

# CHARACTERIZATION OF SHARP GLOBAL GAUSSIAN ESTIMATES FOR SCHRÖDINGER HEAT KERNELS

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ABSTRACT. We investigate when the fundamental solution of the Schrödinger equation  $\partial_t = \Delta + V$  possesses sharp Gaussian bounds global in space and time. We give a characterization for  $V \leq 0$  and a sufficient condition for general  $V$ .

## 1. INTRODUCTION AND MAIN RESULTS

Let  $d \in \mathbb{N}$ . For  $x, y \in \mathbb{R}^d$  and  $t > 0$  we consider the Gaussian kernel

$$g(t, x, y) = g(t, y - x) = (4\pi t)^{-d/2} e^{-|y-x|^2/(4t)}.$$

It is the fundamental solution of the heat equation  $\partial_t = \Delta$ . For a function  $V: \mathbb{R}^d \rightarrow \mathbb{R}$  we let  $G$  be the fundamental solution of  $\partial_t = \Delta + V$ , determined by the following Duhamel or perturbation formula for  $t > 0$ ,  $x, y \in \mathbb{R}^d$ ,

$$G(t, x, y) = g(t, x, y) + \int_0^t \int_{\mathbb{R}^d} G(s, x, z) V(z) g(t-s, z, y) dz ds.$$

We aim at the *sharp global Gaussian bounds* of  $G$ , which mean that there are numbers  $0 < c_1 \leq 1 \leq c_2$  such that

$$(1) \quad c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

Clearly, (1) implies the plain global Gaussian bounds, which only require numbers  $0 < \varepsilon_1, c_1 \leq 1 \leq \varepsilon_2, c_2 < \infty$  such that for all  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$(2) \quad c_1 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t\varepsilon_1}} \leq G(t, x, y) \leq c_2 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t\varepsilon_2}}.$$

To characterize (1) we let

$$S(V, t, x, y) = \int_0^t \int_{\mathbb{R}^d} \frac{g(s, x, z) g(t-s, z, y)}{g(t, x, y)} |V(z)| dz ds, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

This will often be abbreviated to  $S(V, t)$ ,  $S(V)$  or  $S$ , and we always assume that  $V$  is Borel measurable. Denote, as usual,

$$\|S(V)\|_\infty = \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y).$$

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The results of Bogdan, Hansen and Jakubowski [3] and Zhang [10] give enough evidence in favor of using  $S(V)$  in this and more general contexts.

We will say that  $V$  has bounded potential for bridges globally in time, if  $\|S(V)\|_\infty < \infty$ , in which case we can largely resolve (1) thanks to the following folklore result.

**Lemma 1.1.** *If  $\eta := \|S(V^+)\|_\infty < 1$  and  $S(V^-)$  is locally bounded, then*

$$(3) \quad e^{-S(V^-, t, x, y)} \leq \frac{G(t, x, y)}{g(t, x, y)} \leq \frac{1}{1 - \eta}, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

*If  $V \leq 0$ , then (1) holds if and only if  $\|S(V)\|_\infty < \infty$ . If  $V \geq 0$ , then (1) implies  $\|S(V)\|_\infty < \infty$ .*

Here, as usual,  $V^+ = \max(0, V)$  and  $V^- = \max(0, -V)$ . The last statement of the lemma easily follows from Duhamel formula. The rest of the lemma is an excerpt from [2, Lemma 1.1 and Lemma 1.2], where it is proved based on [3, 4]. We note that  $S(V) = \infty$  for every nontrivial  $V$  in dimensions  $d = 1$  and 2, see, e.g., [2, Lemma 1.3], and so (1) is impossible for nontrivial  $V \geq 0$  and nontrivial  $V \leq 0$  in these dimensions. To characterize the boundedness of  $S(V)$ , for  $d \geq 3$  and  $x, y \in \mathbb{R}^d$  we define

$$(4) \quad K(x, y) = \frac{e^{-(|x||y|-x \cdot y)/2}}{|x|^{d-2}} (1 + |x||y|)^{d/2-3/2},$$

where  $x \cdot y$  is the usual scalar product, and we let

$$K(V, x, y) = \int_{\mathbb{R}^d} |V(z)| K(z - x, y) dz.$$

We also denote

$$\|V\|_K = \|K(V)\|_\infty.$$

Here is our main result.

