REPRESENTATION GROWTH OF THE HEISENBERG GROUP OVER $\mathcal{O}[x]/(x^3)$

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Abstract. We present a conjectured formula for the representation zeta function of the Heisenberg group over $\mathcal{O}[x]/(x^n)$ where $\mathcal{O}$ is the ring of integers of some number field. We confirm the conjecture for $n \leq 3$.

1. Introduction

Let G be an infinite finitely generated torsion-free nilpotent group (or a $\mathcal{T}$-group for short). Two complex representations ρ and σ of G are called *twist-equivalent* if there exists a 1-dimensional representation λ of G such that $\rho = \lambda \otimes \sigma$. Twist-equivalence is an equivalence relation on the set of finite dimensional isomorphic irreducible complex representations of G and its classes are called *twist-isoclasses*. The numbers $r_n(G)$ of twist-isoclasses of dimension n are finite for all n, cf. [3, Theorem 6.6]. The *representation zeta function* of G is defined to be the Dirichlet generating function

$$\zeta_G(s) := \sum_{n=1}^{\infty} \frac{r_n(G)}{n^s},$$

where s is a complex variable. The sequence $(r_n(G))$ grows polynomially and thus $\zeta_G(s)$ converges on a complex half-plane $\mathrm{Re}(s) > \alpha$, cf. [7, Lemma 2.1]. The infimum of such α is the *abscissa of convergence* $\alpha(G)$ of $\zeta_G(s)$ which gives the precise degree of polynomial growth; i.e., $\alpha(G)$ is the smallest value such that $\sum_{n=1}^{N} r_n(G) = O(N^{\alpha(G)+\epsilon})$ for every $\epsilon \in \mathbb{R}_{>0}$.

Let $\mathbf{H}$ be the Heisenberg group scheme associated to the Heisenberg $\mathbb{Z}$-Lie lattice of strict upper-triangular 3×3 matrices. For every ring R, the group $\mathbf{H}(R)$ is isomorphic to the group of upper-unitriangular 3×3 matrices over R. If R is a torsion-free finitely generated $\mathbb{Z}$-module, then $\mathbf{H}(R)$ is a $\mathcal{T}$-group of nilpotency class 2 and Hirsch length $3 \cdot \mathrm{rk}_{\mathbb{Z}}(R)$. When $R = \mathcal{O}$ is the ring of integers of a number field K, the zeta function of $\mathbf{H}(\mathcal{O})$ is

$$\zeta_{\mathbf{H}(\mathcal{O})}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)} = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})} \frac{1 - |\mathcal{O}/\mathfrak{p}|^{-s}}{1 - |\mathcal{O}/\mathfrak{p}|^{1-s}}, \tag{1.1}$$

where $\zeta_K(s)$ is the Dedekind zeta function of K and $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$. This is proved in [4] for $K = \mathbb{Q}$, in [2] for quadratic number fields, and in [7, Theorem B] for

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arbitrary number fields. The zeta function $\zeta_{\mathbf{H}(\mathcal{O})}(s)$ has abscissa of convergence $\alpha(\mathbf{H}(\mathcal{O})) = 2$, which is independent of K, and may be meromorphically continued to the whole complex plane.

In this paper, we consider the Heisenberg group over rings of the form $\mathcal{O}[x]/(x^n)$. If $n = 1$ then it is the Heisenberg group over $\mathcal{O}$. Snocken computed in his PhD thesis [6, Example 6.5] that the zeta function of $\mathbf{H}(\mathcal{O}[x]/(x^2))$ is

$$\zeta_{\mathbf{H}(\mathcal{O}[x]/(x^2))}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)} \cdot \frac{\zeta_K(2s-3)}{\zeta_K(2s-2)}. \tag{1.2}$$

In this paper, we carry out the computations for the zeta function of $\mathbf{H}(\mathcal{O}[x]/(x^3))$. The result is the following.

Theorem 1.1. *The zeta function of* $\mathbf{H}(\mathcal{O}[x]/(x^3))$ *is*

$$\zeta_{\mathbf{H}(\mathcal{O}[x]/(x^3))}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)} \cdot \frac{\zeta_K(2s-3)}{\zeta_K(2s-2)} \cdot \frac{\zeta_K(3s-5)}{\zeta_K(3s-4)}. \tag{1.3}$$

Corollary 1.2. *The zeta function of* $\mathbf{H}(\mathcal{O}[x]/(x^n))$*, with* $n \leq 3$*, has abscissa of convergence* 2 *and meromorphic continuation to the whole complex plane.*

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{H}$ and $\mathfrak{g}[\varepsilon_n] := \mathfrak{g} \otimes_{\mathbb{Z}} \mathbb{Q}[\varepsilon_n]$ regarded as a $\mathfrak{g}$-Lie lattice. Let $\mathbf{H}[\varepsilon_n]$ denote the group attached to $\mathfrak{g}[\varepsilon_n]$(cf. [5, Section 7]), then $\mathbf{H}[\varepsilon_n]$ is a unipotent group scheme and $\mathbf{H}[\varepsilon_n](\mathcal{O}) = \mathbf{H}(\mathcal{O}[x]/(x^n))$. We remark that the uniformity of the analytic invariants determined in Corollary 1.2, namely their independence of $\mathcal{O}$, is the general feature of representation zeta functions of $\mathcal{T}$-groups obtained from unipotent group schemes; see [1] for more details. We believe that Corollary 1.2 also holds for the Heisenberg group $\mathbf{H}(R)$ over arbitrary ring R which is also finitely-generated torsion-free as an $\mathbb{Z}$-module.

