such lattices whose generic local ideal zeta functions do *not* satisfy the kind of functional equations established by Theorem 1.2. To our knowledge, in all cases of ideal zeta functions of nilpotent Lie lattices which are known to satisfy generic local functional equations, Condition 1.1 is satisfied, supporting the speculation that it may actually be necessary for such functional equations.

Theorem 1.2 may be viewed as an analogue of the functional equation for the generic Igusa local zeta functions associated to a homogeneous polynomial over $\mathcal{O}$, as proven by Denef and Meuser [7]. In the light of this analogy, Condition 1.1 may be viewed as a "homogeneity condition" for nilpotent algebras of endomorphisms. For a further discussion of the connection with Igusa's local zeta function, necessary vs. sufficient conditions for local functional equations for submodule zeta functions, and potential interpretations of the left hand side of (1.3), see Section 3.

The proof of Theorem 1.2 is given in Section 2.1.

1.2. **Applications I: normal zeta functions of finitely generated nilpotent groups.**

Results such as Theorem 1.2 about ideal zeta functions of nilpotent Lie lattices have corollaries pertaining to normal subgroup zeta functions of finitely generated nilpotent groups, enumerating the groups' normal subgroups of finite index. Indeed, by the Mal'cev correspondence, for every finitely generated torsion-free nilpotent group G there exists a nilpotent Lie ring $\mathcal{L}$ such that, for almost all primes p, the local ideal zeta function $\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s)$ coincides with the local normal subgroup zeta function

$$\zeta^{\triangleleft}_{G,p}(s) = \sum_{H \triangleleft_p G} |G : H|^{-s}$$

of G at p, enumerating normal subgroups of G of p-power index in G; see [13, Theorem 4.1]. Moreover, every nilpotent Lie ring arises in this way.

Applications of Theorem 1.2 to ideal zeta functions of nilpotent Lie lattices are discussed in Section 4. Via the Mal'cev correspondence, all of them have analogues for normal subgroup zeta functions of finitely generated nilpotent groups. We only spell out the following corollary of Theorem 4.4 on free nilpotent Lie lattices. Let $c, d \in \mathbb{N}$

and $F_{c,d}$ be the free class-c-nilpotent group on d generators. For $i \in \{0, 1, \dots, c\}$, set

$$N_i = \sum_{1 \leq j \leq c-i} \frac{1}{j} \sum_{k|j} \mu(k) d^{j/k}, \tag{1.6}$$

where μ is the Möbius function. It is well-known that the N_i are the Hirsch lengths of the quotients of $F_{c,d}$ by the terms of the upper (or, equivalently, lower) central series. They are also equal to the $\mathbb{Z}$-ranks of the quotients of free class-c-nilpotent Lie ring on d generators $\mathfrak{f}_{c,d}$ by the terms of the upper central series; cf. Section 4.2. Note that $\mathfrak{f}_{c,d}$ is an nilpotent Lie ring associated to $F_{c,d}$ by the Mal'cev correspondence. Our Theorem 4.4 on the generic local ideal zeta functions of these Lie rings has the following consequence.

Corollary 1.6. *For almost all primes p, the following functional equation holds:*

$$\zeta^{\triangleleft}_{F_{c,d},p}(s)\Big|_{p \to p^{-1}} = (-1)^{N_0} p^{\binom{N_0}{2} - s\left(\sum_{i=0}^{c-1} N_i\right)} \zeta^{\triangleleft}_{F_{c,d},p}(s).$$

1.3. **Applications II: degree in q^{-fs} and behaviour at $s = -\infty$.** Let $(\mathcal{L}, \mathcal{E})$ be as above with $\mathcal{E}$ nilpotent, not necessarily satisfying Condition 1.1. The following definition is analogous to the concept of *uniform representability* of families of local zeta functions developed in [18, Section 3]; see also [10, § 1.2.4].

Definition 1.7. We call the pair $(\mathcal{L}, \mathcal{E})$ *almost uniform* if there exists a rational function $W \in \mathbb{Q}(X, Y)$ such that, for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}$ and all finite extensions $\mathfrak{O}$ of $\mathcal{O}_{\mathfrak{p}}$, with residue field cardinality q^f, say, $\zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s) = W(q^f, q^{-fs})$. By abuse of notation we also call $\zeta_{\mathcal{E} \curvearrowright \mathcal{L}}(s)$ *almost uniform* in this case.

Recall that, for general reasons, for all $\mathfrak{p}$ and all $\mathfrak{O}$, the local submodule zeta function $\zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s)$ is a rational function in q^{-fs}. The *degree* of a rational function $W = P/Q \in \mathbb{Q}(Z)$ is $\deg_Z W = \deg_Z P - \deg_Z Q$.

Conjecture 1.8. *For almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}$ and all finite extensions $\mathfrak{O}$ of $\mathcal{O}_{\mathfrak{p}}$, with residue field cardinality q^f, say,*

$$\deg_{q^{-fs}} \left(\zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s)\right) = -\sum_{i=0}^{c-1} N_i, \tag{1.7}$$

$$\lim_{s \to -\infty} \left(q^{-fs \sum_{i=0}^{c-1} N_i} \zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s)\right) = (-1)^n q^{-f\binom{n}{2}}. \tag{1.8}$$

If $(\mathcal{L}, \mathcal{E})$ is almost uniform, say $\zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s) = W(q^f, q^{-fs})$ for almost all $\mathfrak{p}$ and all $\mathfrak{O}$ for some $W \in \mathbb{Q}(X, Y)$, then $\deg_X W = -\binom{n}{2}$.

Informally speaking, equation (1.8) pins down the quotient of the leading coefficients of the polynomials in q^{-fs} giving numerator and denominator of the rational function $\zeta_{\mathcal{E}(\mathfrak{O}) \curvearrowright \mathcal{L}(\mathfrak{O})}(s) \in \mathbb{Q}(q^{-fs})$. The functional equation (1.3) established in Theorem 1.2 allows us to confirm this conjecture in the case that Condition 1.1 is satisfied. The following is proven in Section 2.2.

Corollary 1.9. *Conjecture 1.8 holds if $(\mathcal{L}, \mathcal{E})$ satisfies Condition 1.1.*

Further evidence for Conjecture 1.8 is provided by the numerous examples of ideal zeta functions of nilpotent Lie rings in [10] which do not satisfy generic local functional equations; cf. Section 4.

Recall that if the degree $\deg_Y W$ of a rational *generating* function $W = P/Q \in \mathbb{Q}(Y)$ is nonpositive, then $\deg_Y Q$ is the length of a shortest linear recurrence relation satisfied by the coefficients of W when expanded as a power series in Y; cf. [25, Theorem 4.1.1].

Equation (1.7) thus yields a lower bound on the length of such a linear recurrence relation. Determining this shortest length precisely seems to be a challenging problem.

Remark 1.10. Our conjecture (1.8) on the behaviour of $\zeta_{\mathcal{E}(\mathfrak{O})\curvearrowright\mathcal{L}(\mathfrak{O})}(s)$ at $s = -\infty$ may be compared with the conjectural behaviour at $s = 0$. In [17, Conjecture IV ($\mathfrak{P}$-adic form)] Rossmann conjectures that, for all (!) $\mathfrak{p}$ and all $\mathfrak{O}$,

$$(1 - q^{-fs})\zeta_{\mathcal{E}(\mathfrak{O})\curvearrowright\mathcal{L}(\mathfrak{O})}(s)\Big|_{s=0} = \frac{1}{(1-q^f)(1-q^{2f})\dots(1-q^{(n-1)f})};$$

cf. also [17, Section 8.3].

1.4. **Context and related work – zeta functions of groups, rings, and algebras.** Our proof of Theorem 1.2, presented in Section 2.1, proceeds by adapting the $\mathfrak{p}$-adic machinery developed in [29]. There, this technique is applied to establish generic local functional equations for a range of zeta functions of groups and rings. The most general of these applications is to subring zeta functions of arbitrary rings of finite additive rank, i.e. finitely generated abelian groups with some bi-additive multiplicative structure; see [29, Theorem A]. Via the Mal'cev correspondence, this translates into results for the generic Euler factors of the subgroup zeta functions of finitely generated nilpotent groups, i.e. Dirichlet generating series enumerating all finite index subgroups of such a group; see [29, Corollary 1.1]. In [29, Theorem C] we prove functional equations for generic local ideal zeta functions of nilpotent Lie rings of class 2 (or, equivalently, generic local normal zeta functions of finitely generated nilpotent groups of class 2). Theorem 1.2 generalizes this result.

A variant of the subgroup zeta function of a finitely generated nilpotent group G is its *pro-isomorphic zeta function* $\zeta^{\wedge}_G(s)$, enumerating the finite index subgroups of G whose profinite completions are isomorphic to the profinite completion of G. These zeta functions, too, enjoy an Euler product $\zeta^{\wedge}_G(s) = \prod_{p \text{ prime}} \zeta^{\wedge}_{G,p}(s)$ whose factors are rational functions in p^{-s}. There are numerous examples of groups whose local pro-isomorphic zeta functions satisfy functional equations akin to (1.3) (see [9, 2] and [5, 4]), but also an example showing that this symmetry phenomenon is not universal for pro-isomorphic zeta functions (see [3]). Formulating necessary and sufficient criteria for generic local functional equations in this context remains an interesting challenge.

In [31], Woodward computed the ideal zeta functions of the full upper triangular $n \times n$-matrices over $\mathbb{Z}$, as well as a number of combinatorially defined quotients of these algebras. He gives sufficient criteria for local functional equations, as well as some examples suggesting that these criteria might be necessary.

Local functional equations akin to (1.3) are also ubiquitous in the theory of representation zeta functions associated to arithmetic (and related pro-p) groups; see, for instance, [1, 26].

1.5. **Notation.** We write $\mathbb{N}$ for the natural numbers $\{1, 2, \dots\}$. Given a subset $I \subseteq \mathbb{N}$, we write I_0 for $I \cup \{0\}$. Given $m, n \in \mathbb{N}_0$, we set $[m] = \{1, 2, \dots, m\}$ and $]m, n] = \{m + 1, \dots, n\}$. We write $\operatorname{diag}\left(\lambda_1^{(e_1)}, \dots, \lambda_m^{(e_m)}\right)$ for the diagonal matrix composed of the matrices $\lambda_1 \operatorname{Id}_{e_1}, \dots, \lambda_m \operatorname{Id}_{e_m}$. Given matrices $A_1, \dots, A_n$ with the same number of rows, we denote by $(A_1|\dots|A_n)$ the matrix obtained by juxtaposition.

Throughout, p is a rational prime, $\mathcal{O}$ the ring of integers of a number field, and $\mathfrak{p}$ a nonzero prime ideal of $\mathcal{O}$. We write $\mathfrak{o}$ to denote a local ring of the form $\mathcal{O}_{\mathfrak{p}}$, the completion of $\mathcal{O}$ at $\mathfrak{p}$, with maximal ideal $\mathfrak{p}$, residue field $\mathfrak{o}/\mathfrak{p}$ of cardinality q and characteristic p, and $\mathfrak{p}$-adic valuation v. Note that these rings are exactly the finite extensions of the p-adic integers $\mathbb{Z}_p$. We denote by $\mathfrak{O}$ a finite extension of $\mathfrak{o}$, with maximal ideal $\mathfrak{P}$ and residue field cardinality $|\mathfrak{O}/\mathfrak{P}| = q^f$. In other words, $f = f(\mathfrak{O}, \mathfrak{o})$ is the relative degree of inertia. Given a matrix $A = (A_{ij}) \in \mathrm{Mat}_{m,n}(\mathfrak{o})$ we write $v(A) = \min\{v(A_{ij}) \mid i \in [m], j \in [n]\}$ for the minimal valuation of its entries.

The "Kronecker delta" δ_P associated to a property P is equal to 1 if P holds and equal to 0 otherwise.