**Theorem 1.2.** *There are constants  $M_1, M_2$  depending only on  $d$ , such that*

$$(5) \quad M_1 \|V\|_K \leq \|S(V)\|_\infty \leq M_2 \|V\|_K.$$

Here by constants we mean positive numbers. The proof of Theorem 1.2 is given in Section 2. In view of (5) and of the second and the third statements of Lemma 1.1, the condition  $\|V\|_K < \infty$  may replace  $\|S(V)\|_\infty < \infty$  in characterizing (1), which will be often used without mention. Similarly, sufficient smallness of  $\|V\|_K$  yields (3) in view of the first statement of Lemma 1.1.

**Corollary 1.3.** *If  $V \leq 0$ , then (1) holds if and only if  $K(V)$  is bounded.*

Compared with  $S(V)$ ,  $K(V)$  is easier to investigate, because  $K(V)$  has one argument less than  $S(V)$ . This leads to considerable progress in analysis of (1), which we now present. For  $d \geq 3$  we let  $C_d = \Gamma(d/2 - 1)/(4\pi^{d/2})$  and

$$-\Delta^{-1}V(x) = \int_0^\infty \int_{\mathbb{R}^d} g(t, x, z)V(z) dz du = C_d \int_{\mathbb{R}^d} \frac{1}{|z - x|^{d-2}} V(z) dz.$$

For  $d = 3$  the formula for  $K$  simplifies and we easily obtain

$$(6) \quad \|V\|_K = C_d^{-1} \|\Delta^{-1}|V|\|_\infty,$$

thus  $\|\Delta^{-1}|V|\|_\infty$  resolves (1) in the same way as  $\|S(V)\|_\infty$ . For instance, if  $d = 3$  and  $V \leq 0$ , then the sharp global Gaussian bounds (1) are equivalent to

the condition  $\|\Delta^{-1}V\|_\infty < \infty$ . This equivalence was first proved by Milman and Semenov [7, Remark (3) on p. 4].

The main focus of the present paper is on the case of  $d \geq 4$ . Let

$$\|V\|_{d/2} = \left( \int_{\mathbb{R}^d} |V(z)|^{d/2} dz \right)^{2/d}.$$

**Proposition 1.4.** *If  $d \geq 4$ , then*

$$(7) \quad C_d^{-1} \|\Delta^{-1}|V|\|_\infty \leq \|V\|_K \leq 2^{(d-3)/2} \left( C_d^{-1} \|\Delta^{-1}|V|\|_\infty + \kappa_d \|V\|_{d/2} \right).$$

The result is an analogue of [6, Corollary 1]. In Section 2 we give the proof and specify the constant  $\kappa_d$ . As a consequence,  $\|\Delta^{-1}V\|_\infty < \infty$  is necessary for (1) if  $V \leq 0$  and if  $V \geq 0$ , cf. Lemma 1.1. On the other hand for every  $d \geq 3$  there is  $V \leq 0$  such that  $\|V\|_K < \infty$ , i.e., (1) holds, but  $V \notin L^1(\mathbb{R}^d) \cup \bigcup_{p>1} L^p_{loc}(\mathbb{R}^d)$ , in particular  $\|V\|_{d/2} = \infty$ , see [2].

A long-standing open problem on (1) for  $V \leq 0$  posed by Liskevich and Semenov [6, p. 602] reads as follows: “The validity of the two-sided estimates for the case  $d > 3$  without the additional assumption  $V \in L^{d/2}$  is an open question.” The question is whether  $\|V\|_K$  and  $\|\Delta^{-1}V\|_\infty$  are comparable for  $d > 3$ . It turns out that the answer is negative, as follows.