Formulae (1.1), (1.2) and (1.3) suggest the following conjecture for the zeta function of $\mathbf{H}(\mathcal{O}[x]/(x^n))$.

Conjecture 1.3. *The representation zeta function of* $\mathbf{H}(\mathcal{O}[x]/(x^n))$ *is*

$$\zeta_{\mathbf{H}(\mathcal{O}[x]/(x^n))}(s) = \prod_{i=1}^{n} \frac{\zeta_K(is-2i+1)}{\zeta_K(is-2i+2)}. \tag{1.4}$$

In [5], Rossmann introduces and studies topological representation zeta functions associated to unipotent group schemes Informally, this is the constant term of the local zeta function expanded as a series in $p-1$, which is a rational function in the parameter s; cf. [5, Definition 3.5]. It follows from (1.1), (1.2) and (1.3) that the topological zeta function of $\mathbf{H}[\varepsilon_n]$ with $n \leq 3$ is

$$\prod_{i=1}^{n} \frac{is-2i+2}{is-2i+1}. \tag{1.5}$$

Simple computations show that all questions in [5, Section 7], except Question 7.3 which is not yet known, have positive answers for $\mathbf{H}[\varepsilon_n]$ with $n \leq 3$.

Organization and notation. In Section 2, we recall formulae of local representation zeta functions in terms of p-adic integrals. The zeta function for the case $n = 3$ is computed in Section 3. The calculation in Section 3.2.2 yields a lot of terms and is then rather long, we record the results in the Appendix.

We fix some notation. Denote by $\mathbb{N}$ the set of positive integers. Let $\mathcal{O}$ be the ring of integers of some number field K and $\mathfrak{p}$ a non-zero prime ideal of $\mathcal{O}$. Denote by $\mathcal{O}_\mathfrak{p}$ the p-adic completion of $\mathcal{O}$ and let $q := |\mathcal{O}_\mathfrak{p}/\mathfrak{p}|$ the residue field cardinality. Let v be the valuation on $\mathcal{O}_\mathfrak{p}$ and for $x \in \mathcal{O}_\mathfrak{p}$, let $|x| = q^{v(x)}$ denote the $\mathfrak{p}$-adic absolute value.

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2. Preliminaries

2.1. **Local representation zeta functions.** In this section, we will present the local zeta functions of $\zeta_{\mathbf{H}(\mathcal{O}[x]/(x^n))}(s)$ in terms of p-adic integrals. The zeta functions of $\mathcal{T}$-groups obtained from unipotent group schemes (eg. $\mathbf{H}$) are developed by Stasinski and Voll in [7]. Readers are referred to [7] for background information.

The group $\mathbf{H}(\mathcal{O}[x]/(x^n))$ is a $\mathcal{T}$-group of nilpotency class 2 and Hirsch length $3n \cdot \mathrm{rk}_\mathbb{Z}(\mathcal{O})$. The zeta function $\zeta_{\mathbf{H}(\mathcal{O}[x]/(x^n))}(s)$ has an Euler factorization (cf. [7, Proposition 2.2])

$$\zeta_{\mathbf{H}(\mathcal{O}[x]/(x^n))}(s) = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})} \zeta_{\mathbf{H}(\mathcal{O}_\mathfrak{p}[x]/(x^n))}(s),$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals in $\mathcal{O}$ and $\mathcal{O}_\mathfrak{p}$ is the completion of $\mathcal{O}$ at $\mathfrak{p}$. The local factors $\zeta_{\mathbf{H}(\mathcal{O}_\mathfrak{p}[x]/(x^n))}$ are rational in $|\mathcal{O}/\mathfrak{p}|^{-s}$ and almost all of them satisfy a functional equation (cf. [7, Theorem A]).

The $\mathcal{O}$-Lie lattice associated to $\mathbf{H}(\mathcal{O}[x]/(x^n))$ has the following presentation; see [7, Section 2.4]:

$$\left\langle \begin{matrix} x_0, x_1, \cdots, x_{n-1} \\ y_0, y_1, \cdots, y_{n-1} \\ z_0, z_1, \cdots, z_{n-1} \end{matrix} \;\middle|\; [x_i, y_j] = \begin{cases} z_{i+j} & \text{if } i+j < n, \\ 0 & \text{otherwise.} \end{cases} \right\rangle.$$

The associated commutator matrix with respect to the chosen $\mathcal{O}$-basis is defined by

$$\mathcal{R}_n(\mathbf{Y}) = \left(\begin{array}{c|c} 0 & Q_n(\mathbf{Y}) \\ \hline -Q_n(\mathbf{Y})^t & 0 \end{array} \right),$$

where

$$Q_n(\mathbf{Y}) = \begin{pmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_{n-1} & Y_n \\ Y_2 & Y_3 & \cdots & & Y_n & 0 \\ Y_3 & \cdots & & \cdots & & 0 \\ \vdots & & \cdots & & & \vdots \\ Y_{n-1} & Y_n & \cdots & \cdots & \cdots & \vdots \\ Y_n & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_n(\mathcal{O}[Y_1, \cdots, Y_n]).$$

