

TRANSFERENCE OF BILINEAR RESTRICTION ESTIMATES TO QUADRATIC VARIATION NORMS AND THE DIRAC-KLEIN-GORDON SYSTEM

TIMOTHY CANDY AND SEBASTIAN HERR

ABSTRACT. Firstly, bilinear Fourier Restriction estimates –which are well-known for free waves– are extended to adapted spaces of functions of bounded quadratic variation, under quantitative assumptions on the phase functions. This has applications to nonlinear dispersive equations, in particular in the presence of resonances. Secondly, critical global well-posedness and scattering results for massive Dirac-Klein-Gordon systems in dimension three are obtained, in resonant as well as in non-resonant regimes. The results apply to small initial data in scale-invariant Sobolev spaces exhibiting a small amount of angular regularity.

1. INTRODUCTION

The Fourier restriction conjecture was shaped in the 1970s by work of Stein, among others, and has generated significant advances in the field of harmonic analysis and dispersive partial differential equations since then, see e.g. [37, 44] for a survey and references.

As an example, let $n \geq 2$ and C be a compact subset of the cone, say $C = \{(|\xi|, \xi) \mid \frac{1}{2} \leq |\xi| \leq 2\} \subset \mathbb{R}^{n+1}$, and g be a Schwartz function on \mathbb{R}^{n+1} . Equivalently to the Fourier restriction operator $\mathcal{R} : g \mapsto \widehat{g}|_C$, consider its adjoint, the Fourier extension operator

$$\mathcal{E}f(t, x) = \int_{\mathbb{R}^n} e^{-i(t,x) \cdot (|\xi|, \xi)} f(\xi) d\xi,$$

for smooth f with $\text{supp}(f)$ contained in the unit annulus. The function $\mathcal{E}f$ can be viewed as the inverse Fourier transform of a surface-measure supported on the cone C , and defines a function on \mathbb{R}^{n+1} which solves the wave equation. The Fourier restriction conjecture for the cone is equivalent to establishing the corresponding Fourier extension estimate

$$\|\mathcal{E}f\|_{L_{t,x}^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^q(\mathbb{R}^n)}$$

within the optimal range of p, q . In the special case $q = 2$ this holds iff $p \geq \frac{2n+2}{n-1}$, and in the literature on dispersive equations this is stated as

$$\|e^{-it|\nabla|} f\|_{L_{t,x}^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{L_x^2}$$

and called a Strichartz estimate for the wave equation [41], see also [22] for more information.

In the course of proving Fourier extension estimates for the cone, it became apparent that a key role was played by bilinear estimates. Indeed, a major breakthrough was achieved by Wolff [49], when he proved that for every $p > \frac{n+3}{n+1}$, $n \geq 2$, we have

$$\|e^{-it|\nabla|} f e^{-it|\nabla|} g\|_{L_{t,x}^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{L_x^2} \|g\|_{L_x^2},$$

provided the supports of \widehat{f} and \widehat{g} are angularly separated and contained in the unit annulus. As a result Wolff was able to prove the linear restriction conjecture for C in dimensions $n = 3$. It is important to note that, in the presence of angular separation, a larger set of p can be covered in the bilinear estimate than would follow from a simple application of Hölder's inequality together with the linear estimates.

In parallel to these developments, bilinear estimates proved useful in the context of nonlinear dispersive equations, see e.g. [23, 14, 19]. The perturbative approach to dispersive equations is based on constructing

2010 *Mathematics Subject Classification*. Primary: 42B37, 35Q41; Secondary: 42B20, 42B10, 81Q05.

Key words and phrases. Bilinear Fourier Restriction, Adapted Function Spaces, Quadratic Variation, Atomic Space, Dirac-Klein-Gordon System, Resonance, Global Well-Posedness, Scattering.

The authors acknowledge support from the German Research Foundation via Collaborative Research Center 701.

adapted function spaces in which nonlinear terms can be effectively estimated. Bilinear estimates for solutions to the homogeneous equation, which go beyond simple almost orthogonality considerations, give precise control over dynamic interactions of products of linear solutions. However, to apply these homogeneous estimates to the nonlinear problem, necessitates the transfer of such genuinely bilinear estimates to adapted function spaces.

Such a *Transference Principle* was implemented first in $X^{s,b}$ spaces, see [20, Lemma 2.3] and [25, Proposition 3.7]. Let us briefly illustrate it by looking at the following example. Suppose that $u, v \in L_t^\infty L_x^2$ are superpositions of modulated solutions of the homogeneous equation, i.e.

$$u(t) = \int_{\mathbb{R}} e^{it\lambda} e^{it|\nabla|} F_\lambda d\lambda, \quad v(t) = \int_{\mathbb{R}} e^{it\lambda'} e^{it|\nabla|} G_{\lambda'} d\lambda'.$$

which is true for $u, v \in X^{0,b}$ if $b > \frac{1}{2}$. Suppose in addition, that the spatial Fourier supports of u, v are angularly separated. Then, for any $p > \frac{n+3}{n+1}$, Wolff's estimate transfers to

$$\|uv\|_{L_{t,x}^p(\mathbb{R}^{n+1})} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \|e^{it|\nabla|} F_\lambda e^{it|\nabla|} G_{\lambda'}\|_{L_{t,x}^p(\mathbb{R}^{n+1})} d\lambda d\lambda' \lesssim \left(\int_{\mathbb{R}} \|F_\lambda\|_{L_x^2} d\lambda \right) \left(\int_{\mathbb{R}} \|G_{\lambda'}\|_{L_x^2} d\lambda' \right)$$

which is equivalent to the bilinear estimate holding for functions in $X^{0,b}$. Another strategy involves certain atomic function spaces introduced in [27]. Suppose that

$$u(t) = \sum_{J \in \mathcal{I}} \mathbb{1}_J(t) e^{it|\nabla|} f_J, \quad v(t) = \sum_{J' \in \mathcal{I}'} \mathbb{1}_{J'}(t) e^{it|\nabla|} g_{J'}.$$

for finite partitions $\mathcal{I}, \mathcal{I}'$ of \mathbb{R} and $f_J, g_{J'} \in L_x^2$. Then, under the above angular separation assumption, Wolff's bound implies

$$\|uv\|_{L_{t,x}^p(\mathbb{R}^{n+1})} \leq \left(\sum_{J \in \mathcal{I}} \sum_{J' \in \mathcal{I}'} \|e^{it|\nabla|} f_J e^{it|\nabla|} g_{J'}\|_{L_{t,x}^p(\mathbb{R}^{n+1})}^p \right)^{\frac{1}{p}} \lesssim \left(\sum_{J \in \mathcal{I}} \|f_J\|_{L_x^2}^p \right)^{\frac{1}{p}} \left(\sum_{J' \in \mathcal{I}'} \|g_{J'}\|_{L_x^2}^p \right)^{\frac{1}{p}}.$$

As a consequence, we deduce that Wolff's bilinear estimate holds for angularly separated functions in the atomic space U^p , see Definition 3.4 below. This is one instance of the transference principle in U^p , which has been formalised in [21, Proposition 2.19].

For many applications, the above superposition requirements are too strong, partly due to the duality theory for the spaces $X^{0,b}$ for $b > \frac{1}{2}$ and U^p for $p \leq 2$. Nevertheless, variations of the above strategies have been successfully employed in numerous applications to nonlinear global-in-time problems in the case $p \geq 2$. In the case $p < 2$, the only result we are aware of is [39, Lemma 5.7 and its proof], where this approach is used in conjunction with an interpolation argument to give a partial result only, see Remark 6.2 for further details.

It turned out that one of the most powerful function spaces in the context of adapted function spaces, is the space of functions of bounded quadratic variation V^2 , which is slightly bigger than U^2 . Our first main result of this paper is the corresponding transference principle in V^2 for a quite general class of surfaces in Theorem 1.1 below.

We start with some definitions. Define $\mathcal{Z} = \{(t_j)_{j \in \mathbb{Z}} \mid t_j \in \mathbb{R} \text{ and } t_j < t_{j+1}\}$ to be the set of increasing sequences of real numbers and $1 \leq p < \infty$. Given a function $\rho : \mathbb{R} \rightarrow L_x^2$, we define the p -variation of ρ to be

$$|\rho|_{V^p} = \sup_{(t_j) \in \mathcal{Z}} \left(\sum_{j \in \mathbb{Z}} \|\rho(t_j) - \rho(t_{j-1})\|_{L_x^2}^p \right)^{\frac{1}{p}}.$$

The Banach space V^p is then defined to be all right continuous functions $\rho : \mathbb{R} \rightarrow L_x^2$ such that

$$\|\rho\|_{V^p} = \|\rho\|_{L_t^\infty L_x^2} + |\rho|_{V^p} < \infty.$$

Given a phase $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ we let V_Φ^p denote the space of all functions u such that $e^{-it\Phi(-i\nabla)} u \in V^p$ equipped with the obvious norm $\|u\|_{V_\Phi^p} = \|e^{-it\Phi(-i\nabla)} u\|_{V^p}$. In other words, the space V_Φ^p contains all functions $u \in L_t^\infty L_x^2$ such that the pull-back along the linear flow has bounded p -variation, in particular we have

$$\|e^{it\Phi(-i\nabla)} f\|_{V_\Phi^p} = \|f\|_{L_x^2}.$$

Before stating Theorem 1.1, we need to introduce the assumptions that we impose on our phases, which are motivated by [33, 2]. Examples will be discussed in Section 2. Let $\Phi_j : \mathbb{R}^n \rightarrow \mathbb{R}$ and Λ_j be a convex subset of $\{\frac{1}{16} \leq |\xi| \leq 16\}$. Given $\mathfrak{h} = (a, h) \in \mathbb{R}^{1+n}$ and $\{j, k\} = \{1, 2\}$ we define the hypersurfaces

$$\Sigma_j(\mathfrak{h}) = \{\xi \in \Lambda_j \cap (\Lambda_k + h) \mid \Phi_j(\xi) = \Phi_k(\xi - h) + a\}.$$

With this notation, we are ready to state the main assumption, cp. [2, 33].

Assumption 1 (Transversality/Curvature/Regularity). *There exist $\mathbf{D}_1, \mathbf{D}_2 > 0$ and $N \in \mathbb{N}$ such that for $\Phi_1, \Phi_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ the following holds true:*

(i) *for every $\{j, k\} = \{1, 2\}$, $\mathfrak{h} \in \mathbb{R}^{1+n}$, $\xi, \xi' \in \Sigma_j(\mathfrak{h})$, and $\eta \in \Lambda_k$ we have the estimate*

$$|(\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \wedge (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta))| \geq \mathbf{D}_1|\xi - \xi'|,$$

(ii) *we have $\Phi_j \in C^N(\Lambda_j)$ with the derivative bound*

$$\sup_{1 \leq |\kappa| \leq N} \|\partial^\kappa \Phi_j\|_{L^\infty(\Lambda_j)} \leq \mathbf{D}_2.$$

The condition (i) in Assumption 1 is somewhat difficult to interpret, but one immediate consequence is the bound

$$|\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')| \geq \frac{\mathbf{D}_1|\xi - \xi'|}{\|\nabla\Phi_1\|_{L^\infty} + \|\nabla\Phi_2\|_{L^\infty}}. \quad (1.1)$$

which holds for every $\xi, \xi' \in \Sigma_j(\mathfrak{h})$. To some extent, this is a *curvature* condition, as it shows that the normal direction varies on $\Sigma_j(\mathfrak{h})$. Another consequence of (i) is that for every $\xi \in \Lambda_1$, $\eta \in \Lambda_2$ we have the *transversality* bound

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| \geq \frac{\mathbf{D}_1}{\min\{\|\nabla^2\Phi_1\|_{L^\infty}, \|\nabla^2\Phi_2\|_{L^\infty}\}}. \quad (1.2)$$

This follows by simply observing that for every $\xi \in \Lambda_1$ there is $\mathfrak{h} \in \mathbb{R}^{1+n}$ such that $\xi \in \Sigma_1(\mathfrak{h})$. Our first main result can now be stated as follows.

Theorem 1.1. *Let $n \geq 2$, $p > \frac{n+3}{n+1}$, and $\mathbf{D}_1, \mathbf{D}_2, \mathbf{R}_0 > 0$. For $j = 1, 2$, let $\Lambda_j, \Lambda_j^* \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ with Λ_j convex and $\Lambda_j^* + \frac{1}{\mathbf{R}_0} \subset \Lambda_j$. There exists $N \in \mathbb{N}$ and a constant $C > 0$ such that, for any phases Φ_1 and Φ_2 satisfying Assumption 1, and any $u \in V_{\Phi_1}^2$, $v \in V_{\Phi_2}^2$ with $\text{supp } \widehat{u}(t) \subset \Lambda_1^*$, $\text{supp } \widehat{v}(t) \subset \Lambda_2^*$, we have*

$$\|uv\|_{L_{t,x}^p(\mathbb{R}^{1+n})} \leq C \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

Note that the constants N and C depend on the parameters $p > \frac{n+3}{n+1}$, $n \geq 2$, and $\mathbf{D}_1, \mathbf{D}_2, \mathbf{R}_0 > 0$, but are otherwise independent of the phase Φ_j , the sets Λ_j, Λ_j^* , and the functions u and v . Moreover, as the conditions in Assumption 1 are invariant under translations, the condition that $\Lambda_j \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ can be replaced with the condition that the sets Λ_j are simply contained in balls of radius 16. In other words, the *location* of the sets Λ_j plays no role. We refer the reader to Corollary 6.1 for a generalisation of Theorem 1.1 to mixed norms. Further, we refer to Corollary 6.4 for a generalisation to more general frequency scales in the case of hyperboloids, which is also shown to be sharp.

Let us summarize the developments for solutions to the homogeneous equation, i.e.

$$u = e^{it\Phi_1(-i\nabla)} f, \quad v = e^{it\Phi_2(-i\nabla)} g.$$

First estimates of this type for nontrivial $p < 2$ are due to Bourgain [12, 13] in the case of the cone, i.e. $\Phi_1(\xi) = \Phi_2(\xi) = |\xi|$. Subsequently, these have been improved by Tao–Vargas–Vega [47], Moyua–Vargas–Vega [35], Tao–Vargas [45], before finally Tao [42] proved the endpoint case $p = \frac{n+3}{n+1}$, see also Remark 5.1. Actually, we observe that the vector-valued inequality in [42] is strong enough to deduce the estimate in U^2 in the case of the wave equation, see Remark 5.2. Related estimates for null-forms have been proved by Tao–Vargas [46], Klainerman–Rodnianski–Tao [24], Lee–Vargas [32], and Lee–Rogers–Vargas [31]. In the case of the paraboloid, i.e. $\Phi_1(\xi) = \Phi_2(\xi) = |\xi|^2$, the result for homogeneous solutions is due to Tao [43], with generalisations by Lee [29, 30], Lee–Vargas [33], and Bejenaru [2] under more general curvature and transversality conditions, as well as by Buschenhenke–Müller–Vargas [15] for surfaces of finite type. For our approach, the most important references are [43] concerning notation and general line of proof and [33, 2], concerning the assumptions on the phases and its consequences. Throughout the paper, we shall point out similarities and differences in more detail.

We would like to highlight the fact that we explicitly track the dependence of the constants on the phases in Theorem 1.1 based on the global, quantitative Assumption 1, in particular we avoid abstract localisation arguments. This is helpful for applications to dispersive equations, as we will see below. The main novelty of this result, however, lies in the fact that it holds for $V_{\Phi_j}^2$ -functions in the range $p \leq 2$.

Now, we turn to the application of Theorem 1.1 to nonlinear dispersive equations with a quadratic nonlinearity which exhibit resonances. Roughly speaking, by a resonance we mean the scenario that a product of two solutions to the homogeneous equations creates another solution of the homogeneous equation, see Section 8 for details. This leads to the lack of oscillations in the Duhamel integral and hence to strong nonlinear effects. In many instances, one finds that the Fourier supports intersect transversally in the resonant sets. As an example, we mention the local well-posedness theory for the Zakharov system [6, 4], where this is exploited in terms of a nonlinear Loomis-Whitney inequality [10, 7, 8, 26]. This is a special case of the multilinear restriction theory [9, 8]. Here, we will exploit transversality in resonant sets via Theorem 1.1 and prove global-in-time estimates which go beyond the range of linear Strichartz estimates.

With this approach, we address the Dirac-Klein-Gordon system

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= \phi\psi \\ \square\phi + m^2\phi &= \psi^\dagger \gamma^0 \psi \end{aligned} \tag{1.3}$$

Here, $\psi : \mathbb{R}^{1+3} \rightarrow \mathbb{C}^4$ is a spinor field, $\psi^\dagger = \overline{\psi}^t$, $\phi : \mathbb{R}^{1+3} \rightarrow \mathbb{R}$ is a scalar field, $\square := \partial_t^2 - \Delta_x$ is the d'Alembertian operator, and $M, m \geq 0$. We use the summation convention with respect to $\mu = 0, \dots, 4$ and the Dirac matrices $\gamma^\mu \in \mathbb{C}^{4 \times 4}$ are given by

$$\gamma^0 = \text{diag}(1, 1, -1, -1), \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are interested in the system (1.3) with the initial condition

$$\psi(0) = \psi_0 : \mathbb{R}^3 \rightarrow \mathbb{C}^4 \text{ and } (\phi(0), \partial_t \phi(0)) = (\phi_0, \phi_1) : \mathbb{R}^3 \rightarrow \mathbb{R} \times \mathbb{R}. \tag{1.4}$$

In the massless case (1.3) can be rescaled and the scale invariant Sobolev space for (ψ_0, ϕ_0, ϕ_1) is

$$L^2(\mathbb{R}^3; \mathbb{C}^4) \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}).$$

Let $\langle \Omega \rangle^\sigma$ denote σ angular derivatives, see Subsection 7.2 for precise definitions. Our second main result is the following.

Theorem 1.2. *Suppose that either $2M \geq m > 0$ and $\sigma > 0$, or that $m > 2M > 0$ and $\sigma > \frac{7}{30}$. Then, for initial data satisfying*

$$\|\langle \Omega \rangle^\sigma \psi_0\|_{L^2(\mathbb{R}^3)} + \|\langle \Omega \rangle^\sigma \phi_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \|\langle \Omega \rangle^\sigma \phi_1\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \ll 1,$$

the system (1.3)–(1.4) is globally well-posed and solutions (ψ, ϕ) scatter to free solutions as $t \rightarrow \pm\infty$.

As the proof relies on contraction arguments in adapted function spaces, the notion of global well-posedness in Theorem 1.2 includes persistence of regularity and the local Lipschitz continuity of the flow map and it provides a certain uniqueness class. Note that the angular regularity does not affect the scaling of the spaces. In summary, Theorem 1.2 establishes global well-posedness and scattering in the critical Sobolev space for small initial data with a bit of angular regularity.

In the case $2M > m > 0$, which we call *non-resonant regime* due to Lemma 8.7, this theorem improves Wang's result [48] by both relaxing the angular regularity hypothesis and replacing Besov spaces by Sobolev spaces. We also mention the previous subcritical result [3] without additional angular regularity, where the possibility of a Besov endpoint result with an $\epsilon > 0$ of angular regularity was discussed [3, Remark 4.2]. In the case $m > 2M > 0$, which we call the *resonant regime* due to Lemma 8.7, this appears to be the first global well-posedness and scattering result in critical spaces for (1.3). A similar comment applies to the case $2M = m > 0$, which we call the *weakly resonant regime*. It is the resonant regime where we employ Theorem 1.1, see also Remark 7.6. Concerning further comments on the number of angular derivatives required in the resonant case, we refer to Remark 8.4.

We shall only mention a few selected results on this well-studied system (1.3). We refer the reader to [18] for previous local results and to [16, 1, 3, 48] for previous global results on this system, also to the references therein. Concerning its relevance in physics we refer the reader to [11].

The organisation of the paper is as follows: In Section 2, we discuss a sufficient condition on the phases, verify Assumption 1 in the case of the Schrödinger, the wave, and the Klein-Gordon equation, and derive important consequences, in particular the dispersive inequality, and a bilinear estimate for homogeneous solutions in $L^2_{t,x}$. In Section 3, we study wave packets, atomic spaces and tubes. In Section 4, we state and prove a crucial localised version of Theorem 1.1. The proof proceeds by performing an induction on scales argument, and reducing the problem to obtaining a crucial L^2 -bound which in turn follows from a combinatorial estimate. Section 5 is devoted to the globalisation lemma, which removes the localisation assumption used in Section 4, and hence concludes the proof of Theorem 1.1. In Section 6, we generalise Theorem 1.1 to mixed norms and, in the case of hyperboloids, give an extension to general scales and discuss counterexamples. In Section 7 we prepare the analysis of the Dirac-Klein-Gordon System and prove Theorem 1.2 under the hypothesis that certain bilinear estimates hold true. In Section 8 we discuss some auxilliary estimates and finally provide proofs of the bilinear estimates used in Section 7.

2. ON ASSUMPTION 1: EXAMPLES AND CONSEQUENCES

In this section we discuss examples, and consider in detail a number of key consequences of Assumption 1. All of this is known to experts at least in the specific cases we are interested in. The main objective is to verify that Assumption 1 allows for a unified treatment which allows to track the dependence of constants on the phases.

2.1. A Sufficient Condition. Let $\text{diam}(\Lambda_j) = \sup_{\xi, \xi' \in \Lambda_j} |\xi - \xi'|$. The condition (i) in Assumption 1 is somewhat difficult to check (essentially since we insist on a *global* condition rather than just a local condition using the Hessian of Φ_j). In practise it is easier to check the following marginally stronger conditions.

Lemma 2.1. *Assume that the following three conditions hold:*

(i) *For all $\xi \in \Lambda_1$ and $\eta \in \Lambda_2$*

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| \geq \mathbf{A}_1. \quad (2.1)$$

(ii) *For $j = 1, 2$, and every $\mathfrak{h} \in \mathbb{R}^{1+n}$ and $\xi, \xi' \in \Sigma_j(\mathfrak{h})$*

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \geq \mathbf{A}_2 |\xi - \xi'|. \quad (2.2)$$

(iii) *The sets Λ_1 and Λ_2 satisfy*

$$\text{diam}(\Lambda_1) + \text{diam}(\Lambda_2) \leq \frac{\mathbf{A}_1 \mathbf{A}_2}{2(\|\nabla^2\Phi_1\|_{L^\infty(\Lambda_1)} + \|\nabla^2\Phi_2\|_{L^\infty(\Lambda_2)})^2}. \quad (2.3)$$

Then, condition (i) in Assumption 1 holds with $\mathbf{D}_1 = \frac{1}{2}\mathbf{A}_1\mathbf{A}_2$.

Proof. The first step is to observe that for vectors $x, y \in \mathbb{R}^n$, and $\omega \in \mathbb{S}^{n-1}$ we have

$$|x \wedge y| \geq |y||x \cdot \omega| - |x||y \cdot \omega|. \quad (2.4)$$

Indeed, this follows from $|x \wedge y|^2 = |x|^2|y|^2 - (x \cdot y)^2 = |y|^2|x - \frac{x \cdot y}{|y|^2}y|^2$, which implies

$$|x \wedge y| = |y| \left| x - \frac{x \cdot y}{|y|^2}y \right| \geq |y| \left| x \cdot \omega - \frac{x \cdot y}{|y|^2}y \cdot \omega \right| \geq |y| \left(|x \cdot \omega| - \frac{|x|}{|y|}|y \cdot \omega| \right).$$

In particular, if we let $x = \nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')$, $y = \nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)$, and $\omega = \frac{\xi - \xi'}{|\xi - \xi'|}$, then since $|x| \leq \|\nabla^2\Phi_j\|_{L^\infty(\Lambda_j)}|\xi - \xi'|$ (using the convexity of Λ_j) the lower bound (i) in Assumption 1 would follow from (2.2), (2.4), and the transversality condition (2.1), provided that

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \leq \frac{\mathbf{A}_1 \mathbf{A}_2}{2\|\nabla^2\Phi_j\|_{L^\infty(\Lambda_j)}}. \quad (2.5)$$

The proof of (2.5) requires the condition $\xi, \xi' \in \Sigma_j(\mathbf{h})$ together with the assumption (2.3) on the size of the sets Λ_j . Let

$$\sigma_j(x, z) = \Phi_j(x) - \Phi_j(z) - \nabla\Phi_j(z) \cdot (x - z).$$

A computation gives

$$\begin{aligned} \nabla\Phi_j(z) \cdot (x - y) &= (\Phi_j(x) - \sigma_j(x, z) - \Phi_j(z) - \nabla\Phi_j(z) \cdot z) - (\Phi_j(y) - \sigma_j(y, z) - \Phi_j(z) - \nabla\Phi_j(z) \cdot z) \\ &= \Phi_j(x) - \Phi_j(y) + \sigma_j(y, z) - \sigma_j(x, z) \end{aligned}$$

and hence, using the assumption $\xi, \xi' \in \Sigma_j(\mathbf{h})$, we see that

$$\begin{aligned} &(\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)) \cdot (\xi - \xi') \\ &= \Phi_j(\xi) - \Phi_j(\xi') + \sigma_j(\xi', \xi) - \left(\Phi_j(\xi - h) - \Phi_k(\xi' - h) + \sigma_k(\xi' - h, \eta) - \sigma_k(\xi - h, \eta) \right) \\ &= \sigma_j(\xi', \xi) + \sigma_k(\xi - h, \eta) - \sigma_k(\xi' - h, \eta). \end{aligned}$$

If we now observe that

$$\sigma_j(x, z) - \sigma_j(y, z) = \int_0^1 [\nabla\Phi_j(y + t(x - y)) - \nabla\Phi_j(z)] \cdot (x - y) dt \leq \|\nabla^2\Phi_j\|_{L^\infty(\Lambda_j)} \text{diam}(\Lambda_j) |x - y|$$

we then deduce the bound

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta)) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \leq \text{diam}(\Lambda_1) \|\nabla^2\Phi_1\|_{L^\infty(\Lambda_1)} + \text{diam}(\Lambda_2) \|\nabla^2\Phi_2\|_{L^\infty(\Lambda_2)}.$$

Consequently (2.5) follows from (2.3). \square

2.2. The Schrödinger, the Wave and the Klein-Gordon Equation. We now consider some examples of phases satisfying Assumption 1. It is enough to check the conditions in Lemma 2.1. In particular, by making the sets Λ_j slightly smaller if necessary, it suffices to ensure that the transversality condition (2.1) and curvature condition (2.2) hold.

Firstly, consider the Schrödinger case

$$\Phi_j(\xi) = \frac{1}{2}|\xi|^2.$$

Then the condition (2.1) in Lemma 2.1 becomes

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| = |\xi - \eta|,$$

thus we simply require that the sets Λ_j have some separation. Assuming that the diameters of the sets Λ_j are sufficiently small, we just need to ensure that (2.2) holds. However (2.2) is just

$$\left| (\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| = |\xi - \xi'|$$

and so (2.2) clearly holds (with constant $\mathbf{A}_2 = 1$).

Secondly, consider the case

$$\Phi_j(\xi) = \langle \xi \rangle_{m_j} = (m_j^2 + |\xi|^2)^{\frac{1}{2}}$$

where the mass satisfies $m_j \geq 0$. To simplify notation, we assume that for $\xi \in \Lambda_j$ we there is a constant $A > 0$ such that

$$\frac{1}{A} \leq \langle \xi \rangle_{m_j} \leq A.$$

To check the transversality condition (2.1) we note that

$$\begin{aligned} |\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)|^2 &= \left| \frac{\xi}{\langle \xi \rangle_{m_1}} - \frac{\eta}{\langle \eta \rangle_{m_2}} \right|^2 \\ &= \left(\frac{|\xi|}{\langle \xi \rangle_{m_1}} - \frac{|\eta|}{\langle \eta \rangle_{m_2}} \right)^2 + \frac{2|\xi||\eta|}{\langle \xi \rangle_{m_1} \langle \eta \rangle_{m_2}} \left(1 - \frac{\xi \cdot \eta}{|\xi||\eta|} \right) \\ &= \left(\frac{(m_2|\xi| + m_1|\eta|)(m_2|\xi| - m_1|\eta|)}{\langle \xi \rangle_{m_1} \langle \eta \rangle_{m_2} (|\xi| \langle \eta \rangle_{m_2} + |\eta| \langle \xi \rangle_{m_1})} \right)^2 + \frac{2|\xi||\eta|}{\langle \xi \rangle_{m_1} \langle \eta \rangle_{m_2}} \left(1 - \frac{\xi \cdot \eta}{|\xi||\eta|} \right) \end{aligned} \quad (2.6)$$

(in particular, we *always* have transversality if $|\xi| \approx |\eta| \approx 1$ and $m_1 \ll m_2$).

On the other hand, to check the condition (2.2), we use the following elementary bound.

Lemma 2.2. *Let $\ell \geq 2$ and $(a, h) \in \mathbb{R}^{1+\ell}$. If $x, y \in \{z \in \mathbb{R}^\ell \mid |z| = |z - h| + a\}$ we have the inequality*

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \geq |x - y|^2 \left| \frac{x}{|x|} - \frac{x - h}{|x - h|} \right|^4 \frac{|x - h|^2}{16|x||y||x - h|^2 + 4(|x - h| + |x|)^2|y|^2}.$$

Proof. The condition $x \in \{z \in \mathbb{R}^\ell \mid |z| = |z - h| + a\}$ implies that $|x - h|^2 = (|x| - a)^2$ and hence $\frac{x}{|x|} \cdot h = \frac{|h|^2 - a^2}{2|x|} + a$. Therefore

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right| \geq \frac{|h|^2 - a^2}{2|h|} \left| \frac{1}{|x|} - \frac{1}{|y|} \right| = \frac{|x - h|}{2|h||y|} \left| \frac{x}{|x|} - \frac{x - h}{|x - h|} \right|^2 |x| - |y|$$

where we used the identities $h = x - (x - h)$ and $a = |x| - |x - h|$. Lemma now follows by noting that $|x - y|^2 = |x||y| \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 + ||x| - |y||^2$. \square

We now show that (2.2) holds. A computation gives

$$\begin{aligned} |(\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')) \cdot (\xi - \xi')| &= \left| \frac{|\xi|^2}{\langle \xi \rangle_{m_j}} + \frac{|\xi'|^2}{\langle \xi' \rangle_{m_j}} - \frac{\xi \cdot \xi'}{\langle \xi \rangle_{m_j}} - \frac{\xi \cdot \xi'}{\langle \xi' \rangle_{m_j}} \right| \\ &= \left| \langle \xi \rangle_{m_j} + \langle \xi' \rangle_{m_j} - \frac{\xi \cdot \xi' + m_j^2}{\langle \xi \rangle_{m_j}} - \frac{\xi \cdot \xi' + m_j^2}{\langle \xi' \rangle_{m_j}} \right| \\ &= \frac{\langle \xi \rangle_{m_j} + \langle \xi' \rangle_{m_j}}{2} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \end{aligned}$$

where we let $x = (m_j, \xi)$ and $y = (m_j, \xi')$. If we now note that the surface $\Phi_j(\xi) = \Phi_k(\xi - h) + a$ can be written as $|x| = |y - h'| + a$ with $h' = (m_k - m_j, h)$, then an application of Lemma 2.2 gives

$$|(\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')) \cdot (\xi - \xi')| \geq \frac{\mathbf{A}_1^4}{32A^6} |\xi - \xi'|^2.$$

Therefore, by Lemma 2.1, we see that (i) in Assumption 1 holds with $\mathbf{D}_1 = \frac{\mathbf{A}_1^5}{64A^6}$. Note that the above argument also applies in the case of the wave equation $m_1 = m_2 = 0$.

2.3. The Dispersive Inequality. To simplify the statements to follow, we fix constants $\mathbf{R}_0 \geq 1$, $\mathbf{D}_1, \mathbf{D}_2 > 0$ and $N > n + 1$, and assume that we have phases Φ_1, Φ_2 satisfying Assumption 1 and sets Λ_j, Λ_j^* with Λ_j convex and $\Lambda_j^* + \frac{1}{\mathbf{R}_0} \subset \Lambda_j \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$.

As a consequence of the curvature type bound (1.1) relative to the $(n - 1)$ -dimensional surface $\Sigma_j(\mathbf{h})$, we expect that we should have the dispersive inequality

$$\|e^{it\Phi_j(-i\nabla)} f\|_{L_x^\infty} \lesssim t^{-\frac{n-1}{2}} \|f\|_{L_x^1} \quad (2.7)$$

for $f \in L^1$ with $\text{supp } \widehat{f} \subset \Lambda_j$. To prove this decay in practise, the standard approach would involve a stationary phase argument. However, as we only have curvature information on the surfaces $\Sigma_j(\mathbf{h})$, and these surfaces are somewhat involved to work with, the standard approach via stationary phase arguments, keeping track of the constants, seems difficult to implement. Consequently, we instead present a different argument, using an approach via wave packets. Roughly speaking, fixing some large time $t \approx R$, the idea is to cover Λ_j with balls of size $R^{-\frac{1}{2}}$ and decompose

$$e^{it\Phi_j(-i\nabla)} f = \sum_{\xi_0 \in R^{-\frac{1}{2}} \mathbb{Z}^n \cap \text{supp } \widehat{f}} K_{\xi_0} * f$$

for some smooth kernels $K_{\xi_0}(t, x)$ with $\|K_{\xi_0}(t)\|_{L_x^\infty} \leq R^{-\frac{n}{2}}$. Then since $\Sigma_j(\mathfrak{h})$ is a hypersurface, by restricting to points close to $\Sigma_j(\mathfrak{h})$ we should have

$$\begin{aligned} \|e^{it\Phi_j(-i\nabla)}f\|_{L_x^\infty} &\leq \|f\|_{L_x^1} \left\| \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap \text{supp } \widehat{f}} K_{\xi_0}(t, x) \right\|_{L_x^\infty} \\ &\lesssim \|f\|_{L_x^1} R^{\frac{1}{2}} \sup_{\mathfrak{h}} \left\| \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_j(\mathfrak{h}) + R^{-\frac{1}{2}})} K_{\xi_0}(t, x) \right\|_{L_x^\infty}. \end{aligned}$$

The condition (i) in Assumption 1 then shows that, for times $t \approx R^{-\frac{1}{2}}$, the spatial supports of the kernels $K_{\xi_0}(t, x)$ are essentially disjoint, and hence

$$\left\| \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_j(\mathfrak{h}) + R^{-\frac{1}{2}})} K_{\xi_0}(t, x) \right\|_{L_x^\infty} \approx \sup_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_j(\mathfrak{h}) + R^{-\frac{1}{2}})} \|K_{\xi_0}(t)\|_{L_x^\infty} \lesssim R^{-\frac{n}{2}} \approx t^{-\frac{n}{2}}$$

which would then give the desired dispersive estimate (2.7).