2. Proofs of Theorem 1.2 and Corollary 1.9

2.1. **Proof of Theorem 1.2.**

2.1.1. *Overview of the proof.* Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}$. We write $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ and $K_\mathfrak{p}$ for the field of fractions of $\mathfrak{o}$. We are looking to establish the functional equation (1.3) for almost all zeta functions $\zeta_{\mathcal{E}(\mathfrak{o})\curvearrowright\mathcal{L}(\mathfrak{o})}(s)$, enumerating the $\mathfrak{o}$-submodules Λ of finite index in $\mathcal{L}(\mathfrak{o}) \cong \mathfrak{o}^n$ which are also $\mathcal{E}(\mathfrak{o})$-submodules, written $\Lambda \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})$. Clearly the latter property is really a property of the integral members of the *homothety class* $[\Lambda] = \{x\Lambda \mid x \in K_\mathfrak{p}^*\}$ of Λ in $K_\mathfrak{p}^n$: either all elements of $[\Lambda]$ are $\mathcal{E}(\mathfrak{o})$-submodules, or none are. We write $[\Lambda] \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})$ in the former case and set

$$\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})} = \{[\Lambda] \mid \Lambda \text{ } \mathfrak{o}\text{-lattice in } K_\mathfrak{p}^n,\ [\Lambda] \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})\}.$$

Evidently, every homothety class $[\Lambda]$ of $\mathfrak{o}$-lattices in $K_\mathfrak{p}^n$ contains a unique element $\Lambda_{\max}$ which is contained in $\mathcal{L}(\mathfrak{o})$ and maximal with respect to this property. As the intersection of $[\Lambda]$ with the set of all full sublattices of $\mathcal{L}(\mathfrak{o})$ equals $\{\mathfrak{p}^m\Lambda_{\max} \mid m \in \mathbb{N}_0\}$ it thus suffices – in principle – to describe the elements of $\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}$ and to keep track of their maximal integral members' indices in $\mathcal{L}(\mathfrak{o})$. Indeed,

$$\zeta_{\mathcal{E}(\mathfrak{o})\curvearrowright\mathcal{L}(\mathfrak{o})}(s) = \frac{1}{1-q^{-ns}} \sum_{[\Lambda]\in\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}} |\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|^{-s}. \tag{2.1}$$

Geometrically, one may view the set of homothety classes of $\mathfrak{o}$-lattices in $K_\mathfrak{p}$ as the set of vertices $\mathcal{V}_n$ of the affine Bruhat-Tits building $\Delta(\mathrm{SL}_n(K_\mathfrak{p}))$; see, for instance, [12, Chapter 19]. Keeping track of the indices $|\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|$ is easy (cf. (2.5)), so the problem of computing the right hand side of (2.1) is to identify $\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}$ as a subset of $\mathcal{V}_n$. We give an informal overview of our way to address this challenge, deferring details for the time being.

Firstly, we define an equivalence relation $\sim$ on $\mathcal{V}_n$ (viz. *δ-equivalence*; cf. Definition 2.5) with the property that – provided $\mathcal{E} \neq \{0\}$, as we may assume without loss of generality – each $\sim$-class $\mathcal{C}$ is a totally ordered set naturally isomorphic to $(\mathbb{Z}, \leq)$. Moreover, the sets $\mathcal{C}_{\geq 0} := \mathcal{C} \cap \mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}$ correspond, under these isomorphisms, to $(\mathbb{N}_0, \leq)$. Setting

$$Z_{\mathcal{C}_{\geq 0}}(s) = \sum_{[\Lambda]\in\mathcal{C}_{\geq 0}} |\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|^{-s} \tag{2.2}$$

we thus obtain

$$\zeta_{\mathcal{E}(\mathfrak{o})\curvearrowright\mathcal{L}(\mathfrak{o})}(s) = \frac{1}{1-q^{-ns}} \sum_{\mathcal{C}\in\mathcal{V}_n/\sim} Z_{\mathcal{C}_{\geq 0}}(s). \tag{2.3}$$

Rather than to analyze the functions $Z_{\mathcal{C}_{\geq 0}}(s)$ directly, we extend the sums (2.2) defining them to the whole of $\mathcal{C}$ in a way that allows us to recover the original sum algebraically, uniformly over all $\mathcal{C}$. The motivation to consider these extensions in the first place is that they give rise to a Dirichlet generating series $A^{\triangleleft}(s)$ satisfying a functional equation of the desired type; cf. (2.9).

To this end we introduce, secondly, judiciously chosen functions $\widetilde{m}_1, m_2 : \mathcal{V}_n \to \mathbb{N}_0$ with the properties

$$[\Lambda] \in \mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})} \Leftrightarrow \widetilde{m}_1([\Lambda]) = 0, \quad [\Lambda] \in \mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})} \Rightarrow m_2([\Lambda]) = 0.$$

The function $\widetilde{m}_1$ may be thought of as measuring a kind of “distance” to $\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}$. We set, for each $\mathcal{C} \in \mathcal{V}_n/\sim$,

$$Z_{\mathcal{C}}(s) = \sum_{[\Lambda]\in\mathcal{C}} |\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|^{-s} q^{-s((c-1)\widetilde{m}_1([\Lambda]) - m_2([\Lambda]))},$$

naturally extending the sum (2.2) defining $Z_{\mathcal{C}_{\geq 0}}(s)$. Our choices of $\widetilde{m}_1$ and m_2 will ensure that – up to a power of q^{-s} depending on $\mathcal{C}$ – the function $Z_{\mathcal{C}}(s)$ is the sum of two geometric progressions, covering respectively the “nonnegative” part $\mathcal{C}_{\geq 0}$ and the “negative” part $\mathcal{C}_{<0} := \mathcal{C} \setminus \mathcal{C}_{\geq 0}$ of $\mathcal{C}$, which only depend on the data $(N_i)_{i=0}^{c-1}$ but, crucially, not on $\mathcal{C}$; cf. Corollary 2.7 and Lemma 2.10. This entails that, with $\xi(s) := \frac{1-q^{-s\sum_{i=1}^{c-1}(n-N_i)}}{1-q^{-s(c-1)n}}$, we have $Z_{\mathcal{C}_{\geq 0}}(s) = Z_{\mathcal{C}}(s)\xi(s)$ for all $\sim$-classes $\mathcal{C}$. As indicated above, the point of these constructions is that the functions $\widetilde{m}_1$ and m_2 are defined in such a way that – at least for almost all $\mathfrak{p}$ – the function

$$A^{\triangleleft}(s) := \sum_{\mathcal{C}\in\mathcal{V}_n/\sim} Z_{\mathcal{C}}(s) = \sum_{[\Lambda]\in\mathcal{V}_n} |\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|^{-s} q^{-s((c-1)\widetilde{m}_1([\Lambda]) - m_2([\Lambda]))}$$

may be expressed in terms of $\mathfrak{p}$-adic integrals which fit the templates provided by [29]. In particular, it satisfies, for almost all $\mathfrak{p}$, the functional equation (2.9). Moreover,

$$\zeta_{\mathcal{E}(\mathfrak{o})\curvearrowright\mathcal{L}(\mathfrak{o})}(s) = \frac{1}{1-q^{-ns}} \sum_{\mathcal{C}\in\mathcal{V}_n/\sim} Z_{\mathcal{C}_{\geq 0}}(s) =$$
$$\frac{1}{1-q^{-ns}}\xi(s) \sum_{\mathcal{C}\in\mathcal{V}_n/\sim} Z_{\mathcal{C}}(s) = \frac{1}{1-q^{-ns}}\xi(s) A^{\triangleleft}(s).$$

The factor $\frac{1}{1-q^{-ns}}\xi(s)$ trivially satisfies a functional equation of the required type; see (2.10). Together with (2.9), this will prove Theorem 1.2.

2.1.2. *Cocentral bases.* For $i \in [c]_0$ we write $n_i = \mathrm{rk}_{\mathcal{O}}(\mathcal{L}_i)$, so that $N_i = \mathrm{rk}_{\mathcal{O}}(\mathcal{L}/Z_i) = \sum_{j\leq c-i} n_j$. An $\mathcal{O}$-basis $\mathbf{e} = (e_1, \dots, e_n)$ of $\mathcal{L}$ is called *cocentral* if

$$Z_i = \langle e_{N_i+1}, \dots, e_n \rangle_{\mathcal{O}}$$

for all $i \in [c]$. (This terminology extends the one introduced in [10, Definition 4.37].) By Assumption 1.1, cocentral bases clearly exist. Condition 1.1 is equivalent to the following condition.

Condition 2.1. *There exist generators $c_1, \dots, c_d$ of $\mathcal{E}$ and a cocentral $\mathcal{O}$-basis $\mathbf{e}$ of $\mathcal{L}$ such that, for all $k \in [d]$, the matrix C_k representing c_k with respect to $\mathbf{e}$ (acting from the right on row vectors) has the form*

$$C_k = \left(C_k^{(ij)}\right)_{i,j\in[c]} \in \mathrm{Mat}_n(\mathcal{O})$$

for blocks $C_k^{(ij)} \in \mathrm{Mat}_{n_i,n_j}(\mathcal{O})$ which are zero unless $j = i + 1$.

2.1.3. *Lattices, matrices, and the submodule condition.* Let $\mathbf{e}$ be a cocentral $\mathcal{O}$-basis of $\mathcal{L}$ as in Condition 2.1. It yields an $\mathfrak{o}$-basis of $\mathcal{L}(\mathfrak{o})$ which we also denote by $\mathbf{e}$ and which allows us to identify the $\mathfrak{o}$-module $\mathcal{L}(\mathfrak{o})$ with $\mathfrak{o}^n$ and $\mathcal{E}(\mathfrak{o})$ with a nilpotent subalgebra of $\mathrm{Mat}_n(\mathfrak{o})$, generated by matrices $C_1, \dots, C_d$ representing the $\mathfrak{o}$-linear operators $c_1, \dots, c_d$. Set $\Gamma = \mathrm{GL}_n(\mathfrak{o})$. A full $\mathfrak{o}$-sublattice Λ of $\mathcal{L}(\mathfrak{o})$ may be identified with a coset ΓM for a matrix $M \in \mathrm{GL}_n(K_{\mathfrak{p}}) \cap \mathrm{Mat}_n(\mathfrak{o})$, whose rows encode the coordinates with respect to $\mathbf{e}$ of a set of generators of Λ. Let $\pi \in \mathfrak{p}$ be a uniformizer of $\mathfrak{o}$. By the elementary divisor theorem there exist $I = \{i_1, \dots, i_l\}_< \subseteq [n-1]$, $r_n \in \mathbb{N}_0$, and $\mathbf{r} = (r_{i_1}, \dots, r_{i_l}) \in \mathbb{N}^l$, all uniquely determined by Λ, and $\alpha \in \Gamma$ such that $M = D\alpha^{-1}$, where

$$D = \pi^{r_n} \operatorname{diag}((\pi^{\sum_{\iota \in I} r_\iota})^{(i_1)}, (\pi^{\sum_{\iota \in I \setminus \{i_1\}} r_\iota})^{(i_2 - i_1)}, \dots, (\pi^{r_{i_l}})^{(i_l - i_{l-1})}, 1^{(n - i_l)}) \in \mathrm{Mat}_n(\mathfrak{o}). \tag{2.4}$$

We write $\nu([\Lambda]) = (I, \mathbf{r}_I)$. Note that $r_n = v(M)$ in the notation introduced in Section 1.5.

The matrix $\alpha \in \Gamma$ is only unique up to right multiplication by an element of a finite index subgroup $\Gamma_{I,\mathbf{r}}$ of Γ; see [29, Section 3.1] for details. Obviously,

$$|\mathcal{L}(\mathfrak{o}) : \Lambda| = \det D = q^{\sum_{\iota \in I \cup \{n\}} \iota r_\iota}. \tag{2.5}$$

We call Λ *maximal* if $r_n = 0$ and denote by $\Lambda_{\max}$ the unique maximal integral element of $[\Lambda]$. In the sequel we will often – and sometimes without explicit mentioning – toggle between lattices Λ and representing matrices M, extending notation for lattices to matrices representing them. We write, for instance, $[M]$ for the homothety class $[\Lambda]$ of the lattice Λ determined by ΓM and $M \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})$ if $\Lambda \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})$. We also write, given matrices A and B over $\mathfrak{o}$ with the same number of columns, $A \leq B$ if each row of A is contained in the $\mathfrak{o}$-row span of B.

Define the diagonal matrix

$$\delta := \operatorname{diag}((\pi^{c-1})^{(n_1)}, \dots, (\pi)^{(n_{c-1})}, 1^{(n_c)}) \in \mathrm{Mat}_n(\mathfrak{o}). \tag{2.6}$$

We remark that, up to a scalar factor, δ represents a map closely related to the map $\pi_{\mathcal{B}}$ in [10, Definition 4.40]. Note that $\det \delta = \pi^{\sum_{i=1}^{c-1} N_i}$. The following is a trivial consequence of Condition 2.1; for our purposes it is key nevertheless.

Lemma 2.1. *For all $k \in [d]$, $\delta C_k \delta^{-1} = \pi C_k$.*

For $i \in [n]$ and $k \in [d]$, write $c_k(e_i) = \sum_{l=1}^n \lambda_{ik}^l e_l$ for $\lambda_{ik}^l \in \mathcal{O}$. Then C_k satisfies $(C_k)_{rs} = \lambda_{rk}^s$ for $r, s \in [n]$. Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ be independent variables and set

$$\mathcal{R}(\mathbf{Y}) = \left(\sum_{l=1}^n \lambda_{ik}^l Y_l \right)_{ik} \in \mathrm{Mat}_{n \times d}(\mathcal{O}[\mathbf{Y}]).$$

Note that $\mathcal{R} = 0 \Leftrightarrow \mathcal{E} = 0 \Leftrightarrow c = 1$. In this case, Theorem 1.2 holds (cf. Example 1.3), so we may assume that $c > 1$. We write $\alpha[i]$ for the i-th column of a matrix $\alpha \in \Gamma$, so that $\mathcal{R}(\alpha[i]) \in \mathrm{Mat}_{n \times d}(\mathfrak{o})$. The following lemma is verified by a trivial computation.