Fix a nonzero prime ideal $\mathfrak{p}$ and denote $\mathfrak{o} := \mathcal{O}_{\mathfrak{p}}$. Let $q := |\mathfrak{o}/\mathfrak{p}|$ be the residue field cardinality and p its characteristic. Let $W_n(\mathfrak{o}) = \mathfrak{o}^n \setminus \mathfrak{p}^n$. Set

$$\mathcal{Z}_{\mathfrak{p}}(\rho, \tau) := \int_{(u,\mathbf{y})\in\mathfrak{p}\times W_n(\mathfrak{o})} |u|^\tau \prod_{j=1}^{n} \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y})u^2\|^\rho}{\|F_{j-1}(\mathbf{y})\|^\rho} d\mu, \tag{2.1}$$

where the additive Haar measure μ on $\mathfrak{o}^{n+1}$ is normalized such that $\mu(\mathfrak{o}^{n+1}) = 1$, and

$$F_j(\mathbf{Y}) = \{f \mid f = f(\mathbf{Y}) \text{ a principal } 2j \times 2j \text{ minor of } \mathcal{R}_n(\mathbf{Y})\},$$
$$\|H(X, \mathbf{Y})\| = \max\{|h(X, \mathbf{Y})| \mid h \in H\} \text{ for a finite set } H \subset \mathfrak{o}[X, \mathbf{Y}].$$

The local factor $\zeta_{\mathbf{H}(\mathfrak{o}[x]/(x^n))}(s)$ can be expressed in terms of the p-adic integral (2.1) as the following (cf. [7, Corollary 2.11]):

$$\zeta_{\mathbf{H}(\mathfrak{o}[x]/(x^n))} = 1 + (1 - q^{-1})^{-1}\mathcal{Z}_{\mathfrak{p}}(-s/2, ns - n - 1). \tag{2.2}$$

2.2. Auxiliary lemma.

Lemma 2.1. *The following identities hold in the ring of power series* $\mathbb{Q}[[a, b, c]]$.

(1) $\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}} = \frac{abc(1-ab)}{(1-abc)(1-a)(1-b)}$.

(2) $\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,2Y\}} = \frac{abc(1-a+ac-a^2bc)}{(1-a)(1-b)(1-a^2bc^2)}$.

(3) $\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}} c^{\min\{X,2Y\}} = \frac{abc^2(1-a+ac-abc-a^2bc^3+a^3b^2c^3)}{(1-a)(1-b)(1-abc^2)(1-a^2bc^3)}$.

(4)

$$\sum_{(X,Y,Z)\in\mathbb{N}^3} a^X b^Y c^Z d^{\min\{X,Y+2Z\}} d^{\min\{X,2Y+4Z\}} = \frac{abd^2}{1-abd^2}\frac{1}{1-b}\frac{c}{1-c} +$$
$$+ \frac{acd^2}{1-acd^2}\frac{abd^2}{1-abd^2}\frac{1}{1-c} + \frac{a^2cd^4}{1-a^2cd^4}\frac{abd^2}{1-abd^2}\frac{1}{1-acd^2} + \frac{a^2bd^3}{1-a^2bd^3}\frac{1}{1-abd^2}\frac{a^2cd^4}{1-a^2cd^4}$$
$$+ \frac{a^2bd^3}{1-a^2bd^3}\frac{1-a+ad-a^4cd^5}{(1-a)(1-a^2cd^4)(1-a^4cd^6)}.$$

Proof. The identity (1) is from [8, Lemma 2.2]. We present the proofs of (2) and (3) while (4) is proven similarly.

For (2), consider the case $X \le Y$ and let $Y = X + Y'$ with $Y' \in \mathbb{N}_0$. Then

$$\sum_{\substack{(X,Y)\in\mathbb{N}^2\\ X\le Y}} a^X b^Y c^{\min\{X,2Y\}} = \sum_{(X,Y')\in\mathbb{N}\times\mathbb{N}_0} a^X b^{X+Y'} c^X = \frac{abc}{1-abc}\frac{1}{1-b}.$$

Consider the case $X > Y$ and let $X = Y + X'$ with $X' \in \mathbb{N}$. Then

$$\sum_{\substack{(X,Y)\in\mathbb{N}^2\\ X>Y}} a^X b^Y c^{\min\{X,2Y\}} = \sum_{(X',Y)\in\mathbb{N}^2} a^{X'}(abc)^Y c^{\min\{X',Y\}} = \frac{a^2bc^2(1-a^2bc)}{(1-a^2bc^2)(1-a)(1-abc)}$$

by (1). Hence

$$\begin{aligned}\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,2Y\}} &= \frac{abc}{1-abc}\frac{1}{1-b} + \frac{a^2bc^2(1-a^2bc)}{(1-a^2bc^2)(1-a)(1-abc)}\\ &= \frac{abc(1-a+ac-a^2bc)}{(1-a)(1-b)(1-a^2bc^2)}.\end{aligned}$$

For (3), first consider the case $X \leq Y$ and let $Y = X + Y'$ with $Y' \in \mathbb{N}_0$. Then

$$\sum_{\substack{(X,Y)\in\mathbb{N}^2\\ X\leq Y}} a^X b^Y c^{\min\{X,Y\}} c^{\min\{X,2Y\}} = \sum_{(X,Y')\in\mathbb{N}\times\mathbb{N}_0} a^X b^{X+Y'} c^X c^X = \frac{abc^2}{1-abc^2}\frac{1}{1-b}.$$