In the remainder of this subsection, we fill in the details of the argument sketched above. We first require a technical lemma involving the surfaces $\Sigma_j(\mathfrak{h})$.

Lemma 2.3. *Let $\{j, k\} = \{1, 2\}$, $\mathfrak{h} = (a, h) \in \mathbb{R}^{1+n}$, and $r \geq 2\frac{\mathbf{D}_2}{\mathbf{D}_1}\mathbf{R}_0$. Assume $\xi_0 \in (\Lambda_j^* + \frac{1}{2\mathbf{R}_0}) \cap (\Lambda_k^* + h + \frac{1}{2\mathbf{R}_0})$ and*

$$|\Phi_j(\xi_0) - \Phi_k(\xi_0 - h) - a| \leq \frac{1}{r}.$$

Then $|\xi_0 - \Sigma_j(\mathfrak{h})| \leq \frac{\mathbf{D}_2}{\mathbf{D}_1 r}$.

Proof. Define $F(\xi) = \Phi_1(\xi) - \Phi_2(\xi - h) - a$, by replacing F with $-F$ if necessary, we may assume that $F(\xi_0) \geq 0$. We need to show there exists $|\xi - \xi_0| \leq \frac{\mathbf{D}_2}{\mathbf{D}_1 r}$ such that $F(\xi) = 0$. To this end, let $\xi(s)$ be the solution to

$$\begin{aligned} \partial_s \xi(s) &= -\frac{\nabla F(\xi(s))}{|\nabla F(\xi(s))|} \\ \xi(0) &= \xi_0. \end{aligned}$$

Note that, for times $s \in [0, \frac{\mathbf{D}_2}{r\mathbf{D}_1}]$, we have $|\xi(s) - \xi_0| \leq s$. On the other hand, since $|F(\xi_0)| \leq \frac{1}{r}$ by assumption, the transversality property (1.2) implies

$$F(\xi(s)) = F(\xi_0) - \int_0^s |\nabla F(\xi(s'))| ds' \leq \frac{1}{r} - s \frac{\mathbf{D}_1}{\mathbf{D}_2}.$$

Consequently $F(\xi(s))$ must be zero for some $s \in [0, \frac{\mathbf{D}_2}{r\mathbf{D}_1}]$ and hence result follows. \square

We now come to the proof of the dispersive inequality.

Lemma 2.4 (Dispersion). *Let $j = 1, 2$. For any $f \in L_x^1$ with $\text{supp } \widehat{f} \subset \Lambda_j^* + \frac{1}{2\mathbf{R}_0}$ and any $t \geq 1$ we have*

$$\|e^{it\Phi_j(-i\nabla)}f\|_{L_x^\infty} \lesssim t^{-\frac{n-1}{2}} \|f\|_{L_x^1}$$

where the implied constant depends only $\mathbf{R}_0, \mathbf{D}_1, \mathbf{D}_2$, and $n \geq 2$.

Proof. It is enough to consider the case $j = 1$ and $R \leq t \leq 2R$ with $R \geq (10\mathbf{R}_0)^2$. Since $\Lambda_2^* + \frac{1}{2\mathbf{R}_0}$ contains a ball of size $(2\mathbf{R}_0)^{-1}$, we can find a finite set $H \subset \mathbb{R}^n$ such that $\#H \lesssim \mathbf{R}_0^n$ and $\Lambda_1 = \cup_{h \in H} \Lambda_1 \cap (\Lambda_2^* + \frac{1}{2\mathbf{R}_0}h)$. In particular, by decomposing \widehat{f} into $\mathcal{O}(\mathbf{R}_0^n)$ sets, it is enough to consider the case $\text{supp } \widehat{f} \subset (\Lambda_1^* + \frac{1}{2\mathbf{R}_0}) \cap (\Lambda_2^* + \frac{1}{2\mathbf{R}_0}h)$. Let $\rho \in C_0^\infty(|\xi| \leq 1)$ such that

$$\sum_{k \in \mathbb{Z}^n} \rho(\xi - k) = 1.$$

The support assumption on \widehat{f} , together with the fact that $R \geq (10\mathbf{R}_0)^2$, implies that

$$(e^{it\Phi_1(-i\nabla)}f)(x) = \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\text{supp } \widehat{f} + \frac{1}{10\mathbf{R}_0})} K_{\xi_0}(t) * f(x)$$

where $K_{\xi_0}(t, x) = \int_{\mathbb{R}^n} \rho(R^{\frac{1}{2}}(\xi - \xi_0)) e^{it\Phi_1(\xi)} e^{ix \cdot \xi} d\xi$. Since $R \leq t \leq 2R$, our goal is to show that

$$\left\| \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\text{supp } \widehat{f} + \frac{1}{10\mathbf{R}_0})} |K_{\xi_0}(t, x)| \right\|_{L_x^\infty} \lesssim R^{-\frac{n-1}{2}}.$$

We would like to write this sum in a way which involves the hypersurfaces $\Sigma_1(\mathfrak{h})$. Fix $0 < \delta \ll \frac{\mathbf{D}_1}{\mathbf{D}_1 + \mathbf{D}_2}$ and let $\delta^* = \frac{\mathbf{D}_1}{\mathbf{D}_2} \delta$. Given $\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z} \cap (\text{supp } \widehat{f} + \frac{1}{10\mathbf{R}_0})$, we can find $a \in \delta^* R^{-\frac{1}{2}}\mathbb{Z}$ with $|a| \leq 2\mathbf{D}_2$ such that

$$|\Phi_1(\xi_0) - \Phi_2(\xi_0 - h) - a| \leq \delta^* R^{-\frac{1}{2}}.$$

Therefore, an application of Lemma 2.3 with $r = R^{\frac{1}{2}}/\delta^*$, implies that $\xi_0 \in \Sigma_1(a, h) + \delta R^{-\frac{1}{2}}$ and hence we have

$$\begin{aligned} \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\text{supp } \widehat{f} + \frac{1}{10\mathbf{R}_0})} |K_{\xi_0}(t, x)| &\leq \sum_{\substack{a \in \delta^* R^{-\frac{1}{2}}\mathbb{Z} \\ |a| \leq 2\mathbf{D}_2}} \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_1(a, h) + \delta R^{-\frac{1}{2}})} |K_{\xi_0}(t, x)| \\ &\lesssim R^{\frac{1}{2}} \sup_{\mathfrak{h}} \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_1(\mathfrak{h}) + \delta R^{-\frac{1}{2}})} |K_{\xi_0}(t, x)|. \end{aligned}$$

We now exploit the localisation of the kernel, together with the partial curvature condition (1.1). Write

$$K_{\xi_0}(t, x) = R^{-\frac{n}{2}} \int_{\mathbb{R}^n} \rho(\xi) e^{it[\Phi_1(R^{-\frac{1}{2}}\xi + \xi_0) - R^{-\frac{1}{2}}\nabla\Phi_1(\xi_0) \cdot \xi]} e^{iR^{-\frac{1}{2}}(x + t\nabla\Phi_1(\xi_0)) \cdot \xi} d\xi.$$

Integrating by parts $n+1$ times gives

$$|K_{\xi_0}(t, x)| \lesssim R^{-\frac{n}{2}} \left(1 + R^{-\frac{1}{2}}|x + t\nabla\Phi_1(\xi_0)|\right)^{-n-1}. \quad (2.8)$$

Let $\xi'_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_1(a, h) + R^{-\frac{1}{2}})$ denote the minimum of $|x + t\nabla\Phi_1(\xi_0)|$. We claim that for every $\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_1(a, h) + R^{-\frac{1}{2}})$ we have

$$|x + t\nabla\Phi_1(\xi_0)| \geq \frac{\mathbf{D}_1}{4} R |\xi_0 - \xi'_0|. \quad (2.9)$$

Assuming this holds for the moment, we would then obtain

$$\begin{aligned} \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\text{supp } \widehat{f} + \frac{1}{10\mathbf{R}_0})} |K_{\xi_0}(t, x)| &\lesssim R^{\frac{1}{2}} \sup_{\mathfrak{h}} \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n \cap (\Sigma_1(\mathfrak{h}) + R^{-\frac{1}{2}})} |K_{\xi_0}(t, x)| \\ &\lesssim R^{-\frac{n-1}{2}} \sum_{\xi_0 \in R^{-\frac{1}{2}}\mathbb{Z}^n} (1 + R^{\frac{1}{2}}|\xi_0 - \xi'_0|)^{-n-1} \\ &\lesssim R^{-\frac{n-1}{2}} \end{aligned}$$

as required. Thus it only remains to verify (2.9). This is immediate if $R\mathbf{D}_1|\xi_0 - \xi'_0| \leq 2|x + t\nabla\Phi_1(\xi'_0)|$. Thus we may assume that $R\mathbf{D}_1|\xi_0 - \xi'_0| \geq 2|x + t\nabla\Phi_1(\xi'_0)|$. Note that this implies that $|\xi - \xi_0| \geq R^{-\frac{1}{2}}$. By construction, there exists $\xi, \xi' \in \Sigma_1(\mathfrak{h})$ such that $|\xi - \xi_0| \leq \delta R^{-\frac{1}{2}}$, $|\xi' - \xi'_0| \leq \delta R^{-\frac{1}{2}}$. Therefore, applying the lower bound (1.1), we deduce that

$$\begin{aligned} |x + t\nabla\Phi_1(\xi_0)| &\geq t|\nabla\Phi(\xi) - \nabla\Phi(\xi')| - |x + t\nabla\Phi_1(\xi'_0)| - t|\nabla\Phi_1(\xi_0) - \nabla\Phi_1(\xi)| - t|\nabla\Phi_1(\xi'_0) - \nabla\Phi_1(\xi')| \\ &\geq R\mathbf{D}_1|\xi - \xi'| - |x + t\nabla\Phi_1(\xi'_0)| - 4\mathbf{D}_2\delta R^{\frac{1}{2}} \\ &\geq R\frac{\mathbf{D}_1}{2}|\xi_0 - \xi'_0| - 4(\mathbf{D}_1 + \mathbf{D}_2)\delta R^{\frac{1}{2}} \\ &\geq R\frac{\mathbf{D}_1}{4}|\xi_0 - \xi'_0| \end{aligned}$$

provided that we choose $\delta \ll \frac{\mathbf{D}_1}{\mathbf{D}_1 + \mathbf{D}_2}$. Hence we obtain (2.9) and thus result follows. \square

Remark 2.5. By the standard TT^* -argument, this implies the linear Strichartz type estimates for wave admissible pairs. We omit the details and refer to [22].

2.4. Classical Bilinear Estimate in $L_{t,x}^2$. The main use of the transversality property (1.2) contained in Assumption 1 is to deduce the following well-known bilinear estimate, which dates back at least to Bourgain [14, Lemma 111] in the case of the Schrödinger equation and $n = 2$.

Lemma 2.6. *Let $0 < r < 1$ and $f, g \in L_x^2$. Assume that the supports of \widehat{f} and \widehat{g} are contained in balls of radius r intersected with Λ_1 and Λ_2 respectively, and for all $\xi \in \Lambda_1$ and $\eta \in \Lambda_2$*

$$|\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| \geq \mathbf{C}_0. \quad (2.10)$$

Then,

$$\|e^{it\Phi_1(-i\nabla)} f e^{it\Phi_2(-i\nabla)} g\|_{L_{t,x}^2(\mathbb{R}^{1+n})} \lesssim \left(\frac{r^{n-1}}{\mathbf{C}_0}\right)^{\frac{1}{2}} \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

Proof. For $m = 1, \dots, n$ let $\Omega_m = \{(\xi, \eta) \in \Lambda_1 \times \Lambda_2 \mid |\partial_m\Phi_1(\xi) - \partial_m\Phi_2(\eta)| \geq \frac{\mathbf{C}_0}{2n}\}$. Condition (2.10) and the support assumptions on \widehat{f} and \widehat{g} implies that we can decompose

$$(e^{it\Phi_1(-i\nabla)} \widehat{f e^{it\Phi_2(-i\nabla)} g})(\xi) = \sum_{m=1}^n \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta) \widehat{g}(\eta) \mathbb{1}_{\Omega_m}(\xi - \eta, \eta) e^{it(\Phi_1(\xi - \eta) + \Phi_2(\eta))} d\eta.$$

Consider the $m = 1$ term and write $\eta = (\eta_1, \eta') \in \mathbb{R} \times \mathbb{R}^{n-1}$. The change of variables $(\eta_1, \eta') \mapsto (\tau, \eta')$ where $\tau = \Phi_1(\xi - \eta) + \Phi_2(\eta)$ gives

$$\begin{aligned} & \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta) \widehat{g}(\eta) \mathbb{1}_{\Omega_1}(\xi - \eta, \eta) e^{it(\Phi_1(\xi - \eta) + \Phi_2(\eta))} d\eta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{\widehat{f}(\xi - \eta^*) \widehat{g}(\eta^*)}{\partial_1\Phi_1(\xi - \eta^*) - \partial_1\Phi_2(\eta^*)} \mathbb{1}_{\Omega_1}(\xi - \eta^*, \eta^*) d\eta' e^{it\tau} d\tau, \end{aligned}$$

where $\eta^* = (\eta_1[\tau, \xi, \eta'], \eta')$. Thus an application of Plancherel, followed by Hölder in η' , shows that

$$\begin{aligned} & \left\| \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta) \widehat{g}(\eta) \mathbb{1}_{\Omega_m}(\xi - \eta, \eta) e^{it(\Phi_1(\xi - \eta) + \Phi_2(\eta))} d\eta \right\|_{L_{t,\xi}^2} \\ &= \left\| \int_{\mathbb{R}^{n-1}} \frac{\widehat{f}(\xi - \eta) \widehat{g}(\eta)}{\partial_1\Phi_1(\xi - \eta) - \partial_1\Phi_2(\eta)} \mathbb{1}_{\Omega_1}(\xi - \eta, \eta) d\eta' \right\|_{L_{\tau,\xi}^2} \\ &\leq (2r)^{\frac{n-1}{2}} \frac{2n}{\mathbf{C}_0} \left\| \widehat{f}(\xi - \eta^*) \widehat{g}(\eta^*) \right\|_{L_{\tau,\xi,\eta'}^2} \\ &\leq 2n \left(\frac{(2r)^{n-1}}{\mathbf{C}_0}\right)^{\frac{1}{2}} \|f\|_{L_x^2} \|g\|_{L_x^2} \end{aligned}$$

where the last inequality follows by undoing the change of variables. Since the terms with $1 < m \leq n$ are identical the lemma follows. \square

2.5. Geometric Consequences. The last step in the proof of Theorem 1.1 requires a combinatorial Keakey type bound. This bound relies on the fact that certain tubes intersect transversally, and is the main reason for introducing the condition (i) in Assumption 1. The following is motivated by [33, 2], see also Section 9 of [43].

Let $\mathfrak{h} \in \mathbb{R}^{1+n}$ and define the conic hypersurface

$$\mathcal{C}_j(\mathfrak{h}) = \{(r, -r\nabla\Phi_j(\xi)) \mid r \in \mathbb{R}, \xi \in \Sigma_j(\mathfrak{h})\}.$$

A computation shows that the tangent plane to $\mathcal{C}_j(\mathfrak{h})$ is spanned by the vectors

$$(1, -\nabla\Phi_j(\xi)) \text{ and } H\Phi_j(\xi)v \text{ for } v \in T_\xi\Sigma_j(\mathfrak{h}).$$

On the other hand, as we will see in the proof the Lemma 2.7 below, the condition (i) in Assumption 1 implies that

$$|(1, -\nabla\Phi_j(\xi)) \wedge (1, -\nabla\Phi_k(\eta)) \wedge (0, \nabla\Phi_j(\xi) - \nabla\Phi_j(\xi'))| \gtrsim |\xi - \xi'|$$

for every $\xi, \xi' \in \Sigma_j(\mathfrak{h})$. Hence, letting $\xi' \rightarrow \xi$ in $\Sigma_j(\mathfrak{h})$, we can interpret (i) in Assumption 1 as saying that, for every $v \in T_\xi \Sigma_j(\mathfrak{h})$ we have

$$|(1, -\nabla\Phi_j(\xi)) \wedge (1, -\nabla\Phi_k(\eta)) \wedge (0, H\Phi(\xi)v)| \gtrsim |v|$$

where $H\Phi_j(\xi)$ denotes the Hessian of Φ_j at ξ . In particular, the vector $(1, -\nabla\Phi_k(\eta))$ must be transversal to the surface $\mathcal{C}_j(\mathfrak{h})$ for every $\eta \in \Lambda_k$. A more quantitative version of this statement – and the one we make use of in practise – is given by the following.

Lemma 2.7. *Let $\mathfrak{h} \in \mathbb{R}^{1+n}$ and $\{j, k\} = \{1, 2\}$. For every $\eta \in \Lambda_j$ and $p, q \in \mathcal{C}_k(\mathfrak{h})$ we have*

$$|(p - q) \wedge (1, -\nabla\Phi_j(\eta))| \geq \frac{\mathbf{D}_1 |p - q|}{(1 + \|\nabla\Phi_k\|_{L^\infty(\Lambda_k)}) \|\nabla^2\Phi_k\|_{L^\infty(\Lambda_k)}}.$$

Proof. Let $w, w', w'' \in \mathbb{R}^n$. The identity $|x \wedge y \wedge z| = \inf_{v \in \text{span}\{x, y\}} \frac{|v \wedge z|}{|v|} |x \wedge y|$ implies that

$$\begin{aligned} |(1, w'') \wedge (1, w) \wedge (0, w - w')| &= |(1, w'') \wedge (0, w - w'') \wedge (0, w - w')| \\ &= \inf_{v \in W} \frac{|v \wedge (1, w'')|}{|v|} |(0, w - w'') \wedge (0, w - w')| \\ &\geq |(w - w'') \wedge (w - w')|, \end{aligned}$$

where $W = \text{span}\{(0, w - w''), (0, w - w')\}$. Consequently, applying the wedge product identity once more, we deduce that for every $v \in \text{span}\{(1, w), (0, w - w')\}$

$$|v \wedge (1, w'')| \geq \frac{|(w - w'') \wedge (w - w')|}{(1 + |w|)|w - w'|} |v|. \quad (2.11)$$

Fix $\eta \in \Lambda_j$ and $p, q \in \mathcal{C}_k(\mathfrak{h})$. By definition, this implies that we have $\xi, \xi' \in \Sigma_j(\mathfrak{h})$ and $r, r' > 0$ such that $p = (r, -r\nabla\Phi_k(\xi))$ and $q = (r', -r'\nabla\Phi_k(\xi'))$. Clearly, due to the convexity of Λ_k we have $|\nabla\Phi_k(\xi) - \nabla\Phi_k(\xi')| \leq \|\nabla^2\Phi_k\|_{L^\infty(\Lambda_k)} |\xi - \xi'|$. If we now let $w = -\nabla\Phi_k(\xi)$, $w' = -\nabla\Phi_k(\xi')$, and $w'' = -\nabla\Phi_j(\eta)$ in (2.11), then we deduce from (i) in Assumption 1 that

$$|v \wedge (1, -\nabla\Phi_j(\eta))| \geq \frac{\mathbf{D}_1 |v|}{(1 + \|\nabla\Phi_k\|_{L^\infty(\Lambda_k)}) \|\nabla^2\Phi_k\|_{L^\infty(\Lambda_k)}}$$

for every $v \in \text{span}\{(1, -\nabla\Phi_k(\xi)), (0, \nabla\Phi_k(\xi) - \nabla\Phi_k(\xi'))\}$. Taking $v = p - q$ and observing that we can write

$$(p - q) = (r - r')(1, -\nabla\Phi_k(\xi)) + r'(0, \nabla\Phi_k(\xi) - \nabla\Phi_k(\xi')),$$

the required bound now follows. \square

3. WAVE PACKETS, ATOMIC SPACES, AND TUBES

In this section we discuss the wave packet decomposition. To some extent, we follow the arguments in [43], but use a slightly different notation by using projections labelled by phase space points as in [33]. Again, this helps us to carefully track constants. In addition, we consider certain atomic decompositions. Concerning the phases Φ_j , it turns out that the only property we require in the construction of wave packets below, is (ii) in Assumption 1. Consequently, throughout this section, we fix constants $\mathbf{R}_0 \geq 1$, $\mathbf{D}_2 > 0$ and $N > n + 1$, and assume that for $j = 1, 2$ we have sets Λ_j, Λ_j^* with Λ_j convex and $\Lambda_j^* + \frac{1}{\mathbf{R}_0} \subset \Lambda_j \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$, and phases $\Phi_j : \Lambda_j \rightarrow \mathbb{R}$ such that

$$\sup_{1 \leq |\kappa| \leq N} \|\partial^\kappa \Phi_j\|_{L^\infty(\Lambda_j)} \leq \mathbf{D}_2.$$

3.1. Wave Packets. Let $R \geq 1$ and define the cylinder

$$Q_R = \left\{ (t, x) \in \mathbb{R}^{1+n} \mid \frac{R}{2} < t < R, |x| < R \right\},$$

and $\mathcal{X} = R^{\frac{1}{2}}\mathbb{Z}^n \times R^{-\frac{1}{2}}\mathbb{Z}^n$. Define

$$\mathcal{X}_j = \{(x_0, \xi_0) \in \mathcal{X} \mid \xi_0 \in \Lambda_j^* + 3R^{-\frac{1}{2}}\}$$

to be the set of phase points which are within $3R^{-\frac{1}{2}}$ of Λ_j^* . Note that provided $R \geq (3R_0)^2$, if $\gamma = (x_0, \xi_0) \in \mathcal{X}_j$, then $\xi_0 \in \Lambda_j$. Given a point $\gamma = (x_0, \xi_0) \in \mathcal{X}$ in phase-space, we let $x(\gamma) = x_0$ and $\xi(\gamma) = \xi_0$ denote the projections onto the first and second components respectively. Fix $\eta, \rho \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \widehat{\eta} \subset \{|\xi| \leq 1\}$, $\text{supp } \rho \subset \{|\xi| \leq 1\}$, and for all $x, \xi \in \mathbb{R}^n$

$$\sum_{k \in \mathbb{Z}^n} \eta(x - k) = \sum_{k \in \mathbb{Z}^n} \rho(\xi - k) = 1.$$

Given $\gamma \in \mathcal{X}$ and $f \in L_x^2(\mathbb{R}^n)$, define the phase-space localisation operator

$$(L_\gamma f)(x) = \eta\left(\frac{x - x(\gamma)}{R^{\frac{1}{2}}}\right) \left[\rho\left(\frac{-i\nabla - \xi(\gamma)}{R^{-\frac{1}{2}}}\right) f \right](x).$$

Note that by definition we have

$$f = \sum_{\gamma \in \mathcal{X}} L_\gamma f, \quad \text{supp } \widehat{L_\gamma f} \subset \{\xi \in \mathbb{R}^n \mid |\xi - \xi(\gamma)| \leq 2R^{-\frac{1}{2}}\}.$$

Moreover, letting $w_\gamma(x) = \left(1 + \frac{|x - x(\gamma)|}{R^{\frac{1}{2}}}\right)^{N-1 + \frac{n+1}{2}}$, for any $\Gamma \subset \mathcal{X}$ we have the orthogonality bounds

$$\left\| \sum_{\gamma \in \Gamma} L_\gamma f \right\|_{L_x^2} \lesssim \left(\sum_{\gamma \in \Gamma} \|w_\gamma(x) L_\gamma f(x)\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L_x^2}. \quad (3.1)$$

To simplify notation slightly, we define the slightly larger phase-space localisation operators $L_\gamma^\sharp = \omega_\gamma(x) L_\gamma$. It is worth noting that $L_\gamma^\sharp f$ no longer has compact Fourier support, this does not pose any problems in the arguments to follow, as the only properties that we require are the trivial bound $\|L_\gamma f\|_{L_x^2} \leq \|L_\gamma^\sharp f\|_{L_x^2}$ and the orthogonality bound in (3.1).

To define wave packets, we conjugate the phase-space localisation operator L_γ with the flow $e^{it\Phi_j(-i\nabla)}$.

Definition 3.1 (Wave Packets). Let $j = 1, 2$, $R \geq (3R_0)^2$, and $u \in L_t^\infty L_x^2(\mathbb{R}^{1+n})$. Given a point $\gamma_j \in \mathcal{X}_j$, we define

$$(\mathcal{P}_{\gamma_j} u)(t) = e^{it\Phi_j(-i\nabla)} L_{\gamma_j} \left(e^{-it\Phi_j(-i\nabla)} u(t) \right).$$

Similarly, we define

$$(\mathcal{P}_{\gamma_j}^\sharp u)(t) = e^{it\Phi_j(-i\nabla)} L_{\gamma_j}^\sharp \left(e^{-it\Phi_j(-i\nabla)} u(t) \right).$$

We also require the associated tubes T_γ .

Definition 3.2 (Tubes). Let $j = 1, 2$ and $\gamma_j \in \mathcal{X}_j$. Then we define the tube $T_{\gamma_j} \subset \mathbb{R}^{1+n}$ as

$$T_{\gamma_j} = \left\{ (t, x) \in \mathbb{R}^{1+n} \mid \frac{R}{2} \leq t \leq R, |x - x(\gamma) + t\nabla\Phi_j(\xi(\gamma))| \leq R^{\frac{1}{2}} \right\}.$$

The most important properties of the wave packets $\mathcal{P}_{\gamma_j} u$ are summarised in the following.

Proposition 3.3 (Properties of Wave Packets). *Let $j = 1, 2$. For any $R \geq (3R_0)^2$, $f \in L_x^2$ with $\text{supp } \widehat{f} \subset \Lambda_j^*$, and $u = e^{it\Phi_j(-i\nabla)} f$, we have $u = \sum_{\gamma_j \in \mathcal{X}_j} \mathcal{P}_{\gamma_j} u$, $\text{supp } \widehat{\mathcal{P}_{\gamma_j} u} \subset \{|\xi - \xi(\gamma)| \leq 2R^{-\frac{1}{2}}\}$, and given any $\Gamma_j \subset \mathcal{X}_j$ we have the orthogonality bound*

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_j} u \right\|_{L_t^\infty L_x^2} \lesssim \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^\sharp f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L_x^2}. \quad (3.2)$$

Moreover, the wave packets $\mathcal{P}_{\gamma_j} u$ are concentrated on the tubes T_{γ_j} in the sense that for every $r \geq R^{\frac{1}{2}}$, and any ball $B \subset \mathbb{R}^{1+n}$, we have the bound

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \text{dist}(T_{\gamma_j}, B) > r}} \mathcal{P}_{\gamma_j} u \right\|_{L_{t,x}^\infty(B \cap Q_R)} \lesssim \left(\frac{r}{R^{\frac{1}{2}}} \right)^{\frac{n+3}{2} - N} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^\# f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \quad (3.3)$$

Here, the implied constants depend only on $\mathbf{R}_0, \mathbf{D}_2, N$ and $n \geq 2$.

Proof. This result is somewhat standard, see for instance [43, Lemma 4.1] and [29, Lemma 2.2] for related estimates. We only prove the localisation property (3.3), as the remaining properties follow directly from the definition of \mathcal{P}_γ , together with the analogous properties of the phase-space localisation operator L_γ . Let $\gamma_j = (x_0, \xi_0)$ and write

$$\begin{aligned} \overline{\mathcal{P}_{\gamma_j} u}(t, x) &= \int_{\mathbb{R}^n} \widehat{(L_{\gamma_j} f)}(\xi) e^{it\Phi_j(\xi)} e^{ix \cdot \xi} d\xi \\ &= \int_{\mathbb{R}^n} K_{\xi_0}(t, x - y) (L_{\gamma_j} f)(y) dy \end{aligned}$$

where, as in the proof of Lemma 2.4, the kernel is given by $K_{\xi_0}(t, x) = \int_{\mathbb{R}^n} \rho(R^{\frac{1}{2}}(\xi - \xi_0)) e^{it\Phi_j(\xi)} e^{ix \cdot \xi} d\xi$. Note that, as in (2.8), integrating by parts $N - 1$ times, and using the fact that $|t| \leq R$, $R \gg 1$, we deduce that

$$K_{\xi_0}(t, x) \lesssim R^{-\frac{n}{2}} \left(1 + \frac{|x + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{1-N}.$$

Plugging this bound into the identity for $\mathcal{P}_{\gamma_j} u(t, x)$, we deduce that

$$\begin{aligned} |\mathcal{P}_{\gamma_j} u(t, x)| &\lesssim R^{-\frac{n}{2}} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{1-N} \int_{\mathbb{R}^n} \left(1 + \frac{|y - x_0|}{R^{\frac{1}{2}}} \right)^{N-1} |L_{\gamma_j} f(y)| dy \\ &\lesssim R^{-\frac{n}{4}} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{1-N} \|L_{\gamma_j}^\# f\|_{L_x^2} \end{aligned}$$

Since there are $\mathcal{O}(R^{\frac{n}{2}})$ choices of ξ_0 , and

$$|x - x_0 + t\nabla\Phi_j(\xi_0)| = |(t, x) - (t, x_0 - t\nabla\Phi_j(\xi_0))| \geq \text{dist}((t, x), T_{\gamma_j}),$$

an application of Hölder's inequality gives for any $(t, x) \in B$

$$\begin{aligned} \sum_{\substack{\gamma_j \in \Gamma_j \\ \text{dist}(T_{\gamma_j}, B) \geq r}} |\mathcal{P}_{\gamma_j} u(t, x)| &\lesssim R^{-\frac{n}{4}} \left(\sum_{\substack{\gamma_j \in \mathcal{X}_j \\ \text{dist}(T_{\gamma_j}, B) \geq r}} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{2-2N} \right)^{\frac{1}{2}} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^\# f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{r}{R^{\frac{1}{2}}} \right)^{\frac{n+3}{2} - N} \sup_{\xi_0} \left(\sum_{x_0 \in R^{\frac{1}{2}} \mathbb{Z}^n} \left(1 + \frac{|x - x_0 + t\nabla\Phi_j(\xi_0)|}{R^{\frac{1}{2}}} \right)^{-n-1} \right)^{\frac{1}{2}} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^\# f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{r}{R^{\frac{1}{2}}} \right)^{\frac{n+3}{2} - N} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^\# f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \end{aligned}$$

as required. \square

3.2. Atomic Spaces and Wave Packets. Closely related to the V^p spaces, are the slightly smaller U^p spaces, see [27, 21, 28].

Definition 3.4. Let $1 \leq p < \infty$. A function $\rho : \mathbb{R} \rightarrow L_x^2$ is called a U^p atom if there exists a decomposition $\rho = \sum_{J \in \mathcal{I}} \mathbb{1}_J(t) f_J$ subordinate to a finite partition $\mathcal{I} = \{(-\infty, t_1), [t_2, t_3), \dots, [t_N, \infty)\}$ of \mathbb{R} , such that

$$\|f_J\|_{\ell_p^J L_x^2} := \left(\sum_{J \in \mathcal{I}} \|f_J\|_{L_x^2}^p \right)^{\frac{1}{p}} \leq 1.$$

The atomic Banach space U^p is then defined as

$$U^p = \left\{ \sum_j c_j \rho_j \mid (c_j) \in \ell^1(\mathbb{N}), \rho_j \text{ a } U^p \text{ atom} \right\}$$

with the induced norm

$$\|\rho\|_{U^p} = \inf_{\substack{\rho = \sum_k c_k \phi_k \\ \phi_k \text{ } U^p \text{ atom}}} \sum_k |c_k|.$$

The space U_{Φ}^p is the set of all $u : \mathbb{R} \rightarrow L_x^2$ such that $e^{-it\Phi(-i\nabla)}u \in U^p$ with the obvious norm.

Let $u = \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} f_J$ be a $U_{\Phi_j}^2$ atom. Since $\mathbb{1}_J(t)$ commutes with spatial Fourier multipliers, we have

$$\mathcal{P}_{\gamma_j} u = \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} L_{\gamma_j} f_J, \text{ and } \mathcal{P}_{\gamma_j}^{\sharp} u = \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} L_{\gamma_j}^{\sharp} f_J.$$

Proposition 3.3 gives the following.

Corollary 3.5 (Wave Packets for $U_{\Phi_j}^2$ atoms). *Let $j = 1, 2$. For any $R \geq (3\mathbf{R}_0)^2$ and $U_{\Phi_j}^2$ atom $u = \sum_J \mathbb{1}_J(t) e^{it\Phi_j(-i\nabla)} f_J$ with $\text{supp } \widehat{u} \subset \Lambda_j^*$, we have $u = \sum_{\gamma_j \in \mathcal{X}_j} \mathcal{P}_{\gamma_j} u$, $\text{supp } \widehat{\mathcal{P}_{\gamma_j} u} \subset \{|\xi - \xi(\gamma)| \leq 2R^{-\frac{1}{2}}\}$, and given any $\Gamma_j \subset \mathcal{X}_j$ we have the orthogonality bound*

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_j} u \right\|_{L_t^\infty L_x^2} \lesssim \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^{\sharp} f_J\|_{\ell_3^2 L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \|f_J\|_{\ell_3^2 L_x^2}. \quad (3.4)$$

Moreover, the wave packets $\mathcal{P}_{\gamma_j} u$ are concentrated on the tubes T_{γ_j} in the sense that for every $r \geq R^{\frac{1}{2}}$, and any ball $B \subset \mathbb{R}^{1+n}$, we have the bound

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \text{dist}(T_{\gamma_j}, B) > r}} \mathcal{P}_{\gamma_j} u \right\|_{L_{t,x}^\infty(B \cap Q_R)} \lesssim \left(\frac{r}{R^{\frac{1}{2}}} \right)^{\frac{n+3}{2} - N} \left(\sum_{\gamma_j \in \Gamma_j} \|L_{\gamma_j}^{\sharp} f_J\|_{\ell_3^2 L_x^2}^2 \right)^{\frac{1}{2}}. \quad (3.5)$$

Here, the implied constants depend only on $\mathbf{R}_0, \mathbf{D}_2, N$ and $n \geq 2$.