**Proposition 1.5.** *Let  $d \geq 4$ . For  $\mathbf{z} = (z_1, z_2, \dots, z_d) \in \mathbb{R}^d$  we write  $\mathbf{z} = (z_1, \mathbf{z}_2)$ , where  $\mathbf{z}_2 = (z_2, \dots, z_d) \in \mathbb{R}^{d-1}$ . We define*

$$\begin{aligned} A &= \{(z_1, \mathbf{z}_2) \in \mathbb{R}^d : z_1 > 4, |\mathbf{z}_2| \leq \sqrt{z_1}\}, \quad \text{and} \\ V(z_1, \mathbf{z}_2) &= -\frac{1}{z_1} \mathbf{1}_A(z_1, \mathbf{z}_2). \end{aligned}$$

*Then  $\|\Delta^{-1}V\|_\infty < \infty$  and  $\|V\|_K = \infty$ . There even is function  $V \leq 0$  with compact support and such that  $\|\Delta^{-1}V\|_\infty < \infty$  but  $\|V\|_K = \infty$ .*

Generally, for  $d \geq 4$ , neither finiteness nor smallness of  $\|\Delta^{-1}V\|_\infty$  are sufficient for the comparability of  $g$  and  $G$ , even for  $V$  with fixed sign and compact support.

Here are a few more comments to relate our result to existing literature. In [8] Milman and Semenov denote  $e(V, 0) = \|\Delta^{-1}|V|\|_\infty$  and introduce  $e_*(V, 0) = \sup_{\alpha \in \mathbb{R}^d} \|V(-\Delta + 2\alpha \cdot \nabla)^{-1}\|_{1 \rightarrow 1}$  to describe (1) – see [8, Theorem 1C]. The spatial anisotropy introduced by  $\alpha \cdot \nabla$  has a similar role as that seen in the integral defining  $S(V, t, x, y)$  and there are constants  $c_1, c_2$  depending only on  $d \geq 3$  such that

$$c_1 \|V\|_K \leq e_*(V, 0) \leq c_2 \|V\|_K.$$

This is proved in (9) below. For  $d = 3$  we have  $e(V, 0) = e_*(V, 0)$ . On the contrary, for  $d \geq 4$  by Proposition 1.5 there is  $V \leq 0$  such that  $e(V, 0) < \infty$  but  $e_*(V, 0) = \infty$ .

For the last remark we restrict ourselves to  $V \leq 0$ . Then the condition  $\|\Delta^{-1}V\|_\infty < \infty$  characterizes the plain global Gaussian bounds (2), see [9]. By (6), for  $d = 3$  (and  $V \leq 0$ ) the plain global Gaussian bounds (2) hold if and only if the sharp global Gaussian bounds (1) hold. In contrast, by Proposition 1.5 for  $d \geq 4$  the property (2) is weaker than (1).

The remainder of the paper is structured as follows. In Section 2 we prove Theorem 1.2, Proposition 1.4 and Proposition 1.5. Section 3 gives auxiliary

results, in particular the following crucial estimate of an inverse-Gaussian type integral.

**Theorem 1.6.** *Let  $c > 0$ ,  $\beta > 1$  and*

$$f(a, b) = \int_0^\infty u^{-\beta} e^{-c \left[ \sqrt{ub} - \frac{a}{\sqrt{u}} \right]^2} du, \quad a, b > 0.$$

We have

$$f(a, b) \stackrel{\beta, c}{\approx} \frac{(1 + 4ab)^{\beta-3/2}}{a^{2(\beta-1)}}.$$

Here  $\stackrel{\beta, c}{\approx}$  means that the ratio of both sides is bounded above and below by constants depending only on  $\beta$  and  $c$ .

## 2. PROOFS OF MAIN RESULTS

For  $t > 0$ ,  $x, y \in \mathbb{R}^d$ , we consider

$$(8) \quad \begin{aligned} N(V, t, x, y) &:= \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ &+ \int_{t/2}^t \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4(t-\tau))}}{(t-\tau)^{d/2}} |V(z)| dz d\tau = N(V, t, y, x). \end{aligned}$$

Clearly,  $S(V) = S(|V|)$  and  $N(V) = N(|V|)$ . Because of the work of Zhang [10],  $N$  is a convenient approximation of  $S$ . Namely, by [10, Lemma 3.1, Lemma 3.2], there are constants  $m_1, m_2$  depending only on  $d$  such that

$$(L) \quad S(V, t, x, y) \geq m_1 N(V, t/2, x, y), \quad t > 0, x, y \in \mathbb{R}^d,$$

$$(U) \quad S(V, t, x, y) \leq m_2 N(V, t, x, y), \quad t > 0, x, y \in \mathbb{R}^d.$$

As seen in [10], the comparability even holds for the kernels of  $S$  and  $N$ .