Consider now the case $X > Y$ and let $X = X' + Y$ with $X' \in \mathbb{N}$. Then, by (1)

$$\begin{aligned}\sum_{\substack{(X,Y)\in\mathbb{N}^2\\ X\leq Y}} a^X b^Y c^{\min\{X,Y\}} c^{\min\{X,2Y\}} &= \sum_{(X',Y)\in\mathbb{N}^2} a^{X'+Y} b^Y c^Y c^{\min\{X'+Y,2Y\}}\\ = \sum_{(X',Y)\in\mathbb{N}^2} a^{X'}(abc^2)^Y c^{\min\{X',Y\}} &= \frac{a^2bc^3(1-a^2bc^2)}{(1-a^2bc^3)(1-a)(1-abc^2)}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}} c^{\min\{X,2Y\}} &= \frac{abc^2}{1-abc^2}\frac{1}{1-b} + \frac{a^2bc^3(1-a^2bc^2)}{(1-a^2bc^3)(1-a)(1-abc^2)}\\ &= \frac{abc^2(1-a+ac-abc-a^2bc^3+a^3b^2c^3)}{(1-a)(1-b)(1-abc^2)(1-a^2bc^3)}.\end{aligned}$$

□

2.3. **The zeta function of $\mathbf{H}(\mathcal{O}[x]/(x^2))$.** In this case $\tau = 2s-3$ and $F_0(\mathbf{Y}) = 1, F_1(\mathbf{Y}) = \{X^2, Y^2\}$, $F_2(\mathbf{Y}) = \{Y^4\}$, with $\mathbf{Y} = (X, Y)$; cf. (2.2) and (2.1). We have

$$\mathcal{Z}_\mathfrak{p}(-s/2, 2s-3) = \int_{\substack{u\in\mathfrak{p}\\ \mathbf{y}=(x,y)\in W_2(\mathfrak{o})}} |u|^{2s-3}\|ux, uy, y^2\|^{-s} d\mu.$$

One can easily compute $\mathcal{Z}_\mathfrak{p}(-s/2, 2s-3)$ and obtain (1.2).

3. The zeta function of $\mathbf{H}(\mathcal{O}[x]/(x^3))$

In this case one has $\tau = 3s-4$ (cf. (2.2)). Denote $\mathbf{Y} = (X, Y, X)$, one has (cf. (2.1)) $F_0(\mathbf{Y}) = 1$, $F_1(\mathbf{Y}) = \{X^2, Y^2, Z^2\}$, $F_2(\mathbf{Y}) = \{Z^4, Y^2Z^2, (XZ-Y^2)^2\}$, $F_3(\mathbf{Y}) = \{Z^6\}$. By abuse of notation, we set the following functions in u, x, y, z:

$$A := \|z^2, yz, xz-y^2\|^s,$$
$$B := \|z^2, yz, xz-y^2, xu, yu, uz\|^{-s},$$
$$C := \|z^3, z^2u, yzu, (xz-y^2)u\|^{-s}.$$

Then

$$\zeta_{\mathbf{H}(\mathfrak{o}[x]/(x^3))}(s) = 1 + (1-q^{-1})^{-1}\mathcal{Z}(s),$$

where

$$\mathcal{Z} := \mathcal{Z}_{\mathfrak{p}}(-s/2, 3s-4) = \int_{\substack{u\in\mathfrak{p}\\ \mathbf{y}=(x,y,z)\in W_3(\mathfrak{o})}} |u|^{3s-4}ABCd\mu.$$

Write $\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3$ where

$$\mathcal{Z}_1 := \int_{\substack{u\in\mathfrak{p}, z\in W_1(\mathfrak{o})\\ x,y\in\mathfrak{o}}} |u|^{3s-4}ABCd\mu,$$
$$\mathcal{Z}_2 := \int_{\substack{u,z\in\mathfrak{p}\\ x\in\mathfrak{o}\\ y\in W_1(\mathfrak{o})}} |u|^{3s-4}ABCd\mu,$$
$$\mathcal{Z}_3 := \int_{\substack{u,y,z\in\mathfrak{p}\\ x\in W_1(\mathfrak{o})}} |u|^{3s-4}ABCd\mu.$$

3.1. **Computation of $\mathcal{Z}_1$.** Since $z \in W_1(\mathfrak{o})$, it follows that $A = B = C = 1$. Hence

$$\mathcal{Z}_1 = \int_{\substack{u\in\mathfrak{p}, z\in W_1(\mathfrak{o})\\ x,y\in\mathfrak{o}}} |u|^{3s-4}d\mu = (1-q^{-1})^2\frac{q^3t^3}{1-q^3t^3}\ , \text{ where } t := q^{-s}.$$

3.2. **Computation of $\mathcal{Z}_2$.** Since $y \in W_1(\mathfrak{o})$ and $z \in \mathfrak{p}$, it follows that $xz - y^2 \in W_1(\mathfrak{o})$ and $A = 1,\ B = 1,\ C = \|u, z^3\|^{-s}$. Thus

$$\mathcal{Z}_2 = (1-q^{-1})\int_{u,z\in\mathfrak{p}} |u|^{3s-4}\|u, z^3\|^{-s}d\mu.$$

Write $\mathcal{Z}_2 = \mathcal{Z}_{21} + \mathcal{Z}_{22}$ where

$$\mathcal{Z}_{21} := \int_{\substack{u,z\in\mathfrak{p}\\ v(u)\le v(z)}} |u|^{3s-4}\|u, z^3\|^{-s}d\mu,$$
$$\mathcal{Z}_{22} := \int_{\substack{u,z\in\mathfrak{p}\\ v(u)>v(z)}} |u|^{3s-4}\|u, z^3\|^{-s}d\mu.$$