Lemma 2.2. *For all $\alpha \in \Gamma$ and $\Delta \in \mathrm{Mat}_n(\mathfrak{o})$, and D as in* (2.4)*,*

$$(\forall k \in [d] : \ \Delta C_k \alpha \leq D) \Leftrightarrow (\forall i \in [n] : \ \Delta \mathcal{R}(\alpha[i]) \equiv 0 \bmod D_{ii}).$$

Proposition 2.3. *Given $M \in \mathrm{GL}_n(K_{\mathfrak{p}}) \cap \mathrm{Mat}_n(\mathfrak{o})$, there exists a unique $\widetilde{m}_1 = \widetilde{m}_1(M) \in \mathbb{N}_0$ such that, for all $m \in \mathbb{N}_0$,*

$$M\delta^m \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o}) \text{ if and only } m \geq \widetilde{m}_1.$$

In particular, $M \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})$ if and only if $\widetilde{m}_1 = 0$. Moreover, $\widetilde{m}_1 \leq \sum_{\iota \in I} r_\iota$.

Proof. Write $M = D\alpha^{-1}$ as above. Without loss we may assume $r_n = 0$. Using Lemmas 2.1 and 2.2 and the fact that $D_{ii} = \pi^{\sum_{i\leq\iota\in I} r_\iota}$ for $i \in [n]$, we obtain

$$\begin{aligned}
&M\delta^m \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})\\
&\qquad\Leftrightarrow \forall k \in [d] : M\delta^m C_k \leq M\delta^m\\
&\qquad\Leftrightarrow \forall k \in [d] : D\alpha^{-1}\delta^m C_k \delta^{-m}\alpha \leq D\\
&\qquad\Leftrightarrow \forall k \in [d] : \pi^m D\alpha^{-1} C_k \alpha \leq D\\
&\qquad\Leftrightarrow \forall i \in [n] : \pi^m D\alpha^{-1}\mathcal{R}(\alpha[i]) \equiv 0 \bmod D_{ii}\\
&\qquad\Leftrightarrow \forall i \in [n] : \pi^m D\alpha^{-1}\mathcal{R}(\alpha[i])\pi^{\sum_{i>\iota\in I} r_\iota} \equiv 0 \bmod \pi^{\sum_{\iota\in I} r_\iota}\\
&\qquad\Leftrightarrow \pi^m D\alpha^{-1}\left(\mathcal{R}(\alpha[1]) \mid \cdots \mid \mathcal{R}(\alpha[n])\right)\cdot\\
&\qquad\quad \operatorname{diag}(1^{(di_1)}, (\pi^{r_{i_1}})^{(d(i_2-i_1))}, \dots, (\pi^{\sum_{\iota\in I} r_\iota})^{(d(n-i_l))}) \equiv 0 \bmod \pi^{\sum_{\iota\in I} r_\iota}.
\end{aligned}$$

In the last congruence, we may replace α^{-1} by the adjunct matrix α^{adj}. Setting, for $i, r \in [n]$,

$$\begin{aligned}
\mathcal{R}_{(i)}(\alpha) &= \alpha^{\mathrm{adj}}\mathcal{R}(\alpha[i]),\\
v_{ir}^{(1)}(\alpha) &= \min\left\{v(\mathcal{R}_{(\iota)}(\alpha)_{\rho\sigma}) \mid \iota \leq i, \rho \geq r, \sigma \in [d]\right\},
\end{aligned}$$

and

$$m_1(M) = \min\left\{\sum_{\iota\in I} r_\iota, \sum_{r\leq\iota\in I} r_\iota + \sum_{i>\iota\in I} r_\iota + v_{ir}^{(1)}(\alpha) \;\middle|\; (i,r) \in [n]^2\right\},$$

we may rephrase the above equivalence as follows:

$$M\delta^m \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o}) \Leftrightarrow m \geq \sum_{\iota\in I} r_\iota - m_1(M) =: \widetilde{m}_1(M). \tag{2.7}$$

□

Definition 2.4. For a lattice Λ corresponding to a coset ΓM, we set $\widetilde{m}_1([\Lambda]) = \widetilde{m}_1(M)$.

Informally, $\widetilde{m}_1([\Lambda])$ is a 'distance' in $\mathcal{V}_n$ between $[\Lambda]$ and $\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}$.

2.1.4. *δ-equivalence.* Recall the diagonal matrix δ defined in (2.6).

Definition 2.5. Lattice classes $[\Lambda_1], [\Lambda_2] \in \mathcal{V}_n$ are called *δ-equivalent*, written $[\Lambda_1] \sim [\Lambda_2]$, if there exists $m \in \mathbb{Z}$ such that $[\Lambda_1] = [\Lambda_2\delta^m]$.

In the sequel, we use the terms *lattice class* for a homothety class of lattices and *δ-class* for a $\sim$-equivalence class of lattice classes. Proposition 2.3 asserts that every δ-class of lattice classes intersects $\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}$ nontrivially. Its proof also shows that, more precisely, in each δ-class $\mathcal{C}$ there is a unique lattice class $[\Lambda_0]$ such that $[\Lambda_0\delta^m] \leq_{\mathcal{E}(\mathfrak{o})} \mathcal{L}(\mathfrak{o})$ if and only if $m \in \mathbb{N}_0$. We shall say that $[\Lambda_0]$ *generates* $\mathcal{C}_{\geq 0}$ and write $\Lambda_{0,\max}$ for the unique maximal element of $[\Lambda_0]$. Setting

$$\begin{aligned}
\mathcal{C}_{\geq 0} &= \{[\Lambda_0\delta^m] \mid m \geq 0\} = \mathcal{C} \cap \mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})},\\
\mathcal{C}_{<0} &= \{[\Lambda_0\delta^m] \mid m < 0\} = \mathcal{C} \setminus \mathcal{C}_{\geq 0},
\end{aligned}$$

we obtain a partition $\mathcal{C} = \mathcal{C}_{\geq 0} \cup \mathcal{C}_{<0}$ and (2.3) holds with $Z_{\mathcal{C}_{\geq 0}}(s)$ defined as in (2.2).

Note that clearly $v(M) \leq v(M\delta)$ for all $M \in \mathrm{GL}_n(K_\mathfrak{p}) \cap \mathrm{Mat}_n(\mathfrak{o})$.

Lemma 2.6. *For almost all prime ideals $\mathfrak{p}$, the following holds for all $M \in \mathrm{GL}_n(K_\mathfrak{p}) \cap \mathrm{Mat}_n(\mathcal{O}_\mathfrak{p})$: if $M \leq_{\mathcal{E}(\mathcal{O}_\mathfrak{p})} \mathcal{L}(\mathcal{O}_\mathfrak{p})$, then $v(M) = v(M\delta)$.*

Proof. We proceed by induction on c, including the case $c = 1$, which we excluded in the previous arguments. Indeed, for this base case the statement holds trivially (and for all $\mathfrak{p}$) as $\delta = \mathrm{Id}_n$. Assume thus that $c \geq 2$ and that the induction hypothesis holds. Given $\mathfrak{p}$ and M, write Λ for the lattice defined by M and set $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$. Without loss we may assume that M is a block-upper triangular matrix, i.e. composed of blocks $M^{(ij)} \in \mathrm{Mat}_{n_i,n_j}(\mathfrak{o})$, $i, j \in [c]$, with $M^{(ij)} = 0$ unless $j \geq i$. The claim is that, for almost all $\mathfrak{p}$, the minimal $\mathfrak{p}$-adic valuation of the entries of M is equal to that of its last block column: if π divides one of the block matrices $M^{(ic)}$, $i \in [c]$, then it divides the whole matrix M. By induction hypothesis, the matrix $(M^{(ij)})_{i,j\in[2,c]} \in \mathrm{Mat}_{n-N_{c-1}}(\mathfrak{o})$, defining the lattice $\Lambda \cap Z_{c-1}(\mathfrak{o})$, where $Z_{c-1}(\mathfrak{o}) = Z_{c-1} \otimes_{\mathcal{O}} \mathfrak{o}$, has the desired property. So assume that π divides the last block column of M and thus that $M^{(ij)} \equiv 0 \bmod \pi$ for $i \geq 2$ or $j = c$, but that there exists $j \in [c-1]$ such that $v(M^{(1j)}) = 0$. Thus one of the first n_1 rows of M is nonzero modulo π, defining an element $x \in \mathcal{L}(\mathfrak{o})/Z_1(\mathfrak{o})$ which is nonzero modulo π. But as the reduction modulo $\mathfrak{p}$ of $Z_1(\mathfrak{o})$ is, for all but finitely many $\mathfrak{p}$, the centraliser of $\mathcal{L}(\mathfrak{o}/\mathfrak{p}) = \mathcal{L} \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{p}$, this yields the desired contradiction. □

Assume from now that $\mathfrak{p}$ satisfies the conclusions of Lemma 2.6.

Corollary 2.7. *For every $\mathcal{C} \in \mathcal{V}_n/\sim$,*

$$Z_{\mathcal{C}_{\geq 0}}(s) = |\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}|^{-s} \frac{1}{1 - q^{-s\sum_{i=1}^{c-1} N_i}}.$$

Proof. For all $m \in \mathbb{N}_0$ we have $\Lambda_{0,\max}\delta^m = (\Lambda_{0,\max}\delta^m)_{\max}$ by Lemma 2.6. Hence

$$|\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}\delta^m| = |\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}| q^{m\sum_{i=1}^{c-1} N_i}$$

and therefore

$$Z_{\mathcal{C}_{\geq 0}}(s) = \sum_{[\Lambda]\in\mathcal{C}_{\geq 0}} |\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|^{-s} = \sum_{m=0}^{\infty} |\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}\delta^m|^{-s} =$$

$$|\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}|^{-s} \sum_{m=0}^{\infty} q^{-sm\sum_{i=1}^{c-1} N_i} = |\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}|^{-s} \frac{1}{1 - q^{-s\sum_{i=1}^{c-1} N_i}}. \qquad \square$$

We set $\widetilde{\delta} = \pi^{c-1}\delta^{-1}$ and note that $\det \widetilde{\delta} = \pi^{\sum_{i=1}^{c-1}(n-N_i)}$ and $\mathcal{C}_{<0} = \{[\Lambda_0\widetilde{\delta}^m] \mid m > 0\}$. We seek to describe the 'weight function' $w : \mathcal{V}_n \to \mathbb{N}_0$ defined by the property that $w|_{\mathrm{SubMod}_{\mathcal{E}(\mathfrak{o})}} = 0$ and, for each δ-class $\mathcal{C}$,

$$Z_{\mathcal{C}_{<0}}(s) := \sum_{[\Lambda]\in\mathcal{C}_{<0}} |\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|^{-s} q^{-snw([\Lambda])} = \sum_{m=1}^{\infty} |\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}\widetilde{\delta}^m|^{-s}. \tag{2.8}$$

Note that then

$$Z_{\mathcal{C}_{<0}}(s) = |\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}|^{-s} \sum_{m=1}^{\infty} q^{-sm\sum_{i=1}^{c-1}(n-N_i)} = |\mathcal{L}(\mathfrak{o}) : \Lambda_{0,\max}|^{-s} \frac{q^{-s\sum_{i=1}^{c-1}(n-N_i)}}{1 - q^{-s\sum_{i=1}^{c-1}(n-N_i)}}.$$

To obtain this analogue of the formula for $Z_{\mathcal{C}_{\geq 0}}(s)$ established in Corollary 2.7, we need to take care to define w judiciously: the point here is that, whilst with $\Lambda_{0,\max}$ also $\Lambda_{0,\max}\delta^m$ is maximal (cf. Lemma 2.6), the lattice $\Lambda_{0,\max}\widetilde{\delta}^m$ is not, in general, as the following example illustrates.

Example 2.8. Let $\mathcal{L}(\mathfrak{o}) = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 0\rangle_{\mathfrak{o}}$ be the Heisenberg $\mathfrak{o}$-Lie lattice. Consider the matrices

$$M_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & 1 \\ & 1 & 1 \\ & & \pi \end{pmatrix} \in \mathrm{Mat}_3(\mathfrak{o}),$$

encoding sublattices of index q in $\mathcal{L}(\mathfrak{o}) \cong \mathfrak{o}^3$. Clearly neither of them are ideals of $\mathcal{L}(\mathfrak{o})$ (in fact, $\widetilde{m}_1(M_1) = \widetilde{m}_1(M_2) = 1$) and their homothety classes are δ-inequivalent. Considering

$$M_1\delta = \begin{pmatrix} \pi & & \\ & \pi & \\ & & \pi \end{pmatrix} = \pi \,\mathrm{Id}_3, \quad M_2\delta = \begin{pmatrix} \pi & & 1 \\ & \pi & 1 \\ & & \pi \end{pmatrix},$$

it is clear that whilst $M_2\delta$ is maximal, $M_1\delta$ is not. Thus $w([M_1]) = 0$ but $w([M_2]) = 1$.