In this section we prove our main result, i.e., Theorem 1.2. We start by using  $N(V, t)$ , (U) and (L), to estimate  $S(V, t)$ .

**Lemma 2.1.** *Let  $t > 0$ . We have*

$$\int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \leq N(V, t)(x, y), \quad x, y \in \mathbb{R}^d,$$

and

$$\sup_{x, y} N(V, t)(x, y) \leq 2 \sup_{x, y} \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau.$$

*Proof.* The first inequality follows by the definition of  $N(V, t)(x, y)$ . For the proof of the second one we note that

$$\begin{aligned} &\int_{t/2}^t \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4(t-\tau))}}{(t-\tau)^{d/2}} |V(z)| dz d\tau \\ &= \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-x+(\tau/t)(x-y)|^2/(4\tau)}}{\tau} |V(z)| dz d\tau. \end{aligned}$$

□

For  $x, y \in \mathbb{R}^d$  we let

$$J(x, y) = \int_0^\infty \tau^{-d/2} e^{-\frac{|x-\tau y|^2}{4\tau}} d\tau.$$

In view of the discussion in Section 1 we have

$$\begin{aligned} e_*(V, 0) &= \sup_{\alpha \in \mathbb{R}^d} \|(-\Delta + 2\alpha \cdot \nabla)^{-1}|V|\|_\infty \\ &= (4\pi)^{-d/2} \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - x, y) |V(z)| dz. \end{aligned}$$

**Lemma 2.2.** *We have*

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \geq m_1 \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - x, y) |V(z)| dz.$$

and

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq 2m_2 \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - x, y) |V(z)| dz,$$

*Proof.* By (L) and Lemma 2.1,

$$\begin{aligned} \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) &\geq m_1 \sup_{t>0, x, y \in \mathbb{R}^d} N(|V|, t/2)(x, y) \\ &\geq m_1 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(2\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ &= m_1 \sup_{t>0, y, w \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} \frac{e^{-|z-y+\tau w|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ &= m_1 \sup_{y, w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - y, w) |V(z)| dz. \end{aligned}$$

By (U) and Lemma 2.1,

$$\begin{aligned} \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) &\leq m_2 \sup_{t>0, x, y \in \mathbb{R}^d} N(V, t)(x, y) \\ &\leq 2m_2 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ &\leq 2m_2 \sup_{t>0, y, w \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+\tau w|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ &= 2m_2 \sup_{y, w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - y, w) |V(z)| dz. \end{aligned}$$

□

*Proof of Theorem 1.2.* We claim (5) holds with  $M_1 > 0$  that depends only on  $d$ , and  $M_2 = m_2 2^d \int_0^\infty (1 \vee r)^{d/2-3/2} r^{-1/2} e^{-r} dr$ . To this end, according to Lemma 2.2, we analyze

$$J(z - x, y) = \int_0^\infty \tau^{-d/2} e^{-\frac{|z-x-\tau y|^2}{4\tau}} d\tau.$$

Obviously,  $J = \infty$  if  $d = 1$  or  $d = 2$ . For  $d \geq 3$  we observe that

$$\frac{|z-x-\tau y|^2}{4\tau} = \frac{1}{4} \left[ \frac{|z-x|}{\sqrt{\tau}} - \sqrt{\tau}|y| \right]^2 + \frac{1}{2} (|z-x||y| - (z-x) \cdot y),$$

and thus

$$J(z-x, y) = e^{-\frac{1}{2}(|z-x||y| - (z-x) \cdot y)} \int_0^\infty \tau^{-d/2} e^{-\frac{1}{4} \left[ \sqrt{\tau}|y| - \frac{|z-x|}{\sqrt{\tau}} \right]^2} d\tau.$$