Computation of $\mathcal{Z}_{21}$. Let $z = uz_1$ with $z_1 \in \mathfrak{o}$. Then

$$\mathcal{Z}_{21} = \int_{\substack{u\in\mathfrak{p}\\ z_1\in\mathfrak{o}}} |u|^{3s-4}|u|^{-s+1}d\mu = (1-q^{-1})\frac{q^2t^2}{1-q^2t^2}.$$

Computation of $\mathcal{Z}_{22}$. Let $u = zu_1$ with $u_1 \in \mathfrak{p}$. Then

$$\mathcal{Z}_{22} = \int_{u_1, z \in \mathfrak{p}} |u_1|^{3s-4}|z|^{2s-3}\|u_1, z^2\|^{-s} d\mu.$$

Since $\mu(\{(u_1, z) \in \mathfrak{p}^2 \mid v(u_1) = X, v(z) = Y\}) = (1-q^{-1})^2 q^{-X-Y}$, one has by Lemma 2.1 (2)

$$\begin{aligned}
\mathcal{Z}_{22} &= (1-q^{-1})^2 \sum_{(X,Y)\in\mathbb{N}^2} q^{-X-Y} q^{(-3s+4)X} q^{(-2s+3)Y} q^{s\min\{X,2Y\}} \\
&= (1-q^{-1})^2 \sum_{(X,Y)\in\mathbb{N}^2} q^{(-3s+3)X} q^{(-2s+2)Y} q^{s\min\{X,2Y\}} \\
&= (1-q^{-1})^2 \frac{q^{-3s+3-2s+2+s}(1-q^{-3s+3}+q^{-3s+3+s}-q^{-6s+6-2s+2+s})}{(1-q^{-3s+3})(1-q^{-2s+2})(1-q^{-6s+6-2s+2+2s})} \\
&= (1-q^{-1})^2 \frac{q^5t^4(1-q^3t^3+q^3t^2-q^8t^7)}{(1-q^2t^2)(1-q^3t^3)(1-q^8t^6)}.
\end{aligned}$$

Hence

$$\mathcal{Z}_2 = \mathcal{Z}_{21} + \mathcal{Z}_{22} = (1-q^{-1}) \frac{q^2t^2(1+q^3t^2-q^3t^3+q^6t^4-q^6t^5-q^8t^7)}{(1-q^3t^3)(1-q^8t^6)}.$$

3.3. **Computation of $\mathcal{Z}_3$.** In this case, $x \in W_1(\mathfrak{o}), y, z \in \mathfrak{p}$, whence $A = \|z^2, yz, xz - y^2\|^s$, $B = \|u, z^2, yz, xz - y^2\|^{-s}$, $C = \|z^3, z^2u, yzu, (xz-y)^2u\|^{-s}$. Write $\mathcal{Z}_3 = \mathcal{Z}_{31} + \mathcal{Z}_{32}$, where

$$\mathcal{Z}_{31} := \int_{\substack{u,y,z\in\mathfrak{p} \\ x\in W_1(\mathfrak{o}) \\ v(y)>v(z)}} |u|^{3s-4} ABC d\mu,$$

$$\mathcal{Z}_{32} := \int_{\substack{u,y,z\in\mathfrak{p} \\ x\in W_1(\mathfrak{o}) \\ v(y)\leq v(z)}} |u|^{3s-4} ABC d\mu.$$

3.3.1. *Computation of* $\mathcal{Z}_{31}$. Let $y = zy_1$ with $y_1 \in \mathfrak{p}$. Then

$$\begin{aligned}
&A = \|z^2, xz - y_1^2 z^2\|^s = |z|^s \|z, x - y_1^2 z\|^s = |z|^s \text{ (since } x - y_1^2 z \in W_1(\mathfrak{o})), \\
&B = \|u, z\|^{-s}, \ C = |z|^{-s}\|u, z^2\|^{-s}.
\end{aligned}$$

Thus

$$\mathcal{Z}_{31} = q^{-1}(1-q^{-1}) \int_{u,z\in\mathfrak{p}} |z||u|^{3s-4}\|u, z\|^{-s}\|u, z^2\|^{-s} d\mu.$$

Since $\mu\{(u, z) \in \mathfrak{p} \mid v(u) = X, v(z) = Y\} = (1-q^{-1})^2 q^{-X-Y}$, Lemma 2.1 (3) implies that

$$\begin{aligned}
\mathcal{Z}_{31} &= q^{-1}(1-q^{-1}) \int_{u,z\in\mathfrak{p}} |u|^{3s-4}|z|\|u, z\|^{-s}\|u, z^2\|^{-s} d\mu \\
&= q^{-1}(1-q^{-1})^3 \sum_{(X,Y)\in\mathbb{N}^2} q^{-X-Y} q^{(-3s+4)X} q^{-Y} q^{s\min\{X,Y\}} q^{s\min\{X,2Y\}} \\
&= q^{-1}(1-q^{-1})^3 \sum_{(X,Y)\in\mathbb{N}^2} q^{(-3s+3)X} q^{-2Y} q^{s\min\{X,Y\}} q^{s\min\{X,2Y\}} \\
&= (1-q^{-1})^3 \frac{t(1-q^3t^3+q^3t^2-qt^2-q^4t^3+q^5t^6)}{(1-q^{-2})(1-qt)(1-q^3t^3)(1-q^4t^3)}.
\end{aligned}$$