Given w to the requirements of (2.8), it suffices to show that the function

$$A^{\lhd}(s) := \sum_{[\Lambda]\in\mathcal{V}_n} |\mathcal{L}(\mathfrak{o}) : \Lambda_{\max}|^{-s} q^{-snw([\Lambda])}$$

satisfies the functional equation

$$A^{\lhd}(s)|_{q\to q^{-1}} = (-1)^{n-1} q^{\binom{n}{2}} A^{\lhd}(s). \tag{2.9}$$

Indeed, $A^{\lhd}(s)$ is, by design of w, as Dirichlet generating series with nonnegative coefficients, divisible by

$$\underbrace{\frac{1}{1 - q^{-s\sum_{i=1}^{c-1} N_i}}}_{\text{sum over } \mathcal{C}_{\geq 0}} + \underbrace{\frac{q^{-s\sum_{i=1}^{c-1}(n-N_i)}}{1 - q^{-s\sum_{i=1}^{c-1}(n-N_i)}}}_{\text{sum over } \mathcal{C}_{<0}} = \frac{1 - q^{-s(c-1)n}}{(1 - q^{-s\sum_{i=1}^{c-1} N_i})(1 - q^{-s\sum_{i=1}^{c-1}(n-N_i)})}.$$

We want, however, to isolate the geometric progression $(1 - q^{-s\sum_{i=1}^{c-1} N_i})^{-1}$ taking care of the enumeration over the $\mathcal{C}_{\geq 0}$. Thus, by (2.3),

$$\begin{aligned} \zeta_{\mathcal{E}(\mathfrak{o})\curvearrowright\mathcal{L}(\mathfrak{o})}(s) &= \frac{1}{1 - q^{-ns}} \cdot \frac{1}{1 - q^{-s\sum_{i=1}^{c-1} N_i}} \cdot \frac{(1 - q^{-s\sum_{i=1}^{c-1} N_i})(1 - q^{-s\sum_{i=1}^{c-1}(n-N_i)})}{1 - q^{-s(c-1)n}} A^{\lhd}(s) \\ &= \frac{1 - q^{-s\sum_{i=1}^{c-1}(n-N_i)}}{(1 - q^{-ns})(1 - q^{-s(c-1)n})} A^{\lhd}(s) = \frac{1}{1 - q^{-ns}} \xi(s) A^{\lhd}(s). \end{aligned}$$

Given (2.9), the functional equation (1.3) follows as, trivially,

$$\left.\frac{1 - q^{-s\sum_{i=1}^{c-1}(n-N_i)}}{(1 - q^{-ns})(1 - q^{-s(c-1)n})}\right|_{q\to q^{-1}} = -q^{-s\sum_{i=0}^{c-1} N_i} \frac{1 - q^{-s\sum_{i=1}^{c-1}(n-N_i)}}{(1 - q^{-ns})(1 - q^{-s(c-1)n})}. \tag{2.10}$$

It remains to devise the weight function w to the requirement of (2.8).

Definition 2.9. Given $M \in \mathrm{GL}_n(K_{\mathfrak{p}}) \cap \mathrm{Mat}_n(\mathfrak{o})$ corresponding to a maximal lattice Λ, write $M = \left(M^{(ij)}\right)_{i,j\in[c]}$ as in the proof of Lemma 2.6. Define

$$m_2([\Lambda]) = \min\{v(M^{(ic)}) \mid i \in [c]\}.$$

Informally speaking, $m_2([\Lambda])$ is the valuation of the last n_c columns of a matrix M representing $\Lambda_{\max}$. Recall the "distance" function $\widetilde{m}_1 : \mathcal{V}_n \to \mathbb{N}_0$; see Definition 2.4.

Lemma 2.10. *Equation* (2.8) *holds with* $w([\Lambda]) = (c-1)\widetilde{m}_1([\Lambda]) - m_2([\Lambda])$.

Proof. Denote by $[\Lambda_0]$ the generator of $\mathcal{C}_{\geq 0}$ and by $\Lambda_{0,\max} \in [\Lambda_0]$ its unique maximal element. It suffices to observe that $v(M\delta^{\widetilde{m}_1([\Lambda])}) = m_2([\Lambda])$. Hence the matrix $M\delta^{\widetilde{m}_1([\Lambda])}$ corresponds to the lattice $\pi^{m_2([\Lambda])}\Lambda_{0,\max}$, whence

$$\pi^{-m_2([\Lambda])}\left(M\delta^{\widetilde{m}_1([\Lambda])}\right)\widetilde{\delta}^{\widetilde{m}_1([\Lambda])} = M\pi^{(c-1)\widetilde{m}_1([\Lambda])-\widetilde{m}_2([\Lambda])}$$

corresponds to $\Lambda_{0,\max}\widetilde{\delta}^{\widetilde{m}_1([\Lambda])}$. □

For later reference we record another formula for the invariant m_2. Write $M = D\alpha^{-1}$ as above. Setting, for $r \in [n]$,

$$v_r^{(2)}(\alpha) := \min\left\{v\left((\alpha^{\mathrm{adj}})_{\rho\sigma}\right) \mid \rho \geq r, \sigma \in \left]N_1, n\right]\right\},$$

we obtain

$$m_2([\Lambda]) = \min\left\{\sum_{\iota\in I} r_\iota, \sum_{r\leq\iota\in I} r_\iota + v_r^{(2)}(\alpha) \mid r \in [n]\right\}.$$

To compute $A^{\triangleleft}(s)$ we need, given a lattice class $[\Lambda] \in \mathcal{V}_n$ with $\nu([\Lambda]) = (I, \mathbf{r}_I)$, to keep track of the quantity

$$q^{-s\left((\sum_{\iota\in I}\iota r_\iota)+nw([\Lambda])\right)} = q^{-s\left((\sum_{\iota\in I}\iota r_\iota)+n((c-1)\widetilde{m}_1([\Lambda])-m_2([\Lambda]))\right)} =$$
$$q^{-s\left((\sum_{\iota\in I} r_\iota(\iota+n(c-1)))-n((c-1)m_1([\Lambda])+m_2([\Lambda]))\right)}.$$

(Here we used (2.5), Lemma 2.10, and (2.7).) To this end we define, given $I \subseteq [n-1]_0$ and $\mathbf{r} \in \mathbb{N}^{|I|}$ as above, for $\mathbf{m} = (m_1, m_2) \in \mathbb{N}_0^2$,

$$\mathcal{N}^{\triangleleft}_{I,\mathbf{r},\mathbf{m}} = |\{[\Lambda] \in \mathcal{V}_n \mid \nu([\Lambda]) = (I, \mathbf{r}),\, m_i([\Lambda]) = m_i,\, i \in \{1,2\}\}|$$

and set

$$A^{\triangleleft}_I(s) = \sum_{\mathbf{r}\in\mathbb{N}^{|I|}} q^{-s\sum_{\iota\in I} r_\iota(\iota+n(c-1))} \sum_{\mathbf{m}=(m_1,m_2)\in\mathbb{N}_0^2} \mathcal{N}^{\triangleleft}_{I,\mathbf{r},\mathbf{m}} q^{sn((c-1)m_1+m_2)}, \tag{2.11}$$

so that $A^{\triangleleft}(s) = \sum_{I\subseteq[n-1]} A^{\triangleleft}_I(s)$.

2.1.5. $\mathfrak{p}$-*Adic integration.* To establish the functional equation (2.9) for $A^{\triangleleft}(s)$ we express each of the functions $A^{\triangleleft}_I(s)$ in terms of $\mathfrak{p}$-adic integrals of the form [29, (6)] that satisfy the hypotheses of [29, Theorem 2.3]. To this end we define, for $i, r \in [n]$,

$$\mathbf{f}^{(1)}_{ir}(\mathbf{y}) = \left\{\left(\mathcal{R}_{(\iota)}(\mathbf{y})\right)_{\rho\sigma} \mid \iota \leq i, \rho \geq r, \sigma \in [d]\right\},$$
$$\mathbf{f}^{(2)}_{r}(\mathbf{y}) = \left\{\left(\mathbf{y}^{\mathrm{adj}}\right)_{\rho\sigma} \mid \rho \geq r, \sigma \in \left]N_1, n\right]\right\},$$

and set, for $I \subseteq [n-1]$,

$$\mathbf{g}^{(1)}_{n,I}(\boldsymbol{x}, \mathbf{y}) = \left\{\prod_{\iota\in I} x_\iota\right\} \cup \bigcup_{(i,r)\in[n]^2} \left(\prod_{\iota\in I} x_\iota^{\delta_{r\leq\iota}+\delta_{i>\iota}}\right)\mathbf{f}^{(1)}_{ir}(\mathbf{y}),$$
$$\mathbf{g}^{(2)}_{n,I}(\boldsymbol{x}, \mathbf{y}) = \left\{\prod_{\iota\in I} x_\iota\right\} \cup \bigcup_{r\in[n]} \left(\prod_{\iota\in I} x_\iota^{\delta_{r\leq\iota}}\right)\mathbf{f}^{(2)}_{r}(\mathbf{y}),$$

and, for $\kappa \in [n-1]$,

$$\mathbf{g}_{\kappa,I}(\boldsymbol{x}, \mathbf{y}) = \left\{\prod_{\iota\in I} x_\iota^{\delta_{\iota\kappa}}\right\}.$$

With this data we define the $\mathfrak{p}$-adic integral

$$Z_I^{\triangleleft}(\mathbf{s}) = Z_I^{\triangleleft}((s_\iota)_{\iota\in I}, s_n^{(1)}, s_n^{(2)}) := \int_{\mathfrak{p}^{|I|}\times\Gamma} \|\mathbf{g}_{n,I}^{(1)}(\boldsymbol{x},\mathbf{y})\|^{s_n^{(1)}} \|\mathbf{g}_{n,I}^{(2)}(\boldsymbol{x},\mathbf{y})\|^{s_n^{(2)}} \prod_{\kappa\in[n-1]} \|g_{\kappa,I}(\boldsymbol{x},\mathbf{y})\|^{s_\kappa} |\mathrm{d}\boldsymbol{x}_I||\mathrm{d}\mathbf{y}|.$$

Here, $\mathbf{s} = (s_1,\dots,s_{n-1},s_n^{(1)},s_n^{(2)})$ is a vector of complex variables; note, however, that s_κ occurs on the right hand side if and only if $\kappa\in I$. Moreover, $|\mathrm{d}\boldsymbol{x}_I||\mathrm{d}\mathbf{y}|$ denotes the Haar measure, normalized such that the domain of integration has measure $q^{-|I|}\mu(\Gamma)$, where $\mu(\Gamma) = \prod_{i=1}^n (1-q^{-i})$. The integral $Z_I^{\triangleleft}(\mathbf{s})$ is, by design, of the form [29, (6)]. Discarding at most finitely many primes, we may assume that the assumptions of [29, Theorem 2.2] are satisfied. This will imply that the normalized integrals

$$\widetilde{Z_I^{\triangleleft}}(\mathbf{s}) := \frac{Z_I^{\triangleleft}(\mathbf{s})}{(1-q^{-1})^{|I|}\mu(\Gamma)}$$

(cf. [29, (10)]) satisfy the 'inversion properties' established in [29, Theorem 2.3]. Hence the sum $\widetilde{Z^{\triangleleft}}(\mathbf{s}) := \sum_{I\subseteq[n-1]} \binom{n}{I}_{q^{-1}} \widetilde{Z_I^{\triangleleft}}(\mathbf{s})$ (cf. [29, (16)]) satisfies the functional equation

$$\widetilde{Z^{\triangleleft}}(\mathbf{s})\Big|_{q\to q^{-1}} = (-1)^{n-1} q^{\binom{n}{2}} \widetilde{Z^{\triangleleft}}(\mathbf{s}); \tag{2.12}$$

cf. [29, Cor. 2.3]. Here, $\binom{n}{I}_X \in \mathbb{Z}[X]$ denotes the Gaussian multinomial coefficient.