Finally, by Theorem 1.6 with  $a = |z-x|/2$ ,  $b = |y|/2$ ,  $\beta = d/2$  and  $c = 1$ ,

$$(9) \quad J(z-x, y) \stackrel{d}{\approx} K(z-x, y).$$

This also gives the explicit constants, as a consequence of Remark 3.4. For instance we can take  $M_2 = 8m_2\sqrt{\pi}$  if  $d = 3$ .  $\square$

*Proof of Proposition 1.4.* The left hand side inequality follows from the identity  $K(V)(x, 0) = C_d^{-1}(-\Delta^{-1})|V|(x)$ . If  $y = 0$ , then the upper bound trivially holds. For  $y \neq 0$  we consider two domains of integration. We have

$$\begin{aligned} \int_{|z-x||y| \leq 1} K(z-x, y)|V(z)| dz &\leq 2^{(d-3)/2} \int_{|z-x||y| \leq 1} \frac{1}{|z-x|^{d-2}} |V(z)| dz \\ &\leq \frac{2^{(d-3)/2}}{C_d} \|\Delta^{-1}|V|\|_\infty. \end{aligned}$$

Furthermore, by a change of variables and Hölder inequality,

$$\begin{aligned} &\int_{|z-x||y| \geq 1} K(z-x, y)|V(z)| dz \\ &\leq 2^{(d-3)/2} \int_{|z-x||y| \geq 1} \frac{e^{-\frac{1}{2}(|z-x||y| - (z-x) \cdot y)}}{(|z-x||y|)^{(d-1)/2}} |y|^{d-2} |V(z)| dz \leq 2^{(d-3)/2} \kappa_d \|V\|_{d/2}, \end{aligned}$$

where

$$\kappa_d = \left( \int_{|w| > 1} \left( e^{-\frac{1}{2}(|w|-w \cdot 1)} |w|^{-(d-1)/2} \right)^{d/(d-2)} dw \right)^{(d-2)/d}.$$

The finiteness of  $\kappa_d$  follows from Lemma 3.5 below.  $\square$

*Proof of Proposition 1.5.* We use the notation introduced in the formulation of the theorem. First we prove that  $\|V\|_K = \infty$ . Let  $\mathbf{y} = (1, \mathbf{0}) \in \mathbb{R}^d$ ,  $\mathbf{x} = 0$ . Observe that for  $\mathbf{z} \in A$  we have

$$0 \leq |\mathbf{z}||\mathbf{y}| - \mathbf{z} \cdot \mathbf{y} = |\mathbf{z}| - z_1 = \frac{|\mathbf{z}_2|^2}{\sqrt{z_1^2 + |\mathbf{z}_2|^2} + z_1} \leq \frac{z_1}{\sqrt{z_1^2 + |\mathbf{z}_2|^2} + z_1} \leq 1$$

and thus also  $z_1 \leq |\mathbf{z}| \leq 2z_1$ . Then,

$$\begin{aligned} \|V\|_K &\geq \int_{\mathbb{R}^d} e^{-\frac{1}{2}(|\mathbf{z}||\mathbf{y}| - \mathbf{z} \cdot \mathbf{y})} |V(\mathbf{z})| \frac{(1 + |\mathbf{z}||\mathbf{y}|)^{\frac{d}{2} - \frac{3}{2}}}{|\mathbf{z}|^{d-2}} d\mathbf{z} \geq c \int_A \frac{1}{z_1} \frac{z_1^{\frac{d}{2} - \frac{3}{2}}}{z_1^{d-2}} d\mathbf{z} \\ &= c \int_4^\infty \int_{|\mathbf{z}_2| < \sqrt{z_1}} z_1^{-1+2-d+\frac{d}{2}-\frac{3}{2}} d\mathbf{z}_2 dz_1 = c_1 \int_4^\infty z_1^{-1+2-d+\frac{d}{2}-\frac{3}{2}+\frac{1}{2}(d-1)} dz_1 \\ &= c_1 \int_4^\infty \frac{1}{z_1} dz_1 = \infty. \end{aligned}$$