3.3.2. *Computation of $\mathcal{Z}_{32}$.* Let $z = yz_1$ with $z_1 \in \mathfrak{o}$. We have

$$A = |y|^s \underbrace{\|yz_1, xz_1 - y\|^s}_{:=A_1},\ B = \|u, y^2 z_1, y(xz_1 - y)\|^{-s},$$
$$C = |y|^{-s} \underbrace{\|y^2 z_1^3, yz_1 u, u(xz_1 - y)\|^{-s}}_{:=C_1}.$$

Thus

$$\mathcal{Z}_{32} = \int_{\substack{x \in W_1(\mathfrak{o}) \\ u,y \in \mathfrak{p} \\ z_1 \in \mathfrak{o}}} |u|^{3s-4} |y| A_1 B C_1 d\mu.$$

Write $\mathcal{Z}_{32} = \mathcal{Z}_{321} + \mathcal{Z}_{322}$, where

$$\mathcal{Z}_{321} := \int_{\substack{x \in W_1(\mathfrak{o}) \\ u,y \in \mathfrak{p} \\ z_1 \in W_1(\mathfrak{o})}} |u|^{3s-4} |y| A_1 B C_1 d\mu,$$
$$\mathcal{Z}_{322} := \int_{\substack{x \in W_1(\mathfrak{o}) \\ u,y \in \mathfrak{p} \\ z_1 \in \mathfrak{p}}} |u|^{3s-4} |y| A_1 B C_1 d\mu.$$

Computation of $\mathcal{Z}_{321}$. Since $z_1 \in W_1(\mathfrak{o})$, it follows that $xz_1 - y \in W_1(\mathfrak{o})$ and so

$$\mathcal{Z}_{321} = (1-q^{-1})^2 \int_{u,y \in \mathfrak{p}} |u|^{3s-4} |y| \|u, y\|^{-s} \|u, y^2\|^{-s} d\mu = (q-1)\mathcal{Z}_{31}.$$

Computation of $\mathcal{Z}_{322}$. Write $\mathcal{Z}_{322} = \mathcal{Z}_{322a} + \mathcal{Z}_{322b}$, where

$$\mathcal{Z}_{322a} := \int_{\substack{x \in W_1(\mathfrak{o}) \\ u,y,z_1 \in \mathfrak{p} \\ v(y) > v(z_1)}} |u|^{3s-4} |y| A_1 B C_1 d\mu,$$
$$\mathcal{Z}_{322b} := \int_{\substack{x \in W_1(\mathfrak{o}) \\ u,y,z_1 \in \mathfrak{p} \\ v(y) \leq v(z_1)}} |u|^{3s-4} |y| A_1 B C_1 d\mu.$$

Computation of $\mathcal{Z}_{322a}$. Let $y = z_1 y_1$ with $y_1 \in \mathfrak{p}$. We have

$$A_1 = |z_1|^s \text{ (since } x - y_1 \in W_1(\mathfrak{o})),\ B = \|u, y_1 z_1^2\|^{-s},$$
$$C_1 = |z_1|^{-s} \underbrace{\|y_1^2 z_1^4, u\|^{-s}}_{:=C_2}.$$

Thus

$$\mathcal{Z}_{322a} = (1-q^{-1}) \int_{u,y_1,z_1 \in \mathfrak{p}} |u|^{3s-4} |y_1| |z_1|^2 B C_2 d\mu.$$

Since $\mu\{(u, y_1, z_1) \in \mathfrak{p} \mid v(u) = X, v(y_1) = Y, v(z_1) = Z\} = (1-q^{-1})^3 q^{-X-Y-Z}$, one has

$$\begin{aligned}\mathcal{Z}_{322a} &= (1-q^{-1})^4 \sum_{(X,Y,Z) \in \mathbb{N}^3} q^{-X-Y-Z} q^{(-3s+4)X} q^{-Y} q^{-2Z} q^{s\min\{X, Y+2Z\}} q^{s\min\{X, 2Y+4Z\}} \\ &= (1-q^{-1})^4 \sum_{(X,Y,Z) \in \mathbb{N}^3} q^{(-3s+3)X} q^{-2Y} q^{-3Z} q^{s\min\{X, Y+2Z\}} q^{s\min\{X, 2Y+4Z\}}.\end{aligned}$$

One now can apply Lemma 2.1 (4) with $a = q^{-3s+3},\ b = q^{-2},\ c = q^{-3}$ and $d = q^s$ to obtain $\mathcal{Z}_{322a}$. We record the result in the Appendix.

Computation of $\mathcal{Z}_{322b}$. Let $z_1 = yz_2$ with $z_2 \in \mathfrak{o}$. We have

$$A_1 = |y|^s \underbrace{\|yz_2, xz_2 - 1\|^s}_{:=A_2},\ B = \|u, y^3z_2, y^2(xz_2-1)\|^{-s},$$
$$C_1 = |y|^{-s} \underbrace{\|y^4z_2^3, yz_2u, u(xz_2-1)\|^{-s}}_{:=C_2}.$$

Thus

$$\mathcal{Z}_{322b} = \int_{\substack{x\in W_1(\mathfrak{o})\\ z_2\in\mathfrak{o}\\ u,y\in\mathfrak{p}}} |u|^{3s-4}|y|^2 A_2 B C_2 d\mu = \mathcal{Z}_{322b1} + \mathcal{Z}_{322b2},$$

where

$$\mathcal{Z}_{322b1} := \int_{\substack{x\in W_1(\mathfrak{o})\\ z_2\in\mathfrak{p}\\ u,y\in\mathfrak{p}}} |u|^{3s-4}|y|^2 A_2 B C_2 d\mu,$$
$$\mathcal{Z}_{322b2} := \int_{\substack{x\in W_1(\mathfrak{o})\\ z_2\in W_1(\mathfrak{o})\\ u,y\in\mathfrak{p}}} |u|^{3s-4}|y|^2 A_2 B C_2 d\mu.$$