3.3. Sets and Relations of Tubes. We repeat the definitions and notation used by Tao [43], but as above we adopt the point of view that the basic objects are the phase space elements $\gamma \in \mathcal{X}_j$, rather than the associated tubes T_{γ_j} .

For $\delta > 0$, let \mathcal{B} be a collection of (space-time) balls of radius $R^{1-\delta}$ which form a finitely overlapping cover of Q_R . Similarly let \mathbf{q} denote a collection of finitely overlapping cubes q of radius $R^{\frac{1}{2}}$ which cover the cylinder Q_R . Let $R^\delta q$ denote a cube of radius $R^{\delta+\frac{1}{2}}$ with the same centre as q . Given a collection $\Gamma_j \subset \mathcal{X}_j$, and a cube $q \in \mathbf{q}$, we define

$$\Gamma_j(q) = \{\gamma_j \in \Gamma_j \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset\}$$

so $\Gamma_j(q)$ is the subcollection of our phase-space decomposition, such that the associated tube T_{γ_j} intersects a slight enlargement of the cube $q \in \mathbf{q}$. In the remainder of this subsection, the implied constants may depend on $n \geq 2$ only. Given $1 \leq \mu_1, \mu_2 \lesssim R^{100n}$, define

$$\mathbf{q}(\mu_1, \mu_2) = \{q \in \mathbf{q} \mid \mu_j \leq \#\Gamma_j(q) < 2\mu_j, j = 1, 2\}.$$

Thus, roughly, $\mathbf{q}(\mu_1, \mu_2)$ restricts to those elements of \mathbf{q} which are intersected by μ_j tubes T_{γ_j} , $\gamma_j \in \Gamma_j$. Given $\gamma_j \in \Gamma_j$, we let

$$\lambda(\gamma_j, \mu_1, \mu_2) = \#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset\}$$

and for every $1 \leq \lambda_j \lesssim R^{100n}$ we define

$$\Gamma_j[\lambda_j, \mu_1, \mu_2] = \{\gamma_j \in \Gamma_j \mid \lambda_j \leq \lambda(\gamma_j, \mu_1, \mu_2) < 2\lambda_j\}.$$

So $\Gamma_j[\lambda_j, \mu_1, \mu_2]$ essentially restricts to $\gamma_j \in \Gamma_j$, such that the associated tubes T_{γ_j} intersect λ_j cubes in $\mathbf{q}(\mu_1, \mu_2)$. Clearly

$$\bigcup_{1 \leq \lambda_j, \mu_1, \mu_2 \lesssim R^{100n}} \Gamma_j(\lambda_j, \mu_1, \mu_2) = \Gamma_j.$$

The following relation \sim between balls in \mathcal{B} and $\gamma_j \in \Gamma_j$ plays a key role in the arguments to follow.

Definition 3.6. Given $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$, we let $B(\gamma_j, \lambda_j, \mu_1, \mu_2) \in \mathcal{B}$ denote a ball which maximises

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_j, \mu_1, \mu_2) \neq \emptyset\}.$$

If $B \in \mathcal{B}$, and $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$, we then define $\gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B$, if $B \subset 10B(\gamma_j, \lambda_j, \mu_1, \mu_2)$. To extend this definition to general points $\gamma_j \in \Gamma_j$, we simply say that $\gamma_j \sim B$ if there exists some $1 \leq \lambda_j, \mu_1, \mu_2 \lesssim R^{100n}$ such that $\gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B$.

Remark 3.7. This definition has the following important consequences.

- (i) Let $\gamma_j \in \Gamma_j$ and consider the set $\{B \in \mathcal{B} \mid \gamma_j \sim B\}$. Since there are at most $\mathcal{O}(R^\epsilon)$ dyadic $1 \leq \lambda_j, \mu_1, \mu_2 \leq R^{100n}$ such that $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$, and only $\mathcal{O}(1)$ balls B such that $\gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B$, we have

$$\#\{B \in \mathcal{B} \mid \gamma_j \sim B\} \leq \sum_{\substack{1 \leq \lambda_j, \mu_1, \mu_2 \leq R^{100n} \\ \gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]}} \#\{B \in \mathcal{B} \mid \gamma_j \sim_{\lambda_j, \mu_1, \mu_2} B\} \lesssim \sum_{1 \leq \lambda_j, \mu_1, \mu_2 \leq R^{100n}} 1 \lesssim R^\epsilon.$$

- (ii) Fix $1 \leq \lambda_1, \mu_1, \mu_2 \lesssim R^{100n}$ and let $\gamma_j \in \Gamma_j[\lambda_j, \mu_1, \mu_2]$. By definition, we have

$$\begin{aligned} \lambda_j &\leq \#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset\} \\ &\leq \sum_{B \in \mathcal{B}} \#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B \neq \emptyset\} \\ &\leq \#\mathcal{B} \#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_1, \mu_1, \mu_2) \neq \emptyset\} \end{aligned}$$

where we used the maximal property of the ball $B(\gamma_j, \lambda_j, \mu_1, \mu_2)$. Therefore, as $\#\mathcal{B} \lesssim R^{(n+1)\delta}$, we deduce the lower bound

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_j, \mu_1, \mu_2) \neq \emptyset\} \gtrsim R^{-(n+1)\delta} \lambda_j.$$

4. A LOCAL BILINEAR RESTRICTION ESTIMATE

The main step in the proof of Theorem 1.1 is prove the following spatially localised version in U_{Φ}^2 .

Theorem 4.1. *Let $n \geq 2$ and $\alpha > 0$. Let $\mathbf{R}_0 \geq 1$ and $\mathbf{D}_1, \mathbf{D}_2 > 0$. For $j = 1, 2$, let $\Lambda_j, \Lambda_j^* \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ with Λ_j convex and $\Lambda_j^* + \frac{1}{\mathbf{R}_0} \subset \Lambda_j$. There exists $N \in \mathbb{N}$ and a constant $C > 0$ such that, for any phases Φ_1 and Φ_2 satisfying Assumption 1, any $u \in U_{\Phi_1}^2$, $v \in U_{\Phi_2}^2$ with $\text{supp } \widehat{u}(t) \subset \Lambda_1^*$, $\text{supp } \widehat{v}(t) \subset \Lambda_2^*$, and any $R \geq 1$, we have*

$$\|uv\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \leq CR^\alpha \|u\|_{U_{\Phi_1}^2} \|v\|_{U_{\Phi_2}^2}$$

In the remainder of this section we give the proof of Theorem 4.1. The proof is broken up into three key steps. The first step is use an induction on scales argument to reduce to proving an $L_{t,x}^2$ bound. We then use the localisation properties of the wave packet decomposition to show that the $L_{t,x}^2$ bound follows from a combinatorial Keakeya type bound. The final step is prove the combinatorial estimate using a ‘‘bush’’ argument.

4.1. Induction on Scales. Let $\alpha > 0$ and fix the constants $\mathbf{R}_0 \geq 1$, $\mathbf{D}_1, \mathbf{D}_2 > 0$. Fix $N = \frac{\alpha+1}{\alpha}(100n)^2$. For $j = 1, 2$, let $\Lambda_j, \Lambda_j^* \subset \{\frac{1}{16} \leq |\xi| \leq 16\}$ with Λ_j convex and $\Lambda_j^* + \frac{1}{\mathbf{R}_0} \subset \Lambda_j$. It is enough to show that there exists a constant $C > 0$ such that, for any phases Φ_1 and Φ_2 satisfying Assumption 1, any $R \geq (3\mathbf{R}_0)^2$, and any $U_{\Phi_j}^2$ atoms $u = \sum_J \mathbf{1}_J(t) e^{it\Phi_1(-i\nabla)} f_J$, $v = \sum_{J'} \mathbf{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{J'}$ with $\text{supp } \widehat{f} \subset \Lambda_1^*$, $\text{supp } \widehat{g}_{J'} \subset \Lambda_2^*$, we have

$$\|uv\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \leq CR^{2\alpha}. \quad (4.1)$$

To simplify the notation to follow, we now work under the assumption that any implicit constants may now depend on $\alpha, n \geq 2$, and the constants $\mathbf{R}_0, \mathbf{D}_1, \mathbf{D}_2$, but will be independent of R and the particular choice of phases Φ_j satisfying Assumption 1.

The proof of (4.1) proceeds along the same lines as Tao’s argument for the paraboloid [43]. Namely, we use an induction on scales argument to deduce the estimate at scale R , by applying a weaker estimate

at a smaller scale $R^{1-\delta}$. We start by observing that it suffices to show that, for every $\Gamma_j \subset \mathcal{X}_j$ such that $\#\Gamma_j \leq R^{10n}$, and any $\beta \geq \alpha$, we have

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \lesssim R^\beta (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_{J'}^2 L_x^2}. \quad (4.2)$$

To deduce (4.1) from (4.2), we let

$$\mathcal{X}_1(\nu_1) = \{\gamma_1 \in \mathcal{X}_1 \mid \nu_1 \leq \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \leq 2\nu_1, T_{\gamma_1} \cap 2Q_R \neq \emptyset\}$$

and $\mathcal{X}_2(\nu_2)$ similarly where $\nu_j \in 2^{\mathbb{Z}}$. An application of Corollary 3.5 gives the decomposition $u = \sum_{\gamma_j \in \mathcal{X}_j} \mathcal{P}_{\gamma_j} u$ as well as the bounds

$$\left\| \sum_{\substack{\gamma_1 \in \mathcal{X}_1 \\ T_{\gamma_1} \cap 2Q_R = \emptyset}} \mathcal{P}_{\gamma_1} u \right\|_{L_{t,x}^\infty(Q_R)} \lesssim R^{-99n}$$

and

$$\left(\sum_{\gamma_j \in \mathcal{X}_j} \|\mathcal{P}_{\gamma_1} u\|_{L_t^\infty L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{\gamma_j \in \mathcal{X}_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2}^2 \right)^{\frac{1}{2}} \lesssim 1.$$

The analogous bounds hold for v . Moreover $\#\{\gamma_j \in \mathcal{X}_j \mid T_{\gamma_j} \cap 2Q_R \neq \emptyset\} \lesssim R^{n+1}$. Collecting these properties together, we deduce that $\mathcal{X}_1(\nu_1) = \emptyset$ for $\nu_1 \gg 1$ and

$$\left\| u - \sum_{R^{-100n} \leq \nu_1 \lesssim 1} \sum_{\gamma_1 \in \mathcal{X}_1(\nu_1)} \mathcal{P}_{\gamma_1} u \right\|_{L_{t,x}^\infty(Q_R)} \lesssim R^{-90n}.$$

A similar argument shows that

$$\left\| v - \sum_{R^{-100n} \leq \nu_2 \lesssim 1} \sum_{\gamma_2 \in \mathcal{X}_2(\nu_2)} \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^\infty(Q_R)} \lesssim R^{-90n}.$$

Therefore, applying the bound (4.2) with $\Gamma_j = \mathcal{X}_j(\nu_j)$ and $\beta = \alpha$, we obtain

$$\begin{aligned} \|uv\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} &\leq \left\| uv - \sum_{R^{-100n} \leq \nu_j \lesssim 1} \sum_{\gamma_j \in \mathcal{X}_j(\nu_j)} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \\ &\quad + \sum_{R^{-100n} \leq \nu_j \lesssim 1} \left\| \sum_{\gamma_j \in \mathcal{X}_j(\nu_j)} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \\ &\lesssim 1 + \log(R) R^\alpha \sup_{\nu_j} \left((\#\mathcal{X}_1(\nu_1) \#\mathcal{X}_2(\nu_2))^{\frac{1}{2}} \sup_{\gamma_j \in \mathcal{X}_j(\nu_j)} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_{J'}^2 L_x^2} \right) \\ &\lesssim R^{2\alpha} \end{aligned}$$

where the last line follows from the orthogonality properties of the phase space localisation operators (3.1). Hence (4.1) follows.

The proof of (4.2) proceeds via an induction on scales argument. The first step is to note that we already have (4.2) provided we take $\beta > 0$ sufficiently large. Indeed, a crude argument by Hölder and Bernstein inequalities implies the bound with $\beta = \frac{n+1}{n+3}$ (which could be improved by using linear Strichartz estimates as indicated in Remark 2.5). Suppose we could show that, if (4.2) holds for some $\beta > \alpha$, then for every $\epsilon > 0$ we have

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \lesssim R^{2\epsilon} (R^{(1-\delta)\beta} + R^{D\delta}) (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^\# g_{J'}\|_{\ell_{J'}^2 L_x^2}. \quad (4.3)$$

where $\delta = \frac{\alpha}{D+\alpha}$ and $D \geq 0$ is some constant which depends only on the dimension n . Then, since $D\delta < \alpha$, by taking $\epsilon > 0$ sufficiently small, we deduce that we must have (4.2) for some $\beta' < \beta$. Iterating this argument then gives (4.2) for $\beta = \alpha$. Consequently, our aim is to prove (4.3), under the assumption that we already have (4.2) for some $\beta > \alpha$.

We now fix $\Gamma_j \subset \mathcal{X}_j$ such that $\#\Gamma_j \leq R^{10n}$, and $\beta > \alpha$. Let \mathcal{B} denote a collection of balls B of radius $R^{1-\delta}$ which form a finitely overlapping cover of Q_R . Let \sim denote the relation between points $\gamma_j \in \Gamma_j$ and

balls $B \in \mathcal{B}$ given by Definition 3.6. It is important to note that the relation \sim depends only on the fixed sets Γ_j , and *not* on u and v . Decompose

$$\left\| \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \leq \sum_{B \in \mathcal{B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_j \sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(B)} + \sum_{B \in \mathcal{B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(B)}.$$

For the first term, which contains the tubes which are concentrated on B , we apply the induction assumption at scale $R^{1-\delta}$ to deduce that

$$\begin{aligned} & \sum_{B \in \mathcal{B}} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_j \sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(B)} \\ & \lesssim R^{(1-\delta)\beta} \sum_{B \in \mathcal{B}} (\#\{\gamma_1 \in \Gamma_1 \mid \gamma_1 \sim B\} \#\{\gamma_2 \in \Gamma_2 \mid \gamma_2 \sim B\})^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^{\#} f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^{\#} g_{J'}\|_{\ell_{J'}^2 L_x^2} \\ & \lesssim R^{\epsilon} R^{(1-\delta)\beta} (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^{\#} f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^{\#} g_{J'}\|_{\ell_{J'}^2 L_x^2} \end{aligned}$$

where the last line followed from (i) in Remark 3.7. For the second term, as we can now safely lose factors of R^{δ} , we may ignore the sum over the balls B (as there are only $\mathcal{O}(R^{\delta(n+1)})$ balls). Thus, after replacing D with $D - n - 1$, we need to prove the bound

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(B)} \lesssim R^{\epsilon+D\delta} (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^{\#} f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^{\#} g_{J'}\|_{\ell_{J'}^2 L_x^2}. \quad (4.4)$$

To this end, an application of Hölder together with the orthogonality property of the tube decomposition gives

$$\begin{aligned} \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^1(B)} & \lesssim R \left(\sum_{\gamma_1 \in \Gamma_1} \|L_{\gamma_1}^{\#} f_J\|_{\ell_J^2 L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{\gamma_2 \in \Gamma_2} \|L_{\gamma_2}^{\#} g_{J'}\|_{\ell_{J'}^2 L_x^2}^2 \right)^{\frac{1}{2}} \\ & \lesssim R (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^{\#} f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^{\#} g_{J'}\|_{\ell_{J'}^2 L_x^2} \end{aligned}$$

In particular, the convexity of the L^p norms implies that (4.4) would follow from the $L_{t,x}^2$ bound

$$\begin{aligned} & \left\| \sum_{\substack{\gamma_j \in \Gamma_j \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^2(B)} \\ & \lesssim R^{\epsilon+D\delta-\frac{n-1}{4}} (\#\Gamma_1 \#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^{\#} f_J\|_{\ell_J^2 L_x^2} \|L_{\gamma_2}^{\#} g_{J'}\|_{\ell_{J'}^2 L_x^2}. \end{aligned} \quad (4.5)$$

Thus we have reduced the problem of obtaining the $L_{t,x}^{\frac{n+3}{n+1}}$ estimate (4.3), to proving the $L_{t,x}^2$ bound (4.5).

Remark 4.2. The fact that the above reduction can be done in U_{Φ}^2 , is the key reason why we can extend the homogeneous bilinear Fourier restriction estimates to U_{Φ}^2 .

Our goal in the following two subsections is to prove the bound (4.5), and thus complete the proof of Theorem 4.1. As in the previous subsections, we essentially follow the argument of Tao [43], but apply the results of Section 2 in place of analogous results for the paraboloid. The general strategy is to first use the transversality via Lemma 2.6 to reduce to counting intersections of tubes. The number of tubes is then controlled by using (i) in Assumption 1 via Lemma 2.7 together with a ‘‘bush’’ argument. The notation for various cubes and tubes introduced in Subsection 3.3 is used heavily in what follows.

4.2. The L^2 Bound: Initial Reductions and Transversality. Recall that the ball $B \in \mathcal{B}$ is now fixed. Write

$$\sum_{\substack{\gamma_j \in \Gamma_j, \\ \gamma_1 \not\sim B \text{ or } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v = \sum_{\substack{\gamma_j \in \Gamma_j, \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v + \sum_{\substack{\gamma_j \in \Gamma_j, \\ \gamma_1 \sim B \text{ and } \gamma_2 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v.$$

We only prove the bound for the first term, as an identical argument can handle the second term (just replace Γ_1 with $\{\gamma_1 \in \Gamma_1 \mid \gamma_1 \sim B\}$ and reverse the roles of u and v). The first step is make a number of reductions exploiting the spatial localisation properties of the wave packets, together with a dyadic pigeon hole argument to fix various quantities. To this end, decompose into cubes $q \in \mathbf{q}$

$$\left\| \sum_{\substack{\gamma_j \in \Gamma_j, \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(B)} \leq \left(\sum_{q \in \mathbf{q}, q \subset 2B} \left\| \sum_{\substack{\gamma_j \in \Gamma_j, \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(q)} \right)^{\frac{1}{2}}.$$

Note that the concentration property of the wave packet decomposition implies that

$$\left\| \sum_{\gamma_1 \in \Gamma_1, T_{\gamma_1} \cap R^\delta q = \emptyset} \mathcal{P}_{\gamma_1} u \right\|_{L^\infty_{t,x}(q)} \lesssim R^{-\delta(N - \frac{n+3}{2})} (\#\Gamma_1)^{\frac{1}{2}} \sup_{\gamma_1 \in \Gamma_1} \|L_{\gamma_1}^\# f_J\|_{\ell^2 J L^2_x}.$$

A similar bound holds for v . By our choice of N , we have $\delta(N - \frac{n+3}{2}) \geq 100n$. Therefore, as $\#\Gamma_j \lesssim R^{10n}$ and $\#\mathbf{q} \lesssim R^{2n}$, it suffices to prove

$$\begin{aligned} & \left(\sum_{q \in \mathbf{q}, q \subset 2B} \left\| \sum_{\substack{\gamma_j \in \Gamma_j(q), \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(q)} \right)^{\frac{1}{2}} \\ & \lesssim R^{\epsilon + D\delta - \frac{n-1}{4}} (\#\Gamma_1)^{\frac{1}{2}} (\#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell^2 J L^2_x} \|L_{\gamma_2}^\# g_{J'}\|_{\ell^2_{J'} L^2_x}. \end{aligned} \quad (4.6)$$

Let $\Gamma_1^{\not\sim B}(q) = \{\gamma_1 \in \Gamma_1(q) \mid \gamma_1 \not\sim B\}$ and decompose into

$$\begin{aligned} & \left(\sum_{q \in \mathbf{q}, q \subset 2B} \left\| \sum_{\substack{\gamma_j \in \Gamma_j(q), \\ \gamma_1 \not\sim B}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(q)} \right)^{\frac{1}{2}} \\ & \leq \sum_{1 \leq \lambda_1, \mu_1, \mu_2 \lesssim R^{100n}} \left(\sum_{q \in \mathbf{q}(\mu_1, \mu_2), q \subset 2B} \left\| \sum_{\substack{\gamma_1 \in \Gamma_1^{\not\sim B}(q) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2] \\ \gamma_2 \in \Gamma_2(q)}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(q)} \right)^{\frac{1}{2}}. \end{aligned}$$

Clearly, as we can freely lose R^ϵ , (4.6) would follow from proving the estimate for fixed λ_1, μ_1, μ_2 ,

$$\begin{aligned} & \left(\sum_{q \in \mathbf{q}(\mu_1, \mu_2), q \subset 2B} \left\| \sum_{\substack{\gamma_1 \in \Gamma_1^{\not\sim B}(q) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2] \\ \gamma_2 \in \Gamma_2(q)}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L^2_{t,x}(q)} \right)^{\frac{1}{2}} \\ & \lesssim R^{\epsilon + D\delta - \frac{n-1}{4}} (\#\Gamma_1)^{\frac{1}{2}} (\#\Gamma_2)^{\frac{1}{2}} \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^\# f_J\|_{\ell^2 J L^2_x} \|L_{\gamma_2}^\# g_{J'}\|_{\ell^2_{J'} L^2_x}. \end{aligned} \quad (4.7)$$

To make the notation slightly less cumbersome, we introduce the short hand

$$\Gamma_1^*(q) = \Gamma_1^{\not\sim B}(q) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2].$$

Given $q \in \mathbf{q}$ and $\mathbf{h} \in \mathbb{R}^{1+n}$, we define the set

$$\Gamma_1^{**}(q, \mathbf{h}) = \Gamma_1^{**}[\lambda_1, \mu_1, \mu_2](q, \mathbf{h}) = \{\gamma_1 \in \Gamma_1^*(q) \mid \xi(\gamma_1) \in \Sigma_1(\mathbf{h}) + \mathcal{O}(R^{-\frac{1}{2}})\}.$$

Thus $\Gamma_1^{**}(q, \mathfrak{h})$ consists of all $\gamma_1 \in \Gamma_1^*(q)$ such that $\xi(\gamma_1)$ lies within $CR^{-\frac{1}{2}}$ of the surface $\Sigma_1(\mathfrak{h})$. If we expand the square of the $L_{t,x}^2$ in (4.7) we get

$$\left\| \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma_2 \in \Gamma_2(q)}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^2}^2 \leq \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma_2' \in \Gamma_2(q)}} \sum_{\gamma_1' \in \Gamma_1^*(q)} \sum_{\gamma_2 \in \Gamma_2(q)} \left| \langle \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v, \mathcal{P}_{\gamma_1'} u \mathcal{P}_{\gamma_2'} v \rangle_{L_{t,x}^2} \right|.$$

We now exploit the Fourier localisation properties of the wave packets to deduce that the inner product vanishes unless

$$\begin{aligned} \xi(\gamma_1) + \xi(\gamma_2) &= \xi(\gamma_1') + \xi(\gamma_2') + \mathcal{O}(R^{-\frac{1}{2}}) \\ \Phi_1(\xi(\gamma_1)) + \Phi_2(\xi(\gamma_2)) &= \Phi_1(\xi(\gamma_1')) + \Phi_2(\xi(\gamma_2')) + \mathcal{O}(R^{-\frac{1}{2}}) \end{aligned} \quad (4.8)$$

In particular, if we take $\mathfrak{h}_{\gamma_1, \gamma_2'} = (\Phi_1(\xi(\gamma_1)) - \Phi_2(\xi(\gamma_2')), \xi(\gamma_1) - \xi(\gamma_2'))$, then an application of Lemma 2.3 implies that

$$\begin{aligned} &\left\| \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma_2 \in \Gamma_2(q)}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^2}^2 \\ &\leq \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma_2' \in \Gamma_2(q)}} \sum_{\substack{\gamma_1' \in \Gamma_1^{**}(q, \mathfrak{h}_{\gamma_1, \gamma_2'}) \\ \xi(\gamma_2) = \xi(\gamma_1') + \xi(\gamma_2') - \xi(\gamma_1) + \mathcal{O}(R^{-\frac{1}{2}})}} \sum_{\gamma_2 \in \Gamma_2(q)} \left| \langle \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v, \mathcal{P}_{\gamma_1'} u \mathcal{P}_{\gamma_2'} v \rangle_{L_{t,x}^2} \right|. \end{aligned}$$

On the other hand, an application of Lemma 2.6 easily gives the $U_{\mathbb{F}}^2$ bound

$$\|\mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v\|_{L_{t,x}^2} \lesssim R^{-\frac{n-1}{4}} \|L_{\gamma_1}^{\sharp} f_J\|_{\ell_j^2 L_x^2} \|L_{\gamma_2}^{\sharp} g_{J'}\|_{\ell_j^2, L_x^2}.$$

If we now note that, for fixed γ_1, γ_2' , and γ_1' , and any $q \in \mathfrak{q}$ we have

$$\#\{\gamma_2 \in \Gamma_2 \mid T_{\gamma_2} \cap R^{\delta} q \neq 0, \xi(\gamma_2) = \xi(\gamma_1') + \xi(\gamma_2') - \xi(\gamma_1) + \mathcal{O}(R^{-\frac{1}{2}})\} \lesssim R^{n\delta}$$

then an application of Cauchy-Schwarz gives

$$\left\| \sum_{\substack{\gamma_1 \in \Gamma_1^*(q) \\ \gamma_2 \in \Gamma_2(q)}} \mathcal{P}_{\gamma_1} u \mathcal{P}_{\gamma_2} v \right\|_{L_{t,x}^2}^2 \lesssim R^{D\delta - \frac{n-1}{2}} \#\Gamma_1^*(q) \#\Gamma_2(q) \sup_{\mathfrak{h}} \#\Gamma_1^{**}(q, \mathfrak{h}) \sup_{\gamma_j \in \Gamma_j} \|L_{\gamma_1}^{\sharp} f_J\|_{\ell_j^2 L_x^2}^2 \|L_{\gamma_2}^{\sharp} g_{J'}\|_{\ell_j^2, L_x^2}^2.$$

Consequently the bound (4.7) would follow from the combinatorial estimate

$$\sum_{\substack{q \in \mathfrak{q}(\mu_1, \mu_2) \\ q \subset 2B}} \#\Gamma_1^*(q) \#\Gamma_2(q) \sup_{\mathfrak{h} \in \mathbb{R}^{1+n}} \#\Gamma_1^{**}(q, \mathfrak{h}) \lesssim R^{D\delta} \#\Gamma_1 \#\Gamma_2. \quad (4.9)$$

We now simplify this bound slightly by exploiting the dyadic localisations we performed earlier. More precisely, by definition, for every $q \in \mathfrak{q}(\mu_1, \mu_2)$, we have $\#\Gamma_2(q) \leq 2\mu_2$. On the other hand, by exchanging the order of summation, recalling the short hand $\Gamma_1^*(q) = \Gamma_1^{\mathcal{B}}(q) \cap \Gamma_1[\lambda_1, \mu_1, \mu_2]$, and using the definition of the set $\Gamma_1[\lambda_1, \mu_1, \mu_2]$, we deduce that

$$\begin{aligned} \sum_{\substack{q \in \mathfrak{q}(\mu_1, \mu_2) \\ q \subset 2B}} \#\Gamma_1^*(q) &\leq \sum_{q \in \mathfrak{q}(\mu_1, \mu_2)} \#(\Gamma_1(q) \cap \Gamma[\lambda_1, \mu_1, \mu_2]) \\ &= \sum_{\gamma_1 \in \Gamma[\lambda_1, \mu_1, \mu_2]} \#\{q \in \mathfrak{q}(\mu_1, \mu_2) \mid T_{\gamma_1} \cap R^{\delta} q \neq 0\} \\ &\leq 2\lambda_1 \#\Gamma_1 \end{aligned}$$

Therefore, we have reduced the bound (4.9) to proving the combinatorial Keakeya type estimate

$$\sup_{\substack{\mathfrak{h} \in \mathbb{R}^{1+n} \\ q \in \mathfrak{q}(\mu_1, \mu_2), q \subset 2B}} \Gamma_1^{**}[\lambda_1, \mu_1, \mu_2](q, \mathfrak{h}) \lesssim R^{D\delta} \frac{\#\Gamma_2}{\lambda_1 \mu_2}. \quad (4.10)$$

The proof of this bound is the focus of the next subsection.

4.3. The L^2 Bound: The Combinatorial Estimate. We have reduced the proof of Theorem 4.1 to obtaining the combinatorial bound (4.10), which is essentially well-known to experts as it does not see the difference between homogeneous solutions and $V_{\Phi_j}^2$ -functions. For completeness, we include the proof here. We follow the ‘‘bush’’ argument used in [43], making some minor adjustments only to relate it to Assumption 1. Recall that we have fixed a ball $B \in \mathcal{B}$. Fix any $\mathfrak{h} \in \mathbb{R}^{1+n}$ and $q_0 \in \mathbf{q}(\mu_1, \mu_2)$ with $q_0 \subset 2B$. Our goal is to prove

$$\#\Gamma_1^{**}(q_0, \mathfrak{h}) \lesssim R^{D\delta} \frac{\#\Gamma_2}{\lambda_1 \mu_2}.$$

The first step is to exploit the fact that γ_1 is not concentrated on B . Recall from Subsection 3.3 that for $\gamma_1 \in \Gamma_1$ we have defined the ball $B(\gamma_1, \lambda_1, \mu_1, \mu_2) \in \mathcal{B}$ to be (a) maximiser for the quantity

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_j} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_j, \lambda_j, \mu_1, \mu_2) \neq \emptyset\}.$$

Let $\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})$. By construction this implies that $\gamma_1 \in \Gamma_1^{\neq B}(q_0)$, and hence by definition of the relation \sim , we have $B \not\subset 10B(\gamma_1, \lambda_1, \mu_1, \mu_2)$. Since $q_0 \subset 2B$ and the balls in \mathcal{B} have radius $R^{1-\delta}$, we must have $\text{dist}(q_0, B(\gamma_1, \lambda_1, \mu_1, \mu_2)) \gtrsim R^{1-\delta}$. In particular, by (ii) in Remark 3.7, we have for every $\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})$

$$\begin{aligned} & \#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_1} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \\ & \gtrsim \#\{q \in \mathbf{q}(\mu_1, \mu_2) \mid T_{\gamma_1} \cap R^\delta q \neq \emptyset, q \cap B(\gamma_1, \lambda_1, \mu_1, \mu_2) \neq \emptyset\} \\ & \gtrsim R^{-D\delta} \lambda_1. \end{aligned}$$

On the other hand, since for $q \in \mathbf{q}(\mu_1, \mu_2)$ we have $\#\Gamma_2(q) \geq \mu_2$, we deduce that

$$\#\{(q, \gamma_2) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_2 \mid T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \gtrsim R^{-D\delta} \lambda_1 \mu_2.$$

Summing up over $\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})$ and then changing the order of summation gives

$$\begin{aligned} & \lambda_1 \mu_2 \#\Gamma_1^{**}(q_0, \mathfrak{h}) \\ & \lesssim R^{D\delta} \sum_{\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})} \#\{(q, \gamma_2) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_2 \mid T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \\ & = R^{D\delta} \sum_{\gamma_2 \in \Gamma_2} \#\{(q, \gamma_1) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_1^{**}(q_0, \mathfrak{h}) \mid T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\}. \end{aligned}$$

Therefore the required bound (4.10) follows from the following lemma, cf. [43, Lemma 8.1].

Lemma 4.3. *Let $q_0 \in \mathbf{q}$, $\mathfrak{h} \in \mathbb{R}^{1+n}$, and $\gamma_2 \in \Gamma_2$. Then*

$$\#\{(q, \gamma_1) \in \mathbf{q}(\mu_1, \mu_2) \times \Gamma_1^{**}(q_0, \mathfrak{h}) \mid T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_2} \cap R^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim R^{1-\delta}\} \lesssim R^{D\delta}.$$

Proof. Define the bush (or ‘‘fan’’) at q_0 by

$$\text{Bush}(q_0) = \bigcup_{\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})} T_{\gamma_1}.$$

Thus $\text{Bush}(q_0) \subset \mathbb{R}^{1+n}$ is the union of all tubes T_{γ_1} (associated to phase space elements $\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})$) passing through a neighbourhood of the cube q_0 . Our goal is then to bound the sum

$$\sum_{\substack{q \in \mathbf{q}(\mu_1, \mu_2), \\ q \subset \text{Bush}(q_0) \cap T_{\gamma_2} + \mathcal{O}(R^{\frac{1}{2}+\delta}) \\ \text{dist}(q, q_0) \gtrsim R^{1-\delta}}} \#\{\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h}) \mid T_{\gamma_1} \cap R^\delta q \neq \emptyset\}. \quad (4.11)$$

We first count the number of possible cubes in the outer summation. The idea is to first show that

$$\text{Bush}(q_0) \subset (t_0, x_0) + \mathcal{C}_1(\mathfrak{h}) + \mathcal{O}(R^{\frac{1}{2}+D\delta}) \quad (4.12)$$

where (t_0, x_0) denotes the centre of the cube q_0 , and the conic hypersurface $\mathcal{C}_1(\mathfrak{h})$ is given by

$$\mathcal{C}_1(\mathfrak{h}) = \{(r, -r\nabla\Phi_1(\xi)) \mid r \in \mathbb{R}, \xi \in \Sigma_1(\mathfrak{h})\}.$$

Since if we had (4.12), an application of Lemma 2.7 would then show that $\text{Bush}(q_0) \cap T_{\gamma_2}$ is contained in a ball of radius $R^{\frac{1}{2}+D\delta}$, and hence the outer summation in (4.11) only contains $\mathcal{O}(R^{D\delta})$ terms. To show the

inclusion (4.12), suppose that $(t, x) \in \text{Bush}(q_0)$. Then $(t, x) \in T_{\gamma_1}$ for some $\gamma_1 \in \Gamma_1^{**}(q_0, \mathfrak{h})$. By construction, we have $\xi(\gamma) = \xi^* + \mathcal{O}(R^{-\frac{1}{2}})$ for some $\xi^* \in \Sigma_1(\mathfrak{h})$. On the other hand, since $T_{\gamma_1} \cap R^\delta q_0 \neq \emptyset$, we have

$$x - x_0 + (t - t_0)\nabla\Phi_1(\xi(\gamma_1)) = [x - x(\gamma) + t\nabla\Phi_1(\xi(\gamma_1))] - [x_0 - x(\gamma) + t_0\nabla\Phi_1(\xi(\gamma_1))] = \mathcal{O}(R^{\frac{1}{2}+\delta}).$$

Therefore, since $|t - t_0| \lesssim R$, we can write

$$\begin{aligned} & (t, x) - (t_0, x_0) \\ &= (t - t_0, -(t - t_0)\nabla\Phi_1(\xi^*)) + (0, x - x_0 + (t - t_0)\nabla\Phi_1(\xi(\gamma_1))) + (0, (t - t_0)[\nabla\Phi_1(\xi^*) - \nabla\Phi_1(\xi(\gamma_1))]) \\ &= (t - t_0, -(t - t_0)\nabla\Phi_1(\xi^*)) + \mathcal{O}(R^{\frac{1}{2}+\delta}) \end{aligned}$$

and hence we have (4.12). Consequently, the outer sum in (4.11) is only over $\mathcal{O}(R^{C\delta})$ cubes.