It remains to show that, for each $I\subseteq[n-1]$, the generating function $A_I^{\triangleleft}(s)$ is indeed obtainable from the $\mathfrak{p}$-adic integral $Z_I^{\triangleleft}(\mathbf{s})$ by a suitable specialization of the variables $\mathbf{s}$. We start by measuring the sets on which the integrand of $Z_I^{\triangleleft}(\mathbf{s})$ is constant. More precisely we set, for $\mathbf{m} = (m_1,m_2)\in\mathbb{N}_0^2$ and $\mathbf{r}\in\mathbb{N}^{|I|}$,

$$\mu_{I,\mathbf{r},\mathbf{m}}^{\triangleleft} := \mu\left\{(\boldsymbol{x},\mathbf{y})\in\mathfrak{p}^{|I|}\times\Gamma \mid \forall\iota\in I: v(x_\iota) = r_\iota, \mathbf{m}(\boldsymbol{x},\mathbf{y}) = (m_1,m_2)\right\},$$

where $\mathbf{m}(\boldsymbol{x},\mathbf{y}) = (\mathbf{m}(\boldsymbol{x},\mathbf{y})_1, \mathbf{m}(\boldsymbol{x},\mathbf{y})_2)$ and

$$\mathbf{m}(\boldsymbol{x},\mathbf{y})_1 = \min\left\{\sum_{\iota\in I} r_\iota, \sum_{r\leq\iota\in I} v(x_\iota) + \sum_{i>\iota\in I} v(x_\iota) + v_{ir}^{(1)}(\mathbf{y}) \mid (i,r)\in[n]^2\right\},$$

$$\mathbf{m}(\boldsymbol{x},\mathbf{y})_2 = \min\left\{\sum_{\iota\in I} r_\iota, \sum_{r\leq\iota\in I} v(x_\iota) + v_r^{(2)}(\mathbf{y}) \mid r\in[n]\right\}.$$

Then, by design,

$$\widetilde{Z_I^{\triangleleft}}(\mathbf{s}) = \frac{1}{(1-q^{-1})^{|I|}\mu(\Gamma)} \sum_{\mathbf{r}\in\mathbb{N}^{|I|}} q^{-\sum_{\iota\in I} s_\iota r_\iota} \sum_{\mathbf{m}=(m_1,m_2)\in\mathbb{N}_0^2} \mu_{I,\mathbf{r},\mathbf{m}}^{\triangleleft} q^{-s_n^{(1)}m_1 - s_n^{(2)}m_2}. \tag{2.13}$$

The numbers $\mu_{I,\mathbf{r},\mathbf{m}}^{\triangleleft}$ are closely related to the natural numbers $\mathcal{N}_{I,\mathbf{r},\mathbf{m}}^{\triangleleft}$ we are looking to control.

Lemma 2.11.

$$\mathcal{N}_{I,\mathbf{r},\mathbf{m}}^{\triangleleft} = \frac{\binom{n}{I}_{q^{-1}}}{(1-q^{-1})^{|I|}\mu(\Gamma)} \mu_{I,\mathbf{r},\mathbf{m}}^{\triangleleft} q^{\sum_{\iota\in I} r_\iota(\iota(n-\iota)+1)}. \tag{2.14}$$

Proof. Analogous to [29, Lemma 3.1]. □

Thus, combining (2.11), (2.14), and (2.13), we obtain

$$\begin{aligned} A_I^{\triangleleft}(s) &= \sum_{\mathbf{r}\in\mathbb{N}^{|I|}} q^{-s\sum_{\iota\in I} r_\iota(\iota+n(c-1))} \sum_{\mathbf{m}=(m_1,m_2)\in\mathbb{N}_0^2} \mathcal{N}^{\triangleleft}_{I,\mathbf{r},\mathbf{m}} q^{sn((c-1)m_1+m_2)} \\ &= \frac{\binom{n}{I}_{q^{-1}}}{(1-q^{-1})^{|I|}\mu(\Gamma)} \sum_{\mathbf{r}\in\mathbb{N}^{|I|}} q^{-\sum_{\iota\in I} r_\iota(s(\iota+n(c-1))-\iota(n-\iota)-1)} \sum_{\mathbf{m}\in\mathbb{N}_0^2} \mu^{\triangleleft}_{I,\mathbf{r},\mathbf{m}} q^{sn((c-1)m_1+m_2)} \\ &= \binom{n}{I}_{q^{-1}} \widetilde{Z}_I^{\triangleleft}\left((s(\iota+n(c-1))-\iota(n-\iota)-1)_{\iota\in I}, -ns(c-1), -ns\right). \end{aligned}$$

The functional equation (2.9) for $A^{\triangleleft}(s) = \sum_{I\subseteq[n-1]} A_I^{\triangleleft}(s)$ now follows from (2.12). This concludes the proof of Theorem 1.2.

2.2. **Proof of Corollary 1.9.** The proof of Theorem 1.2 expresses the relevant local submodule zeta functions in terms of $\mathfrak{p}$-adic integrals which are known to be representable by formulae of *Denef-type*. In other words, there exist algebraic varieties V_i, defined over $\mathcal{O}$, and rational functions $W_i(X,Y) \in \mathbb{Q}(X,Y)$ for $i = 1,\dots,M$, such that the following holds. For almost all $\mathfrak{p}$ and all $\mathfrak{O}$ we have

$$\zeta_{\mathcal{E}(\mathfrak{O})\curvearrowright\mathcal{L}(\mathfrak{O})}(s) = \sum_{i=1}^{M} |\overline{V_i}(\mathfrak{O}/\mathfrak{P})| W_i(q^f, q^{-fs}) = \frac{P_{\mathfrak{O}}(q^{-fs})}{Q_{\mathfrak{O}}(q^{-fs})}$$

for coprime polynomials $P_{\mathfrak{O}}(Z), Q_{\mathfrak{O}}(Z) \in \mathbb{Z}[Z]$, say

$$P_{\mathfrak{O}}(Z) = \sum_{i=0}^{\deg_Z P_{\mathfrak{O}}} \alpha_i Z^i, \quad Q_{\mathfrak{O}}(Z) = \prod_{j=1}^{r}(1-q^{fa_j}Z^{b_j}) = \sum_{i=0}^{\deg_Z Q_{\mathfrak{O}}} \beta_i Z^i$$

for integers $a_j \in \mathbb{N}_0$, $b_j \in \mathbb{N}$, $j \in [r]$. Note that the coefficients β_i of $Q_{\mathfrak{O}}$ are uniformly given by polynomials in q^f with integral coefficients which are independent of $\mathfrak{O}$. (The pair $(\mathcal{L},\mathcal{E})$ is almost uniform if and only the same holds for the coefficients α_i of $P_{\mathfrak{O}}$.) Moreover, the degree $\delta_2 := \deg_Z Q_{\mathfrak{O}}$ is independent of $\mathfrak{O}$. Note that $\alpha_0 = \beta_0 = 1$, as the generating series $\zeta_{\mathcal{E}(\mathfrak{O})\curvearrowright\mathcal{L}(\mathfrak{O})}(s)$ has evidently constant term 1. The functional equation (1.3) implies that $\delta_1 := \deg_Z P_{\mathfrak{O}}$, too, is independent of $\mathfrak{O}$ and there exist $c_1, c_2 \in \mathbb{N}_0$, $\sigma_1, \sigma_2 \in \{0,1\}$ satisfying $c_2 - c_1 = \binom{n}{2}$ and $\sigma_1 + \sigma_2 \equiv n \bmod (2)$ such that

$$\begin{aligned} \alpha_i|_{q\to q^{-1}} &= (-1)^{\sigma_1} q^{-fc_1} \alpha_{\delta_1 - i}, &&\text{for } i \in [\lfloor \delta_1/2 \rfloor], \\ \beta_i|_{q\to q^{-1}} &= (-1)^{\sigma_2} q^{-fc_2} \beta_{\delta_2 - i}, &&\text{for } i \in [\lfloor \delta_2/2 \rfloor], \end{aligned}$$

and $\delta_1 - \delta_2 = \deg_{q^{-fs}}\left(\zeta_{\mathcal{E}(\mathfrak{O})\curvearrowright\mathcal{L}(\mathfrak{O})}(s)\right) = -\sum_{i=0}^{c-1} N_i$. Corollary 1.9 follows.

3. Necessary vs. sufficient conditions for local functional equations

Theorem 1.2 gives sufficient criteria for generic local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms. Moreover, all examples of such submodule zeta functions for which we know explicit formulae for the generic Euler factors are consistent with the speculation that the hypotheses of Theorem 1.2 are also necessary for local functional equations of the form (1.3) to hold. An analogy with Igusa's local zeta function, however, sketched in Section 3.1, suggests caution. An insufficient but potentially necessary criterion for local functional equations for zeta functions in terms of so-called reduced zeta functions is discussed in Section 3.2.

3.1. **Igusa's local zeta function.** The proof of Theorem 1.2 is ultimately inspired by Denef's and Meuser's proof of a functional equation for Igusa's local zeta functions associated to a homogeneous polynomial. Let $F \in \mathcal{O}[X_1, \dots, X_n]$, let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}$ of index q in $\mathcal{O}$ and $\mathfrak{O}$ be a finite extension of $\mathcal{O}_\mathfrak{p}$, with maximal ideal $\mathfrak{P}$ of index q^f in $\mathfrak{O}$. Igusa's local zeta function associated to F at $\mathfrak{P}$ is

$$Z_{F,\mathfrak{P}}(s) = \int_{\mathfrak{O}^n} |F(\boldsymbol{x})|_\mathfrak{P}^s |\mathrm{d}\boldsymbol{x}|,$$

where $|\ |_\mathfrak{P}$ is the $\mathfrak{P}$-adic norm and $|\mathrm{d}\boldsymbol{x}|$ denotes the additive Haar measure on $\mathfrak{O}^n$, normalized such that $\mathfrak{O}^n$ has measure 1. It is a rational function in q^{-fs}. Denef and Meuser proved that, if F is homogeneous of degree d, then, for almost all $\mathfrak{p}$ and all $\mathfrak{O}$, the functional equation

$$Z_{F,\mathfrak{P}}(s)|_{q \to q^{-1}} = q^{-fds} Z_{F,\mathfrak{P}}(s) \tag{3.1}$$

holds; cf. [7, Theorem 4]. (Note that in [7] the inertia degree $f = f(\mathfrak{O}, \mathcal{O}_\mathfrak{p})$ is denoted by e.) Here, as in Theorem 1.2, equation (3.1) is explained in terms of a suitable formula of the form (1.5) and the Weil conjectures for smooth *projective* algebraic varieties over finite fields; cf. Remark 1.4. Projectivity of the relevant varieties is a consequence of the homogeneity of F. This line of argument breaks down if F is not homogeneous (or becomes homogeneous after an affine transformation of polynomial variables), and simple examples illustrate that functional equations such as (3.1) will not hold in general. We do not know of general necessary conditions.

Note that the functional equation (3.1) implies an analogue of (1.7), viz. that

$$\deg_{q^{-fs}}(Z_{F,\mathfrak{P}}(s)) = -d, \tag{3.2}$$

a fact first proven in [6]. We are not aware of general results on the degree of Igusa's local zeta function for nonhomogeneous polynomials, but again simple examples show that (3.2) is not universal. This contrasts with Conjecture 1.8, which predicts the degree of generic local submodule zeta functions associated with nilpotent endomorphism algebras, regardless of whether or not they satisfy functional equations.

3.2. **Reduced zeta functions.** In [11], Evseev introduced "reduced zeta functions" associated to various enumeration problems pertaining to finite-dimensional Lie algebras. His constructions apply quite generally to zeta functions $Z(s) = \prod_\mathfrak{p} Z_\mathfrak{p}(s)$ which satisfy Euler products, indexed by the nonzero prime ideals $\mathfrak{p}$ of a number ring, whose factors are rational functions in q^{-s} whose coefficients can be interpreted in terms of suitably "geometric functions" in the respective residue fields, such as formulae of the form (1.5). Informally speaking, the reduced zeta function $Z_{\mathrm{red}}(T)$ is a rational function in T obtained by "setting $q = 1$" in such formulae, whilst treating the parameter q^{-s} as an independent formal variable T. If $Z(s)$ is "almost uniform" (cf. Definition 1.7), i.e. if there exists $W \in \mathbb{Q}(X, Y)$ such that $Z_\mathfrak{p}(s) = W(q, q^{-s})$ for almost all $\mathfrak{p}$, then $Z_{\mathrm{red}}(T) = W(1, T)$. The formal definition given in [11] uses Euler-Poincaré characteristics in a motivic setting. We remark that the concept of reduced zeta function seems subtly related to the concept of topological zeta function; cf. [17].

Under certain conditions, reduced zeta functions satisfy functional equations upon inversion of T which reflect geometric properties of polyhedral cones. Indeed, in [11], Evseev describes sufficient conditions for a reduced zeta function to be the Hilbert series $H(T)$ of a graded ring $R_\mathcal{C}$ associated to a rational polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n_{\geq 0}$; cf. [11, Proposition 4.1]. Up to a sign, the rational function resulting from the inversion of T is the Hilbert series of the graded ring $R_{\mathcal{C}^\circ}$ associated to the *interior* of $\mathcal{C}$. A functional equation of the form

$$H(1/T) = \pm T^{\beta} H(T)$$

occurs if and only $\mathcal{C}$ has a unique minimal integral interior vector. For further details and an interpretation of these facts in terms of commutative algebra, viz. the language of Cohen-Macaulay and Gorenstein rings and modules, see [24, Chapter 1]. For an application in the context of ideal zeta functions of nilpotent Lie lattices, see Section 4.3.