Computation of $\mathcal{Z}_{322b1}$. Since $z_2 \in \mathfrak{p}$, it follows that $xz_2 - 1 \in W_1(\mathfrak{o})$. Thus $A_2 = 1, B = \|u, y^2\|^{-s}$ and $C_2 = \|y^4z_2^3, u\|^{-s}$. It's now easy to compute

$$\mathcal{Z}_{322b1} = (1-q^{-1}) \int_{u,y,z_2\in\mathfrak{p}} |u|^{3s-4}|y|^2 B C_2 d\mu.$$

Since $\mu\{(u,y,z_2)\in\mathfrak{p}^3 \mid v(u)=X,\ v(y)=Y,\ v(z_2)=Z\} = (1-q^{-1})^3 q^{-X-Y-Z}$, one has

$$\begin{aligned}\mathcal{Z}_{322b1} &= (1-q^{-1})^4 \sum_{(X,Y,Z)\in\mathbb{N}^3} q^{-X-Y-Z} q^{(-3s+4)X} q^{-2Y} q^{s\min\{X,2Y\}} q^{s\min\{X,4Y+3Z\}} \\ &= (1-q^{-1})^4 \sum_{(X,Y,Z)\in\mathbb{N}^3} q^{(-3s+3)X} q^{-3Y} q^{-Z} q^{s\min\{X,2Y\}} q^{s\min\{X,4Y+3Z\}}.\end{aligned}$$

One needs first to compute $\sum_{(X,Y,Z)\in\mathbb{N}^3} a^X b^Y c^Z d^{\min\{X,2Y\}} d^{\min\{X,4Y+3Z\}}$ similarly to Lemma 2.1 (4) and then apply to $a = q^{-3s+3},\ b = q^{-3},\ c = q^{-1}$ and $d = q^s$ to obtain $\mathcal{Z}_{322b1}$. The result is recorded in the Appendix.

Computation of $\mathcal{Z}_{322b2}$. The equation $xz_2 \equiv 1 \mod \mathfrak{p}$ has $q-1$ roots $(a_1,a_2)\in(\mathbb{F}_q^*)^2$. We have

$$\begin{aligned}\mathcal{Z}_{322b2} &= \int_{\substack{x,z_2\in W_1(\mathfrak{o})\\ u,y\in\mathfrak{p}}} |u|^{3s-4}|y|^2 A_2 B C_2 d\mu = \sum_{(a_1,a_2)\in(\mathbb{F}_q^*)^2} \int_{\substack{(x,z_2)\in(a_1,a_2)+\mathfrak{p}^2\\ u,y\in\mathfrak{p}}} |u|^{3s-4}|y|^2 A_2 B C_2 d\mu \\ &= (q-1)(q-2)J_1 + (q-1)J_2,\end{aligned}$$

where

$$J_1 := \int_{\substack{(x,z_2)\in(a_1,a_2)+\mathfrak{p}^2\\ a_1a_2\not\equiv 1 \mod \mathfrak{p}\\ u,y\in\mathfrak{p}}} |u|^{3s-4}|y|^2 A_2 B C_2 d\mu,$$
$$J_2 := \int_{\substack{(x,z_2)\in(a_1,a_2)+\mathfrak{p}^2\\ a_1a_2\equiv 1 \mod \mathfrak{p}\\ u,y\in\mathfrak{p}}} |u|^{3s-4}|y|^2 A_2 B C_2 d\mu.$$

In computing J_1, notice that in this case $xz_1 \not\equiv 1 \mod \mathfrak{p}$, and so $A_2 = 1, B = \|u, y^2\|^{-s}$ and $C_2 = \|u, y^4\|^{-s}$, and thus we have

$$J_1 = q^{-2} \int_{u,y\in\mathfrak{p}} |u|^{3s-4}|y|^2 BC_2 d\mu.$$

Since $\mu\{(u,y) \in \mathfrak{p}^2 \mid v(u) = X,\ v(y) = Y\} = (1-q^{-1})q^{-X-Y}$, one has

$$\begin{aligned} J_1 &= q^{-2}(1-q^{-1})^2 \sum_{(X,Y)\in\mathbb{N}^2} q^{-X-Y} q^{(-3s+4)X} q^{-2Y} q^{s\min\{X,2Y\}} q^{s\min\{X,4Y\}} \\ &= q^{-2}(1-q^{-1})^2 \sum_{(X,Y)\in\mathbb{N}^2} q^{(-3s+3)X} q^{-3Y} q^{s\min\{X,2Y\}} q^{s\min\{X,4Y\}}. \end{aligned}$$