Fix $q \in \mathbf{q}(\mu_1, \mu_2)$ with $\text{dist}(q, q_0) \gtrsim R^{1-\delta}$. As the outer sum in (4.11) only adds $\mathcal{O}(R^{D\delta})$, the required bound would now follow from

$$\#\{\gamma_1 \in \Gamma_1 \mid \xi(\gamma_1) \in \Sigma_1(\mathfrak{h}) + \mathcal{O}(R^{-\frac{1}{2}}), T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_1} \cap R^\delta q_0 \neq \emptyset\} \lesssim R^\delta. \quad (4.13)$$

The point is that since the cubes q and q_0 are at a distance $R^{1-\delta}$ apart, the condition that T_{γ_1} must intersect *both* cubes, essentially fixes the tube T_{γ_1} . Since $\xi(\gamma_1) \in \Sigma_1(\mathfrak{h}) + \mathcal{O}(R^{-\frac{1}{2}})$, the bound (1.1) implies that fixing the tube T_{γ_1} , also more or less fixes the phase space element γ_1 (note that without the bound (1.1), the set in (4.13) could potentially contain far more than $\mathcal{O}(R^\delta)$ points). In more detail, let

$$\gamma_1, \gamma'_1 \in \{\gamma_1 \in \Gamma_1 \mid \xi(\gamma_1) \in \Sigma_1(\mathfrak{h}) + \mathcal{O}(R^{-\frac{1}{2}}), T_{\gamma_1} \cap R^\delta q \neq \emptyset, T_{\gamma_1} \cap R^\delta q_0 \neq \emptyset\}.$$

In light of (1.1), the estimate (4.13) would follow from the bounds

$$|x(\gamma_1) - x(\gamma'_1)| \lesssim R^{\frac{1}{2}+\delta}, \quad |v(\gamma_1) - v(\gamma'_1)| \lesssim R^{-\frac{1}{2}+\delta} \quad (4.14)$$

where ease of notation we define the *velocity* $v(\gamma_1) = \Phi_1(\xi(\gamma_1))$. We now exploit the condition that the tubes T_{γ_1} and $T_{\gamma'_1}$ intersect the cubes q and q_0 . Let (t_q, x_q) denote the centre of the cube q and (t_0, x_0) the centre of q_0 . Since $|v(\gamma_1)| \leq \mathbf{D}_2$ and

$$x_0 - x_q + (t_0 - t_q)v(\gamma_1) = (x_0 - x(\gamma_1) + t_0v(\gamma_1)) - (x_q - x(\gamma_1) + t_qv(\gamma_1)) = \mathcal{O}(R^{\frac{1}{2}+D\delta}),$$

the separation of the cubes q and q_0 implies that $R^{1-C\delta} \lesssim |t_0 - t_q| \lesssim R$. A computation shows that

$$(t_0 - t_q)(v(\gamma_1) - v(\gamma'_1)) = \mathcal{O}(R^{\frac{1}{2}+D\delta}), \quad x(\gamma_1) - x(\gamma'_1) = t_0(v(\gamma'_1) - v(\gamma_1)) + \mathcal{O}(R^{\frac{1}{2}+D\delta})$$

and hence the bound on $|t_0 - t_q|$ gives (4.14). \square

5. THE GLOBALISATION LEMMA

In this section, we complete the proof of Theorem 1.1 by showing that it follows from the localised bound in Theorem 4.1. The proof of Theorem 1.1 proceeds by using a strategy sketched in Section 8 of [24], together with interpolation argument to replace $U_{\Phi_j}^2$ with $V_{\Phi_j}^2$.

Proof of Theorem 1.1. The first step is to show that by, exploiting the (approximate) finite speed of propagation of frequency localised waves, the bilinear estimate on Q_R implies the same estimate holds on $I_R \times \mathbb{R}^n$ with $I_R = [0, R]$. The second step is to remove the remaining temporal localisation and R^α factor by using duality, together with the dispersive decay in Lemma 2.4. Finally we use a simple interpolation argument to replace $U_{\Phi_j}^2$ with the larger $V_{\Phi_j}^2$ space.

Step 1: From Q_R to $I_R \times \mathbb{R}^n$. Let $R \geq (10\mathbf{R}_0)^2$, $u \in U_{\Phi_j}^2$, and $v \in U_{\Phi_j}^2$ with $\text{supp } \widehat{u} \subset \Lambda_1^*$ and $\text{supp } \widehat{v} \subset \Lambda_2^*$. Assuming Theorem 4.1, our goal is to prove that for every $\alpha > 0$ we have

$$\|uv\|_{L_{t,x}^{\frac{n+3}{n+1}}(I_R \times \mathbb{R}^n)} \lesssim R^\alpha \|u\|_{U_{\Phi_j}^2} \|v\|_{U_{\Phi_j}^2}. \quad (5.1)$$

It is enough to consider the case where u , and v are atoms, thus we have a decomposition

$$u = \sum_J \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_J, \quad v = \sum_{J'} \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{J'}$$

with

$$\sum_J \|f_J\|_{L^2}^2 + \sum_{J'} \|g_{J'}\|_{L^2}^2 \leq 1$$

and we may assume that $\text{supp } \widehat{f}_J \subset \Lambda_1^*$ and $\text{supp } \widehat{g}_{J'} \subset \Lambda_2^*$ (using sharp Fourier cutoffs). By translation invariance, the bound (5.1) would then follow from

$$\|uv\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \lesssim R^\alpha \left(\sum_J \|(1+R^{-1}|x|)^{-(n+1)} f_J\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{J'} \|(1+R^{-1}|x|)^{-(n+1)} g_{J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \quad (5.2)$$

since we can then sum up over the centres of balls (or cubes) of radius R which cover \mathbb{R}^n . The inequality (5.2) is a reflection of the fact that, as u and v are localised to frequencies of size ≈ 1 , we expect that the waves $e^{it\Phi_j(-i\nabla)} f_J$ should travel with velocity 1. In particular, u and v on Q_R , should only depend on the data in $\{|x| \lesssim R\}$. It turns out that this is true, modulo a rapidly decreasing tail.

Let $\rho \in \mathcal{S}$ with $\text{supp } \widehat{\rho} \subset \{|\xi| \leq 1\}$ and $\rho \gtrsim 1$ on $|x| \leq 1$. To prove (5.2), we start by noting that since the left hand integral is only over Q_R , we may replace uv with $\rho(R^{-1}x)u(t,x)\rho(R^{-1}x)v(y)$. Note that we can write

$$\begin{aligned} \rho\left(\frac{x}{R}\right) \left(e^{it\Phi_j(-i\nabla)} f \right)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R^n \widehat{\rho}(R(\xi - \eta)) e^{it\Phi_j(\eta)} \widehat{f}(\eta) d\eta e^{i\xi \cdot x} d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R^n \widehat{\rho}(R(\xi - \eta)) \widehat{f}(\eta) F(t, R(\xi - \eta), \eta) d\eta e^{i\xi \cdot x} e^{i\Phi_j(\xi)} d\xi \end{aligned} \quad (5.3)$$

where $F(t, \xi, \eta) = \chi(\xi, \eta) e^{it(\Phi_j(\frac{\xi}{R} + \eta) - \Phi_j(\eta))}$ and $\chi \in C_0^\infty(\{|\xi| \leq 2\} \times (\Lambda_j^* + \frac{1}{R_0}))$ with $\chi = 1$ on $\{|\xi| \leq 2\} \times \Lambda_j^*$. The oscillating component of F is essentially constant for $|t| \leq R$. To exploit this, we expand F using a Fourier series to get

$$F(t, \xi, \eta) = \sum_{k \in \mathbb{Z}^{2n}} c_k(t) e^{ik \cdot (\xi, \eta)}, \quad c_k(t) = \int_{\mathbb{R}^{2n}} \overline{F(t, \xi, \eta)} e^{ik \cdot (\xi, \eta)} d\xi d\eta$$

and by (ii) in Assumption 1, the coefficients satisfy $|c_k(t)| \lesssim_{\mathbf{R}_0, \mathbf{D}_2} (1 + |k_1|)^{-2(n+1)} (1 + |k_2|)^{-2(n+1)}$ with $k = (k_1, k_2)$. Applying this expansion to $\rho(R^{-1}x)u$ and $\rho(R^{-1}x)v$ we obtain the decompositions

$$\rho(R^{-1}x)u = \sum_J \sum_k c_k(t) \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_{k,J}, \quad \rho(R^{-1}x)v = \sum_{J'} \sum_k c'_k(t) \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{k,J'} \quad (5.4)$$

where the coefficients c_k, c'_k are independent of J and J' , and the functions $f_{k,J}$ and $g_{k,J'}$ are given by

$$f_{k,J}(x) = \rho\left(\frac{x}{R} + k_1\right) f_J(x + k_2), \quad g_{k,J'}(x) = \rho\left(\frac{x}{R} + k_1\right) g_{J'}(x + k_2)$$

with $k = (k_1, k_2)$. Note that $\text{supp } \widehat{f}_{k,J} \subset \Lambda_1^* + \frac{1}{2R_0}$ since $R \geq (10R_0)^2$, thus the $f_{k,J}$ satisfy the support conditions in Theorem 4.1 (with Λ_j^* replaced with $\Lambda_j^* + \frac{1}{R_0}$, and \mathbf{R}_0 replaced with $2\mathbf{R}_0$). A similar comment applies to the $g_{k,J'}$. Therefore, plugging the decomposition (5.4) into the left hand side of (5.2), we deduce via an application of Theorem 4.1 that

$$\begin{aligned} &\|uv\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \\ &\lesssim \sum_{k,k' \in \mathbb{Z}^n \times \mathbb{Z}^n} (1 + |k|)^{-2(n+1)} (1 + |k'|)^{-2(n+1)} \left\| \sum_{J,J'} \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_{k,J} \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{k',J'} \right\|_{L_{t,x}^{\frac{n+3}{n+1}}(Q_R)} \\ &\lesssim R^\alpha \sum_{k,k'} (1 + |k|)^{-2(n+1)} (1 + |k'|)^{-2(n+1)} \\ &\quad \times \left(\sum_J \|(1+R^{-1}|x - k_1 + Rk_2|)^{-(n+1)} f_J\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{J'} \|(1+R^{-1}|x - k'_1 + Rk'_2|)^{-(n+1)} g_{J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim R^\alpha \left(\sum_J \|(1+R^{-1}|x|)^{-(n+1)} f_J\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{J'} \|(1+R^{-1}|x|)^{-(n+1)} g_{J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \end{aligned}$$

Thus we obtain (5.2) and hence (5.1).

Step 2: From $I_R \times \mathbb{R}^n$ to \mathbb{R}^{1+n} . Let $u \in U_{\Phi_1}^2$ and $v \in U_{\Phi_2}^2$ with $\text{supp } \widehat{u} \subset \Lambda_1^*$ and $\text{supp } \widehat{v} \subset \Lambda_2^*$. Our goal is to show that for every $p > \frac{n+3}{n+1}$

$$\|uv\|_{L_{t,x}^p} \lesssim \|u\|_{U_{\Phi_1}^2} \|v\|_{U_{\Phi_2}^2}. \quad (5.5)$$

In fact the argument below gives the marginally stronger (though essentially equivalent) bound

$$\|uv\|_{L_t^p L_x^{\frac{n+3}{n+1}}} \lesssim \|u\|_{U_{\Phi_1}^2} \|v\|_{U_{\Phi_2}^2}. \quad (5.6)$$

To deduce (5.5) from (5.6), note that dispersive estimate in Lemma 2.4, together with the abstract Strichartz estimates of Keel-Tao [22, Theorem 1.2], implies there exists $1 < a < b < \infty$ such that $\|uv\|_{L_t^a L_x^b} \lesssim 1$. On the other hand, the Fourier support assumptions imply that we have the trivial bound $\|uv\|_{L_t^\infty L_x^p(\mathbb{R}^{1+n})} \lesssim 1$ for every $p \geq 1$. Thus interpolation gives (5.5) from (5.6).

We now turn to the proof of (5.6). As in step 1, we may assume that u and v are atoms with the decomposition

$$u = \sum_J \mathbb{1}_J(t) e^{it\Phi_1(-i\nabla)} f_J, \quad v = \sum_{J'} \mathbb{1}_{J'}(t) e^{it\Phi_2(-i\nabla)} g_{J'}$$

with $\text{supp } \widehat{f}_J \subset \Lambda_1^*$, $\text{supp } \widehat{g}_{J'} \subset \Lambda_2^*$, and

$$\sum_J \|f_J\|_{L^2}^2 + \sum_{J'} \|g_{J'}\|_{L^2}^2 \leq 1.$$

By real interpolation it is enough to show that for every $q > \frac{n+3}{n+1}$ we have

$$\|uv\|_{L_t^q L_x^{\frac{n+3}{n+1}}} \lesssim 1$$

where $L_t^{q,\infty}$ is the Lorentz norm. Applying duality, this would follow from the estimate

$$\int_\Omega \|uv\|_{L_x^{\frac{n+3}{n+1}}} dt \lesssim |\Omega|^{\frac{1}{q'}} \quad (5.7)$$

for every measurable $\Omega \subset \mathbb{R}$. Define the Fourier localised solution operator $\mathcal{U}_j(t)[h] = e^{it\Phi_j(-i\nabla)} P_{\Lambda_j^*} h$ where we let $\widehat{P_{\Lambda_j^*} h}(\xi) = \rho_{\Lambda_j^*}(\xi) \widehat{h}(\xi)$ with $\rho \in C_0^\infty(\Lambda_j^* + \frac{1}{10\mathbf{R}_0})$ and $\rho = 1$ on Λ_j^* . If we interpolate Lemma 2.4 with the trivial $L_t^\infty L_x^2$ bound and apply duality, we deduce that for every $1 \leq a \leq 2$

$$\int_{\substack{(t,t') \in \Omega \times \Omega \\ |t-t'| \gtrsim R}} \left\langle \mathcal{U}_j^*(t)[G(t)], \mathcal{U}_j(t')[G(t')] \right\rangle_{L_x^a} dt dt' \lesssim |\Omega|^2 R^{-\frac{n-1}{2}(\frac{2}{a}-1)} \|G\|_{L_t^\infty L_x^a}^2 \quad (5.8)$$

where \mathcal{U}_j^* denotes the L_x^2 adjoint of \mathcal{U}_j . The dispersive bound (5.8) together with the bilinear estimate (5.1) are the key inequalities required in the proof of (5.7).

We now begin the proof of (5.7). If $|\Omega| \lesssim 1$, then (5.7) follows by putting $uv \in L_t^\infty L_x^{\frac{n+3}{n+1}}$ and using Sobolev embedding. Thus we may assume that $|\Omega| \gg 1$. Let us set $J'_\Omega := \Omega \cap J'$. An application of duality gives

$$\begin{aligned} \int_\Omega \|uv\|_{L_x^{\frac{n+3}{n+1}}} dt &\leq \sup_{\substack{\|F\| \\ L_t^\infty L_x^{\frac{n+3}{2}}}} \left| \int_\Omega \langle F, uv \rangle_{L_x^2} dt \right| \\ &= \sup_{\substack{\|F\| \\ L_t^\infty L_x^{\frac{n+3}{2}}}} \left| \sum_{J'} \int_{J'_\Omega} \langle F, u \mathcal{U}_2(t)[g_{J'}] \rangle_{L_x^2} dt \right| \\ &\lesssim \sup_{\substack{\|F\| \\ L_t^\infty L_x^{\frac{n+3}{2}}}} \left(\sum_{J'} \left\| \int_{J'_\Omega} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 \right)^{\frac{1}{2}} \end{aligned}$$

If we expand the square of the L_x^2 norm, we have via (5.8) with $\frac{1}{a} = \frac{2}{n+3} + \frac{1}{2}$

$$\begin{aligned}
\sum_{J'} \left\| \int_{J'_\Omega} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 &= \sum_{J'} \int_{t,t' \in J'_\Omega} \langle \mathcal{U}_2^*(t)[F\bar{u}], \mathcal{U}_2^*(t')[F\bar{u}] \rangle_{L_x^2} dt dt' \\
&= \sum_{J'} \int_{\substack{t,t' \in J'_\Omega \\ |t-t'| \gtrsim R}} \langle \mathcal{U}_2^*(t)[F\bar{u}], \mathcal{U}_2^*(t')[F\bar{u}] \rangle_{L_x^2} dt dt' \\
&\quad + \sum_{J'} \sum_{|I-I'| \leq R} \int_{J'_\Omega \cap I} \int_{J'_\Omega \cap I'} \langle \mathcal{U}_2^*(t)[F\bar{u}], \mathcal{U}_2^*(t')[F\bar{u}] \rangle_{L_x^2} dt dt' \\
&\lesssim |\Omega|^2 R^{-\frac{n-1}{2}(\frac{2}{a}-1)} \|F\bar{u}\|_{L_t^\infty L_x^a}^2 + \sum_{J',I} \left\| \int_{J'_\Omega \cap I} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 \\
&\lesssim |\Omega|^2 R^{-\frac{2(n-1)}{n+3}} \|F\|_{L_t^\infty L_x^{\frac{n+3}{2}}}^2 \|u\|_{L_t^\infty L_x^2}^2 + \sum_{J',I} \left\| \int_{J'_\Omega \cap I} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2
\end{aligned}$$

here we always take I (and I') to be a decomposition of \mathbb{R} into intervals of size R . We now essentially repeat the previous argument, but expand u instead of v to obtain

$$\begin{aligned}
\sum_{J',I} \left\| \int_{J'_\Omega \cap I} \mathcal{U}_2^*(t)[F\bar{u}] dt \right\|_{L_x^2}^2 &\leq \sup_{\sum_{J',I} \|g_{J',I}\|_{L_x^2}^2 \leq 1} \left| \sum_{J',I} \int_{J'_\Omega \cap I} \langle F, \bar{u} \mathcal{U}_2(t) g_{J',I} \rangle_{L_x^2} dt \right|^2 \\
&\lesssim \sup_{\sum_{J',I} \|g_{J',I}\|_{L_x^2}^2 \leq 1} \left| \sum_{J,I} \int_{J_\Omega \cap I} \langle \mathcal{U}_1^*(t)[F\bar{v}_I], f_J \rangle_{L_x^2} dt \right|^2 \\
&\lesssim \sup_{\sum_{J',I} \|g_{J',I}\|_{L_x^2}^2 \leq 1} \sum_J \left\| \sum_I \int_{J_\Omega \cap I} \mathcal{U}_1^*(t)[F\bar{v}_I] dt \right\|_{L_x^2}^2
\end{aligned}$$

where we take $v_I = \sum_{J'} \mathbb{1}_{J'}(t) \mathcal{U}_2(t) g_{J',I}$. Again expanding out the L_x^2 norm, and applying (5.8), we have

$$\begin{aligned}
\sum_J \left\| \sum_I \int_{J_\Omega \cap I} \mathcal{U}_1^*(t)[F\bar{v}_I] dt \right\|_{L_x^2}^2 &= \sum_J \sum_{|I-I'| \gg R} \int_{J_\Omega \cap I} \int_{J_\Omega \cap I'} \langle \mathcal{U}_1^*(t)[F\bar{v}_I], \mathcal{U}_1^*(t')[F\bar{v}_{I'}] \rangle_{L_x^2} dt dt' \\
&\quad + \sum_J \sum_{|I-I'| \lesssim R} \int_{J_\Omega \cap I} \int_{J_\Omega \cap I'} \langle \mathcal{U}_1^*(t)[F\bar{v}_I], \mathcal{U}_1^*(t')[F\bar{v}_{I'}] \rangle_{L_x^2} dt dt' \\
&\lesssim |\Omega|^2 R^{-\frac{2(n-1)}{n+3}} \|F\|_{L_t^\infty L_x^{\frac{n+3}{2}}}^2 \sup_I \|v_I\|_{L_t^\infty L_x^2}^2 + \sum_{J,I} \left\| \int_{J_\Omega \cap I} \mathcal{U}_1(t)[Fv_I] dt \right\|_{L_x^2}^2.
\end{aligned}$$

Collection the above chain of estimates together, and using the fact that

$$\|v_I\|_{L_t^\infty L_x^2}^2 \leq \sum_{I,J'} \|g_{J',I}\|_{L_x^2}^2 \leq 1$$

together with another application of duality, we see that

$$\begin{aligned}
\int_\Omega \|uv\|_{L_x^{\frac{n+3}{n+1}}} dt &\lesssim |\Omega| R^{-\frac{n-1}{n+3}} + \sup_{\substack{\|F\|_{L_t^\infty L_x^{\frac{n+3}{2}}} \leq 1 \\ \sum_{I,J'} \|g_{I,J'}\|_{L_x^2}^2 \leq 1}} \left(\sum_{J,I} \left\| \int_{J_\Omega \cap I} \mathcal{U}_1(t)[F\bar{v}_I] dt \right\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\
&\leq |\Omega| R^{-\frac{n-1}{n+3}} + \sup_{\sum_{I,J'} \|g_{I,J'}\|_{L_x^2}^2, \sum_{I,J} \|f_{I,J}\|_{L_x^2}^2 \leq 1} \sum_I \int_{J_\Omega \cap I} \|u_I v_I\|_{L_x^{\frac{n+3}{n+2}}} dt
\end{aligned}$$

where we define $u_I = \sum_{J'} \mathbb{1}_{J'}(t) \mathcal{U}_1(t)[f_{I,J}]$. Observe that $\sum_I \|u_I\|_{V_{\Phi_1}^2}^2 \leq \sum_{I,J} \|f_{I,J}\|_{L_x^2}^2 \leq 1$, and that u_I satisfies the support properties in Theorem 4.1 (with Λ_j^* replaced by $\Lambda_j^* + \frac{1}{10\mathbf{R}_0}$, and \mathbf{R}_0 replaced by $2\mathbf{R}_0$).

A similar comment applies to v_I . Consequently, an application of (5.1) gives for any $\alpha > 0$

$$\begin{aligned} \sum_I \int_{\Omega \cap I} \|u_I v_I\|_{L_x^{\frac{n+3}{n+1}}} dt &\leq |\Omega|^{\frac{2}{n+3}} \sum_I \|u_I v_I\|_{L_{t,x}^{\frac{n+3}{n+1}}(I \times \mathbb{R}^n)} \\ &\lesssim |\Omega|^{\frac{2}{n+3}} R^\alpha \left(\sum_{I,J} \|f_{I,J}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \left(\sum_{I,J'} \|g_{I,J'}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \leq |\Omega|^{\frac{2}{n+3}} R^\alpha \end{aligned}$$

and therefore

$$\int_{\Omega} \|uv\|_{L_x^{\frac{n+3}{n+1}}} dt \lesssim |\Omega| R^{-\frac{n-1}{n+3}} + |\Omega|^{\frac{2}{n+3}} R^\alpha.$$

To complete the proof, we choose $R = |\Omega|^C$ with $C > 0$ sufficiently large so that $|\Omega| R^{-\frac{n-1}{n+3}} \leq |\Omega|^{\frac{1}{q}}$. On the other hand, since $q > \frac{n+3}{n+1}$, we can take $\alpha = \frac{1}{2C}(\frac{n+1}{n+3} - \frac{1}{q})$ which implies that $|\Omega|^{\frac{2}{n+3}} R^\alpha = |\Omega|^{\frac{2}{n+3} + \alpha C} \leq |\Omega|^{\frac{1}{q}}$. Therefore we obtain (5.7) as required.

Step 3: From U_{Φ}^2 to V_{Φ}^2 . Let $p > \frac{n+3}{n+1}$, $u \in V_{\Phi_1}^2$, $v \in V_{\Phi_2}^2$, and $\text{supp } \widehat{u} \subset \Lambda_1^*$ and $\text{supp } \widehat{v} \subset \Lambda_2^*$. An application of [27, Lemma 6.4], see also [21, Proposition 2.5 and Proposition 2.20], gives a decomposition $u = \sum_{k \in \mathbb{N}} u_k$ and $v = \sum_{k \in \mathbb{N}} v_k$ such that u_k, v_k retain the correct Fourier support properties (we can just use sharp Fourier cutoffs here) and for any $r \geq 2$ we have the bounds

$$\|u_k\|_{U_{\Phi_1}^r} \lesssim 2^{k(\frac{2}{r}-1)} \|u\|_{V_{\Phi_1}^2}, \quad \|v_k\|_{U_{\Phi_2}^r} \lesssim 2^{k(\frac{2}{r}-1)} \|v\|_{V_{\Phi_2}^2}.$$

Let $\frac{n+3}{n+1} < q < p$, and take $\theta = \frac{q}{p} < 1$. Then an application of (5.5) (with $p = q$) together with the convexity of L^p norms, gives

$$\begin{aligned} \|uv\|_{L_{t,x}^p} &\leq \sum_{k,k'} \|u_k v_{k'}\|_{L_{t,x}^p} \leq \sum_{k,k'} \|u_k v_{k'}\|_{L_{t,x}^q}^\theta \|u_k v_{k'}\|_{L_{t,x}^\infty}^{1-\theta} \\ &\leq \sum_{k',k} \left(\|u_k\|_{U_{\Phi_1}^2} \|v_{k'}\|_{U_{\Phi_2}^2} \right)^\theta \left(\|u_k\|_{U_{\Phi_1}^\infty} \|v_{k'}\|_{U_{\Phi_2}^\infty} \right)^{1-\theta} \\ &\lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2} \sum_{k,k'} 2^{-k(1-\theta)} 2^{-k'(1-\theta)} \\ &\lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2} \end{aligned}$$

where we used Sobolev embedding and the fact that the Fourier support of u, v is contained in the unit ball to control the $L_{t,x}^\infty$ norm. Thus Theorem 1.1 follows. \square

Remark 5.1. The argument in Step 3 above, using (5.6), also implies the slightly stronger estimate

$$\|uv\|_{L_t^p L_x^{\frac{n+3}{n+1}}(\mathbb{R}^{1+n})} \leq C \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2},$$

This is well known in the case of homogeneous solutions, see e.g. [43]. However, the estimate in the endpoint $p = q = \frac{n+3}{n+1}$ remains open. For homogeneous solutions it is known only in the case of the cone [42].

Remark 5.2. In fact, since Tao's endpoint result [42, Theorem 1.1] holds for Hilbert space valued waves, we observe that one can deduce the U^2 -estimate for the cone directly. This follows by noting that, given U^2 -atoms $u = \sum_{I \in \mathcal{I}} \mathbb{1}_I u_I$ and $v = \sum_{J \in \mathcal{J}} \mathbb{1}_J v_J$, we have

$$|uv| \leq \left(\sum_{I \in \mathcal{I}} |u_I|^2 \right)^{\frac{1}{2}} \left(\sum_{J \in \mathcal{J}} |v_J|^2 \right)^{\frac{1}{2}} = |U| |V|$$

with ℓ^2 -valued waves U and V .

6. MIXED NORMS AND GENERALISATIONS TO SMALL SCALES

In this section we give some consequences of the bilinear estimate in Theorem 1.1. Namely, we state an extension to mixed $L_t^q L_x^r$ spaces, and, in the case of the hyperboloid, we give a small scale version of Theorem 1.1. The small scale estimate will play a key role in our application to the Dirac-Klein-Gordon system.

6.1. Mixed Norms. Let Φ_1 and Φ_2 be phases satisfying Assumption 1. A standard TT^* argument (see for instance [22]), together Lemma 2.4 implies that, provided $\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}$ and $q > 2$ we have the Strichartz type bound

$$\|e^{it\Phi_j(-i\nabla)}f\|_{L_t^q L_x^r(\mathbb{R}^{1+n})} \lesssim \|f\|_{L_x^2}. \quad (6.1)$$

As in Step 3 of the proof of the globalisation lemma, by decomposing V^2 into U^a atoms (see [27, Lemma 6.4] or [21, Proposition 2.5 and Proposition 2.20]) we see that, for any $\frac{1}{a} + \frac{n-1}{2b} \leq \frac{n-1}{2}$,

$$\|uv\|_{L_t^a L_x^b} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

Interpolating with Theorem 1.1 then gives the following mixed norm version.

Corollary 6.1. *Let $n \geq 2$ and assume that $a > 1$, $\frac{1}{a} + \frac{n+1}{2b} < \frac{n+1}{2}$, and*

$$\frac{1}{a} + \frac{n-1}{4b} < \begin{cases} \frac{n+1}{4} & n \geq 3 \\ \frac{1}{2} + \frac{5}{12b} & n = 2 \end{cases}. \quad (6.2)$$

Let Φ_1, Φ_2 , and u, v be as in the statement of Theorem 1.1. Then

$$\|uv\|_{L_t^a L_x^b} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

Remark 6.2. Let $p > \frac{n+3}{n+1}$. It is possible to deduce a weaker version of Theorem 1.1 and Corollary 6.1 directly from the homogeneous estimate

$$\|e^{it\Phi_1(-i\nabla)}f e^{it\Phi_2(-i\nabla)}g\|_{L_{t,x}^p(\mathbb{R}^{1+n})} \lesssim \|f\|_{L_x^2} \|g\|_{L_x^2} \quad (6.3)$$

where the phases satisfy the conditions in Assumption 1, and $f, g \in L^2$ have the required support conditions. We sketch the argument as follows. By interpolating (6.3) with the trivial $L_t^\infty L_x^2$ bound, we deduce that for every $a > 2$ we have

$$\|e^{it\Phi_1(-i\nabla)}f e^{it\Phi_2(-i\nabla)}g\|_{L_t^a L_x^{\frac{n+1}{n}}} \lesssim \|f\|_{L_x^2} \|g\|_{L_x^2}$$

By decomposing V^2 functions into U^a atoms [27, 21, 28] and using the convexity of the L^p spaces, we see that for $a > 2$

$$\|uv\|_{L_t^a L_x^{\frac{n+1}{n}}} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}.$$

Consequently, as in the proof of Corollary 6.1, by interpolating with the standard Strichartz estimates, we obtain

$$\|uv\|_{L_t^a L_x^b} \lesssim \|u\|_{V_{\Phi_1}^2} \|v\|_{V_{\Phi_2}^2}$$

provided that $a > 1$, $\frac{1}{a} + \frac{n+1}{2b} < \frac{n+1}{2}$, and

$$\frac{1}{a} < \begin{cases} \frac{n-1}{n+3} \left(\frac{n}{2} - \frac{n+1}{2b} \right) + \frac{1}{2} & n \geq 3 \\ \frac{1}{2} & n = 2. \end{cases} \quad (6.4)$$

In particular, the homogeneous bounds contained in the work of Lee-Vargas [33] and Bejenaru [2], implies a weaker version of our main result, with (6.2) in Corollary 6.1 replaced with (6.4). Note that condition (6.4) is much more restrictive than (6.2). This is most apparent in the low dimensional cases, for instance if $n = 2$ then Corollary 6.4 allows $a < 2$ while (6.4) only allows the somewhat trivial (from a V^2 perspective) $a > 2$. To summarise, our main result, Theorem 1.1 not only clarifies the dependence of the constant on the global properties of the phases Φ_1 and Φ_2 , but also presents a significant strengthening of the allowed exponents for the V^2 estimate.

We observe that the above argument, namely deducing a V^2 bound directly from the homogeneous estimate, has been used in [39, Lemma 5.7 and its proof] in the case of the cone.

Remark 6.3. In the special case of the hyperboloid, $\Phi_j = \langle \xi \rangle_{m_j}$, or the paraboloid, $\Phi_j = |\xi|^2$, the Strichartz bound (6.1) holds in the larger region $\frac{1}{q} + \frac{n}{2r} \leq \frac{n}{4}$. This can be used to improve the range of exponents in Corollary 6.1, in particular (6.2) can be replaced with

$$\frac{1}{a} + \frac{n}{3b} < \frac{n+1}{3}.$$

However, it is important to note that, in the case of the hyperboloid, some care has to be taken as the constant will now depend on the masses m_j .

6.2. Small Scale Bilinear Restriction Estimates. In the case of hyperboloids we now generalise Theorem 1.1, similarly to [32] in the case of the cone. Given $0 < \alpha \lesssim 1$, we define \mathcal{C}_α to be a collection of finitely overlapping caps of radius α on the sphere \mathbb{S}^{n-1} . If $\kappa \in \mathcal{C}_\alpha$, we define $\omega(\kappa)$ to be the centre of the cap κ .