We now prove that  $\|\Delta^{-1}|V|\|_\infty < \infty$ . By the symmetric rearrangement inequality (see [5, Chapter 3]) we have

$$\sup_{\mathbf{x} \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{z} - \mathbf{x}|^{d-2}} |V(\mathbf{z})| d\mathbf{z} = \sup_{x_1 \in \mathbb{R}} \int_4^\infty \int_{\mathbb{R}^{d-1}} \frac{\mathbf{1}_{|\mathbf{z}_2| < \sqrt{z_1}}}{[(z_1 - x_1)^2 + |\mathbf{z}_2|^2]^{(d-2)/2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1$$

It suffices then to consider  $\mathbf{x} = (x_1, 0, \dots, 0)$  and we only need to show that the following three integrals are uniformly bounded for  $x_1 \geq 4$ . The first integral is

$$\begin{aligned} I_1 &= \int_{x_1 + \sqrt{x_1}}^\infty \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2} + |\mathbf{z}_2|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\ &\leq \int_{x_1 + \sqrt{x_1}}^\infty \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\ &= c \int_{x_1 + \sqrt{x_1}}^\infty \frac{1}{|z_1 - x_1|^{d-2}} \frac{1}{z_1} z_1^{\frac{1}{2}(d-1)} dz_1 = c \int_{\sqrt{x_1}}^\infty \frac{1}{z_1^{d-2}} (z_1 + x_1)^{\frac{d}{2} - \frac{3}{2}} dz_1 \\ &\leq c' \int_{\sqrt{x_1}}^{x_1} \frac{1}{z_1^{d-2}} x_1^{\frac{d}{2} - \frac{3}{2}} dz_1 + c' \int_{x_1}^\infty \frac{1}{z_1^{d-2}} z_1^{\frac{d}{2} - \frac{3}{2}} dz_1 \leq c'' < \infty. \end{aligned}$$

The second integral we consider is

$$\begin{aligned} I_2 &= \int_2^{x_1 - \sqrt{x_1}} \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2} + |\mathbf{z}_2|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\ &\leq \int_2^{x_1 - \sqrt{x_1}} \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 = c \int_2^{x_1 - \sqrt{x_1}} \frac{1}{(x_1 - z_1)^{d-2}} z_1^{\frac{d}{2} - \frac{3}{2}} dz_1 \\ &= c \int_{\sqrt{x_1}}^{x_1 - 2} \frac{1}{w^{d-2}} (x_1 - w)^{\frac{d}{2} - \frac{3}{2}} dw \leq c \int_{\sqrt{x_1}}^{x_1 - 2} \frac{1}{w^{d-2}} x_1^{\frac{d}{2} - \frac{3}{2}} dw \leq c' < \infty. \end{aligned}$$

The remaining integral is

$$\begin{aligned} I_3 &= \int_{x_1 - \sqrt{x_1}}^{x_1 + \sqrt{x_1}} \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|\mathbf{z} - \mathbf{x}|^{d-2}} \frac{1}{z_1} d\mathbf{z} \leq 2 \int_{x_1 - \sqrt{x_1}}^{x_1 + \sqrt{x_1}} \int_{|\mathbf{z}_2| < 2\sqrt{x_1}} \frac{1}{|\mathbf{z} - \mathbf{x}|^{d-2}} \frac{1}{x_1} d\mathbf{z} \\ &\leq 2 \int_{B(\mathbf{x}, 3\sqrt{x_1})} \frac{1}{|\mathbf{z} - \mathbf{x}|^{d-2}} \frac{1}{x_1} d\mathbf{z} \leq c < \infty. \end{aligned}$$

To prove the second statement of Proposition 1.5, for  $s > 0$  we let  $d_s f(x) = sf(\sqrt{s}x)$ . Note that the dilatation does not change the norms:

$$\|\Delta^{-1}(d_s f)\|_\infty = \|\Delta^{-1}f\|_\infty, \quad \|d_s f\|_K = \|f\|_K.$$