A functional equation for the reduced zeta function is, in general, not sufficient for generic $\mathfrak{p}$-adic functional equations; cf. Example 4.14. If, however, $Z(s)$ is almost uniform as defined above, then a functional equation for the reduced zeta function $Z_{\mathrm{red}}(T)$ is a necessary condition for functional equations for the $\mathfrak{p}$-adic zeta functions $Z_{\mathfrak{p}}(s) = W(q, q^{-s})$. We speculate that this also holds without the hypothesis of almost uniformity.

In the light of the analogy with the theory of Hilbert series associated to rational polyhedral cones sketched above, it is tempting to interpret the result of the $\mathfrak{p}$-adic zeta function obtained by "inverting q" in terms of a generating function enumerating "interior points", too. In this spirit, in [10, Definition 4.26] du Sautoy and Woodward introduced the notion of *p-ideal* of a $\mathbb{Z}_p$-Lie lattice, in analogy to the notion of interior point of a polyhedral cone. Their hope clearly was to interpret local functional equations in terms of natural correspondences between the ideal- and p-ideal-lattices. Our proof of Theorem 1.2, however, relies in an essential manner on geometric properties of smooth projective algebraic varieties, as established by the Weil conjectures (see Remark 1.4). It seems therefore that any such interpretation of the local functional equations established in Theorem 1.2 would have to interpolate between these deep algebro-geometric symmetries on the one hand and the symmetries satisfied by Hilbert series associated to cones on the other.

4. Ideal zeta functions of nilpotent Lie lattices

In this section we discuss applications of Theorem 1.2 to ideal zeta functions of nilpotent Lie lattices. Ideal zeta functions of nilpotent Lie rings have been introduced in [13] as tools in the study of normal subgroup growth of finitely generated nilpotent groups. The technical tool facilitating this linearization is the Mal'cev correspondence; cf. Section 1.2. Hence, all results in this section on ideal zeta functions of nilpotent Lie lattices have immediate consequences on normal subgroup zeta functions of finitely generated nilpotent groups. Only in Corollary 1.6 did we choose to spell out such a consequence (of Theorem 4.4, in this instance).

Assume that $\mathcal{L}$ is a nilpotent $\mathcal{O}$-Lie lattice satisfying Assumption 1.1 – which, as we we saw in Remark 1.1 is vacuous for Lie rings –, with Lie bracket $[\,,]$. Recall that $(Z_i)_{i=0}^{c}$ is the upper central series of $\mathcal{L}$. In particular, c is the nilpotency class of $\mathcal{L}$ and $N_i = \mathrm{rk}_{\mathcal{O}}(\mathcal{L}/Z_i)$ for all $i \in [c]_0$. As noted in Remark 1.5, Condition 1.1 is trivially satisfied if $c \leq 2$. In these cases, Theorem 1.2 confirms known results. For $c = 1$, see Example 1.3. For $c = 2$, Theorem 1.2 in this setting is (a mild generalization of) [29, Theorem C].

4.1. Examples without functional equations.

That local functional equations akin to (1.3) may fail in nilpotency class greater than 2 was first discovered by Woodward.

Example 4.1. Consider the class-3-nilpotent Lie ring

$$\mathcal{L} = \mathrm{Fil}_4 = \langle z, x_1, x_2, x_3, x_4 \mid [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4, [x_1, x_2] = x_4 \rangle_{\mathbb{Z}}.$$

Here, as well as in comparable Lie lattice presentations throughout the paper, relations other than those following – by antisymmetry or the Jacobi identity – from the given ones are assumed to be trivial. [10, Theorem 2.39] gives explicit formulae for the local ideal zeta functions $\zeta^{\triangleleft}_{\mathrm{Fil}_4(\mathbb{Z}_p)}(s)$, valid for all primes p. They are all given by a single rational function $W(X,Y) \in \mathbb{Q}(X,Y)$ upon the substitution $X = p$, $Y = p^{-s}$ and do not satisfy a functional equation of the form (1.3). The associative algebra $\mathcal{E}$ generated by $\mathrm{ad}(\mathrm{Fil}_4)$ does not satisfy the conditions of Theorem 1.2. Indeed, it is generated by $\mathrm{ad}(z)$ and $\mathrm{ad}(x_1)$ which, with respect to the chosen cocentral $\mathbb{Z}$-basis (z, x_1, x_2, x_3, x_4) of Fil_4 are represented by the integral 5×5-matrices

$$C_1 = \left(\begin{array}{cc|c|c|c} & & & & \\ & & -1 & & \\ \hline & & & -1 & \\ \hline & & & & -1 \\ \hline & & & & \end{array}\right) \quad \text{and} \quad C_2 = \left(\begin{array}{cc|c|c|c} & & 1 & & \\ & & & & \\ \hline & & & & 1 \\ \hline & & & & \\ \hline & & & & \end{array}\right),$$

respectively. Here, the block structure reflects the decomposition $\mathcal{L} = \bigoplus_{i=1}^{4} \mathcal{L}_i$ with $\mathcal{L}_1 = \langle z, x_1 \rangle_{\mathbb{Z}}$ and $\mathcal{L}_i = \langle x_i \rangle_{\mathbb{Z}}$ for $i \in \{2,3,4\}$. Condition 2.1 is clearly violated as C_2 is not supported solely on the first block-off diagonal: $(C_2)_{34} \neq 0$. The failure of the functional equation reflects the (easily verifiable) fact that no other decomposition or choice of generators for $\mathcal{E}$ mitigates this failure.

Example 4.2. Consider the class-3-nilpotent Lie ring

$$\mathcal{L} = \mathfrak{g}_{6,6} = \langle x_1, \dots, x_6 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_2, x_3] = x_6 \rangle_{\mathbb{Z}}.$$

As observed in [10, Example 4.58], $Z_1 = \langle x_5, x_6 \rangle_{\mathbb{Z}}$, $Z_2 = \langle x_3, x_4, x_5, x_6 \rangle_{\mathbb{Z}}$, $Z_3 = \mathcal{L}$. Set $\mathcal{L}_1 = \langle x_1, x_2 \rangle_{\mathbb{Z}}$, $\mathcal{L}_2 = \langle x_3, x_4 \rangle_{\mathbb{Z}}$, $\mathcal{L}_3 = \langle x_5, x_6 \rangle_{\mathbb{Z}}$. Whilst $c_1 = \mathrm{ad}(x_1)$ and $c_2 = \mathrm{ad}(x_2)$ satisfy condition (1.2), $c_3 = \mathrm{ad}(x_3)$ does not. Moreover, this failure is independent of the specific choice of complements $\mathcal{L}_i$. Crucially, c_3 is not contained in the associative algebra generated by c_1 and c_2. This is consistent with the fact that the ideal zeta function of $\mathfrak{g}_{6,6}$ does not satisfy the conclusions of Theorem 1.2; see [10, Theorem 2.44].

Remark 4.3. Like many others of their kind, the computations in [10] are only carried out for local rings $\mathfrak{o} = \mathbb{Z}_p$. The resulting formulae, however, also cover the case of general finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$; one replaces p by the residue field cardinality q. See [18, Section 3.3] for a formal justification of this stability under local base extension.

Numerous further formulae for local ideal zeta functions of nilpotent Lie rings lacking a generic local functional equation can be found in [10, Section 2]. Like most formulae recorded in [10], they are obtained by computations with p-adic *cone integrals*; cf. [8]. In [10, Conjecture 4.5] du Sautoy and Woodward formulate a sort of conjecture on such cone integrals that would imply functional equations akin to (1.3). The conjecture's hypotheses, however, suffer from a degree of illdefinedness ("Suppose that [two specified conditions] and some as-yet-undetermined conditions hold."). It takes indeed a "cavalier attitude to the incompleteness of Conjecture 4.5" ([10, p. 98]) to use it to speculate about the occurrence of local functional equations of ideal zeta functions of Lie rings.

The shortcomings of [10, Conjecture 4.5] notwithstanding, the present paper owes a great deal of inspiration to [10, Chapter 4]. Our very Condition 1.1 is modelled on the conjunction of the properties (†) and (∗) in [10, Definition 4.56]; our matrix δ (see (2.6)) represents – up to a scalar factor – the map $\pi_{\mathcal{B}}$ in [10, Definition 4.40] with respect to a cocentral basis, a concept generalizing [10, Definition 4.37]. There are, nonetheless, two fundamental differences in approach. Firstly, we do not analyze cone integrals but develop the $\mathfrak{p}$-adic machinery introduced in [29]. Secondly, we realize that the ideal zeta function of a nilpotent Lie lattice $\mathcal{L}$ is determined by the associative algebra generated by $\mathrm{ad}(\mathcal{L})$; rather than hypothesizing about linear bases for the former, we formulate necessary conditions on suitable generators of the latter.

In the remainder of the section we develop a number of (unconditional) applications of Theorem 1.2 establishing generic functional equations for ideal zeta functions of nilpotent Lie lattices, confirming some of the more specific conjectures in [10].

4.2. **Free nilpotent Lie rings.** Given $c, d \in \mathbb{N}$, consider the free class-c-nilpotent Lie ring $\mathfrak{f}_{c,d}$ on free Lie generators $x_1, \dots, x_d$. The well-known formula for the $\mathbb{Z}$-ranks of the sections of the terms $\gamma_j(\mathfrak{f}_{c,d})$ of the lower central series of $\mathfrak{f}_{c,d}$ (which coincides with the upper central series of $\mathfrak{f}_{c,d}$), due to Witt ([30, Satz 3]), implies that, for $i \in [c]_0$,

$$\mathrm{rk}_{\mathbb{Z}}\left(\mathfrak{f}_{c,d}/\gamma_{c-i+1}(\mathfrak{f}_{c,d})\right) = \sum_{1 \le j \le c-i} \frac{1}{j} \sum_{k \mid j} \mu(k) d^{j/k} = N_i,$$

the numbers defined in (1.6). The following consequence of Theorem 1.2 implies the conclusion of [10, Theorem 1.3] without relying on the incomplete [10, Conjecture 4.5].

Theorem 4.4. *For almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\left.\zeta^{\triangleleft}_{\mathfrak{f}_{c,d}(\mathfrak{o})}(s)\right|_{p \to p^{-1}} = (-1)^{N_0} q^{\binom{N_0}{2} - s\left(\sum_{i=0}^{c-1} N_i\right)} \zeta^{\triangleleft}_{\mathfrak{f}_{c,d}(\mathfrak{o})}(s).$$

Proof. We choose a Hall ($\mathbb{Z}$-)basis $\mathcal{H}$ on $\{x_1, \dots, x_d\}$ for $\mathfrak{f}_{c,d}$; cf. [15]. By construction, elements of $\mathcal{H}$ are Lie monomials in $x_1, \dots, x_d$ with a well-defined degree, or *weight* $i \in [c]$. For $i \in [c]$, write $\mathcal{H}^i$ for the set of elements of $\mathcal{H}$ of weight i. For instance, $\mathcal{H}^1 = \{x_1, \dots, x_d\}$. Note that $\bigcup_{\iota \le i} \mathcal{H}^{\iota}$ has cardinality N_i for each $i \in [c]_0$. Setting $\mathcal{L}_i := \langle \mathcal{H}^i \rangle_{\mathbb{Z}}$ for $i \in [c]$ and $\{c_1, \dots, c_d\} = \{\mathrm{ad}(x_1), \dots, \mathrm{ad}(x_d)\}$, Condition 1.1 is clearly satisfied. The result thus follows from Theorem 1.2. □

Explicit formulae for $\zeta^{\triangleleft}_{\mathfrak{f}_{c,d}(\mathfrak{o})}(s)$ are known for $c = 1$ (see (1.4)), $c = 2$ (see [28]), and $(c, d) = (3, 2)$ (see [10, Theorem 2.35]).

Remark 4.5. Theorem 4.4 also implies a corrected version of the conclusion of [10, Theorem 4.73]. Note that the meaning of the numbers N_1 and N_2 there is different from the one in the present paper. In any case, the power of p^{-s} on the right hand side of [10, (4.45)] is not equal to the sum of the coranks of the upper central series of $\mathfrak{f}_{c,d}$.

4.3. **Some Lie rings of maximal class and their amalgams.** Given a partition $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r$, with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r$, define the class-$\lambda_1$-nilpotent Lie ring

$$\mathcal{L}_\lambda = \langle x_0, \{x_{ij}\}_{i \in [r], j \in [\lambda_i]} \mid \forall i \in [r], j \in [\lambda_j - 1] :\ [x_0, x_{ij}] = x_{i\,j+1} \rangle_{\mathbb{Z}}.$$

For $r = 1$ and $\lambda_1 \ge 2$ this yields the Lie ring M_{λ_1} of maximal class λ_1 described in [10, p. 99]. If $\lambda_r \ge 2$, then $\mathcal{L}_\lambda$ is obtained from amalgamating M_{λ_i}, $i \in [r]$, along x_0. If $\lambda = 1^{(r)} = (1, \dots, 1)$, then we obtain the abelian Lie ring $\mathbb{Z}^{1+r}$; cf. Example 1.3. The general case is evidently just an amalgamation of these special cases, again along x_0. For $\lambda = 2^{(r)}$ we get "Grenham's Lie rings" $\mathcal{G}_{1+r}$ on $1 + r$ Lie generators; cf. [10, Section 2.6]. Explicit formulae for the local ideal zeta functions $\zeta^{\triangleleft}_{\mathcal{L}_{2^{(r)}}(\mathbb{Z}_p)}(s)$, valid for all p, are given in [27, Theorem 5]. (On the face of it, the formulae there are for the local normal zeta functions of the torsion-free finitely generated nilpotent groups associated to the Lie rings $\mathcal{L}_{2^{(r)}}$ by the Mal'cev correspondence. As $c = 2$, however, the formulae coincide with those of the ideal zeta functions of $\mathcal{L}_{2^{(r)}}(\mathbb{Z}_p)$ for all p; cf. [13, Remark on p. 206].)