We consider the case $\Phi_j(\xi) = -\pm_j \langle \xi \rangle$ and define the corresponding $V_{\pm, m}^2$ space as $V_{\pm, m}^2 = V_{\Phi_j}^2$, thus

$$\|u\|_{V_{\pm, m}^2} = \|e^{\pm it \langle \nabla \rangle m} u(t)\|_{V^2}. \quad (6.5)$$

We define the corresponding $U_{\pm, m}^2$ space similarly. Rescaling Theorem 1.1 then gives the following optimal result.

Corollary 6.4. *Let $p > \frac{n+3}{n+1}$, $0 \leq m_1, m_2 \leq 1$.*

(i) *For any $\lambda \gtrsim m_1 + m_2$, $\frac{m_1+m_2}{\lambda} \lesssim \alpha \lesssim 1$, $\kappa, \kappa' \in \mathcal{C}_\alpha$ with $\theta(\pm_1 \kappa, \pm_2 \kappa') \approx \alpha$, and*

$$\text{supp } \widehat{u} \subset \{|\xi| \approx \lambda, \frac{\xi}{|\xi|} \in \kappa\}, \quad \text{supp } \widehat{v} \subset \{|\xi| \approx \lambda, \frac{\xi}{|\xi|} \in \kappa'\},$$

we have the bilinear estimate

$$\|uv\|_{L_{t,x}^p} \lesssim \alpha^{n-1-\frac{n+1}{p}} \lambda^{n-\frac{n+1}{p}} \|u\|_{V_{\pm_1, m_1}^2} \|v\|_{V_{\pm_2, m_2}^2}.$$

(ii) *For any $\lambda \gtrsim m_1 + m_2$, $0 < \alpha \ll \frac{m_1+m_2}{\lambda}$, $\kappa, \kappa' \in \mathcal{C}_\alpha$, $c_1 \approx c_2 \approx \lambda$ with*

$$\theta(\pm_1 \kappa, \pm_2 \kappa') \lesssim \alpha, \quad |m_1 c_1 - m_2 c_2| \approx \alpha \lambda^2,$$

and

$$\text{supp } \widehat{u} \subset \{|\xi \cdot \omega(\kappa)| - c_1| \ll \alpha \lambda^2, \frac{\xi}{|\xi|} \in \kappa\}, \quad \text{supp } \widehat{v} \subset \{|\xi \cdot \omega(\kappa')| - c_2| \ll \alpha \lambda^2, \frac{\xi}{|\xi|} \in \kappa'\},$$

we have the bilinear estimate

$$\|uv\|_{L_{t,x}^p} \lesssim \alpha^{n-\frac{n+2}{p}} \lambda^{n+1-\frac{n+2}{p}} \|u\|_{V_{\pm_1, m_1}^2} \|v\|_{V_{\pm_2, m_2}^2}.$$

Proof. Fix $\pm_1 = +$ and $\pm_2 = \pm$, the remaining cases follow from a reflection. We start with (i). If $\alpha \approx 1$, then estimate follows from rescaling in x together with an application of Theorem 1.1. Thus we may assume that $0 < \alpha \ll 1$, and after a rotation, that κ is centred at e_1 and κ' is centred at $\pm(1 - \alpha^2)^{\frac{1}{2}} e_1 + \alpha e_2$. Similarly to [32], we define the rescaled functions

$$u_{\lambda, \alpha}(t, x) = u\left(\frac{t}{\alpha^2 \lambda}, \frac{x_1}{\lambda} + \frac{t}{\alpha^2 \lambda}, \frac{x'}{\alpha \lambda}\right), \quad v_{\lambda, \alpha}(t, x) = v\left(\frac{t}{\alpha^2 \lambda}, \frac{x_1}{\lambda} + \frac{t}{\alpha^2 \lambda}, \frac{x'}{\alpha \lambda}\right)$$

(where we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$) and the phases

$$\Phi_1(\xi) = \frac{-1}{\alpha^2 \lambda} \left((m_1^2 + \lambda^2 \xi_1^2 + \alpha^2 \lambda^2 |\xi'|^2)^{\frac{1}{2}} - \lambda \xi_1 \right), \quad \Phi_2(\xi) = \frac{\mp 1}{\alpha^2 \lambda} \left((m_2^2 + \lambda^2 \xi_1^2 + \alpha^2 \lambda^2 |\xi'|^2)^{\frac{1}{2}} \mp \lambda \xi_1 \right)$$

with associated sets $\Lambda_1 = \{\xi_1 \approx 1, |\xi'| \ll 1\}$ and $\Lambda_2 = \{\xi_1 \approx \pm 1, \xi_2 \approx 1, |\xi''| \ll 1\}$ (where we write $\xi = (\xi_1, \xi_2, \xi'') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$). A computation gives $\text{supp } \widehat{u}_{\alpha, \lambda} \subset \Lambda_1$ and

$$\left[e^{-it \Phi_1(-i \nabla)} u_{\alpha, \lambda}(t) \right](x) = \left[e^{i \frac{t}{\alpha^2 \lambda} \langle \nabla \rangle_{m_1}} u\left(\frac{t}{\alpha^2 \lambda}\right) \right]\left(\frac{x_1}{\lambda}, \frac{x'}{\alpha \lambda}\right).$$

Similarly we can check that $\text{supp } \widehat{v}_{\alpha, \lambda} \subset \Lambda_2$ and

$$\left[e^{-it \Phi_2(-i \nabla)} v_{\alpha, \lambda}(t) \right](x) = \left[e^{\pm i \frac{t}{\alpha^2 \lambda} \langle \nabla \rangle_{m_2}} v\left(\frac{t}{\alpha^2 \lambda}\right) \right]\left(\frac{x_1}{\lambda}, \frac{x'}{\alpha \lambda}\right).$$

Therefore, after rescaling together with an application of Theorem 1.1, it is enough to check that the phases Φ_j satisfy Assumption 1 on the sets Λ_j . To this end, we start by noting that we can write

$$\nabla \Phi_1(\xi) = \frac{1}{(\lambda^{-2} m_1^2 + \xi_1^2 + \alpha^2 |\xi'|^2)^{\frac{1}{2}}} \left(\frac{-(\frac{m_1}{\alpha \lambda})^2 - |\xi'|^2}{(\lambda^{-2} m_1^2 + \xi_1^2 + \alpha^2 |\xi'|^2)^{\frac{1}{2}} + \xi_1}, \xi' \right)$$

which shows that (ii) in Assumption 1 holds with \mathbf{D}_2 depending only on N and n . A similar argument shows that Φ_2 satisfies (ii) in Assumption 1. On the other hand, to check condition (i) in Assumption 1, we invoke Lemma 2.1. First, we observe that for any $\xi \in \Lambda_1$, $\eta \in \Lambda_2$, we have

$$\begin{aligned} |\nabla\Phi_1(\xi) - \nabla\Phi_2(\eta)| &\geq |\partial_2\Phi_1(\xi) - \partial_2\Phi_2(\eta)| \\ &= \left| \frac{\xi_2}{(\lambda^{-2}m_1^2 + \xi_1^2 + \alpha^2|\xi'|^2)^{\frac{1}{2}}} \mp \frac{\eta_2}{(\lambda^{-2}m_2^2 + \eta_1^2 + \alpha^2|\eta'|^2)^{\frac{1}{2}}} \right| \gtrsim 1 \end{aligned}$$

and hence we can take $\mathbf{A}_1 \approx 1$. It remains to check (2.2) in Lemma 2.1. We make use of the following elementary inequality; if $(h^*, a^*) \in \mathbb{R}^{n+1} \times \mathbb{R}^1$ and $x, y \in \{z \in \mathbb{R}^{n+1} \mid |z| = |z - h^*| + a^*\}$, then

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \geq \frac{1}{4|x||y|} \left(\frac{|x \wedge y|^2}{|x||y|} + \frac{|(x - h^*) \wedge (y - h^*)|^2}{|x - h^*||y - h^*|} \right). \quad (6.6)$$

To prove (6.6), we start by observing that since $x, y \in \{|z| = |z - h^*| + a^*\}$, we have

$$\begin{aligned} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 &= \frac{1}{|x||y|} (|x - y|^2 - ||x| - |y||^2) \\ &= \frac{1}{|x||y|} (|(x - h^*) - (y - h^*)|^2 - ||x - h^*| - |y - h^*||^2) \\ &= \frac{|x - h^*||y - h^*|}{|x||y|} \left| \frac{x - h^*}{|x - h^*|} - \frac{y - h^*}{|y - h^*|} \right|^2. \end{aligned}$$

The inequality (6.6) now follows from the identity $|\omega - \omega^*|^2 \geq \frac{|\omega \wedge \omega^*|^2}{2}$ for $\omega, \omega^* \in \mathbb{S}^{n+1}$. We now return to checking (2.2) in Lemma 2.1, we only check the case $j = 1$ as the remaining case is identical. Let $\xi, \eta \in \Sigma_1(a, h)$ for some $(a, h) \in \mathbb{R}^{1+n}$ such that $\xi - h, \eta - h \in \Lambda_2$. A computation gives

$$\begin{aligned} &|(\nabla\Phi_j(\xi) - \nabla\Phi_j(\eta)) \cdot (\xi - \eta)| \\ &= \alpha^{-2} \left| \left(\frac{(\xi_1, \alpha^2\xi')}{|(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')|} - \frac{(\eta_1, \alpha^2\eta')}{|(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')|} \right) \cdot (\xi - \eta) \right| \\ &= \alpha^{-2} \frac{|(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')| + |(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')|}{2} \left| \frac{(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')}{|(\lambda^{-1}m_1, \xi_1, \alpha^2\xi')|} - \frac{(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')}{|(\lambda^{-1}m_1, \eta_1, \alpha^2\eta')|} \right|^2 \\ &\approx \alpha^{-2} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \end{aligned} \quad (6.7)$$

where we take $x = (\lambda^{-1}m_1, \xi_1, \alpha^2\xi')$ and $y = (\lambda^{-1}m_1, \eta_1, \alpha^2\eta')$. Note that the condition $\xi \in \Sigma_1(a, h)$ becomes $|x| = |x - h^*| + a^*$ with $h^* = (\lambda^{-1}m_2 - \lambda^{-1}m_1, h_1, \alpha h')$ and $a^* = \alpha^2 a$. In particular, since $|x| \approx |y| \approx |x - h^*| \approx |y - h^*| \approx 1$, an application of (6.6) gives

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \gtrsim |x \wedge y|^2 + |(x - h^*) \wedge (y - h^*)|^2. \quad (6.8)$$

The required bound (2.2) with $\mathbf{A}_2 \approx 1$ now follows in the region $|\xi_1 - \eta_1| \lesssim |\xi' - \eta'|$ by noting that

$$|x \wedge y| \geq \alpha|\xi_1\eta' - \eta_1\xi'| \geq \alpha(|\xi' - \eta'| |\xi_1| - |\xi'| |\xi_1 - \eta_1|) \approx \alpha|\xi' - \eta'| \approx \alpha|\xi - \eta|$$

and applying the inequalities (6.7) and (6.8). On the other hand, if $|\xi_1 - \eta_1| \gg |\xi' - \eta'|$, then as $\xi - h, \eta - h \in \Lambda_2$, we have

$$\begin{aligned} |(x - h^*) \wedge (y - h^*)| &\geq \alpha(|\xi_1 - h_1|(\eta_2 - h_2) - (\eta_1 - h_1)(\xi_2 - h_2)|) \\ &\geq \alpha(|\xi_1 - \eta_1||\eta_2 - h_2| - |\xi_2 - \eta_2||\eta_1 - h_1|) \approx \alpha|\xi_1 - \eta_1| \approx \alpha|\xi - \eta| \end{aligned}$$

which again gives (2.2) with $\mathbf{A}_2 \approx 1$. Thus the phases Φ_j satisfy Assumption 1 with $\mathbf{D}_1 \approx \mathbf{D}_2 \approx 1$ and therefore Part (i) follows.

We now turn to the proof of Part (ii). The argument is similar to (i), but we need a further rescaling to exploit the radial separation condition. As before, after rotating, we may assume that $\omega(\kappa_1) = e_1$. Define the rescaled functions

$$u_{\lambda,\alpha}^\#(t, x) = u\left(\frac{t}{\alpha^2\lambda}, \frac{x_1}{\alpha\lambda^2} + \frac{tc_1}{\alpha^2\lambda\langle c_1 \rangle_{m_1}}, \frac{x'}{\alpha\lambda}\right) \quad v_{\lambda,\alpha}^\#(t, x) = v\left(\frac{t}{\alpha^2\lambda}, \frac{x_1}{\alpha\lambda^2} + \frac{tc_1}{\alpha^2\lambda\langle c_1 \rangle_{m_1}}, \frac{x'}{\alpha\lambda}\right)$$

(where, as previously, we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$) and the phases

$$\Phi_1(\xi) = \frac{-1}{\alpha^2\lambda} \left((m_1^2 + (\alpha\lambda^2\xi_1)^2 + \alpha^2\lambda^2|\xi'|^2)^{\frac{1}{2}} - \frac{\alpha\lambda^2c_1}{\langle c_1 \rangle_{m_1}} \xi_1 \right)$$

and

$$\Phi_2(\xi) = \frac{\mp 1}{\alpha^2\lambda} \left((m_2^2 + (\alpha\lambda^2\xi_1)^2 + \alpha^2\lambda^2|\xi'|^2)^{\frac{1}{2}} \mp \frac{\alpha\lambda^2c_1}{\langle c_1 \rangle_{m_1}} \xi_1 \right)$$

with associated sets $\Lambda_1 = \{|\xi_1 - \frac{1}{\alpha\lambda^2}c_1| \ll 1, |\xi'| \ll 1\}$ and $\Lambda_2 = \{|\xi_1 \mp \frac{1}{\alpha\lambda^2}c_2| \ll 1, |\xi'| \lesssim 1\}$. As previously, a computation shows that $\text{supp } \widehat{u}_{\alpha,\lambda}^\# \subset \Lambda_1$, $\text{supp } \widehat{v}_{\alpha,\lambda}^\# \subset \Lambda_2$ and we have the identities

$$\left[e^{-it\Phi_1(-i\nabla)} u_{\alpha,\lambda}^\#(t) \right](x) = \left[e^{it\langle \nabla \rangle_{m_1}} u\left(\frac{t}{\alpha^2\lambda}\right) \right]\left(\frac{x_1}{\alpha\lambda^2}, \frac{x'}{\alpha\lambda}\right)$$

and

$$\left[e^{-it\Phi_2(-i\nabla)} v_{\alpha,\lambda}^\#(t) \right](x) = \left[e^{\pm it\langle \nabla \rangle_{m_2}} v\left(\frac{t}{\alpha^2\lambda}\right) \right]\left(\frac{x_1}{\alpha\lambda^2}, \frac{x'}{\alpha\lambda}\right).$$

Thus, as in the proof of (i), after rescaling and an application of Theorem 1.1, it is enough to check that the phases Φ_j satisfy Assumption 1 on the sets Λ_j . To this end, note that we can write

$$\partial_1 \Phi_1 = \frac{\frac{m_1^2}{\alpha\lambda^3} ((\alpha\lambda^2\xi_1)^2 - c_1^2) - \left(\frac{c_1}{\lambda}\right)^2 \alpha\lambda |\xi'|^2}{f(\alpha\lambda\xi_1, \alpha\xi')}$$

for some smooth function f with $f \approx 1$ on Λ_1 . Since $\partial_{\xi_1}^M [(\alpha\lambda^2\xi_1)^2 - c_1^2] \lesssim \alpha\lambda^3$ for all $M \geq 0$ and $\xi_1 \in \Lambda$, we see that Φ_1 satisfies (ii) in Assumption 1 with constant depending only on n and N . A similar argument, using the fact that $\frac{\lambda}{\alpha} \left| \frac{c_1}{\langle c_1 \rangle_{m_1}} - \frac{c_2}{\langle c_2 \rangle_{m_2}} \right| \approx 1$, shows that Φ_2 also satisfies (ii) in Assumption 1. On the other hand, to check (i) in Assumption 1, we use Lemma 2.1. Concerning the transversality condition (2.1), we observe that for $\xi \in \Lambda_1$, $\eta \in \Lambda_2$, we have $|\xi_1| \approx |\eta_1| \approx \frac{1}{\alpha\lambda}$ and

$$|\xi_1^2 m_2^2 - \eta_1^2 m_1^2| \approx \frac{m_1 + m_2}{\alpha\lambda}, \quad \alpha^2 |\xi_1^2 |\eta'|^2 - \eta_1^2 |\xi'|^2| \lesssim \lambda^{-2} \ll \lambda^{-2} \frac{m_1 + m_2}{\alpha\lambda}$$

Therefore

$$\begin{aligned} |\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)| &= \left| \frac{(\lambda^2 \xi_1, \xi')}{(\lambda^{-2} m_1^2 + \alpha^2 \lambda^2 \xi_1^2 + \alpha^2 |\xi'|^2)^{\frac{1}{2}}} \mp \frac{(\lambda^2 \eta_1, \eta')}{(\lambda^{-2} m_2^2 + \alpha^2 \lambda^2 \eta_1^2 + \alpha^2 |\eta'|^2)^{\frac{1}{2}}} \right| \\ &\gtrsim \lambda^3 \alpha |\xi_1^2 (\lambda^{-2} m_2^2 + \alpha^2 \lambda^2 \eta_1^2 + \alpha^2 |\eta'|^2) - \eta_1^2 (\lambda^{-2} m_1^2 + \alpha^2 \lambda^2 \xi_1^2 + \alpha^2 |\xi'|^2)| \\ &\approx m_1 + m_2 \gtrsim 1 \end{aligned}$$

so that (2.1) holds with $\mathbf{A}_1 \approx 1$. We now check the curvature condition (2.2) for $j = 1$. Let $\xi, \eta \in \Sigma_1(a, h)$. Repeating the computation (6.7) we deduce that

$$|(\nabla \Phi_1(\xi) - \nabla \Phi_1(\eta)) \cdot (\xi - \eta)| \approx \alpha^{-2} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^2 \gtrsim \alpha^{-2} (|x \wedge y|^2 + |(x - h^*) \wedge (y - h^*)|^2)$$

where $x = (\lambda^{-1}m_1, \alpha\lambda\xi_1, \alpha\xi')$, $y = (\lambda^{-1}m_1, \alpha\lambda\eta_1, \alpha\eta')$, $h^* = (\lambda^{-1}m_2 - \lambda^{-1}m_1, \alpha\lambda h_1, \alpha h')$, and we used the fact that $x, y, x - h^*, y - h^*$ all have length 1. It thus remains to show that

$$|x \wedge y| + |(x - h^*) \wedge (y - h^*)| \gtrsim \alpha |\xi - \xi'|$$

since then (2.2) holds with $\mathbf{A}_2 \approx 1$. If $|\xi_1 - \eta_1| \lesssim |\xi' - \eta'|$ we simply observe as previously that

$$|x \wedge y| \geq \alpha |\alpha\lambda\xi_1\eta' - \alpha\lambda\eta_1\xi'| \geq \alpha (|\xi' - \eta'| \alpha\lambda |\xi_1| - |\xi'| \alpha\lambda |\xi_1 - \eta_1|) \approx \alpha |\xi' - \eta'| \approx \alpha |\xi - \eta|$$

On the other hand, if $|\xi_1 - \eta_1| \gtrsim |\xi' - \eta'|$, then as $\xi - h, \eta - h \in \Lambda_2$, we have

$$|x \wedge y| + |(x - h^*) \wedge (y - h^*)| \geq \alpha m_1 |\xi_1 - \eta_1| + \alpha m_2 |(\xi_1 - h_1) - (\eta_1 - h_2)| \gtrsim \alpha |\xi - \eta|.$$

An identical argument shows that Φ_2 also satisfies the curvature condition. Thus the phases Φ_j satisfy Assumption 1 with $\mathbf{D}_1 \approx \mathbf{D}_2 \approx 1$ and therefore Part (ii) follows. \square

The α and λ dependence in Corollary 6.4 is sharp. At least for (ii), this can be seen with the following example. Let

$$\Omega_j = \{|\xi_1 - c_j| \ll \alpha\lambda^2, |\xi'| \ll \alpha\lambda\}$$

with $|c_1 - c_2| \lesssim \alpha\lambda^2$, $c_1 \approx c_2 \approx \lambda$, and $\alpha \ll \lambda^{-1}$. Define $\widehat{f}(\xi) = \mathbb{1}_{\Omega_1}(\xi)$, $\widehat{g}(\xi) = \mathbb{1}_{\Omega_2}(\xi)$ and

$$u = e^{it\langle \nabla \rangle} f, \quad v = e^{it\langle \nabla \rangle} g.$$

Then

$$\|u\|_{V_{\langle \nabla \rangle}^2} = \|f\|_{L_x^2} = |\Omega_1|^{\frac{1}{2}}$$

and similarly $\|v\|_{V_{\langle \nabla \rangle}^2} = |\Omega_2|^{\frac{1}{2}}$. On the other hand we have

$$(uv)(t, x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{u}(t, \xi) \widehat{v}(t, \eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta = \int_{\Omega_1} \int_{\Omega_2} e^{it(\langle \xi \rangle + \langle \eta \rangle)} e^{ix \cdot (\xi + \eta)} d\xi d\eta.$$

The idea is to try and find a set $A \subset \mathbb{R}^{1+n}$ such that the phase is essentially constant for $(t, x) \in A$. We start by noting that for $\xi \in \Omega_1$ we have

$$\langle \xi \rangle - \frac{1 + c_1 \xi_1}{\langle c_1 \rangle} \approx \lambda^{-3} |(1 + |\xi|^2)(1 + c_1^2) - (1 + c_1 \xi_1)^2| = \lambda^{-3} |(\xi_1 - c_1)^2 + (1 + c_1^2)|\xi'|^2| \approx \alpha^2 \lambda$$

and hence

$$\left| \langle \xi \rangle - \langle c_1 \rangle^{-1} - \frac{c_1}{\langle c_1 \rangle} \xi_1 \right| \lesssim \alpha^2 \lambda.$$

Similarly, since

$$\left| \frac{c_1}{\langle c_1 \rangle} - \frac{c_2}{\langle c_2 \rangle} \right| \approx \lambda^{-2} |c_1 \langle c_2 \rangle - c_2 \langle c_1 \rangle| \approx \lambda^{-3} |c_1 - c_2| \approx \frac{\alpha}{\lambda}$$

we deduce that for $\eta \in \Omega_2$

$$\left| \langle \eta \rangle - \langle c_2 \rangle^{-1} - \left(\frac{c_2}{\langle c_2 \rangle} - \frac{c_1}{\langle c_1 \rangle} \right) c_2 - \frac{c_1}{\langle c_1 \rangle} \eta_1 \right| \leq \left| \langle \eta \rangle - \langle c_2 \rangle^{-1} - \frac{c_2}{\langle c_2 \rangle} \eta_1 \right| + \left| \frac{c_1}{\langle c_1 \rangle} - \frac{c_2}{\langle c_2 \rangle} \right| |\eta_1 - c_2| \lesssim \alpha^2 \lambda.$$

In particular, for $|t| \ll (\alpha^2 \lambda)^{-1}$, $|x_1 + \frac{c_1}{\langle c_1 \rangle} t| \ll (\alpha \lambda^2)^{-1}$, and $|x'| \ll (\alpha \lambda)^{-1}$, the phase is essentially constant and hence

$$\begin{aligned} & |(uv)(t, x)| \\ &= \left| \int_{\Omega_1} \int_{\Omega_2} e^{it(\langle \xi \rangle - \langle c_1 \rangle^{-1} - \frac{c_1}{\langle c_1 \rangle} \xi_1)} e^{it(\langle \eta \rangle - \langle c_2 \rangle^{-1} - (\frac{c_2}{\langle c_2 \rangle} - \frac{c_1}{\langle c_1 \rangle})c_2 - \frac{c_1}{\langle c_1 \rangle} \eta_1)} e^{i(x_1 + t \frac{c_1}{\langle c_1 \rangle})(\xi_1 + \eta_1 - c_1 - c_2) + x' \cdot (\xi' + \eta')} d\xi d\eta \right| \\ &\gtrsim |\Omega_1| |\Omega_2| \end{aligned}$$

which then implies that

$$\|uv\|_{L_{t,x}^p} \gtrsim (\alpha^{n+2} \lambda^{n+2})^{-\frac{1}{p}} \times |\Omega_1| |\Omega_2|.$$

Therefore, if the estimate

$$\|uv\|_{L_{t,x}^p} \leq C(\alpha, \lambda) \|u\|_{V_{\langle \nabla \rangle}^2} \|v\|_{V_{\langle \nabla \rangle}^2}$$

holds, then we must have

$$(\alpha \lambda)^{-\frac{n+2}{p}} |\Omega_1| |\Omega_2| \lesssim C |\Omega_1|^{\frac{1}{2}} |\Omega_2|^{\frac{1}{2}}.$$

Since $|\Omega_1| \approx |\Omega_2| \approx \alpha^n \lambda^{n+1}$, after rearranging, this becomes $C \gtrsim \alpha^{n - \frac{n+2}{p}} \lambda^{n+1 - \frac{n+2}{p}}$, which matches the bound obtained in Corollary 6.4.

7. THE DIRAC-KLEIN-GORDON SYSTEM

In this section we set up notation and reduce the DKG system to the first order system (7.3). We then give the proof of Theorem 1.2, up to the crucial nonlinear estimates, which are postponed to Section 8. In the remainder of this article, as we now only consider the DKG system, the dimension is fixed $n = 3$.

7.1. Notation and Setup. Fix a smooth function $\rho \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \rho \subset \{\frac{1}{2} < t < 2\}$ and

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \rho\left(\frac{t}{\lambda}\right) = 1,$$

and let $\rho_1 = \sum_{\lambda \leq 1} \rho\left(\frac{t}{\lambda}\right)$ with $\rho_1(0) = 1$. Similarly, we let Q_μ be a finitely overlapping collection of cubes of diameter $\frac{\mu}{1000}$ covering \mathbb{R}^3 , and fix $(\rho_q)_{q \in Q_\mu}$ to be a corresponding subordinate partition of unity. We now define the standard dyadic Fourier cutoffs, for $\lambda \in 2^{\mathbb{N}}$, $\lambda > 1$, $q \in Q$, $d \in 2^{\mathbb{Z}}$

$$P_\lambda = \rho\left(\frac{|\cdot - i\nabla|}{\lambda}\right), \quad P_1 = \rho_1(|\cdot - i\nabla|), \quad P_q = \rho_q(|\cdot - i\nabla|), \quad C_d^{\pm, m} = \rho\left(\frac{-i\partial_t \pm \langle \cdot - i\nabla \rangle_m}{d}\right).$$

We also let $C_{\leq d}^{\pm, m} = \sum_{d' \leq d} C_{d'}^{\pm, m}$, any related multipliers such as $C_{\geq d}^{\pm, m}$ are defined analogously. To simplify notation somewhat, we make the convention that

$$C_d = C_d^{+, 1}, \quad C_d^\pm = \Pi_\pm C_d^{\pm, M}$$

where M will denote the mass of the spinor in (1.3) and Π_\pm as defined below. Given $\alpha \leq 1$, we let $(\rho_\kappa)_{\kappa \in \mathcal{C}_\alpha}$ be a smooth partition of unity subordinate to the conic sectors $\{\xi \neq 0, \frac{\xi}{|\xi|} \in \kappa\}$, and define the angular Fourier localisation multipliers as

$$R_\kappa = \rho_\kappa(-i\nabla).$$

We use the well-known fact that for any $1 \leq p, q \leq \infty$ the modulation cutoff multipliers are uniformly disposable in $L_t^q L_x^r$ for certain scales, namely we have the bounds

$$\|C_d^{\pm, m} P_\lambda R_\kappa u\|_{L_t^q L_x^r} + \|C_{\leq d}^{\pm, m} P_\lambda R_\kappa u\|_{L_t^q L_x^r} \lesssim \|P_\lambda R_\kappa u\|_{L_t^q L_x^r}, \quad (7.1)$$

provided that $\kappa \in \mathcal{C}_\alpha$ and $d \gtrsim \alpha^2 \lambda$. Similarly, by writing $C_d^{\pm, m} = e^{\mp it \langle \nabla \rangle_m} \rho\left(\frac{-i\partial_t}{d}\right) e^{\pm it \langle \nabla \rangle_m}$, and using the fact that convolution with $L_t^1(\mathbb{R})$ functions is bounded on V^2 , we deduce that for every $d \in 2^{\mathbb{Z}}$

$$\|C_{\leq d}^{\pm, m} u\|_{V_{\pm, m}^2} \lesssim \|u\|_{V_{\pm, m}^2}. \quad (7.2)$$

To deal with solutions to the Dirac equation, we follow the by now standard approach used in [18, 3] and define the projections

$$\Pi_\pm(\xi) = \frac{1}{2} \left(I \pm \frac{1}{\langle \xi \rangle_M} (\xi_j \gamma^0 \gamma^j + M \gamma^0) \right)$$

and the associated Fourier multiplier $\widehat{(\Pi_\pm f)}(\xi) = \Pi_\pm(\xi) \widehat{f}(\xi)$. A computation shows that $\Pi_+ \Pi_- = \Pi_- \Pi_+ = 0$ and $\Pi_\pm^2 = \Pi_\pm$. Moreover, given any spinor ψ we have

$$\psi = \Pi_+ \psi + \Pi_- \psi, \quad (-i\gamma^\mu \partial_\mu + M) \Pi_\pm \psi = \gamma^0 (-i\partial_t \pm \langle -i\nabla \rangle_M) \psi.$$

As in the paper of Bejenaru-Herr [3], we can now reduce the original system (1.3) to a first order system as follows. Suppose we have a solution (ψ_\pm, ϕ_+) to

$$\begin{aligned} (-i\partial_t \pm \langle \nabla \rangle_M) \psi_\pm &= \Pi_\pm (\mathfrak{R}(\phi_+) \gamma^0 \psi) \\ (-i\partial_t + \langle \nabla \rangle_m) \phi_+ &= \langle \nabla \rangle_m^{-1} (\psi^\dagger \gamma^0 \psi) \\ \psi_\pm(0) &= f_\pm, \quad \phi_+(0) = g_+ \end{aligned} \quad (7.3)$$

where $\psi = \Pi_+ \psi_+ + \Pi_- \psi_-$ and the data (f_\pm, g_+) satisfies $\Pi_\pm f_\pm = f_\pm$. If we let $\phi = \mathfrak{R}(\phi_+)$, then since $\psi^\dagger \gamma^0 \psi$ is real-valued, we deduce that

$$\begin{aligned} 2(\phi + i\langle \nabla \rangle_m^{-1} \partial_t \phi) &= \phi_+ + i\langle \nabla \rangle_m^{-1} \partial_t \phi_+ + \overline{(\phi_+ - i\langle \nabla \rangle_m^{-1} \partial_t \phi_+)} \\ &= 2\phi_+ - \langle \nabla \rangle_m^{-2} (\psi^\dagger \gamma^0 \psi) + \langle \nabla \rangle_m^{-2} \overline{(\psi^\dagger \gamma^0 \psi)} = 2\phi_+. \end{aligned}$$

Consequently, if we take $g_+ = \phi(0) + i\langle \nabla \rangle_m^{-1} \partial_t \phi(0)$, a simple computation shows that (ψ, ϕ) is a solution to the original DKG system (1.3). Note that, after rescaling, it suffices to consider the case $m = 1$. Therefore, to prove Theorem 1.2, it is enough to construct global solutions to the reduced system (7.3) with $m = 1$.

7.2. Analysis on the Sphere. We require some basic facts on analysis on the sphere \mathbb{S}^2 which can be found in, for instance, [36, 40, 38]. Let Y_ℓ denote the set of homogeneous harmonic polynomials of degree ℓ , and let $y_{\ell,n}$, $n = 0, \dots, 2\ell$ be an orthonormal basis for Y_ℓ with respect to the inner product

$$\langle y_{\ell,n}, y_{\ell',n'} \rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} [y_{\ell,n}(\omega)]^\dagger y_{\ell',n'}(\omega) d\mathbb{S}(\omega).$$

Given $f \in L^2(\mathbb{R}^3)$, we have the orthogonal (in $L^2(\mathbb{R}^3)$) decomposition

$$f(x) = \sum_{\ell} \sum_{n=0}^{2\ell} \langle f(|x|\omega), y_{\ell,n}(\omega) \rangle_{L^2_\omega(\mathbb{S}^2)} y_{\ell,n}\left(\frac{x}{|x|}\right).$$

For $N > 1$, we define the spherical Littlewood-Paley projections

$$(H_N f)(x) = \sum_{\ell \in \mathbb{N}} \sum_{n=0}^{2\ell} \rho\left(\frac{\ell}{N}\right) \langle f(x), y_{\ell,n} \rangle_{L^2(\mathbb{S}^2)} y_{\ell,n}\left(\frac{x}{|x|}\right) \quad H_1 = \sum_{\ell \in \mathbb{N}} \sum_{n=0}^{2\ell} \rho_1(\ell) \langle f(x), y_{\ell,n} \rangle_{L^2(\mathbb{S}^2)} y_{\ell,n}\left(\frac{x}{|x|}\right).$$

Fractional powers of the angular derivatives $\langle \Omega \rangle$ are then defined as

$$\langle \Omega \rangle^\sigma f = \sum_{N \in 2^{\mathbb{N}}} N^\sigma H_N f. \quad (7.4)$$

If we let $\Omega_{ij} = x_j \partial_j - x_j \partial_i$ denote the standard infinitesimal generators of the rotations on \mathbb{R}^3 , then a computation gives

$$\|\Omega_{ij} H_N f\|_{L^2_x(\mathbb{R}^3)} \approx N \|H_N f\|_{L^2_x(\mathbb{R}^3)}.$$

In addition, if $\Delta_{\mathbb{S}^2}$ denotes the Laplacian on the sphere of radius $|x|$, then $\Delta_{\mathbb{S}^2} = \sum_{j < k} \Omega_{ij}^2$. These facts are not explicitly required in the following, and we shall only make use of the spectral definition (7.4). More important for our purposes, are the basic properties of the multipliers H_N .