Moreover,  $\text{supp}(d_s f) \subseteq B(0, r/\sqrt{s})$  if  $\text{supp}(f) \subseteq B(0, r)$ ,  $r > 0$ . Now, let  $V \geq 0$  be a potential such that  $\|\Delta^{-1}V\|_\infty = C < \infty$  and  $\|V\|_K = \infty$ . Then for any  $r > 0$  we have  $\|\Delta^{-1}(V\mathbf{1}_{B_r})\|_\infty \leq C$  and  $\|V\mathbf{1}_{B_r}\|_K \rightarrow \infty$  as  $r \rightarrow \infty$ . For  $n \in \mathbb{N}$  we define

$$V_n = d_{r_n}^2(V\mathbf{1}_{B_{r_n}}),$$

where  $r_n$  is chosen such that  $\|V\mathbf{1}_{B_{r_n}}\|_K \geq 4^n$ . Thus  $\text{supp}(V_n) \subseteq B(0, 1)$ . Finally we consider  $\tilde{V} = \sum_{n=1}^\infty V_n/2^n$ . Then,

$$\|\tilde{V}\|_K \geq \|V_n\|_K/2^n \geq 2^n \rightarrow \infty, \quad n \rightarrow \infty,$$

and

$$\|\Delta^{-1}\tilde{V}\|_\infty \leq \sum_{n=1}^{\infty} \|\Delta^{-1}V_n\|_\infty / 2^n \leq C.$$

□

### 3. APPENDIX

In this section we collect auxiliary calculations.

**Lemma 3.1.** *Let  $\gamma > -1/2$ . Then*

$$h(x) = \int_0^\infty (x+s^2)^\gamma e^{-cs^2} ds \stackrel{\gamma,c}{\approx} (1+x)^\gamma, \quad x \geq 0.$$

*Proof.* By putting  $r = s^2$  we get

$$h(x) = (1+x)^\gamma \int_0^\infty \left(\frac{x+r}{1+x}\right)^\gamma r^{-1/2} \frac{e^{-cr} dr}{2},$$

Since for all  $x, r \geq 0$  we have

$$1 \vee r \geq \frac{x}{1+x} + \frac{r}{1+x} \geq \begin{cases} r/2, & \text{for } x \in (0, 1), \\ 1/2, & \text{for } x \geq 1, \end{cases}$$

the last integral in the above is comparable with a positive constant depending only on  $\gamma$  and  $c$ . □

**Remark 3.2.** If  $\gamma \geq 0$ , then  $h(x) \leq C(1+x)^\gamma$ ,  $x \geq 0$ , where  $C = \frac{1}{2} \int_0^\infty (1 \vee r)^\gamma r^{-1/2} e^{-cr} dr$ .

**Lemma 3.3.** *Let  $c > 0$ ,  $\beta > 1$  and*

$$I_{\text{app}}(a, b) = \int_0^\infty \left(\frac{s + \sqrt{4ab + s^2}}{2a}\right)^{2(\beta-1)} \frac{e^{-cs^2} ds}{\sqrt{4ab + s^2}}, \quad a, b > 0.$$

*Then*

$$I_{\text{app}}(a, b) \stackrel{\beta,c}{\approx} \frac{(1+4ab)^{\beta-3/2}}{a^{2(\beta-1)}}.$$

*Proof.* Observe that  $0 \leq s \leq \sqrt{4ab + s^2}$ . Thus with  $h(x)$  and  $\gamma = \beta - 3/2$  from Lemma 3.1 we have

$$(10) \quad 2^{-2(\beta-1)} a^{-2(\beta-1)} \leq \frac{I_{\text{app}}(a, b)}{h(4ab)} \leq a^{-2(\beta-1)}.$$

The assertion follows by Lemma 3.1. □

*Proof of Theorem 1.6.* By substitution  $u = (a/b)r$  we obtain

$$f(a, b) = (a/b)^{1-\beta} \int_0^\infty r^{-\beta+1} e^{-cab} \left[\sqrt{r} - \frac{1}{\sqrt{r}}\right]^2 \frac{dr}{r}.$$

By change of variables from  $r$  to  $1/r$  we get

$$f(a, b) = (a/b)^{1-\beta} \int_0^\infty r^{\beta-1} e^{-cab} \left[\sqrt{r} - \frac{1}{\sqrt{r}}\right]^2 \frac{dr}{r}.$$