Definition 4.6. We say that λ is a *near rectangle* if it is of the form $\lambda = (c^{(r_1)}, 1^{(r_2)})$ for some $c \in \mathbb{N}$ and $r_1, r_2 \in \mathbb{N}_0$.

A geometric interpretation of this property of λ is given in Proposition 4.11.

Lemma 4.7. *Condition 1.1 is satisfiable for $\mathcal{L}_\lambda$ if and only if λ is a near rectangle.*

Proof. We may assume that $c > 1$. If $\lambda = (c^{(r_1)}, 1^{(r_2)})$ is a near rectangle, then, setting

$$\mathcal{L}_i = \begin{cases} \langle x_0, x_{11}, \dots, x_{r_1 1} \rangle_{\mathbb{Z}} & \text{for } i = 1, \\ \langle x_{1\,i}, \dots, x_{r_1\,i} \rangle_{\mathbb{Z}} & \text{for } i \in]1, c-1], \\ \langle x_{1c}, \dots, x_{r_1 c}, x_{r_1+1\,1}, \dots x_{r_1+r_2\,1} \rangle_{\mathbb{Z}} & \text{for } i = c, \end{cases}$$

and $\{c_1, \dots, c_d\} = \{\mathrm{ad}(x_0), \mathrm{ad}(x_{i1}) \mid i \in [r_1]\}$, Condition 1.1 is satisfied.

Assume now that λ is not a near rectangle. This implies that $c > 2$ and there exists $k \in [r-1]$ which is minimal with respect to the property that $c = \lambda_k > \lambda_{k+1} > 1$. Write $\mathcal{L} = \mathcal{L}_\lambda$, with upper central series $(Z_i)_{i=0}^{c}$. Following [10, Definition 4.49], we define the *depth* of an element $x \in \mathcal{L}$ as

$$\mathrm{dep}(x) = c + 1 - \min\{i \in [c]_0 \mid x \in Z_i\}.$$

The depth-1-elements of the $\mathbb{Z}$-basis $\mathcal{B} = \{x_0, x_{ij} \mid i \in [r], j \in [\lambda_i]\}$ of $\mathcal{L}$ are exactly x_0 and $x_{\kappa 1}$, for $\kappa \in [k]$. To see that Condition 1.1 is not satisfiable, note that the subalgebra $\mathcal{E}_1 = \langle \mathrm{ad}(z) \mid z \in \mathcal{B}, \mathrm{dep}(z) = 1 \rangle$ of the associative algebra $\mathcal{E}$ generated by $\mathrm{ad}(\mathcal{L})$ is a proper subalgebra of $\mathcal{E}$; indeed, the element $x_{k+1\,1}$ has depth $c - \lambda_{k+1} + 1 > 1$ and $\mathrm{ad}(x_{k+1\,1}) \in \mathcal{E} \setminus \mathcal{E}_1$. This follows, for instance, from the fact that $\mathcal{B}$ clearly has the property that if $z, z' \in \mathcal{B}$, then either $[z, z'] = 0$ or $\mathrm{dep}[z, z'] = \mathrm{dep}(z) + \mathrm{dep}(z')$. □

4.3.1. *Local ideal zeta functions.* The following consequence of Theorem 1.2 implies, in particular, the first of the two conclusions of [10, Proposition 4.75]. Note that the latter only considers the case that $\lambda_r \geq 2$ and is conditional on the incomplete [10, Conjecture 4.5].

Theorem 4.8. *Let $c \in \mathbb{N}$ and $r_1, r_2 \in \mathbb{N}_0$. For almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta^{\triangleleft}_{\mathcal{L}_{(c^{(r_1)}, 1^{(r_2)})}(\mathfrak{o})}(s) \bigg|_{p \to p^{-1}} = (-1)^{1 + cr_1 + r_2} q^{\binom{1 + cr_1 + r_2}{2} - s\left(c + \binom{c+1}{2} r_1 + r_2\right)} \zeta^{\triangleleft}_{\mathcal{L}_{(c^{(r_1)}, 1^{(r_2)})}(\mathfrak{o})}(s).$$

In particular, [10, Conjecture 4.24] *about the Lie rings M_c of maximal class c holds.*

Proof. By Lemma 4.7, Condition 1.1 is satisfied. We find $\mathrm{rk}_{\mathbb{Z}}(\mathcal{L}_{(c^{(r_1)}, 1^{(r_2)})}) = N_0 = 1 + cr_1 + r_2$ and, more generally, $N_i = 1 + (c - i) r_1 + r_2 \delta_{i=0}$ for $i \in [c-1]_0$, whence $\sum_{i=0}^{c-1} N_i = c + \binom{c+1}{2} r_1 + r_2$. The result thus follows from Theorem 1.2. □

In agreement with the second, conjectural conclusion of [10, Proposition 4.75] we speculate that the "near rectangle" condition on λ is also necessary for local functional equations for the ideal zeta functions of the Lie rings $\mathcal{L}_\lambda$. Some evidence for this are the explicit formulae for $\lambda = (3, 2)$ – the smallest partition which is not a near rectangle – given in [10, Theorem 2.32] as well as the following results on the reduced zeta functions $\zeta^{\triangleleft}_{\mathcal{L}_\lambda, \mathrm{red}}(T)$; cf. Section 3.2.

4.3.2. *Reduced ideal zeta functions.*

Proposition 4.9. *The* $\mathbb{C}[\![T]\!]$*-Basis* $\mathcal{B}_\lambda = \{x_0, x_{ij} \mid i \in [r], j \in [\lambda_i]\}$ *of the* $\mathbb{C}[\![T]\!]$*-Lie algebra* $\mathcal{L}_\lambda(\mathbb{C}[\![T]\!])$ *is nice and simple in the sense of* [11].

Proof. $\mathcal{B}_\lambda$ is simple as, for all $x, y \in \mathcal{B}_\lambda$, there exist $z \in \mathcal{B}_\lambda$ and $\varepsilon \in \{0, 1, -1\}$ such that $[x, y] = \varepsilon z$. To see that $\mathcal{B}_\lambda$ is nice we show that all pairs $(x, y) \in \mathcal{B}_\lambda^2$ with $x \neq y$ are removable in the sense of [11, Section 3]. (Strictly speaking, Evseev defines removability only for certain pairs of *indices* of basis elements.) Concretely, we need to find pairwise distinct integers l_0, l_{ij}, $i \in [r]$, $j \in [\lambda_i]$, such that, for all $z \in \mathbb{C} \setminus \{0\}$, the maps

$$x_0 \mapsto z^{l_0} x_0, \quad x_{ij} \mapsto z^{l_{ij}} x_{ij}, \quad i \in [r], j \in [\lambda_i],$$

are automorphisms of $\mathcal{L}_\lambda(\mathbb{C}[\![t]\!])$. This is the case whenever $l_0 + l_{ij} = l_{i\,j+1}$ for all $i \in [r]$, $j \in [\lambda_i - 1]$. Setting $l_0 = 1$ and $l_{ij} = 1 + \left(\sum_{\iota < i} \lambda_\iota\right) + j$ is one way to achieve this. □

Define the rational polyhedral cone

$$\mathcal{C}_\lambda = \left\{ \mathbf{n} \in \mathbb{R}_{\geq 0}^{1+\sum_{i=1}^r \lambda_i} \mid \forall i \in [r] : \; n_0, n_{i1} \geq n_{i2} \geq \cdots \geq n_{i\lambda_i} \right\}. \tag{4.1}$$

Corollary 4.10. $\zeta^{\triangleleft}_{\mathcal{L}_\lambda,\mathrm{red}}(T) = \sum_{\mathbf{n} \in \mathcal{C}_\lambda \cap \mathbb{N}_0^{1+\sum_{i=1}^r \lambda_i}} T^{\sum \mathbf{n}}$.

Proof. As $\mathcal{B}_\lambda$ is nice and simple, this follows from [11, Proposition 4.1], where $\mathcal{C}_\lambda$ is called $\mathcal{C}^{\triangleleft}_{\mathcal{B}_\lambda}$. □

The reduced zeta function $\zeta^{\triangleleft}_{\mathcal{L}_\lambda,\mathrm{red}}(T)$ is thus the Hilbert series associated to the graded monoid algebra $R_{\mathcal{C}_\lambda}$ spanned by the integral points in $\mathcal{C}_\lambda$. Such Hilbert series satisfy functional equations upon inversion of T if and only if (!) $R_{\mathcal{C}_\lambda}$ is Gorenstein (cf. [24, Theorem 12.7]). The latter condition is satisfied if and only if there exists a unique minimal integral vector $\boldsymbol{\beta}$ in the interior $\mathcal{C}_\lambda^\circ$ of $\mathcal{C}_\lambda$, i.e.

$$\text{if } \boldsymbol{\gamma} \in \mathcal{C}_\lambda^\circ \cap \mathbb{N}^{1+\sum_{i=1}^r \lambda_i}, \text{ then } \boldsymbol{\gamma} - \boldsymbol{\beta} \in \mathcal{C}_\lambda. \tag{4.2}$$

Note that $\mathcal{C}_\lambda^\circ$ is defined by replacing the inequalities in (4.1) by strict inequalities.

Proposition 4.11. *A unique minimal integral interior vector* $\boldsymbol{\beta} \in \mathcal{C}_\lambda^\circ$ *exists if and only if* λ *is a near rectangle.*

Proof. Assume that

$$\boldsymbol{\beta} = (\beta_0,\, \beta_{11}, \beta_{12}, \dots, \beta_{1\lambda_1},\, \beta_{21}, \dots \beta_{2\lambda_2},\, \dots,\, \beta_{r1}, \dots, \beta_{r\lambda_r}) \in \mathcal{C}_\lambda^\circ \cap \mathbb{N}^{1+\sum_{i=1}^r \lambda_i}$$

has the property (4.2). It is not hard to check that

$$(\beta_0,\, \beta_{11}, \beta_{12}, \dots, \beta_{1\lambda_1}) = (\lambda_1,\, \lambda_1, \lambda_1 - 1, \dots, 2, 1). \tag{4.3}$$

Observe that λ is a near rectangle if and only if this property determines $\boldsymbol{\beta}$ uniquely. Indeed, if λ is a near rectangle, say $\lambda = (c^{(r_1)}, 1^{(r_2)})$ for some $c \in \mathbb{N}$ and $r_1, r_2 \in \mathbb{N}_0$, then

$$\boldsymbol{\beta} = (c,\, c, c-1, \dots, 2, 1,\, c, c-1, \dots, 2, 1,\, \dots\, c, c-1, \dots, 2, 1,\, 1^{(r_2)}) \\ \in \mathcal{C}^\circ_{(c^{(r_1)}, 1^{(r_2)})} \cap \mathbb{N}^{1+cr_1+r_2}$$

satisfies both (4.2) and (4.3) and is clearly unique with this property. If λ is not a near rectangle, there exist more than one ways to "complete" the vector in (4.3) to an element $\beta \in \mathcal{C}_\lambda^\circ \cap \mathbb{N}^{1+\sum_{i=1}^r \lambda_i}$ such that $\beta_0 = \lambda_1$ and $\beta_{i1} = \lambda_1$ for $i \in [r]$. The difference between any two of these completions, however, is zero on these coordinates, hence is zero, contradiction. □

Corollary 4.12. *λ is a near rectangle if and only if there exist $k, l \in \mathbb{Z}$ such that*

$$\zeta^{\triangleleft}_{\mathcal{L}_\lambda,\mathrm{red}}(1/T) = (-1)^l T^k \zeta^{\triangleleft}_{\mathcal{L}_\lambda,\mathrm{red}}(T). \tag{4.4}$$

In this case, $l \equiv 1 + cr_1 + r_2 \bmod (2)$ *and* $k = c + \binom{c+1}{2} r_1 + r_2$.