Lemma 7.1. *Let $N \geq 1$. Then H_N is uniformly (in N) bounded on $L^p(\mathbb{R}^3)$, and H_N commutes with all radial Fourier multipliers. Moreover, if $N' \geq 1$, then either $N \sim N'$ or*

$$H_N \Pi_\pm H_{N'} = 0.$$

Proof. The first claim follows from [40]. To prove the second claim, let T be a radial Fourier multiplier with $\widehat{Tf}(\xi) = \sigma(|\xi|) \widehat{f}(\xi)$. It is enough to show that, if $f(x) = a(|x|) y_\ell(\frac{x}{|x|})$ for some $y_\ell \in Y_\ell$, then $Tf = b(|x|) y_\ell(\frac{x}{|x|})$ for some $b(|x|)$ depending on a and σ . But this follows directly from [36, page 158]. To prove the final claim, suppose that $N \gg N'$ or $N \ll N'$. Our goal is to show that $H_N \Pi_\pm H_{N'} = 0$. Since H_N commutes with radial Fourier multipliers, it is enough to show that $H_N(\partial_j f) = 0$ in the case $f(x) = a(|x|) y_{\ell'}(\frac{x}{|x|})$ with $y_{\ell'} \in Y_{\ell'}$ and $\frac{N'}{2} \leq \ell' \leq 2N'$. Since $\partial_j = \frac{x_j}{|x|} \partial_r + \sum_k \frac{x_k}{|x|^2} \Omega_{jk}$ where $\partial_r = \frac{x}{|x|} \cdot \nabla$, and $\partial_r(y_{\ell'}(\frac{x}{|x|})) = 0$, we can reduce further to just showing that $H_N(x_k \Omega_{jk} y_{\ell'}) = 0$ which corresponds to checking that

$$\langle y_\ell, x_k \Omega_{kj} y_{\ell'} \rangle_{L^2(\mathbb{S}^2)} = 0 \quad (7.5)$$

for every $\frac{N}{2} \leq \ell \leq 2N$. Since $x_k \Omega_{kj} y_{\ell'}$ is a polynomial of order $\ell' + 1$, by the orthogonality of the polynomials y_ℓ , (7.5) clearly holds if $\ell > \ell' + 1$. On the other hand, after an application of integration by parts, we obtain

$$\langle y_\ell, x_k \Omega_{kj} y_{\ell'} \rangle_{L^2(\mathbb{S}^2)} = \langle \Omega_{kj}(x_k y_\ell), y_{\ell'} \rangle_{L^2(\mathbb{S}^2)}$$

since $\Omega_{kj}(x_k y_\ell)$ is a polynomial of order $\ell + 1$, we see that again (7.5) holds if $\ell' > \ell + 1$. \square

An application of Lemma 7.1 shows that H_N commutes with the P_λ and C_d multipliers since we may write $C_d^{\pm, m} = e^{\mp it \langle \nabla \rangle^m} \rho\left(\frac{-i \partial_t}{d}\right) e^{\pm it \langle \nabla \rangle^m}$. On the other hand, it is important to note that H_N does not commute with the cube and cap localisation operators R_κ and P_q .

7.3. Norms and the energy inequality. Fix $0 < \sigma \ll 1$, $\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000}$, and $b = \frac{3}{a} - 1$, and define

$$\|u\|_{Y_{\lambda,N}^{\pm,m}} = \lambda^{\frac{1}{a}-b} \sup_{d \in 2^{\mathbb{Z}}} d^b \|C_d^{\pm,m} P_{\lambda} H_N u\|_{L_t^a L_x^2}$$

and

$$\|u\|_{F_{\lambda,N}^{\pm,m}} = \|P_{\lambda} H_N u\|_{V_{\pm,m}^2} + \|u\|_{Y_{\lambda,N}^{\pm,m}}.$$

We also let

$$\|u\|_{F_{\pm,m}^{s,\sigma}} = \left(\sum_{\lambda \geq 1} \sum_{N \geq 1} \lambda^{2s} N^{2\sigma} \|u\|_{F_{\lambda,N}^{\pm,m}}^2 \right)^{\frac{1}{2}}$$

and define the Banach space

$$F_{\pm,m}^{s,\sigma} = \{u \in C(\mathbb{R}, \langle \Omega \rangle^{-\sigma} H^s) \mid \|u\|_{F_{\pm,m}^{s,\sigma}} < \infty\}.$$

For the remainder of this section, let $\sigma_M = \sigma$ if $M \geq \frac{1}{2}$ and $\sigma_M = \frac{7}{30} + \sigma$ if $0 < M < \frac{1}{2}$. Thus σ_M corresponds to amount of angular regularity in the statement of Theorem 1.2. We will construct a solution $(\psi_{\pm}, \phi_{\pm}) \in F_{\pm,M}^{0,\sigma_M} \times F_{\pm,1}^{\frac{1}{2},\sigma_M}$ to the reduced system (7.3). Thus we work in a frequency localised V^2 space, with the additional component $Y_{\lambda,N}^{\pm,m}$ needed to control the solution in the high modulation region, for the latter cp. [5, Section 4].

There are three basic properties of $V_{\pm,m}^2$ which we exploit in the following. The first is a simple bound in the high modulation region, see [21, Corollary 2.18] for a proof.

Lemma 7.2. *Let $m \geq 0$ and $2 \leq q \leq \infty$. For any $d \in 2^{\mathbb{Z}}$ we have*

$$\|C_d^{\pm,m} u\|_{L_t^q L_x^2} \lesssim d^{-\frac{1}{q}} \|u\|_{V_{\pm,m}^2}.$$

The second key property is a standard energy inequality, which reduces the problem of estimating a Duhamel integral in $F_{\lambda,N}^{\pm,M}$, to controlling a trilinear integral.

Lemma 7.3. *Let $F \in L_t^{\infty} L_x^2$, and suppose that*

$$\sup_{\|P_{\lambda} H_N v\|_{V_{\pm,m}^2} \lesssim 1} \left| \int_{\mathbb{R}} \langle P_{\lambda} H_N v(t), F(t) \rangle_{L_x^2} dt \right| < \infty.$$

If $u \in C(\mathbb{R}, L_x^2)$ satisfies $-i\partial_t u \pm \langle \nabla \rangle_m u = F$, then $P_{\lambda} H_N u \in V_{\pm,m}^2$ and we have the bound

$$\|P_{\lambda} H_N u\|_{V_{\pm,m}^2} \lesssim \|P_{\lambda} H_N u(0)\|_{L^2} + \sup_{\|P_{\lambda} H_N v\|_{V_{\pm,m}^2} \lesssim 1} \int_{\mathbb{R}} \langle P_{\lambda} H_N v(t), F(t) \rangle_{L_x^2} dt. \quad (7.6)$$

Proof. See [26] or [21, Proposition 2.10] for details on the duality. It is also possible to prove this directly as follows. Clearly it is enough to consider the case $u(0) = 0$, thus $u(t) = \int_0^t e^{\mp i(t-s)\langle \nabla \rangle_m} F(s) ds$. Let $K > 0$ and $(t_k) \in \mathcal{Z}$. A computation gives the identity

$$\left(\sum_{|k| < K} \|e^{\pm i t_k \langle \nabla \rangle_m} P_{\lambda} H_N u(t_k) - e^{\pm i t_{k-1} \langle \nabla \rangle_m} P_{\lambda} H_N u(t_{k-1})\|_{L_x^2}^2 \right)^{\frac{1}{2}} = \int_{\mathbb{R}} \langle P_{\lambda} H_N v(s), F(s) \rangle_{L_x^2} ds$$

with

$$v(s) = A^{-1} \sum_{|k| < K} \mathbb{1}_{[t_{k-1}, t_k)}(s) \left(e^{\mp i(s-t_k)\langle \nabla \rangle_m} u(t_k) - e^{\mp i(s-t_{k-1})\langle \nabla \rangle_m} u(t_{k-1}) \right)$$

and

$$A = \left(\sum_{|k| < K} \|e^{\pm i t_k \langle \nabla \rangle_m} P_{\lambda} H_N u(t_k) - e^{\pm i t_{k-1} \langle \nabla \rangle_m} P_{\lambda} H_N u(t_{k-1})\|_{L_x^2}^2 \right)^{\frac{1}{2}}.$$

It is easy to check that $\|P_{\lambda} H_N v\|_{V_{\pm,m}^2} \lesssim 1$. Thus, by taking the sup over $\|P_{\lambda} H_N v\|_{V_{\pm,m}^2} \lesssim 1$, and then letting $K \rightarrow \infty$ we deduce the bound (7.6). Since u is also continuous, we obtain $u \in V_{\pm,m}^2$ as required. \square

Note that the norm on v can in fact be taken to be the stronger $U_{\pm,m}^2$ norm, but we do not require this improvement here.

The final result we require on the $V_{\pm,m}^2$ spaces, concerns the question of scattering.

Lemma 7.4. *Let $u \in V_{\pm, m}^2$. Then there exists $f \in L_x^2$ such that $\|u(t) - e^{\mp it \langle \nabla \rangle} f\|_{L_x^2} \rightarrow 0$ as $t \rightarrow \infty$.*

Clearly, this result can be extended to elements of the space $F_{\pm, m}^{s, \sigma_M}$. In other words, if we construct a solution in $F_{\pm, m}^{s, \sigma_M}$, then we immediately deduce the solution must scatter to a linear solution as $t \rightarrow \pm\infty$.

7.4. Proof of Theorem 1.2. We now come to the proof of Theorem 1.2. In light of Lemma 7.4, it is enough to construct a solution $(\psi_{\pm}, \phi_{\pm}) \in F_{\pm, M}^{0, \sigma_M} \times F_{\pm, 1}^{\frac{1}{2}, \sigma_M}$ to the reduced system (7.3). Note that we may always assume that $\psi_{\pm} = \Pi_{\pm} \psi_{\pm}$, provided that this is satisfied at $t = 0$. Define the Duhamel integral

$$\mathcal{I}_m^{\pm}[F] = \int_0^t e^{\mp i(t-s) \langle \nabla \rangle_m} F(s) ds.$$

Note that $\mathcal{I}_m^{\pm}[F]$ solves the equation

$$(-i\partial_t \pm \langle \nabla \rangle_m) \mathcal{I}_m^{\pm}[F] = F$$

with vanishing data at $t = 0$. Moreover, we can check that for every $1 < p < \infty$ we have

$$\|C_d^{\pm, m} \mathcal{I}_m^{\pm}[F]\|_{L_t^p L_x^2} \lesssim d^{-1} \|C_d^{\pm, m} F\|_{L_t^p L_x^2}. \quad (7.7)$$

If we had the bounds

$$\begin{aligned} \|\Pi_{\pm 1} \mathcal{I}_M^{\pm 1}[\phi \gamma^0 \Pi_{\pm 2} \varphi]\|_{F_{\pm 1, M}^{0, \sigma_M}} &\lesssim \|\phi\|_{F_{\pm 1}^{\frac{1}{2}, \sigma_M}} \|\varphi\|_{F_{M, \pm 2}^{0, \sigma_M}} \\ \|\langle \nabla \rangle^{-1} \mathcal{I}_1^+[(\Pi_{\pm 1} \psi)^{\dagger} \gamma^0 \Pi_{\pm 2} \varphi]\|_{F_{\pm 1}^{\frac{1}{2}, \sigma_M}} &\lesssim \|\psi\|_{F_{M, \pm 1}^{0, \sigma_M}} \|\varphi\|_{F_{M, \pm 2}^{0, \sigma_M}} \end{aligned} \quad (7.8)$$

then a standard fixed point argument in $F_{\pm, M}^{0, \sigma_M} \times F_{\pm, 1}^{\frac{1}{2}, \sigma_M}$ would give the required solution to (7.3), provided of course that the data (f_{\pm}, g_{\pm}) satisfied

$$\|\langle \Omega \rangle^{\sigma_M} f_{\pm}\|_{L^2} + \|\langle \Omega \rangle^{\sigma_M} g_{\pm}\|_{H^{\frac{1}{2}}} \ll 1.$$

Let

$$\phi_{\mu, N} = P_{\mu} H_N \phi, \quad \psi_{\lambda_1, N_1} = P_{\lambda_1} H_{N_1}, \quad \varphi_{\lambda_2, N_2} = P_{\lambda_2} H_{N_2} \varphi.$$

We have the following frequency localised estimates.

Theorem 7.5. *Fix $M > 0$. Then there exists $\epsilon > 0$ such that*

$$\begin{aligned} &\|\Pi_{\pm 1} \mathcal{I}_M^{\pm 1}[\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \\ &\lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\sigma_M} \left(\frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}} \right)^{\epsilon} \|\phi\|_{F_{\mu, N}^{\pm 1, 1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}} \end{aligned} \quad (7.9)$$

and

$$\begin{aligned} &\|\mathcal{I}_1^+[(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^{\dagger} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{F_{\mu, N}^{\pm 1, 1}} \\ &\lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\sigma_M} \left(\frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}} \right)^{\epsilon} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \end{aligned} \quad (7.10)$$

Remark 7.6. The proof of Theorem 7.5 in the resonant regime $0 < M < \frac{1}{2}$ relies on the small scale V^2 estimates in Corollary 6.4. However, it is possible to prove a weaker version of Theorem 7.5, with σ_M replaced with some larger σ , provided only that a *robust* version of the *homogeneous* bilinear restriction estimate (6.3) holds. More precisely, by following the proof of Corollary 6.4, and then interpolating with the K-G Strichartz estimates as in Remarks 6.2 and 6.3, it is possible to show that (6.3) implies the V^2 bound

$$\|uv\|_{L_t^a L_x^b(\mathbb{R}^{1+3})} \lesssim \lambda^{1+\frac{1}{a}-\frac{1}{b}} \|u\|_{V_{\pm 1, m_1}^2} \|v\|_{V_{\pm 2, m_2}^2}$$

in the range $\frac{1}{a} + \frac{2}{b} < 2$, $\frac{1}{a} + \frac{6}{5b} < \frac{7}{5}$ where u and v have Fourier support in 1-separated angular wedges of size $1 \times 1 \times \lambda$ at distance λ from the origin. The case $a = 2-$ and $b = \frac{4}{3}+$ can be used together with the $L_t^{2+} L_x^{4-}$ angular Strichartz bound from [17, Theorem 1.1] instead of the argument used in the high-high case in the proof of Theorem 8.8 below. However, the estimate obtained is weaker than the one in Theorem 7.5. Moreover, it still requires a robust version of the homogeneous bilinear estimate (6.3) for which we can track the dependence of the constant on the phases Φ_j due to the lack of homogeneity of the Klein-Gordon phase. Irrespective of fact the Theorem 1.1 applies to V^2 -functions, a key advantage of our formulation of Theorem 1.1, in comparison to [2, 33], is that it allows us to read off the above mentioned dependence.

The standard Littlewood-Paley trichotomy implies that the lefthand sides of (7.9) and (7.10) are zero unless

$$\max\{\mu, \lambda_1, \lambda_2\} \approx \text{med}\{\mu, \lambda_1, \lambda_2\} \gtrsim \min\{\mu, \lambda_1, \lambda_2\} \quad (7.11)$$

and

$$\max\{N, N_1, N_2\} \approx \text{med}\{N, N_1, N_2\} \gtrsim \min\{N, N_1, N_2\}$$

It is now easy to check that the bilinear estimates (7.8), follow from Theorem 7.5. Consequently, we have reduced the proof of Theorem 1.2, to proving the frequency localised bilinear estimates in Theorem 7.5. As the proof of Theorem 7.5 requires a number of preliminary results, we postpone the proof till to Subsection 8.4 below.

8. LINEAR AND MULTILINEAR ESTIMATES

In this section our goal is give the proof of Theorem 7.5. To this end, we first provide some linear estimates and adapt them to our functional setup, prove an auxiliary trilinear estimate in V^2 , and eventually give the proof of the crucial Theorem 7.5 in Subsection 8.4.

8.1. Auxiliary Estimates. As is well-known, see e.g. [18], the system 7.3 exhibits null structure. To exploit the null structure of the product $\psi^\dagger \gamma^0 \psi$, we start by noting that for any $x, y \in \mathbb{R}^3$, we have the identity

$$\begin{aligned} [\Pi_{\pm_1} f]^\dagger \gamma^0 \Pi_{\pm_2} g &= [(\Pi_{\pm_1} - \Pi_{\pm_1}(x))f]^\dagger \gamma^0 \Pi_{\pm_2} g \\ &\quad + [\Pi_{\pm_1}(x)f]^\dagger \gamma^0 (\Pi_{\pm_2} - \Pi_{\pm_2}(y))g + f^\dagger \Pi_{\pm_1}(x) \gamma^0 \Pi_{\pm_2}(y)g \end{aligned}$$

This is then exploited by using the null form type bound

$$|\Pi_{\pm_1}(x) \gamma^0 \Pi_{\pm_2}(y)| \lesssim \theta(\pm_1 x, \pm_2 y) + \frac{|\pm_1 |x| \pm_2 |y||}{\langle x \rangle \langle y \rangle}, \quad (8.1)$$

which follows from (2.6) by observing that

$$\begin{aligned} \Pi_{\pm_1}(x) \gamma^0 \Pi_{\pm_2}(y) &= \Pi_{\pm_1}(x) \left(\Pi_{\pm_1}(x) \gamma^0 - \gamma^0 \Pi_{\mp_2}(y) \right) \Pi_{\pm_2}(y) \\ &= \Pi_{\pm_1}(x) \left(\left(\frac{\pm_2 \eta_j}{\langle \eta \rangle_M} - \frac{\pm_1 \xi_j}{\langle \xi \rangle_M} \right) \gamma^j + \left(\frac{\pm_1 M}{\langle \xi \rangle_M} + \frac{\pm_2 M}{\langle \eta \rangle_M} \right) I \right) \Pi_{\pm_2}(y), \end{aligned}$$

together with the following lemma, see [2, Lemma 3.3] for a similar statement to Part (i).

Lemma 8.1. *Let $1 < r < \infty$.*

(i) *If $\lambda \geq 1$, $\alpha \gtrsim \lambda^{-1}$, $\kappa \in \mathcal{C}_\alpha$, then*

$$\|(\Pi_{\pm_1} - \Pi_{\pm_1}(\lambda \omega(\kappa))) R_\kappa P_\lambda f\|_{L_x^r} \lesssim \alpha \|R_\kappa P_\lambda u\|_{L_x^r}.$$

(ii) *If $\lambda \geq 1$, $0 < \alpha \lesssim \lambda^{-1}$, $\kappa \in \mathcal{C}_\alpha$, $q \in Q_{\lambda^2 \alpha}$ with center ξ_0 , then*

$$\|(\Pi_{\pm_1} - \Pi_{\pm_1}(\xi_0)) R_\kappa P_q P_\lambda f\|_{L_x^r} \lesssim \alpha \|R_\kappa P_q P_\lambda u\|_{L_x^r}.$$

Proof. Concerning Part (i), see [2, Proof of Lemma 3.3]. Concerning Part (ii), we may assume $|\xi_0| \approx \lambda$ and, due to boundedness, we may replace the symbol of $R_\kappa P_q P_\lambda$ by a smooth cutoff χ_E to the parallelepiped E with center ξ_0 of side lengths $\alpha \mu^2 \times \alpha \mu \times \alpha \mu$ with long side pointing in the direction ξ_0 . After rotating ξ_0 to $\xi_0 = |\xi_0|(1, 0, 0)$, the operator has the symbol

$$m(\xi) = \left(\pm B^j \left(\frac{\xi_j}{\langle \xi \rangle_M} - \frac{\xi_{0,j}}{\langle \xi_0 \rangle_M} \right) \pm \frac{1}{2} \gamma^0 \left(\frac{1}{\langle \xi \rangle_M} - \frac{1}{\langle \xi_0 \rangle_M} \right) \right) \chi_E(\xi),$$

for certain $B^1, B^2, B^3 \in \mathbb{C}^{4 \times 4}$. It suffices to prove the kernel bound

$$|(\mathcal{F}_x^{-1} m)(x)| \lesssim \alpha^4 \lambda^4 (1 + \alpha \lambda^2 |x_1| + \alpha \lambda |x'|)^{-4}, \quad x = (x_1, x'), \quad (8.2)$$

as it implies $\|\mathcal{F}_x^{-1} m\|_{L^1(\mathbb{R}^3)} \lesssim \alpha$. In the support of χ_E we obtain, from (2.6) and a simple computation,

$$|m(\xi)| \lesssim \lambda^{-3} \|\xi\| - |\xi_0| + \theta(\xi, \xi_0) + \lambda^{-2} \|\xi\| - |\xi_0| \lesssim \alpha.$$

From the localisation of χ_E , where $|\partial_{\xi_1}^\ell \frac{\xi_j}{(\xi)_M}| \lesssim \lambda^{-\ell-1}$, and the Leibniz rule, we conclude for $\ell > 0$

$$|\partial_{\xi_1}^\ell m(\xi)| \lesssim \alpha(\alpha\lambda^2)^{-\ell} + \sum_{0 < \ell_1 \leq \ell} \lambda^{-\ell_1-1} (\alpha\lambda^2)^{\ell_1-\ell} \lesssim \alpha(\alpha\lambda^2)^{-\ell}.$$

Integration by parts now implies (8.2) if $\alpha\lambda^2|x_1| \geq \alpha\lambda|x'|$. For $k = 2, 3$, we have $|\partial_{\xi_k}^\ell \frac{\xi_j}{(\xi)_M}| \lesssim \lambda^{-\ell}$ within the support of χ_E , hence we conclude for $\ell > 0$

$$|\partial_{\xi_k}^\ell m(\xi)| \lesssim \alpha(\alpha\lambda)^{-\ell} + \sum_{0 < \ell_1 \leq \ell} \lambda^{-\ell_1} (\alpha\lambda)^{\ell_1-\ell} \lesssim \alpha(\alpha\lambda)^{-\ell}.$$

Integration by parts now implies (8.2) in the region where $\alpha\lambda^2|x_1| \leq \alpha\lambda|x_k|$. \square

The proof of Theorem 7.5 requires a number of standard linear estimates for homogeneous solutions to the Klein-Gordon equation. We start by recalling the Strichartz estimates for the wave and Klein-Gordon equation.

Lemma 8.2 (Wave Strichartz). *Let $m \geq 0$ and $2 < q \leq \infty$. If $0 < \mu \leq \lambda$, $N \geq 1$, and $\frac{1}{r} = \frac{1}{2} - \frac{1}{q}$ then for every $q \in Q_\mu$ we have*

$$\|e^{\mp it\langle \nabla \rangle^m} P_q P_\lambda f\|_{L_t^q L_x^r} \lesssim \mu^{\frac{1}{2}-\frac{1}{r}} \lambda^{\frac{1}{2}-\frac{1}{r}} \|P_q P_\lambda f\|_{L_x^2}.$$

Moreover, by spending additional angular regularity we have

$$\|e^{\mp it\langle \nabla \rangle^m} P_\lambda H_N f\|_{L_t^q L_x^4} \lesssim \lambda^{\frac{3}{4}-\frac{1}{q}} N \|P_\lambda H_N f\|_{L_x^2}.$$

Proof. The proof of the first estimate can be found in [3, Lemma 3.1]. The second follows by simple modification of the argument in the appendix to [38]. More precisely, after interpolating with the $L_t^\infty L_x^2$ estimate, we need to show that

$$\|e^{\mp it\langle \nabla \rangle^m} H_N P_\lambda f\|_{L_t^2 L_x^r} \lesssim N \lambda^{3(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}} \|H_N \lambda f\|_{L_x^2}.$$

After rescaling, and following the argument on [38, pp. 226–227], it is enough to prove that for every $\epsilon > 0$ we have the space-time Morawetz type bound

$$\|(1 + |x|)^{-\frac{1}{2}-\epsilon} \nabla u\|_{L_{t,x}^2} \lesssim \|(\partial_t u(0), \nabla u(0))\|_{L_x^2} \quad (8.3)$$

for functions u with $\square u + mu = 0$, and the constant in (8.3) is independent of m . However the proof of (8.3) follows the same argument as the wave case in [38], the only change is to replace the wave energy-momentum tensor with the Klein-Gordon version

$$Q_{\alpha\beta} = \frac{1}{2} \left(\partial_\alpha \phi \overline{\partial_\beta \phi} + \partial_\beta \phi \overline{\partial_\alpha \phi} - g_{\alpha\beta} (\partial^\gamma \phi \overline{\partial_\gamma \phi} + m^2 |\phi|^2) \right),$$

we omit the details. \square

The amount of angular regularity required for the $L_t^{2+} L_x^4$ Strichartz estimate to hold, is much less than that stated in Lemma 8.2. In fact, in [38], it is shown that the same estimate holds with $N^{\frac{1}{2}+}$. However, as the sharp number of angular derivatives is not required in the arguments we use in the present paper, we have elected to simply state the result with a whole angular derivative. On the other hand, the number of angular derivatives required in the following Klein-Gordon regime, plays a crucial role.

Lemma 8.3 (Klein-Gordon Strichartz). *Let $m > 0$ and $\frac{3}{10} < \frac{1}{r} < \frac{5}{14}$. Then for every $\epsilon > 0$ we have*

$$\|e^{\mp it\langle \nabla \rangle^m} P_\lambda H_N f\|_{L_{t,x}^r} \lesssim \lambda^{2-\frac{5}{r}} N^{7(\frac{1}{r}-\frac{3}{10})+\epsilon} \|P_\lambda H_N f\|_{L_x^2}.$$

Proof. This is a special case of [17, Theorem 1.1]. \square

Remark 8.4. Without angular regularity, the optimal $L_{t,x}^r$ Strichartz estimate for the Klein-Gordon equation is $r = \frac{10}{3}$, see for instance [34]. However, in the resonant region, we are forced to take r slightly below 3, thus the additional angular regularity is essential to obtain the additional integrability in time. In other words, the angular regularity is used not just to obtain the scale invariant endpoint, but also plays a crucial role in controlling the resonant interaction. Note that the number of angular derivatives required in Lemma 8.3 is not expected to be optimal, and any improvement in this direction has an impact on Theorem 1.2.

We have seen that the addition of angular regularity improves the range of available Strichartz estimates. An alternative way to exploit additional angular regularity is given by the following angular concentration type bound.

Lemma 8.5 ([38, Lemma 5.2]). *Let $2 \leq p < \infty$, and $0 \leq s < \frac{2}{p}$. If $\lambda, N \geq 1$, $\alpha \gtrsim \lambda^{-1}$, and $\kappa \in \mathcal{C}_\alpha$ we have*

$$\|R_\kappa P_\lambda H_N f\|_{L_x^p(\mathbb{R}^3)} \lesssim \alpha^s N^s \|P_\lambda H_N f\|_{L_x^p(\mathbb{R}^3)}.$$

Finally, we need to estimate various square sums of norms. As we work in V^2 , this causes a slight loss in certain estimates. However, as we have some angular derivatives to work with, this loss can always be absorbed elsewhere.

Lemma 8.6. *Let $(P_j)_{j \in \mathcal{J}}$ and $(\mathcal{M}_j)_{j \in \mathcal{J}}$ be a collection of spatial Fourier multipliers. Suppose that the symbols of P_j have finite overlap, and*

$$\|\mathcal{M}_j P_j f\|_{L_x^2} \lesssim \delta \|P_j f\|_{L_x^2}$$

for some $\delta > 0$.

(i) *Let $q > 2$, $r \geq 2$. Suppose that there exists $A > 0$ such that for every j we have the bound*

$$\|e^{\mp it \langle \nabla \rangle^m} P_j f\|_{L_t^q L_x^r} \leq A \|P_j f\|_{L_x^2}.$$

Then for every $\epsilon > 0$ we have

$$\left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j v\|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} \lesssim \delta (\#\mathcal{J})^\epsilon A \|v\|_{V_{\pm, m}^2}.$$

(ii) *Fix $p_0 > 1$. Suppose that there exists $A > 0$ such that $\|P_j f\|_{L_x^\infty} \lesssim A \|f\|_{L_x^2}$. Moreover, suppose that for every $p > p_0$ there exists $B_p > 0$, and for any $j \in \mathcal{J}$ there exists $\mathcal{K}_j \subset \mathcal{J}$ with $\#\mathcal{K}_j \lesssim 1$, such that for every $k \in \mathcal{K}_j$*

$$\|P_j u P_k v\|_{L_{t,x}^p} \lesssim B_p \|P_j u\|_{U_{\pm 1, m_1}^2} \|P_k v\|_{U_{\pm 2, m_2}^2}.$$

Then for every $q > p_0$ and $\frac{p_0}{q} < \theta < 1$ we have

$$\sum_{j \in \mathcal{J}, k \in \mathcal{K}_j} \|P_j u \mathcal{M}_k P_k v\|_{L_{t,x}^q} \lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \|u\|_{V_{\pm 1, m_1}^2} \|v\|_{V_{\pm 2, m_2}^2}.$$

Proof. We start with the proof of (i). Let $2 \leq p \leq q$ and suppose that $\phi = \sum_{I \in \mathcal{I}} \mathbb{1}_I(t) e^{\mp it \langle \nabla \rangle^m} f_I$ is a U^p atom, thus $\sum_I \|f_I\|_{L_x^p}^p \leq 1$. The assumed linear estimate, together with the finite overlap of the Fourier multipliers P_j implies that

$$\begin{aligned} \left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j \phi\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} &\leq \left(\sum_{I \in \mathcal{I}} \sum_{j \in \mathcal{J}} \|e^{\mp it \langle \nabla \rangle^m} \mathcal{M}_j P_j f_I\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} \\ &\leq A \left(\sum_{j \in \mathcal{J}} \sum_{I \in \mathcal{I}} \|\mathcal{M}_j P_j f_I\|_{L_x^2}^p \right)^{\frac{1}{p}} \leq \delta A \left(\sum_{I \in \mathcal{I}} \left(\sum_{j \in \mathcal{J}} \|P_j f_I\|_{L_x^2}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \lesssim \delta A. \end{aligned}$$

Consequently the atomic definition of $U_{\pm, m}^p$ then implies that for any $2 \leq p \leq q$

$$\left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j u\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} \lesssim A \delta \|u\|_{U_{\pm, m}^p}. \quad (8.4)$$

Let $v \in V_{\pm, m}^2$. There exists a decomposition $v = \sum_{\ell \in \mathbb{N}} v_\ell$ such that for every $p \geq 2$ we have $\|v_\ell\|_{U_{\pm, m}^p} \lesssim 2^{\ell(\frac{2}{p}-1)} \|v\|_{V_{\pm, m}^2}$, see e.g. [27, Lemma 6.4] or [21, Proposition 2.5 and Proposition 2.20]. An application of

Hölder's inequality, together with (8.4) gives for any $2 < p \leq q$

$$\begin{aligned}
\left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j v\|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} &\lesssim (\#\mathcal{J})^{\frac{1}{2} - \frac{1}{p}} \sum_{\ell \in \mathbb{N}} \left(\sum_{j \in \mathcal{J}} \|\mathcal{M}_j P_j v_\ell\|_{L_t^q L_x^r}^p \right)^{\frac{1}{p}} \\
&\lesssim \delta A (\#\mathcal{J})^{\frac{1}{2} - \frac{1}{p}} \sum_{\ell \in \mathbb{N}} \|v_\ell\|_{U_{\pm, m}^p} \\
&\lesssim \delta A (\#\mathcal{J})^{\frac{1}{2} - \frac{1}{p}} \|v\|_{V_{\pm, m}^2} \sum_{\ell \in \mathbb{N}} 2^{\ell(\frac{2}{p} - 1)} \\
&\lesssim \delta A (\#\mathcal{J})^{\frac{1}{2} - \frac{1}{p}} \|v\|_{V_{\pm, m}^2}.
\end{aligned}$$

Thus (i) follows by taking p sufficiently close to 2.