We first need to compute $\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,2Y\}} c^{\min\{X,4Y\}}$ similarly to Lemma 2.1 (3) and then apply with $a = q^{-3s+3},\ b = q^{-3}$ and $c = q^s$ to obtain J_1. We record J_1 in the Appendix. In computing J_2, notice that in this case, on each coset $(a_1, a_2)+\mathfrak{p}^2$ we have $xz_2 \equiv 1 \mod \mathfrak{p}$. We change variable $v = xz_2 - 1 \in \mathfrak{p}$. Then $A_2 = \|y, v\|^s, B = \|u, y^3, y^2v\|^{-s},\ C_2 = \|y^4, yu, uv\|^{-s}$ and

$$J_2 = q^{-1} \int_{u,y,v\in\mathfrak{p}} |u|^{3s-4}|y|^2 A_2 BC_2 d\mu.$$

Since $\mu\{(u,y,v) \in \mathfrak{p} \mid v(u) = X,\ v(y) = Y,\ v(v) = Z\} = (1-q^{-1})^3 q^{-X-Y-Z}$, one has

$$\begin{aligned} J_2 = q^{-1}(1-q^{-1})^3 \sum_{(X,Y,Z)\in\mathfrak{p}} \Big(& q^{(-3s+3)X} q^{-3Y} q^{-Z} q^{-s\min\{\min\{Y,Z\}} \\ & \times q^{s\min\{X,3Y,2Y+Z\}} q^{s\min\{X+Y,X+Z,4Y\}} \Big). \end{aligned}$$

Again computing $\sum_{(X,Y,Z)\in\mathbb{N}^3} a^X b^Y c^Z d^{-\min\{Y,Z\}} d^{\min\{X,3Y,2Y+Z\}} d^{\min\{X+Y,X+Z,4Y\}}$ and then applying for $a = q^{-3s+3},\ b = q^{-3},\ c = q^{-1}$ and $d = q^s$ yields J_2 which we record in the Appendix.

Summing up $\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3$, we obtain

$$\zeta_{\mathbf{H}(\mathfrak{o}[x]/(x^3))}(s) = 1 + (1-q^{-1})^{-1}\mathcal{Z} = \frac{(1-t)(1-q^2t^2)(1-q^4t^3)}{(1-qt)(1-q^3t^2)(1-q^5t^3)},$$

and hence Theorem 1.1 follows.

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Appendix

$$\mathcal{Z}_{322a}: \quad (1-q^{-1})^4\Big(\tfrac{q^{-2}t}{(1-q^{-2})(1-q^{-3})(1-qt)}+\tfrac{pt^2}{(1-q^{-3})(1-t)(1-qt)}+\tfrac{q^4t^3}{(1-t)(1-qt)(1-q^3t^2)}+\tfrac{q^7t^5}{(1-qt)(1-q^3t^2)(1-q^4t^3)}+\tfrac{q^{10}t^7}{(1-q^3t^2)(1-q^4t^3)(1-q^6t^4)}+\tfrac{q^{13}t^9}{(1-q^4t^3)(1-q^9t^6)(1-q^6t^4)}+\tfrac{q^{16}t^{12}}{(1-q^4t^3)(1-q^3t^3)(1-q^9t^6)}\Big)$$

$$\mathcal{Z}_{322b1}: \quad (1-q^{-1})^4\Big(\tfrac{t}{(q-1)(1-q^{-3})(1-t)}+\tfrac{q^2t^2}{(1-q^{-1})(1-t)(1-q^3t^2)}+\tfrac{q^5t^4}{(1-q^{-1})(1-q^3t^2)(1-q^6t^4)}+\tfrac{q^8t^6}{(1-q^{-1})(1-q^9t^6)(1-q^6t^4)}+\tfrac{q^{11}t^8}{(1-q^{-1})(1-q^9t^6)(1-q^2t^2)}+\tfrac{q^{14}t^{10}}{(1-q^9t^6)(1-q^5t^4)(1-q^2t^2)}+\tfrac{q^{17}t^{12}}{(1-q^9t^6)(1-q^8t^6)(1-q^5t^4)}+\tfrac{q^{20}t^{15}}{(1-q^9t^6)(1-q^3t^3)(1-q^8t^6)}\Big)$$

$$J_1: \quad q^{-2}(1-q^{-1})^2\Big(\tfrac{t}{(1-q^{-3})(1-t)}+\tfrac{q^3t^2}{(1-t)(1-q^3t^2)}+\tfrac{q^6t^4}{(1-q^6t^4)(1-q^3t^2)}+\tfrac{q^9t^6}{(1-q^9t^6)(1-q^6t^4)}+\tfrac{q^{12}t^9}{(1-q^3t^3)(1-q^9t^6)}\Big)$$

$$J_2: \quad q^{-1}(1-q^{-1})^3\Big(\tfrac{q^{-1}t}{(1-q^{-1})(1-q^{-4})(1-q^{-1}t)}+\tfrac{q^2t^2}{(1-q^{-1})(1-q^2t^2)(1-q^{-1}t)}+\tfrac{q^5t^3}{(1-q^{-1})(1-q^5t^3)(1-q^2t^2)}+\tfrac{q^8t^6}{(1-q^{-1})(1-q^3t^3)(1-q^5t^3)}+\tfrac{q^{-4}t}{(1-q^{-3})(1-q^{-4})(1-q^{-1}t)}+\tfrac{q^{-1}t^2}{(1-q^{-3})(1-q^{-1}t)(1-q^2t^2)}+\tfrac{q^2t^3}{(1-q^{-3})(1-q^5t^3)(1-q^2t^2)}+\tfrac{q^5t^4}{(1-q^{-3})(1-t)(1-q^5t^3)}+\tfrac{q^8t^5}{(1-q^5t^3)(1-q^3t^2)(1-t)}+\tfrac{q^{11}t^7}{(1-q^5t^3)(1-q^6t^4)(1-q^3t^2)}+\tfrac{q^{14}t^9}{(1-q^5t^3)(1-q^9t^6)(1-q^6t^4)}+\tfrac{q^{17}t^{12}}{(1-q^5t^3)(1-q^9t^6)(1-q^6t^4)}\Big)$$

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