Remark 4.13. If $\zeta^{\triangleleft}_{\mathcal{L}_\lambda}(s)$ is almost uniform, say $\zeta^{\triangleleft}_{\mathcal{L}_\lambda(\mathbb{Z}_p)}(s) = W_\lambda(p, p^{-s})$ for $W_\lambda(X, Y) \in \mathbb{Q}(X, Y)$ and almost all primes p, then Corollary 4.12 implies that the "near rectangle condition" on λ is also necessary for local $\mathfrak{p}$-adic functional equations. Indeed, in this case, $\zeta^{\triangleleft}_{\mathcal{L}_\lambda,\mathrm{red}}(T) = W_\lambda(1, T)$; cf. [11, Section 3].

Example 4.14. A functional equation of the form (4.4) for reduced zeta functions is not, in general, sufficient for functional equations for generic $\mathfrak{p}$-adic zeta functions. Consider, for instance, the Lie rings $M_4 = \mathcal{L}_{(4)}$ and Fil_4; cf. Example 4.1. Whereas the local ideal zeta functions $\zeta^{\triangleleft}_{M_4(\mathbb{Z}_p)}(s)$ satisfy functional equations (cf. Corollary 4.8), the local ideal zeta functions $\zeta^{\triangleleft}_{\mathrm{Fil}_4(\mathbb{Z}_p)}(s)$ do not (cf. [10, Theorem 2.39]). The respective reduced ideal zeta functions, however, coincide; cf. [11, Example 4.5].

5. FURTHER EXAMPLES

In this section we discuss two applications of Theorem 1.2 to submodule zeta functions which are not ideal zeta functions of nilpotent Lie lattices. In Section 5.1 we consider a class of abelian matrix algebras; Section 5.2 is dedicated to full algebras of strictly upper triangular matrices. Throughout, $\mathfrak{o}$ is a compact discrete valuation ring of characteristic zero and residue field cardinality q and let $n \in \mathbb{N}$.

5.1. **A class of abelian matrix algebras.**

5.1.1. *Local formulae.* Let p be a prime and $\mathfrak{o}$ be a finite extension of $\mathbb{Z}_p$. We set
(5.1)

$$\mathcal{M}_{\mathbf{1}}(\mathfrak{o}) = \mathcal{M}_{\underbrace{(1,\dots,1)}_{n \text{ times}}}(\mathfrak{o}) = \left\{ \begin{pmatrix} 0 & \mathrm{diag}(z_1,\dots,z_n) \\ 0 & 0 \end{pmatrix} \mid z_1,\dots,z_n \in \mathfrak{o} \right\} \leq \mathrm{Mat}_{2n}(\mathfrak{o})$$

and consider $\mathfrak{o}^{2n}$ as an $\mathcal{M}_{\mathbf{1}}(\mathfrak{o})$-module by right multiplication. We record a formula for the submodule zeta functions $\zeta_{\mathcal{M}_{\mathbf{1}}(\mathfrak{o}) \curvearrowright \mathfrak{o}^{2n}}(s)$. It is very similar to that for the ideal zeta functions $\zeta^{\triangleleft}_{H(\mathfrak{o})^n}(s)$ of the n-fold direct product of the Heisenberg $\mathfrak{o}$-Lie lattices $H(\mathfrak{o})$ of strictly upper triangular 3×3-matrices over $\mathfrak{o}$ given in [20, Theorem 3.1]. We formulate this result partly in the notation of [20]. In particular, $\mathcal{D}_{2n}$ denotes the set of Dyck words in letters $\mathbf{0}$ and $\mathbf{1}$ of length $2n$, of cardinality $\frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number. Recall the formula (1.4) for $\zeta_{\{0\} \curvearrowright \mathfrak{o}^n}(s)$.

Theorem 5.1.

$$\zeta_{\mathcal{M}_{\mathbf{1}}(\mathfrak{o}) \curvearrowright \mathfrak{o}^{2n}}(s) = (1 - q^{-s})^n \zeta_{\{0\} \curvearrowright \mathfrak{o}^n}(s) \sum_{w \in \mathcal{D}_{2n}} D_w(q, q^{-s}),$$

where, for $w = \prod_{i=1}^r (\mathbf{0}^{L_i - L_{i-1}} \mathbf{1}^{M_i - M_{i-1}}) \in \mathcal{D}_{2n}$*, the rational function* $D_w(q, q^{-s})$ *is defined as* $D^{\mathbf{1}}_w(p, p^{-s})$ *in* [20, Theorem 3.1] *with* q *in place of* p *and with numerical data*

$$x_j = q^{j(n+L_i-j)-s(L_i+j)} \qquad \text{for } j \in]M_{i-1}, M_i],$$
$$y_j = q^{(n-M_{i-1}+j)M_{i-1}-s(j+M_{i-1})} \qquad \text{for } j \in]L_{i-1}, L_i].$$

Sketch of proof. Mutatis mutandis, the analysis of [20, Theorem 3.1] carries over. The centre $Z(H(\mathfrak{o})) \cong \mathfrak{o}^n$ is replaced by $Z_1(\mathfrak{o}) = \mathrm{Cent}_{\mathcal{M}_{\mathbf{1}}(\mathfrak{o})}(\mathfrak{o}^{2n}) \cong \mathfrak{o}^n$, whereas the role of the cocentre $H(\mathfrak{o})/Z(H(\mathfrak{o})) \cong \mathfrak{o}^{2n}$ is taken by $\mathfrak{o}^{2n}/Z_1(\mathfrak{o}) \cong \mathfrak{o}^n$, explaining the change in numerical data. □

The construction of the algebra $\mathcal{M}_{\mathbf{1}}(\mathfrak{o})$ may be generalized by replacing the diagonal matrices in (5.1) by "generic block-diagonal" matrices with block sizes $f_1, \dots, f_g$, say. Roughly speaking, formulae for the submodule zeta functions associated to the resulting nilpotent algebras

$$\mathcal{M}_{\mathbf{f}}(\mathfrak{o}) = \mathcal{M}_{(f_1,\dots,f_g)}(\mathfrak{o}) = \begin{pmatrix} 0 & \mathrm{diag}(\mathrm{Mat}_{f_1}(\mathfrak{o}), \dots, \mathrm{Mat}_{f_g}(\mathfrak{o})) \\ 0 & 0 \end{pmatrix} \leq \mathrm{Mat}_{2n}(\mathfrak{o}),$$

acting on $\mathfrak{o}^{2n}$ by right multiplication, are obtained by modifying the "numerical data" in the formulae given in [20, Theorem 3.6] and in the formula for the zeta function preceding it. We leave the precise details to the reader, spelling out only the result in the other "extremal" case $\mathbf{f} = (n)$, yielding

$$\mathcal{M}_{(n)}(\mathfrak{o}) = \begin{pmatrix} 0 & \mathrm{Mat}_n(\mathfrak{o}) \\ 0 & 0 \end{pmatrix} \leq \mathrm{Mat}_{2n}(\mathfrak{o}).$$

The following is analogous to [20, Corollary 3.7].

Theorem 5.2.

$$\zeta_{\mathcal{M}_{(n)}(\mathfrak{o}) \curvearrowright \mathfrak{o}^{2n}}(s) = \zeta_{\{0\} \curvearrowright \mathfrak{o}^n}(s) \frac{1}{1-x_n} \sum_{I \subseteq [n-1]} \binom{n}{I}_{q^{-1}} \prod_{i \in I} \frac{x_i}{1-x_i}, \tag{5.2}$$

where $\binom{n}{I}_X \in \mathbb{Z}[X]$ *denotes the Gaussian multinomial coefficient and*

$$x_j = q^{j(2n-j)-s(n+j)} \text{ for } j \in [n].$$

Remark 5.3. Let k be a field. By a theorem of Schur, $\begin{pmatrix} k\,\mathrm{Id}_n & \mathrm{Mat}_n(k) \\ 0 & k\,\mathrm{Id}_n \end{pmatrix}$ is a maximal abelian subalgebra of $\mathrm{Mat}_{2n}(k)$; cf. [22]. The right hand side of (5.2) is a product of two *Igusa functions* in the terminology of [20, Definition 2.5].

Theorem 5.4. *For all $g \in \mathbb{N}$ and $\mathbf{f} = (f_1, \dots, f_g) \in \mathbb{N}^g$, the functional equation*

$$\zeta_{\mathcal{M}_{\mathbf{f}}(\mathfrak{o}) \curvearrowright \mathfrak{o}^{2n}}(s)\big|_{q \to q^{-1}} = q^{\binom{2n}{2} - 3ns} \zeta_{\mathcal{M}_{\mathbf{f}}(\mathfrak{o}) \curvearrowright \mathfrak{o}^{2n}}(s)$$

holds.

Sketch of proof. Analogous to [20, Theorem 1.2] with $3n$ and $5n(= 3n + 2n)$ replaced by $2n$ and $3n(= 2n + n)$, respectively. □

5.1.2. *Global zeta functions and Euler products.* The local formulae in Section 5.1.1 may be put in a global context as follows. For any ring R, consider

$$\mathcal{M}(R) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} \leq \mathrm{Mat}_2(R).$$

Let K be a number field with ring of integers $\mathcal{O}$ and of degree n, say. By restriction of scalars from K to $\mathbb{Q}$, we may consider $\mathcal{M}(\mathcal{O})$ as a subalgebra of $\mathrm{Mat}_{2n}(\mathbb{Z})$, turning $\mathbb{Z}^{2n}$ into an $\mathcal{M}(\mathcal{O})$-module by right multiplication, with associated submodule zeta function

$$\zeta_{\mathcal{M}(\mathcal{O}) \curvearrowright \mathbb{Z}^{2n}}(s) = \prod_{p \text{ prime}} \zeta_{\mathcal{M}(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \curvearrowright \mathbb{Z}_p^{2n}}(s). \tag{5.3}$$

This is a close analogue of the ideal zeta function $\zeta^{\triangleleft}_{H(\mathcal{O})}(s)$ of the Heisenberg Lie ring over $\mathcal{O}$ studied in [20]. If p is unramified in K, i.e. $p\mathcal{O} = \prod_{i=1}^{g} \mathfrak{p}_i$ for pairwise distinct prime ideals $\mathfrak{p}_i$ of $\mathcal{O}$ with residue degrees f_i for $i \in [g]$, then $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{i=1}^{g} \mathbb{Z}_{p^{f_i}}$, where $\mathbb{Z}_{p^{f_i}}$ denotes the unramified extension of $\mathbb{Z}_p$ of degree f_i. Hence

$$\mathcal{M}(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \cong \mathcal{M}_{\mathbf{f}}(\mathbb{Z}_p).$$

Therefore, all but finitely many of the Euler factors in (5.3) are covered by the formulae sketched – and written out in Theorem 5.1 for the primes which split totally in K and in Theorem 5.2 for the primes which stay inert in K – in Section 5.1.1. Formulae for the Euler factors indexed by the rational primes which remain unsplit in K (but may ramify) may be obtained by modifying those for $\zeta^{\triangleleft}_{H(\mathcal{O})}(s)$ described in [21]. Note that Theorem 1.2 is applicable for $\mathcal{E} = \mathcal{M}(\mathcal{O})$ and $\mathcal{L} = \mathbb{Z}^{2n}$ as $c = 2$; cf. Remark 1.5. The functional equations established in Theorem 5.4 strengthen the result in this special case by implying that the set of exceptional primes is contained in (and conjecturally equal to) the set of primes which ramify in K.

5.2. **Strictly upper triangular matrices.** For $m \in \mathbb{N}$, let $\mathfrak{u}_m(\mathfrak{o})$ be the associative algebra of all strictly upper triangular $m \times m$-matrices over $\mathfrak{o}$, acting on $\mathfrak{o}^m$ by right-multiplication, say. Given a partition $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r$ of $n = \sum_{i=1}^{r} \lambda_i$, consider

$$\mathfrak{u}_\lambda(\mathfrak{o}) = \bigoplus_{i=1}^{r} \mathfrak{u}_{\lambda_i}(\mathfrak{o}),$$

diagonally embedded into $\mathfrak{u}_n(\mathfrak{o})$. Theorem 1.2 is clearly applicable and implies the following result.

Theorem 5.5. *For almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathfrak{u}_\lambda(\mathfrak{o}) \curvearrowright \mathfrak{o}^n}(s)\big|_{p \to p^{-1}} = (-1)^n q^{\left(\binom{n}{2} - s \sum_{i=1}^{r} \binom{n_i+1}{2}\right)} \zeta_{\mathfrak{u}_\lambda(\mathfrak{o}) \curvearrowright \mathfrak{o}^n}(s).$$

Explicit formulae for generic submodule zeta functions of the form $\zeta_{\mathfrak{u}_\lambda(\mathfrak{o}) \curvearrowright \mathfrak{o}^n}(s)$ have been computed by Rossmann for $\lambda = (m)$, $m \leq 5$ as well as well as for numerous other partitions of natural numbers $n \leq 7$; see [19, Section 9.4] and the database in the computer algebra package [16]. All these functions are given by rational functions in q and q^{-s}. Theorem 5.5 is consistent with and explains the functional equations recorded in [19, Theorems 9.5, 9.8, 9.9].

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