We now turn to the proof of (ii). As in the proof of (i), we decompose $u = \sum_{\ell \in \mathbb{N}} u_\ell$ and $v = \sum_{\ell \in \mathbb{N}} v_\ell$ with $\|u_\ell\|_{U_{\pm 1, m_1}^r} \lesssim 2^{\ell(\frac{2}{r} - 1)}$ and $\|v_\ell\|_{U_{\pm 2, m_2}^r} \lesssim 2^{\ell(\frac{2}{r} - 1)}$ for every $r \geq 2$. Let $q > p_0$ and $\frac{p_0}{q} < \theta < 1$. Then the convexity of the L^q norms together with Hölder's inequality, our assumed bilinear estimate, and the U^2 summation argument used in (i) implies that

$$\begin{aligned}
&\sum_{j \in \mathcal{J}, k \in \mathcal{K}_j} \|P_j u \mathcal{M}_k P_k v\|_{L_{t,x}^q} \\
&\lesssim (\#\mathcal{J})^{1-\theta} \sum_{\ell, \ell' \in \mathbb{N}} \left(\sum_{j \in \mathcal{J}, k \in \mathcal{K}_j} \|P_j u \mathcal{M}_k P_k v\|_{L_{t,x}^q} \right)^\theta \left(\sup_{j, k \in \mathcal{J}} \|P_j u \mathcal{M}_k P_k v\|_{L_{t,x}^\infty} \right)^{1-\theta} \\
&\lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \sum_{\ell, \ell' \in \mathbb{N}} (\|u_\ell\|_{U_{\pm 1, m_1}^2} \|v_\ell\|_{U_{\pm 2, m_2}^2})^\theta (\|u_{\ell'}\|_{U_{\pm 1, m_1}^\infty} \|v_{\ell'}\|_{U_{\pm 2, m_2}^\infty})^{1-\theta} \\
&\lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \|u\|_{V_{\pm 1, m_1}^2} \|v\|_{V_{\pm 2, m_2}^2} \sum_{\ell, \ell' \in \mathbb{N}} 2^{-\ell(1-\theta)} 2^{-\ell'(1-\theta)} \\
&\lesssim \delta (\#\mathcal{J})^{1-\theta} A^{1-\theta} B_{\theta q}^\theta \|u\|_{V_{\pm 1, m_1}^2} \|v\|_{V_{\pm 2, m_2}^2}.
\end{aligned}$$

Therefore (ii) follows. \square

Clearly the previous lemma allows us to extend Corollary 6.4, and the linear estimates discussed above, to frequency localised functions in $V_{\pm, m}^2$. For instance, for any $1 \leq \mu \lesssim \lambda$, $\alpha \gtrsim \lambda^{-1}$, and $\epsilon > 0$, $q > 2$, we have by Lemma 8.2

$$\left(\sum_{q \in Q_\mu} \sum_{\kappa \in \mathcal{C}_\alpha} \|R_\kappa P_q u_{\lambda, N}\|_{L_{t,x}^4}^2 \right)^{\frac{1}{2}} \lesssim \alpha^{-\epsilon} \left(\frac{\mu}{\lambda} \right)^{\frac{1}{4} - \epsilon} \lambda^{\frac{1}{2}} \|u_{\lambda, N}\|_{V_{\pm, m}^2}, \quad (8.5)$$

$$\left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_\kappa u_{\lambda, N}\|_{L_t^q L_x^4}^2 \right)^{\frac{1}{2}} \lesssim \alpha^{-\epsilon} \lambda^{\frac{3}{4} - \frac{1}{q}} N \|u_{\lambda, N}\|_{V_{\pm, m}^2} \quad (8.6)$$

where we use the shorthand $u_{\lambda, N} = P_\lambda P_N u$. Similarly, an application of Corollary 6.4, Lemma 8.1, and (ii) in Lemma 8.6 gives for every $q > \frac{3}{2}$ and $\epsilon > 0$

$$\begin{aligned}
&\left(\sum_{\kappa, \kappa'' \in \mathcal{C}_{\mu^{-1}}} \sum_{\substack{q, q'' \in Q_\mu \\ |q - q''| \approx \mu \text{ or } |\kappa - \kappa''| \approx \mu^{-1}}} \|R_{\kappa''} P_{q''} \phi_{\mu, N} [(\Pi_+ - \Pi_+(\mu\omega(\kappa))) R_\kappa P_q \psi_{\mu, N_1}]^\dagger\|_{L_{t,x}^q(\mathbb{R}^{1+3})}^2 \right)^{\frac{1}{2}} \\
&\lesssim \mu^\epsilon \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\mu, N_1}\|_{V_{+, M}^2}
\end{aligned} \quad (8.7)$$

where $\omega(\kappa)$ denotes the centre of the cap $\kappa \in \mathcal{C}_{\mu^{-1}}$. This bilinear bound plays a key role in controlling the solution to the DKG system in the resonant region.

8.2. General Resonance Identity. After an application of Lemma 7.3, proving the bilinear estimates in Theorem 7.5 for the V^2 component of the norm, reduces to estimating trilinear expressions of the form

$$\int_{\mathbb{R}^{1+3}} \phi \psi^\dagger \gamma^0 \varphi dx dt. \quad (8.8)$$

Suppose that ϕ , ψ , and φ have small modulation, thus $\text{supp } \tilde{\phi} \subset \{|\tau + \langle \xi \rangle| \leq d\}$, $\text{supp } \tilde{\psi} \subset \{|\tau \pm_1 \langle \xi \rangle_M| \leq d\}$, and $\text{supp } \tilde{\varphi} \subset \{|\tau \pm_2 \langle \xi \rangle_M| \leq d\}$ for some $d \in 2^{\mathbb{Z}}$. If $\xi \in \text{supp } \tilde{\psi}$ and $\eta \in \text{supp } \tilde{\varphi}$, then it is easy to check that the integral (8.8) vanishes unless

$$|\langle \xi - \eta \rangle \mp_1 \langle \xi \rangle_M \pm_2 \langle \eta \rangle_M| \lesssim d.$$

To exploit this, we define the modulation function

$$\mathfrak{M}_{\pm_1, \pm_2}(\xi, \eta) = |\langle \xi - \eta \rangle \mp_1 \langle \xi \rangle_M \pm_2 \langle \eta \rangle_M|.$$

Clearly we have the symmetry properties $\mathfrak{M}_{+,+}(\xi, \eta) = \mathfrak{M}_{-,-}(\eta, \xi)$ and $\mathfrak{M}_{\pm, \mp}(\xi, \eta) = \mathfrak{M}_{\pm, \mp}(\eta, \xi)$. The proof of our global existence results requires a careful analysis of the zero sets of $\mathfrak{M}_{\pm_1, \pm_2}$, the key tool is the following.

Lemma 8.7. *Let $M > 0$.*

(i) *(Nonresonant interactions). We have*

$$\mathfrak{M}_{-,+}(\xi, \eta) \gtrsim \langle \xi \rangle + \langle \eta \rangle, \quad \mathfrak{M}_{\pm, \pm}(\xi, \eta) \gtrsim \frac{1}{\langle \xi - \eta \rangle} \left(\frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle \langle \eta \rangle} + |\xi| |\eta| \theta^2(\xi, \eta) + 1 \right),$$

and

$$\mathfrak{M}_{-,-}(\xi, \eta) \gtrsim \frac{|\xi - \eta| |\xi|}{\langle \xi \rangle + \langle \eta \rangle} \theta^2(\xi - \eta, -\xi), \quad \mathfrak{M}_{+,+}(\xi, \eta) \gtrsim \frac{|\xi - \eta| |\eta|}{\langle \xi \rangle + \langle \eta \rangle} \theta^2(\xi - \eta, \eta).$$

(ii) *(Resonant interactions). We have*

$$\mathfrak{M}_{+,-}(\xi, \eta) \approx \frac{1}{\langle \xi \rangle + \langle \eta \rangle} \left| M^2 \frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle_M \langle \eta \rangle_M + |\xi| |\eta| + M^2} + |\xi| |\eta| + \xi \cdot \eta + \frac{4M^2 - 1}{2} \right|$$

and

$$\mathfrak{M}_{+,-}(\xi, \eta) \gtrsim \frac{1}{\langle \eta \rangle} \left| \frac{(|\xi| - M|\xi - \eta|)^2}{\langle \xi \rangle_M \langle \xi - \eta \rangle + |\xi| |\xi - \eta| + M} + |\xi| |\xi - \eta| - \xi \cdot (\xi - \eta) + \frac{2M - 1}{2} \right|.$$

Proof. We begin by noting that, if we let $m_1, m_2, m_3 \geq 0$, then for any $x, y \in \mathbb{R}^n$ we have the identity

$$\begin{aligned} & | \langle x - y \rangle_{m_3}^2 - (\langle x \rangle_{m_1} \pm \langle y \rangle_{m_2})^2 | \\ &= | \mp 2 \langle x \rangle_{m_1} \langle y \rangle_{m_2} - 2x \cdot y + (m_3^2 - m_1^2 - m_2^2) | \\ &= | 2(\langle x \rangle_{m_1} \langle y \rangle_{m_2} - (|x||y| + m_1 m_2)) + 2(|x||y| \pm x \cdot y) \pm ((m_1 \pm m_2)^2 - m_3^2) | \\ &= 2 \left| \frac{(m_1 |y| - m_2 |x|)^2}{\langle x \rangle_{m_1} \langle y \rangle_{m_2} + |x||y| + m_1 m_2} + |x||y| \pm x \cdot y \pm \frac{(m_1 \pm m_2)^2 - m_3^2}{2} \right|. \end{aligned} \quad (8.9)$$

We now turn to (i). The bound for $\mathfrak{M}_{-,+}$ is clear. On the other hand, by taking $x = \xi$, $y = \eta$, $m_1 = m_2 = M$, $m_3 = 1$ in (8.9), we have

$$\begin{aligned} \mathfrak{M}_{\pm, \pm}(\xi, \eta) &\geq |\langle \xi - \eta \rangle - |\langle \xi \rangle_M - \langle \eta \rangle_M| \approx \frac{1}{\langle \xi - \eta \rangle} |\langle \xi - \eta \rangle^2 - (\langle \xi \rangle_M - \langle \eta \rangle_M)^2| \\ &\approx \frac{1}{\langle \xi - \eta \rangle} \left(\frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle \langle \eta \rangle} + |\xi| |\eta| \theta^2(\xi, \eta) + 1 \right). \end{aligned}$$

Similarly, taking $x = \xi - \eta$ and $y = \xi$, gives

$$\mathfrak{M}_{-,-}(\xi, \eta) = \frac{|\langle \eta \rangle_M^2 - (\langle \xi - \eta \rangle + \langle \xi \rangle_M)^2|}{\langle \eta \rangle_M + \langle \xi - \eta \rangle + \langle \xi \rangle_M} \gtrsim \frac{|\xi - \eta| |\xi|}{\langle \xi \rangle + \langle \eta \rangle} \theta^2(\xi - \eta, -\xi).$$

Using the symmetry $\mathfrak{M}_{-,-}(\xi, \eta) = \mathfrak{M}_{+,+}(\eta, \xi)$ gives the remaining bound in (i). To prove (ii), we note that another application of (8.9) gives

$$\begin{aligned} \mathfrak{M}_{+,-}(\xi, \eta) &\approx \frac{1}{\langle \xi \rangle + \langle \eta \rangle} |\langle \xi - \eta \rangle^2 - (\langle \xi \rangle_M + \langle \eta \rangle_M)^2| \\ &\approx \frac{1}{\langle \xi \rangle + \langle \eta \rangle} \left| M^2 \frac{(|\xi| - |\eta|)^2}{\langle \xi \rangle_M \langle \eta \rangle_M + |\xi| |\eta| + M^2} + |\xi| |\eta| + \xi \cdot \eta + \frac{4M^2 - 1}{2} \right| \end{aligned}$$

from which the first inequality in (ii). The second inequality in (ii) follows from a similar application of (8.9). \square

8.3. The Trilinear Estimates. Suppose we would like to bound an expression of the form $P_\lambda H_N \mathcal{I}_m^\pm[F]$ in $V_{\pm, m}^2$. An application of the energy inequality, Lemma 7.3, implies that we have

$$\|P_\lambda H_N \mathcal{I}_m^\pm[F]\|_{V_{\pm, m}^2} \lesssim \sup_{\|P_\lambda H_N u\|_{V_{\pm, m}^2} \lesssim 1} \left| \int_{\mathbb{R}^{1+3}} (P_\lambda H_N u)^\dagger F dx dt \right|.$$

Thus to bound the V^2 component of $\|\mathcal{I}_m^\pm[F]\|_{F_{\lambda, N}^{\pm, m}}$, it is enough to control the integral $\int_{\mathbb{R}^{1+3}} (P_\lambda H_N u)^\dagger F dx dt$. Consequently, to estimate the V^2 component of norms in Theorem 7.5, the key step is to prove the following trilinear estimate. To simplify notation somewhat, we define $\mathbf{B}_\epsilon = \left(\frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}}\right)^\epsilon$ if $M \geq \frac{1}{2}$, and if $0 < M < \frac{1}{2}$ we let

$$\mathbf{B}_\epsilon = \begin{cases} \left(\frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}}\right)^\epsilon & \mu \ll \max\{\lambda_1, \lambda_2\} \text{ or } \mu \gg \min\{\lambda_1, \lambda_2\} \\ 1 + \mu^{-\frac{1}{6} + \sigma} (\min\{N, N_1, N_2\})^{\frac{7}{30}} & \mu \approx \lambda_1 \approx \lambda_2. \end{cases}$$

Theorem 8.8. *Let $M > 0$. For every $\frac{\sigma}{100} < \delta \ll 1$ we have*

$$\begin{aligned} & \left| \int_{\mathbb{R}^{3+1}} \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^\delta \mathbf{B}_{\min\{\frac{\delta}{8}, \frac{1}{2a} - \frac{1}{4}\}} \|\phi\|_{F_{\mu, N}^{+, 1}} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}} \end{aligned} \quad (8.10)$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}^{3+1}} \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^\delta \mathbf{B}_{\min\{\frac{\delta}{8}, \frac{1}{2a} - \frac{1}{4}\}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \end{aligned} \quad (8.11)$$

In the region $\lambda_2 \gg \lambda_1$ we have the slightly stronger bound

$$\begin{aligned} & \left| \int_{\mathbb{R}^{3+1}} \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^\delta \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\delta}{8}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.12)$$

Similarly, when $\mu \ll \lambda_1$, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^{3+1}} \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} dx dt \right| \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^\delta \left(\frac{\mu}{\lambda_1}\right)^{\frac{\delta}{8}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.13)$$

Proof. We begin by decomposing the modulation (or distance to the relevant characteristic surface) and writing

$$\begin{aligned} & \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} \\ & = \sum_{d \in 2^{\mathbb{Z}}} C_d \phi_{\mu, N} (\mathcal{C}_{\leq d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \mathcal{C}_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2} + C_{< d} \phi_{\mu, N} (\mathcal{C}_d^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \mathcal{C}_{\leq d}^{\pm 2} \varphi_{\lambda_2, N_2} \\ & \quad + C_{< d} \phi_{\mu, N} (\mathcal{C}_{< d}^{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \mathcal{C}_d^{\pm 2} \varphi_{\lambda_2, N_2} \\ & = \sum_{d \in 2^{\mathbb{Z}}} A_0 + A_1 + A_2 \end{aligned}$$

Keeping in mind (7.11), we now divide the proof into cases depending on the relative sizes of the frequency and the modulation. Namely, we consider separately the low modulation cases

$$\lambda_1 \approx \lambda_2 \gg \mu \text{ and } d \ll \lambda_1, \quad \mu \gg \min\{\lambda_1, \lambda_2\} \text{ and } d \ll \mu, \quad \lambda_1 \approx \lambda_2 \approx \mu \text{ and } d \ll \mu$$

and the high modulation cases

$$\lambda_1 \approx \lambda_2 \gtrsim \mu \text{ and } d \gtrsim \lambda_1, \quad \mu \gg \min\{\lambda_1, \lambda_2\} \text{ and } d \gtrsim \mu.$$

Clearly, this covers all possible frequency combinations. The first case in the low modulation regime, where the two spinors are high frequency, is the easiest, as this case interacts very favourably with the null structure. The second case, when $\mu \gg \min\{\lambda_1, \lambda_2\}$, is more difficult, and is the main obstruction to the scale invariant Sobolev result. The final case, when $\mu \approx \lambda_1 \approx \lambda_2$ is the only resonant interaction, and this is where the bilinear estimates in Corollary 6.4 play a crucial role. In the remaining high modulation cases $d \gtrsim \max\{\mu, \lambda_1, \lambda_2\}$, the null structure of the system no longer plays any role, and we need to exploit the $Y_{\lambda, N}^{\pm, m}$ norms to gain the off diagonal decay term.

High-Low I: $\mu \ll \lambda_1 \approx \lambda_2$ and $d \ll \lambda_1$. Our goal is to show that

$$\begin{aligned} \sum_{d \ll \lambda_1} \left| \int_{\mathbb{R}^{1+3}} A_0 dxdt \right| + \left| \int_{\mathbb{R}^{1+3}} A_1 dxdt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 dxdt \right| \\ \lesssim \mu^{\frac{1}{2}} N_{\min}^{\delta} \left(\frac{\mu}{\lambda_1} \right)^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2} \end{aligned} \quad (8.14)$$

where we let $N_{\min} = \min\{N, N_1, N_2\}$. Clearly this gives the bounds (8.10), (8.11), and (8.13) in the region $\mu \ll \lambda_1 \approx \lambda_2$ and $d \ll \lambda_1$.

We now prove the bound (8.14). The condition $d \ll \lambda_1$, together with an application of Lemma 8.7, implies that we must have $\pm_1 = \pm_2$ and moreover, that the sum over the modulation is restricted to the region $\mu^{-1} \lesssim d \lesssim \mu$ (in particular this case is non-resonant). To estimate the first term, A_0 , we note that after another application of Lemma 8.7, we have the almost orthogonal decomposition

$$A_0 = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\alpha \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{q, q' \in \mathcal{Q}_\mu \\ |q - q'| \lesssim \mu}} C_d \phi_{\mu, N} (C_{\leq d}^{\pm_1} R_\kappa P_q \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm_2} R_{\kappa'} P_{q'} \varphi_{\lambda_2, N_2}$$

where $\alpha = (d\mu)^{\frac{1}{2}} \lambda_1^{-1}$. Then, using the null-structure by writing

$$C_{\leq d}^{\pm_1} R_\kappa P_{\lambda_1} = C_{\leq d}^{\pm_1, M} (\Pi_{\pm_1} - \Pi_{\pm_1}(\lambda_1 \omega)) R_\kappa P_{\lambda_1} + C_{\leq d}^{\pm_1, M} \Pi_{\pm_1}(\lambda_1 \omega_\kappa) R_\kappa P_{\lambda_1}$$

(here ω_κ denotes the centre of the cap κ) and applying Lemma 8.1, together with the uniform disposability of $C_{\leq d}^{\pm_1, M}$ from (7.1), we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int A_0 dxdt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\alpha \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{q, q' \in \mathcal{Q}_\mu \\ |q - q'| \lesssim \mu}} \alpha \|C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|R_\kappa P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} P_{q'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \mu^{\frac{1}{2}} \alpha^{-\epsilon} \left(\frac{\mu}{\lambda_1} \right)^{\frac{1}{2} - \epsilon} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}, \end{aligned} \quad (8.15)$$

where we used Lemma 7.2 to control the $L_{t,x}^2$ norm of the high-modulation term, and the bound (8.5). On the other hand, we can decompose

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa - \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} C_d R_{\kappa''} \phi_{\mu, N} (C_{\leq d}^{\pm_1} R_\kappa \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\leq d}^{\pm_2} R_{\kappa'} \varphi_{\lambda_2, N_2}$$

where $\beta = d^{\frac{1}{2}} \mu^{-\frac{1}{2}}$, again by almost orthogonality and Lemma 8.7. As above, we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int A_0 dxdt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa - \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} \beta \|C_d R_{\kappa''} \phi_{\mu, N}\|_{L_{t,x}^2} \|R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \beta^{1-\epsilon} d^{-\frac{1}{2}} \lambda (\beta N_{\min})^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm_1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm_2, M}^2}, \end{aligned} \quad (8.16)$$

where we used the angular concentration Lemma 8.5 on the lowest angular frequency term. Combining (8.15) and (8.16), by taking $\epsilon > 0$ sufficiently small, we obtain for every $0 < \delta \ll 1$

$$\begin{aligned} \sum_{\mu^{-1} \lesssim d \lesssim \mu} \left| \int A_0 dx dt \right| &\lesssim \sum_{\mu^{-1} \lesssim d \lesssim \mu} \left(\frac{d}{\mu} \right)^{\frac{\delta}{4}} N_{\min}^{\delta} \left(\frac{\mu}{\lambda} \right)^{\frac{1}{4}} \mu^{\frac{1}{2}} \|P_{\mu} H_N \phi\|_{V_{\pm,1}^2} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}} \\ &\lesssim N_{\min}^{\delta} \left(\frac{\mu}{\lambda} \right)^{\frac{1}{4}} \mu^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2} \end{aligned}$$

which gives (8.14) for the A_0 term. Next, we deal with the A_1 term. The argument is similar to the above, but the initial decomposition is slightly different as we no longer require the cube decomposition. Instead, we need to decompose the ϕ term into caps to ensure that the $C_{<d}$ multiplier is disposable. In more detail, the resonance bound in Lemma 8.7 gives

$$A_1 = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_{\alpha} \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{\kappa'' \in \mathcal{C}_{\beta} \\ |\kappa'' \pm 2\kappa'| \lesssim \beta}} C_{<d} R_{\kappa''} \phi_{\mu, N} (\mathcal{C}_d^{\pm 1} R_{\kappa} \psi_{\lambda_1, N_1})^{\dagger} \gamma^0 \mathcal{C}_{\leq d}^{\pm 2} R_{\kappa'} \varphi_{\lambda_2, N_2}$$

where $\alpha = \left(\frac{d\mu}{\lambda_1^2}\right)^{\frac{1}{2}}$ and $\beta = \left(\frac{d}{\mu}\right)^{\frac{1}{2}}$. By exploiting the null structure as previously, we then obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int A_1 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_{\alpha} \\ |\kappa - \kappa'| \lesssim \alpha}} \sum_{\substack{\kappa'' \in \mathcal{C}_{\beta} \\ |\kappa'' \pm 2\kappa'| \lesssim \beta}} \alpha \|R_{\kappa''} \phi_{\mu, N}\|_{L_{t,x}^4} \|\mathcal{C}_d^{\pm 1} R_{\kappa} \psi_{\lambda_1, N_1}\|_{L_{t,x}^2} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \alpha^{1-\epsilon} \mu^{\frac{1}{2}} d^{-\frac{1}{2}} \lambda_2^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}, \end{aligned} \quad (8.17)$$

where we used Lemma 7.2 to control the $L_{t,x}^2$ norm of the high-modulation term, and again used (8.5) To gain a power of d , we again exploit the angular concentration estimate by exploiting a similar argument to (8.16) to deduce that

$$\begin{aligned} \left| \int A_1 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_{\beta} \\ |\kappa - \kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \beta \|R_{\kappa''} \phi_{\mu, N}\|_{L_{t,x}^4} \|\mathcal{C}_d^{\pm 1} R_{\kappa} \psi_{\lambda_1, N_1}\|_{L_{t,x}^2} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \beta^{1-\epsilon} d^{-\frac{1}{2}} \lambda_2^{\frac{1}{2}} \mu^{\frac{1}{2}} (\beta N_{\min})^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.18)$$

Combining (8.18) and (8.18) as in the A_0 case, and summing up over $\mu^{-1} \lesssim d \lesssim \mu$ with ϵ sufficiently small, we obtain (8.14). The remaining term A_2 can be handled in an identical manner to the A_1 . Thus the bound (8.14) follows.

High-Low II: $\mu \gg \min\{\lambda_1, \lambda_2\}$ and $d \ll \mu$. Let $\{j, k\} = \{1, 2\}$ and $\lambda_j \geq \lambda_k$. Our goal is to prove that

$$\sum_{d \ll \mu} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_j dx dt \right| \lesssim \mu^{\frac{1}{2}} N_{\min}^{\delta} \left(\frac{\lambda_k}{\mu} \right)^{\frac{1}{8}} \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \quad (8.19)$$

On the other hand, for the A_k term, we have the weaker bounds

$$\sum_{d \ll \mu} \left| \int_{\mathbb{R}^{1+3}} A_k dx dt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_k}{\mu} \right)^{\frac{\delta}{8}} (\min\{N, N_j\})^{\delta} \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2} \quad (8.20)$$

and

$$\sum_{d \ll \mu} \left| \int_{\mathbb{R}^{1+3}} A_k dx dt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_k}{\mu} \right)^{\frac{1}{2a} - \frac{1}{4}} N_k^{\delta} \|\phi_{\mu, N}\|_{V_{\pm,1}^2} \begin{cases} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2} & k = 1 \\ \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, 1}^2} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}} & k = 2 \end{cases} \quad (8.21)$$

where $\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000}$ is as in the definition of the $Y_{\lambda, N}^{\pm, m}$ norm. Clearly (8.19), (8.20), and (8.21) give the estimates claimed in Theorem 8.8 in the region $\mu \gg \min\{\lambda_1, \lambda_2\}$ and $d \ll \mu$. Note that we have a weaker bound when the low frequency term has modulation away from the hyperboloid, and for this interaction, we are forced to exploit the $Y_{\lambda, N}^{\pm, m}$ norms.

We begin the proof of (8.19), (8.20), and (8.21) by observing that since the estimate is essentially symmetric in ψ and φ , it is enough to consider the case $\mu \approx \lambda_1 \gg \lambda_2$, in other words, we only consider the case $j = 1$ and $k = 2$. As in the previous case, as $d \ll \mu$, Lemma 8.7 implies that we only have a non-zero contribution if $\pm_1 = +$ and $\lambda_2^{-1} \lesssim d \lesssim \lambda_2$. To control the A_0 term, we decompose into caps of radius $\beta = (\frac{d}{\lambda_2})^{\frac{1}{2}}$ and cubes of diameter λ_2 . Lemma 8.7 implies that we have the almost orthogonality identity

$$A_0 = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp_2 \kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in Q_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} P_{q'} C_d \phi_{\mu, N} (P_q R_\kappa \mathcal{C}_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} \mathcal{C}_{\leq d}^{\pm_2} \varphi_{\lambda_2, N_2}.$$

Thus exploiting the null structure as previously, disposing of the $C_d^{\pm, m}$ multipliers using (7.1), and applying the $L_{t,x}^4$ Strichartz estimate we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp_2 \kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in Q_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} \beta \|P_{q'} C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|P_q R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \beta^{-\epsilon} \mu^{\frac{1}{2}} \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{4} - \epsilon} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.22)$$

On the other hand, by decomposing into

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp_2 \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} R_{\kappa''} C_d \phi_{\mu, N} (R_\kappa \mathcal{C}_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} \mathcal{C}_{\leq d}^{\pm_2} \varphi_{\lambda_2, N_2}$$

and using the angular concentration bound Lemma 8.5 on the smallest angular frequency term, a similar argument gives

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp_2 \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} \beta \|C_d R_{\kappa''} \phi_{\mu, N}\|_{L_{t,x}^2} \|R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \mu^{\frac{1}{2}} \beta^{\frac{1}{4} - \epsilon} N_{\min}^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.23)$$

As in the previous case, combining (8.22) and (8.23) with ϵ sufficiently small gives (8.19) for the A_0 term. The A_1 term can be estimated by an identical argument (since the high modulation term is again at frequency μ). To control the A_2 component, we start by again applying Lemma 8.7 and decomposing into

$$A_2 = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp_2 \kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in Q_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} P_{q'} C_{< d} \phi_{\mu, N} (P_q R_\kappa \mathcal{C}_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} \mathcal{C}_d^{\pm_2} \varphi_{\lambda_2, N_2}$$

where as usual $\beta = (\frac{d}{\lambda_2})^{\frac{1}{2}}$. Applying the by now standard null form bound, (7.1), and the $L_{t,x}^4$ Strichartz estimate, we conclude that for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\beta \\ |\kappa \mp_2 \kappa'| \lesssim \beta}} \sum_{\substack{q, q' \in Q_{\lambda_2} \\ |q - q'| \lesssim \lambda_2}} \beta \|P_{q'} \phi_{\mu, N}\|_{L_{t,x}^4} \|P_q R_\kappa \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \mathcal{C}_d^{\pm_2} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^2} \\ &\lesssim \mu^{\frac{1}{2}} \beta^{-\epsilon} \left(\frac{\mu}{\lambda_2}\right)^\epsilon \|\phi_{\mu, N}\|_{V_{+,1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{+,M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.24)$$

Note that we get no high frequency gain here (in fact we have a slight loss due to the sum over cubes). On the other hand, by decomposing all three terms into caps of size β , using null structure, the $L_t^q L_x^4$ Strichartz estimate in Lemma 8.2, and Bernstein's inequality followed by Lemma 7.2 for φ_{λ_2, N_2} , we obtain for any

$$2 < q < 2 + \frac{2}{3}$$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \beta \|R_{\kappa''} \phi_{\mu, N}\|_{L_t^{\frac{q}{q-2}} L_x^{\frac{2q}{4-q}}} \|R_{\kappa} \psi_{\lambda_1, N_1}\|_{L_t^q L_x^4} \|R_{\kappa''} \mathcal{C}_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_t^q L_x^{\frac{4q}{5q-8}}} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{d}{\lambda_2}\right)^{\frac{1}{q} - \frac{1}{4} - \epsilon} \left(\frac{\lambda_2}{\mu}\right)^{\frac{5}{q} - \frac{9}{4}} N_1 \|\phi_{\mu, N}\|_{V_{\mp, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\mp, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, 2, M}^2} \end{aligned} \quad (8.25)$$

(schematically, we are putting the product into $L_t^{\infty-} L_x^{2+} \times L_t^{2+} L_x^4 \times L_t^{2+} L_x^{4-}$). Switching the roles of $\phi_{\mu, N}$ and ψ_{λ_1, N_1} , and combining (8.24) and (8.25) with q close to 2, and $\epsilon > 0$ sufficiently small, we obtain (8.20).

It remains to prove (8.21), thus we need to consider the case where φ also has the smallest angular frequency. We begin by again using Lemma 8.7 to decompose

$$A_2 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q'' \in Q_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} R_{\kappa''} P_{q''} C_{< d} \phi_{\mu, N} (R_{\kappa} P_q \mathcal{C}_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} \mathcal{C}_d^{\pm 2} \varphi_{\lambda_2, N_2}$$

where $\beta = \left(\frac{d}{\lambda_2}\right)^{\frac{1}{2}}$. An application of Bernstein's inequality, Lemma 7.2, and the angular concentration lemma for φ , together with the null form bound, and Lemma 8.2, implies that for any $\epsilon > 0$ sufficiently small

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa \mp 2\kappa'|, |\kappa'' \pm 2\kappa'| \lesssim \beta}} \sum_{\substack{q, q'' \in Q_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} \beta \|R_{\kappa''} P_{q''} \phi_{\mu, N}\|_{L_t^{\frac{2\alpha}{\alpha-1}} L_x^{2\alpha}} \|R_{\kappa} P_q \psi_{\lambda_1, N_1}\|_{L_t^{\frac{2\alpha}{\alpha-1}} L_x^{2\alpha}} \|R_{\kappa'} \mathcal{C}_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_t^q L_x^{\frac{\alpha}{\alpha-1}}} \\ &\lesssim \beta^{1-\epsilon} \left(\frac{\mu}{\lambda_2}\right)^\epsilon (\mu \lambda_2)^{1-\frac{1}{\alpha}} (\beta^2 \lambda_2^3)^{\frac{1}{\alpha} - \frac{1}{2}} (\beta N_2)^\delta \|P_\mu H_N \phi\|_{V_{\mp, 1}^2} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\mathcal{C}_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_t^q L_x^2} \\ &\lesssim \mu^{\frac{1}{2}} N_2^\delta \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{2\alpha} - \frac{1}{4}} \left(\frac{d}{\lambda_2}\right)^{\frac{1}{2}(\frac{\delta}{2} - b + \frac{1}{\alpha})} \|\phi_{\mu, N}\|_{V_{\mp, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\mp, M}^2} \|\varphi\|_{Y_{\lambda_2, N_2}^{\pm 2, M}} \end{aligned}$$

which gives (8.21) since $\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000}$, and $b - \frac{1}{a} = \frac{2}{a} - 1 < \frac{\sigma}{500} < \frac{\delta}{5}$.

High-High: $\mu \approx \lambda_1 \approx \lambda_2$ and $d \ll \mu$. Our goal is to prove that if $M \geq \frac{1}{2}$, then for any $\delta > 0$ we have the bound

$$\begin{aligned} \sum_{d \ll \mu} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_1 dx dt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 dx dt \right| &\lesssim \mu^{\frac{1}{2}} N_{\min}^\delta \|\phi_{\mu, N}\|_{V_{\mp, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, 2, M}^2} \end{aligned} \quad (8.26)$$

while if $0 < M < \frac{1}{2}$, for every $s, \delta > 0$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} \sum_{d \ll \mu} A_0 + A_1 + A_2 dx dt \right| &\lesssim \mu^{\frac{1}{2}} N_{\min}^\delta (1 + \mu^{-\frac{1}{6} + s} N_{\min}^{\frac{7}{30}}) \|\phi_{\mu, N}\|_{V_{\mp, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, 2, M}^2}. \end{aligned} \quad (8.27)$$

The key difference to the previous cases, is that if $0 < M \leq \frac{1}{2}$, we no longer have the non-resonant bound $d \gtrsim \mu^{-1}$, and thus we also have to estimate the *resonant* interactions $d \ll \mu^{-1}$. This is particularly challenging in light of the fact that in the *strongly resonant regime*, $0 < M < \frac{1}{2}$, there is no gain from the null structure when $d \ll \mu^{-1}$. However, we do have *transversality* in the region $d \ll \mu^{-1}$, and consequently, we can apply the key bilinear restriction estimate in Corollary 6.4. On the other hand, in the *weakly resonant regime*, $M = \frac{1}{2}$, somewhat surprisingly and in stark contrast to the cases $M \neq \frac{1}{2}$, the null structure gives cancellation for *all* modulation scales.

We start by considering the non-resonant region $\mu^{-1} \lesssim d \lesssim \mu$. By decomposing into caps of radius $\beta = (\frac{d}{\mu})^{\frac{1}{2}}$, an application of Lemma 8.7 gives the identity

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\pm_1 \kappa \mp_2 \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} R_{\kappa''} C_d \phi_{\mu, N} (R_{\kappa} \mathcal{C}_{\leq d}^{\pm_1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} \mathcal{C}_{\leq d}^{\pm_2} \varphi_{\lambda_2, N_2}.$$

Thus by applying the $L_{t,x}^4$ Strichartz bound, exploiting the null structure as previously (here we need the assumption $d \gtrsim \mu^{-1}$), and using the angular concentration bound in Lemma 8.5 on N_{min} , we obtain for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dx \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\pm_1 \kappa \mp_2 \kappa'|, |\kappa'' \pm_2 \kappa'| \lesssim \beta}} \beta \|R_{\kappa''} C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|R_{\kappa} \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \beta^{1-\epsilon} d^{-\frac{1}{2}} \mu (\beta N_{min})^\delta \|\phi_{\mu, N}\|_{V_{\pm, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, 2, M}^2}. \end{aligned}$$

Taking $\delta > 0$ and $\epsilon > 0$ sufficiently small, and summing up over the modulation $\mu^{-1} \lesssim d \lesssim \mu$ then gives (8.26) and (8.27) for A_0 in the region $\mu^{-1} \lesssim d \lesssim \mu$. A similar argument bounds the A_1 and A_2 terms in (8.26) and (8.27) provided the sum over modulation is restricted to $\mu^{-1} \lesssim d \lesssim \mu$.

We now consider the case $d \ll \mu^{-1}$. Note that if $M > \frac{1}{2}$, then using Lemma 8.7, we see that $A_0 = A_1 = A_2 = 0$ and thus (8.26) is immediate. On the other hand, if we are in the weakly resonant regime $M = \frac{1}{2}$, then another application of Lemma 8.7 implies that $\pm_1 = +$, $\pm_2 = -$, and we have the decomposition

$$A_0 = \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa + \kappa'|, |\kappa'' - \kappa| \lesssim \beta}} \sum_{\substack{q, q' \in Q_{\mu^2 \beta} \\ |q + q'| \lesssim \mu^2 \beta}} R_{\kappa''} C_d \phi_{\mu, N} (R_{\kappa} P_q \mathcal{C}_{\leq d}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} P_{q'} \mathcal{C}_{\leq d}^- \varphi_{\lambda_2, N_2}$$

where $\beta = (\frac{d}{\mu})^{\frac{1}{2}}$. Therefore, using the null form type bound (8.1), together with (ii) in Lemma 8.1 to exploit the null structure, the orthogonality estimate in Lemma 8.6, and an application of Lemma 8.2 gives for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dx dt \right| &\lesssim \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_\beta \\ |\kappa - \kappa'|, |\kappa'' - \kappa| \lesssim \beta}} \sum_{\substack{q, q' \in Q_{\mu^2 \beta} \\ |q + q'| \lesssim \mu^2 \beta}} \beta \|R_{\kappa''} C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|R_{\kappa} P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|R_{\kappa'} P_{q'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \beta \times d^{-\frac{1}{2}} \times \mu \times \beta^{-\epsilon} (\mu \beta)^{-\epsilon} \times (\beta N_{min})^\delta \|\phi_{\mu, N}\|_{V_{\pm, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, M}^2} \|\varphi\|_{V_{\pm, M}^2} \end{aligned}$$

where we used the angular concentration bound in Lemma 8.5 on the term with smallest angular frequency. Choosing $\epsilon > 0$ sufficiently small, and summing up over $0 < d \ll \mu^{-1}$ then gives (8.26) for the A_0 term. An identical argument bounds the A_1 and A_2 terms.

It remains to prove (8.27) when $0 < d \ll \mu^{-1}$. Another application of Lemma 8.7, implies that we must have $\pm_1 = +$ and $\pm_2 = -$, as well as the key orthogonality identity

$$\begin{aligned} &\sum_{d \ll \mu^{-1}} A_0 + A_1 + A_2 \\ &= C_{\ll \mu^{-1}} \phi_{\mu, N} (C_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 C_{\ll \mu^{-1}}^- \varphi_{\lambda_2, N_2} \\ &= \sum_{\substack{\kappa, \kappa', \kappa'' \in \mathcal{C}_{\mu^{-1}} \\ |\kappa + \kappa'| \lesssim \mu^{-1}}} \sum_{\substack{q, q'' \in Q_\mu \\ |q - q''| \approx \mu \text{ or } |\kappa - \kappa''| \approx \mu^{-1}}} R_{\kappa''} P_{q''} C_{\ll \mu^{-1}} \phi_{\mu, N} (R_{\kappa} P_q \mathcal{C}_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1})^\dagger \gamma^0 R_{\kappa'} \mathcal{C}_{\ll \mu^{-1}}^- \varphi_{\lambda_2, N_2}. \end{aligned}$$

Note that the summation is restricted to terms for which $R_{\kappa''} P_{q''} C_{\ll \mu^{-1}} \phi_{\mu, N}$ and $R_{\kappa'} P_q \mathcal{C}_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1}$ have either angular orthogonality, or radial orthogonality. In either case, we may apply Corollary 6.4 (via the bound (8.7)), the null structure bound in Lemma 8.1, and the Klein-Gordon angular Strichartz estimate in

Lemma 8.3, to deduce that for every $\frac{3}{2} < q < \frac{7}{10}$ and $\epsilon > 0$ we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{1+3}} \sum_{d \ll \mu^{-1}} A_0 + A_1 + A_2 dxdt \right| \\
& \lesssim \mu^{-1} \sum_{\kappa, \kappa'' \in \mathcal{C}_{\mu^{-1}}} \sum_{\substack{q, q' \in \mathcal{Q}_{\mu} \\ |q - q''| \approx \mu \text{ or } |\kappa - \kappa''| \approx \mu^{-1}}} \|R_{\kappa''} P_{q''} \mathcal{C}_{\ll \mu^{-1}} \phi_{\mu, N} (R_{\kappa} P_q \mathcal{C}_{\ll \mu^{-1}}^+ \psi_{\lambda_1, N_1})^\dagger\|_{L_{t,x}^q} \|\varphi_{\lambda_2, N_2}\|_{L_{t,x}^{q'}} \\
& \lesssim \mu^{\frac{5}{q} - 3 + \epsilon} N_2^{7(\frac{7}{10} - \frac{1}{q}) + \epsilon} \|\phi_{\mu, N}\|_{V_{\pm, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, M}^2}
\end{aligned}$$

where for ease of reading we suppressed the $\Pi_{\pm}(\omega_{\kappa})$ matrices used to extract the null form gain of μ^{-1} . Choosing q sufficiently close to $\frac{3}{2}$, and $\epsilon > 0$ sufficiently small, then gives (8.27) in the case $N_2 = N_{min}$. To deal with remaining cases, we just reverse the roles of ϕ , ψ , and φ , again apply Lemma 8.7 to deduce the required transversality, and always use the angular Strichartz estimate from Lemma 8.3 on the term with smallest angular frequency. This completes the proof of (8.27).

High modulation I: $\mu \lesssim \lambda_1 \approx \lambda_2$ and $d \gtrsim \lambda_1$. We now consider the high modulation case. In this region, the null structure plays no role, and thus the arguments are significantly easier. Our goal is to prove that

$$\sum_{d \gtrsim \lambda_1} \left| \int_{\mathbb{R}^{1+3}} A_1 dxdt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 dxdt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1} \right)^{\frac{1}{8}} \|\phi_{\mu, N}\|_{V_{\pm, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, M}^2} \quad (8.28)$$

and for every $\delta > 0$, the weaker bounds

$$\sum_{d \gtrsim \lambda_1} \left| \int_{\mathbb{R}^{1+3}} A_0 dxdt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1} \right)^{\frac{\delta}{8}} (\min\{N_1, N_2\})^{\delta} \|\phi_{\mu, N}\|_{V_{\pm, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, M}^2} \quad (8.29)$$

and

$$\sum_{d \gtrsim \lambda_1} \left| \int_{\mathbb{R}^{1+3}} A_0 dxdt \right| \lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1} \right)^{\frac{1}{a} - \frac{1}{2}} \|\phi\|_{Y_{\mu, N}^{\pm, 1}} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, M}^2} \quad (8.30)$$

where a is as in the definition of the $Y_{\lambda, N}^{\pm, m}$ norm. We start with the estimates (8.29) and (8.30) for the A_0 component. Decomposing ψ and φ into cubes of size μ , together with an application of the $L_{t,x}^4$ Strichartz estimate gives for all $\epsilon > 0$

$$\begin{aligned}
\left| \int_{\mathbb{R}^{1+3}} A_0 dxdt \right| & \lesssim \sum_{\substack{q, q' \in \mathcal{Q}_{\mu} \\ |q - q'| \lesssim \mu}} \|C_d \phi_{\mu, N}\|_{L_{t,x}^2} \|P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|P_{q'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\
& \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_1}{\mu} \right)^{\epsilon} \left(\frac{\lambda_1}{d} \right)^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{\pm, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, M}^2}.
\end{aligned} \quad (8.31)$$

As in the proof of (8.25), if we instead apply the $L_t^q L_x^4$ bound, together with Bernstein's inequality for ϕ , we obtain for any $2 < q < 2 + \frac{2}{11}$

$$\begin{aligned}
\left| \int_{\mathbb{R}^{1+3}} A_0 dxdt \right| & \lesssim \|C_d \phi_{\mu, N}\|_{L_t^q L_x^{\frac{4q}{5q-8}}} \|\psi_{\lambda_1, N_1}\|_{L_t^q L_x^4} \|\varphi_{\lambda_2, N_2}\|_{L_t^{\frac{q}{q-2}} L_x^{\frac{2q}{4-q}}} \\
& \lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_1}{d} \right)^{\frac{1}{q}} \left(\frac{\mu}{\lambda_1} \right)^{\frac{6}{q} - \frac{1}{4}} N_1 \|\phi_{\mu, N}\|_{V_{\pm, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm, M}^2}
\end{aligned} \quad (8.32)$$

(schematically, we are putting the product into $L_t^{2+} L_x^{4-} \times L_t^{2+} L_x^4 \times L_t^{\infty-} L_x^{2+}$). Switching the roles of ψ_{λ_1, N_1} and φ_{λ_2, N_2} , and combining (8.31) and (8.32) with q sufficiently close to 2 and $\epsilon > 0$ sufficiently small, followed by summing up over $d \gtrsim \lambda_1$, we obtain (8.29). On the other hand, to obtain (8.30), we again use Lemma

8.2 to deduce that

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_0 dxdt \right| &\lesssim \sum_{\substack{q, q' \in Q_\mu \\ |q - q'| \lesssim \mu}} \|C_d \phi_{\mu, N}\|_{L_t^\alpha L_x^{\frac{\alpha}{\alpha-1}}} \|P_q \psi_{\lambda_1, N_1}\|_{L_t^{\frac{2\alpha}{\alpha-1}} L_x^{2\alpha}} \|P_{q'} \varphi_{\lambda_2, N_2}\|_{L_t^{\frac{2\alpha}{\alpha-1}} L_x^{2\alpha}} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_1}{d}\right)^b \left(\frac{\mu}{\lambda}\right)^{b+\frac{1}{\alpha}-1-\epsilon} \|\phi\|_{Y_{\mu, N}^{+, 1}} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2} \end{aligned}$$

which then gives (8.30) if we choose ϵ sufficiently small as $\frac{1}{\alpha} > \frac{1}{2}$ and $b + \frac{1}{\alpha} - 1 = 4(\frac{1}{\alpha} - \frac{1}{2})$ (here a, b are as in the definition of the $Y_\lambda^{\pm, m}$ norm).

We now turn to the estimates for A_1 and A_2 . By symmetry, it is enough to consider the A_1 term. After decomposing into cubes of size μ and applying the $L_{t,x}^4$ Strichartz estimate, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^{1+3}} A_1 dt dx \right| &\lesssim \sum_{\substack{q, q' \in Q_\mu \\ |q - q'| \lesssim \mu}} \|\phi_{\mu, N}\|_{L_{t,x}^4} \|\mathcal{C}_d^{\pm 1} P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^2} \|P_{q'} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\mu}{\lambda_1}\right)^{\frac{1}{4}-\epsilon} \left(\frac{\lambda_1}{d}\right)^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned}$$

Summing up over $d \gtrsim \lambda_1$ and choosing ϵ sufficiently small, then gives (8.28).

High modulation II: $\mu \gg \min\{\lambda_1, \lambda_2\}$ and $d \gtrsim \mu$. Our goal is to prove the bound

$$\begin{aligned} \sum_{d \gtrsim \mu} \left| \int_{\mathbb{R}^{1+3}} A_0 dxdt \right| + \left| \int_{\mathbb{R}^{1+3}} A_1 dxdt \right| + \left| \int_{\mathbb{R}^{1+3}} A_2 dxdt \right| \\ \lesssim \mu^{\frac{1}{2}} \left(\frac{\min\{\lambda_1, \lambda_2\}}{\mu}\right)^{\frac{1}{4}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.33)$$

As the estimate is essentially symmetric in λ_1 and λ_2 , we may assume that $\lambda_1 \geq \lambda_2$. The bound for A_0 follows by decomposing into cubes of size λ_2 and applying the standard $L_{t,x}^4$ Strichartz estimate to obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^{3+1}} A_0 dt dx \right| &\lesssim \sum_{\substack{q, q'' \in Q_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} \|C_d P_{q''} \phi_{\mu, N}\|_{L_{t,x}^2} \|P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|\varphi_{\lambda_2, N_2}\|_{L_{t,x}^4} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_2}{\mu}\right)^{\frac{3}{4}-\epsilon} \left(\frac{\mu}{d}\right)^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2} \end{aligned}$$

which easily gives (8.33) for the A_0 term, provided we choose ϵ sufficiently small. The proof for the A_1 term is identical (as we do not exploit any null structure here). On the other hand, to estimate the A_2 term, we again decompose into cubes of size λ_2 and apply the $L_{t,x}^4$ Strichartz estimate to deduce that

$$\begin{aligned} \left| \int_{\mathbb{R}^{3+1}} A_2 dt dx \right| &\lesssim \sum_{\substack{q, q'' \in Q_{\lambda_2} \\ |q - q''| \lesssim \lambda_2}} \|P_{q''} \phi_{\mu, N}\|_{L_{t,x}^4} \|P_q \psi_{\lambda_1, N_1}\|_{L_{t,x}^4} \|\mathcal{C}_d^{\pm 2} \varphi_{\lambda_2, N_2}\|_{L_{t,x}^2} \\ &\lesssim \mu^{\frac{1}{2}} \left(\frac{\lambda_2}{\mu}\right)^{\frac{1}{2}-\epsilon} \left(\frac{\mu}{d}\right)^{\frac{1}{2}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned}$$

Therefore (8.33) follows. This completes the proof of Theorem 8.8. \square

8.4. Proof of Theorem 7.5. We begin with the proof of (7.9). An application of the energy inequality in Lemma 7.3 gives

$$\begin{aligned} \left\| P_{\lambda_1} H_{N_1} \Pi_{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{V_{\pm 1, M}^2} \\ \lesssim \sup_{\|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \lesssim 1} \left| \int_{\mathbb{R}^{1+3}} \phi_{\mu, N} (\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2} dxdt \right|. \end{aligned}$$

Therefore an application of (8.10) in Theorem 8.8 implies that

$$\begin{aligned} & \left\| P_{\lambda_1} H_{N_1} \Pi_{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{V_{\pm 1, M}^2} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{4}} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2a} - \frac{1}{4}\}} \|\phi\|_{F_{\mu, N}^{+, 1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}} \end{aligned} \quad (8.34)$$

which gives the required bound (7.9) for the $F_{\lambda_1, N_1}^{\pm 1, M}$ component of the norm. To complete the proof of (7.9), it remains show that there exists $\epsilon > 0$ such that

$$\left\| \Pi_{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{Y_{\lambda_1, N_1}^{\pm 1, M}} \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{2}} \mathbf{B}_\epsilon \|\phi\|_{F_{\mu, N}^{+, 1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \quad (8.35)$$

To this end, we consider separately the cases $\lambda_1 \ll \lambda_2$ and $\lambda_1 \gtrsim \lambda_2$. In the former region, note that an application of (8.12) in Theorem 8.8 together with the energy inequality Lemma 7.3, and the $L_{t,x}^2$ bound in Lemma 7.2, gives

$$\begin{aligned} & \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_{t,x}^2} \\ & \lesssim d^{-\frac{1}{2}} \left\| P_{\lambda_1} H_{N_1} \Pi_{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{V_{\pm 1, M}^2} \\ & \lesssim d^{-\frac{1}{2}} \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{4}} \left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{\sigma}{32}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.36)$$

On the other hand, since we are localised away from the hyperboloid we have by (7.7) together with Lemma 8.2

$$\begin{aligned} & \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_t^{\frac{3}{2}} L_x^2} \\ & \lesssim d^{-1} \left\| P_{\lambda_1} (\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}) \right\|_{L_t^{\frac{3}{2}} L_x^2} \\ & \lesssim d^{-1} \|\phi_{\mu, N}\|_{L_{t,x}^4} \|\varphi_{\lambda_2, N_2}\|_{L_t^{\frac{12}{5}} L_x^4} \\ & \lesssim d^{-1} \mu^{\frac{1}{2}} \lambda_2^{\frac{1}{3}} N_2 \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.37)$$

Repeating this argument but instead putting $\phi \in L_t^{\frac{12}{5}} L_x^4$ and $\varphi \in L_{t,x}^4$ we deduce that, since $\lambda_1 \ll \lambda_2 \approx \mu$,

$$\begin{aligned} & d \lambda_1^{-\frac{1}{3}} \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_t^{\frac{3}{2}} L_x^2} \\ & \lesssim \mu^{\frac{1}{2}} \min\{N, N_2\} \left(\frac{\lambda_2}{\lambda_1} \right)^{\frac{1}{3}} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned} \quad (8.38)$$

Note that this bound is far to weak to be useful on its own, as we have $\lambda_1 \ll \lambda_2$. On the other hand, if we combine (8.36) and (8.38), and use the convexity of the L_t^p spaces, we deduce that if we let $0 < \theta < 1$ be given by $\frac{1}{a} = \frac{2\theta}{3} + \frac{1-\theta}{2}$, then, as this forces $b = \frac{1+\theta}{2}$, we deduce that

$$\begin{aligned} & \lambda_1^{\frac{1}{a}-b} d^b \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_t^a L_x^2} \\ & \lesssim \left(d \lambda_1^{-\frac{1}{3}} \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_t^{\frac{3}{2}} L_x^2} \right)^\theta \\ & \quad \times \left(d^{\frac{1}{2}} \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_{t,x}^2} \right)^{1-\theta} \\ & \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{\sigma}{32}(1-\theta) - \frac{1}{3}\theta} \|\phi_{\mu, N}\|_{V_{+, 1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}. \end{aligned}$$

Since $\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000}$, it is easy enough to check that $\frac{\sigma}{32}(1-\theta) - \frac{1}{3}\theta > 0$, and hence (8.35) holds when $\lambda_1 \ll \lambda_2$. We now consider the case $\lambda_1 \gtrsim \lambda_2$. The proof is similar to the previous case, the main difference is that we need a more refined version of the bound (8.38). To this end, by decomposing φ into cubes of size

$\min\{\mu, \lambda_2\}$, we deduce that by Lemma 8.2 and Lemma 8.6, for every $\epsilon' > 0$

$$\begin{aligned}
& \|P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\|_{L_t^{\frac{3}{2}} L_x^2} \\
& \lesssim d^{-1} \left\| \left\| \phi_{\mu, N} \gamma^0 \Pi_{\pm 2} P_q \varphi_{\lambda_2, N_2} \right\|_{L_x^2}^2 \right\|_{L_t^{\frac{3}{2}}}^{\frac{1}{2}} \\
& \lesssim d^{-1} \|\phi_{\mu, N}\|_{L_t^{\frac{12}{5}} L_x^4} \left(\sum_{q \in Q_{\min\{\mu, \lambda_2\}}} \|P_q \varphi_{\lambda_2, N_2}\|_{L_{t,x}^4}^2 \right)^{\frac{1}{2}} \\
& \lesssim d^{-1} \mu^{\frac{1}{3}} N (\min\{\mu, \lambda_2\})^{\frac{1}{4} - \epsilon'} \lambda_2^{\frac{1}{4} + \epsilon'} \|\phi_{\mu, N}\|_{V_{\pm 1, 1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}.
\end{aligned}$$

Since $(\min\{\mu, \lambda_2\})^{\frac{1}{4} - \epsilon'} \leq \mu^{\frac{1}{6}} \lambda_2^{\frac{1}{4} - \frac{1}{6} + \epsilon'}$ (for ϵ' sufficiently small) and $\lambda_2 \lesssim \lambda_1$, by using the bound (8.37), we deduce that

$$\begin{aligned}
& d \lambda_1^{-\frac{1}{3}} \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\right\|_{L_t^{\frac{3}{2}} L_x^2} \\
& \lesssim \mu^{\frac{1}{2}} \min\{N, N_2\} \|\phi_{\mu, N}\|_{V_{\pm 1, 1}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}.
\end{aligned} \tag{8.39}$$

Note that, unlike the bound (8.39), we have no high frequency loss here. As in the case $\lambda_1 \ll \lambda_2$, we now combine the bound (8.34) with (8.39), and deduce by the convexity of the L_t^p norm and Lemma 7.2, that

$$\begin{aligned}
& \lambda_1^{\frac{1}{\alpha} - b} d^b \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\right\|_{L_t^\alpha L_x^2} \\
& \lesssim \left(d \lambda_1^{-\frac{1}{3}} \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\right\|_{L_t^{\frac{3}{2}} L_x^2} \right)^\theta \\
& \quad \times \left(d^{\frac{1}{2}} \left\| P_{\lambda_1} H_{N_1} \mathcal{C}_d^{\pm 1} \mathcal{I}_M^{\pm 1} [\phi_{\mu, N} \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}]\right\|_{L_{t,x}^2} \right)^{1-\theta} \\
& \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2\alpha} - \frac{1}{4}\}}^{1-\theta} \|\phi\|_{F_{\mu, N}^{+, 1}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}.
\end{aligned}$$

Since $0 < \theta \ll \sigma$, we obtain (8.35). Therefore, the bound (7.9) follows.

We now turn to the proof of the second inequality (7.10). The argument is similar to the proof of (7.9) so we will be brief. An application of the energy inequality in Lemma 7.3 together with (8.11) in Theorem 8.8 implies that

$$\left\| P_\mu H_N \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{V_{\pm 1, 1}^2} \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\frac{\sigma}{4}} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2\alpha} - \frac{1}{4}\}} \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \tag{8.40}$$

Therefore it only remains to prove that there exists $\epsilon > 0$ such that

$$\left\| \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{Y_{\mu, N}^{+, 1}} \lesssim \mu^{\frac{1}{2}} (\min\{N, N_2\})^{\frac{\sigma}{2}} \mathbf{B}_\epsilon \|\psi\|_{F_{\lambda_1, N_1}^{\pm 1, M}} \|\varphi\|_{F_{\lambda_2, N_2}^{\pm 2, M}}. \tag{8.41}$$

Similar to the proof of (8.35), we consider separately the cases $\mu \ll \lambda_1$ and $\mu \gtrsim \lambda_1$. In the former case, as in (8.39), since we are localised away from the hyperboloid we have by (7.7) together with Lemma 8.2

$$\begin{aligned}
& \left\| P_\mu H_N \mathcal{C}_d^{\pm 1} \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \right\|_{L_t^{\frac{3}{2}} L_x^2} \\
& \lesssim d^{-1} \left\| P_\mu ((\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}) \right\|_{L_t^{\frac{3}{2}} L_x^2} \\
& \lesssim d^{-1} (\min\{\lambda_1, \lambda_2\})^{\frac{1}{3}} (\max\{\lambda_1, \lambda_2\})^{\frac{1}{2}} \min\{N_1, N_2\} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}.
\end{aligned} \tag{8.42}$$

Since $\lambda_1 \approx \lambda_2$, we can replace the max and min in (8.42) with $\lambda_1^{\frac{1}{3} + \frac{1}{2}}$. If we now combine (8.42) with the energy inequality in Lemma 7.3, the bound (8.13) in Theorem 8.8, and Lemma 7.2, we deduce that by the

convexity of the L_t^p spaces that

$$\begin{aligned}
& \mu^{\frac{1}{a}-b} d^b \|P_\mu H_N C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_t^a L_x^2} \\
& \lesssim \left(d\mu^{-\frac{1}{3}} \|P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_t^{\frac{3}{2}} L_x^2} \right)^\theta \\
& \quad \times \left(d^{\frac{1}{2}} \|P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_{t,x}^2} \right)^{1-\theta} \\
& \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \left(\frac{\lambda_1}{\mu} \right)^{\frac{\sigma}{32}(1-\theta) - \frac{5}{6}\theta} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}
\end{aligned}$$

where as previously, we have $\frac{1}{a} = \frac{2\theta}{3} + \frac{1-\theta}{2}$ (which implies that $b = \frac{1+\theta}{2}$). Since $\frac{1}{2} < \frac{1}{a} < \frac{1}{2} + \frac{\sigma}{1000}$, it is easy enough to check that $\frac{\sigma}{32}(1-\theta) - \frac{5}{6}\theta > 0$, and hence (8.41) holds when $\mu \ll \lambda_1$. We now consider the case $\mu \gtrsim \lambda_1$. Since we now have $(\min\{\lambda_1, \lambda_2\})^{\frac{1}{3}} (\max\{\lambda_1, \lambda_2\})^{\frac{1}{2}} \lesssim \mu^{\frac{1}{3} + \frac{1}{2}}$, an application of (8.42), together with (8.40), Lemma 7.2 gives

$$\begin{aligned}
& \mu^{\frac{1}{a}-b} d^b \|P_\mu H_N C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_t^a L_x^2} \\
& \lesssim \left(d\mu^{-\frac{1}{3}} \|P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_t^{\frac{3}{2}} L_x^2} \right)^\theta \\
& \quad \times \left(d^{\frac{1}{2}} \|P_\mu C_d^+ \mathcal{I}_1^+ [(\Pi_{\pm 1} \psi_{\lambda_1, N_1})^\dagger \gamma^0 \Pi_{\pm 2} \varphi_{\lambda_2, N_2}] \|_{L_{t,x}^2} \right)^{1-\theta} \\
& \lesssim \mu^{\frac{1}{2}} (\min\{N_1, N_2\})^{\theta + \frac{\sigma}{4}(1-\theta)} \mathbf{B}_{\min\{\frac{\sigma}{32}, \frac{1}{2a} - \frac{1}{4}\}}^{1-\theta} \|\psi_{\lambda_1, N_1}\|_{V_{\pm 1, M}^2} \|\varphi_{\lambda_2, N_2}\|_{V_{\pm 2, M}^2}
\end{aligned}$$

Since $0 < \theta \ll \sigma$ and $\frac{1}{a} > \frac{1}{2}$, we obtain (8.41). Therefore, the bound (7.9) follows. This completes the proof of Theorem 7.5.

REFERENCES

1. Alain Bachelot, *Problème de Cauchy global pour des systèmes de Dirac-Klein-Gordon*, Ann. Inst. H. Poincaré Phys. Théor. **48** (1988), no. 4, 387–422. MR 969173 (90i:81024)
2. Ioan Bejenaru, *Optimal bilinear restriction estimates for general hypersurfaces and the role of the shape operator*, 1601.01000.
3. Ioan Bejenaru and Sebastian Herr, *On global well-posedness and scattering for the massive Dirac-Klein-Gordon system*, arXiv:1409.1778, to appear in J. Eur. Math. Soc. (JEMS).
4. ———, *Convolutions of singular measures and applications to the Zakharov system*, J. Funct. Anal. **261** (2011), no. 2, 478–506. MR 2793120
5. ———, *The cubic Dirac equation: small initial data in $H^1(\mathbb{R}^3)$* , Comm. Math. Phys. **335** (2015), no. 1, 43–82. MR 3314499
6. Ioan Bejenaru, Sebastian Herr, Justin Holmer, and Daniel Tataru, *On the 2D Zakharov system with L^2 -Schrödinger data*, Nonlinearity **22** (2009), no. 5, 1063–1089. MR 2501036
7. Ioan Bejenaru, Sebastian Herr, and Daniel Tataru, *A convolution estimate for two-dimensional hypersurfaces*, Rev. Mat. Iberoam. **26** (2010), no. 2, 707–728. MR 2677013
8. Jonathan Bennett and Neal Bez, *Some nonlinear Brascamp-Lieb inequalities and applications to harmonic analysis*, J. Funct. Anal. **259** (2010), no. 10, 2520–2556. MR 2679017
9. Jonathan Bennett, Anthony Carbery, and Terence Tao, *On the multilinear restriction and Kakeya conjectures*, Acta Math. **196** (2006), no. 2, 261–302. MR 2275834
10. Jonathan Bennett, Anthony Carbery, and James Wright, *A non-linear generalisation of the Loomis-Whitney inequality and applications*, Math. Res. Lett. **12** (2005), no. 4, 443–457. MR 2155223
11. James D. Bjorken and Sidney D. Drell, *Relativistic quantum mechanics*, McGraw-Hill Book Co., New York-Toronto-London, 1964. MR 0187641 (32 #5091)
12. Jean Bourgain, *On the restriction and multiplier problems in \mathbf{R}^3* , Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., vol. 1469, Springer, Berlin, 1991, pp. 179–191. MR 1122623
13. ———, *Estimates for cone multipliers*, Geometric aspects of functional analysis (Israel, 1992–1994), Oper. Theory Adv. Appl., vol. 77, Birkhäuser, Basel, 1995, pp. 41–60. MR 1353448
14. ———, *Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notices (1998), no. 5, 253–283. MR 1616917
15. Stefan Buschenhenke, Detlef Müller, and Ana Vargas, *A Fourier Restriction Theorem For A Twodimensional Surface Of Finite Type*, arXiv:1508.00791.
16. John M. Chadam and Robert T. Glassey, *On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions*, Arch. Rational Mech. Anal. **54** (1974), 223–237. MR 0369952

17. Yonggeun Cho and Sanghyuk Lee, *Strichartz estimates in spherical coordinates*, Indiana Univ. Math. J. **62** (2013), no. 3, 991–1020. MR 3164853
18. Piero D’Ancona, Damiano Foschi, and Sigmund Selberg, *Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 4, 877–899.
19. Damiano Foschi and Sergiu Klainerman, *Bilinear space-time estimates for homogeneous wave equations*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 2, 211–274. MR 1755116
20. Jean Ginibre, Yoshio Tsutsumi, and Giorgio Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), no. 2, 384–436. MR 1491547
21. Martin Hadac, Sebastian Herr, and Herbert Koch, *Well-posedness and scattering for the KP-II equation in a critical space*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 3, 917–941. MR 2526409
22. Markus Keel and Terence Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980.
23. Sergiu Klainerman and Matei Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), no. 9, 1221–1268. MR 1231427
24. Sergiu Klainerman, Igor Rodnianski, and Terence Tao, *A physical space approach to wave equation bilinear estimates*, J. Anal. Math. **87** (2002), 299–336, Dedicated to the memory of Thomas H. Wolff. MR 1945287 (2003m:35139)
25. Sergiu Klainerman and Sigmund Selberg, *Bilinear estimates and applications to nonlinear wave equations*, Commun. Contemp. Math. **4** (2002), no. 2, 223–295.
26. Herbert Koch and Stefan Steinerberger, *Convolution estimates for singular measures and some global nonlinear Brascamp-Lieb inequalities*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 6, 1223–1237. MR 3427606
27. Herbert Koch and Daniel Tataru, *Dispersive estimates for principally normal pseudodifferential operators*, Comm. Pure Appl. Math. **58** (2005), no. 2, 217–284.
28. Herbert Koch, Daniel Tataru, and Monica Vişan, *Dispersive equations and nonlinear waves. Generalized Korteweg-de Vries, nonlinear Schrödinger, wave and Schrödinger maps.*, Basel: Birkhäuser/Springer, 2014 (English).
29. Sanghyuk Lee, *Bilinear restriction estimates for surfaces with curvatures of different signs*, Trans. Amer. Math. Soc. **358** (2006), no. 8, 3511–3533 (electronic). MR 2218987 (2007a:42023)
30. ———, *Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces*, J. Funct. Anal. **241** (2006), no. 1, 56–98. MR 2264247
31. Sanghyuk Lee, Keith M. Rogers, and Ana Vargas, *Sharp null form estimates for the wave equation in \mathbb{R}^{3+1}* , Int. Math. Res. Not. IMRN (2008), Art. ID rnn 096, 18. MR 2439536
32. Sanghyuk Lee and Ana Vargas, *Sharp null form estimates for the wave equation*, Amer. J. Math. **130** (2008), no. 5, 1279–1326. MR 2450209
33. ———, *Restriction estimates for some surfaces with vanishing curvatures*, J. Funct. Anal. **258** (2010), no. 9, 2884–2909. MR 2595728
34. Shuji Machihara, Kenji Nakanishi, and Tohru Ozawa, *Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation*, Rev. Mat. Iberoamericana **19** (2003), no. 1, 179–194.
35. Adela Moyua, Ana Vargas, and Luis Vega, *Restriction theorems and maximal operators related to oscillatory integrals in \mathbb{R}^3* , Duke Math. J. **96** (1999), no. 3, 547–574. MR 1671214
36. E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.
37. Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192
38. Jacob Sterbenz, *Global regularity and scattering for general non-linear wave equations. II. $(4+1)$ dimensional Yang-Mills equations in the Lorentz gauge*, Amer. J. Math. **129** (2007), no. 3, 611–664. MR 2325100 (2008e:58037)
39. Jacob Sterbenz and Daniel Tataru, *Energy dispersed large data wave maps in $2+1$ dimensions*, Comm. Math. Phys. **298** (2010), no. 1, 139–230. MR 2657817
40. Robert S. Strichartz, *Multipliers for spherical harmonic expansions*, Trans. Amer. Math. Soc. **167** (1972), 115–124. MR 0306823
41. ———, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714. MR 0512086
42. Terence Tao, *Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates*, Math. Z. **238** (2001), no. 2, 215–268. MR 1865417
43. ———, *A sharp bilinear restrictions estimate for paraboloids*, Geom. Funct. Anal. **13** (2003), no. 6, 1359–1384.
44. ———, *Some recent progress on the restriction conjecture*, Fourier analysis and convexity, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2004, pp. 217–243. MR 2087245
45. Terence Tao and Ana Vargas, *A bilinear approach to cone multipliers. I. Restriction estimates*, Geom. Funct. Anal. **10** (2000), no. 1, 185–215.
46. ———, *A bilinear approach to cone multipliers. II. Applications*, Geom. Funct. Anal. **10** (2000), no. 1, 216–258.
47. Terence Tao, Ana Vargas, and Luis Vega, *A bilinear approach to the restriction and Keakeya conjectures*, J. Amer. Math. Soc. **11** (1998), no. 4, 967–1000. MR 1625056
48. Xuecheng Wang, *On Global Existence of 3D Charge Critical Dirac-Klein-Gordon System*, International Mathematics Research Notices **2015** (2015), no. 21, 10801–10846.
49. Thomas Wolff, *A sharp bilinear cone restriction estimate*, Ann. of Math. (2) **153** (2001), no. 3, 661–698.

(T. Candy) UNIVERSITÄT BIELEFELD, FAKULTÄT FÜR MATHEMATIK, POSTFACH 100131, 33501 BIELEFELD, GERMANY
E-mail address: `tcandy@math.uni-bielefeld.de`

(S. Herr) UNIVERSITÄT BIELEFELD, FAKULTÄT FÜR MATHEMATIK, POSTFACH 100131, 33501 BIELEFELD, GERMANY
E-mail address: `herr@math.uni-bielefeld.de`