Finally, we let  $\sqrt{r} - 1/\sqrt{r} = s/\sqrt{ab}$ , then  $(\sqrt{r} - s/\sqrt{4ab})^2 = 1 + s^2/(4ab)$ . Note that  $\sqrt{r} > s/\sqrt{4ab}$ , hence

$$r = \left( s/\sqrt{4ab} + \sqrt{1 + s^2/(4ab)} \right)^2 = \left( s + \sqrt{4ab + s^2} \right)^2 / (4ab),$$

and

$$\begin{aligned} dr &= 2 \left( s + \sqrt{4ab + s^2} \right) \left( 1 + s/\sqrt{4ab + s^2} \right) ds / (4ab) \\ &= 2 \left( s + \sqrt{4ab + s^2} \right)^2 ds / \left( 4ab\sqrt{4ab + s^2} \right) \\ &= 2r ds / \sqrt{4ab + s^2}. \end{aligned}$$

This gives

$$f(a, b) = 2 \int_{-\infty}^{\infty} \left( \frac{s + \sqrt{4ab + s^2}}{2a} \right)^{2(\beta-1)} \frac{e^{-cs^2} ds}{\sqrt{4ab + s^2}}.$$

By splitting the last integral we have

$$\begin{aligned} f(a, b) &= 2 \int_0^{\infty} \left( \frac{s + \sqrt{4ab + s^2}}{2a} \right)^{2(\beta-1)} \frac{e^{-cs^2} ds}{\sqrt{4ab + s^2}} \\ &\quad + 2 \int_0^{\infty} \left( \frac{-s + \sqrt{4ab + s^2}}{2a} \right)^{2(\beta-1)} \frac{e^{-cs^2} ds}{\sqrt{4ab + s^2}}. \end{aligned}$$

Since  $\beta > 1$  and  $0 \leq -s + \sqrt{4ab + s^2} \leq s + \sqrt{4ab + s^2}$ , we have

$$(11) \quad 2 I_{\text{app}}(a, b) \leq f(a, b) \leq 4 I_{\text{app}}(a, b).$$

The proof is ended by an applications of Lemma 3.3.  $\square$

**Remark 3.4.** Using (11), (10) and Remark 3.2 we get an explicit constant in the upper bound in Theorem 1.6 for  $\beta \geq 3/2$ :

$$f(a, b) \leq C \frac{(1 + 4ab)^{\beta-3/2}}{a^{2(\beta-1)}},$$

where

$$C = 2 \int_0^{\infty} (1 \vee r)^{\beta-3/2} r^{-1/2} e^{-cr} dr.$$

In particular if  $\beta = 3/2$ , then  $C = \sqrt{4\pi/c}$ .

We now verify the finiteness of  $\kappa_d$  from the statement of Proposition 1.4.

**Lemma 3.5.** *Let  $d \geq 3$ . Then,*

$$\int_{\mathbb{R}^d \setminus B(0,1)} e^{-(|w|-w \cdot 1)} |w|^{-\beta} dw < \infty \iff \beta > (d+1)/2.$$

*Proof.* We always have

$$\int_{\{w \in \mathbb{R}^d \setminus B(0,1) : w \cdot 1 \leq |w|/\sqrt{2}/2\}} e^{-(|w|-w \cdot 1)} |w|^{-\beta} dw < \infty,$$

therefore we only need to characterize the finiteness of the complementary integral. We will follow the usual notation for spherical coordinates in  $\mathbb{R}^d$  [1].

In particular,  $w \cdot 1 = r \cos \varphi_1$  and the Jakobian is  $r^{d-1} \prod_{k=1}^{d-2} \sin^k(\varphi_{d-1-k})$ . We denote  $\varphi = \varphi_1$ , and we consider

$$\begin{aligned} & \int_1^\infty \int_0^{\pi/4} e^{-r(1-\cos \varphi)} r^{-\beta+d-1} \sin^{d-2} \varphi \, d\varphi \, dr \\ &= \int_0^{\pi/4} h(\varphi) \frac{\sin^{d-2} \varphi}{(1-\cos \varphi)^{d-\beta}} \, d\varphi, \end{aligned}$$

where  $h(\varphi) = \int_{1-\cos \varphi}^\infty e^{-s} s^{-\beta+d-1} \, ds$ . If  $\beta = d$ , then  $h(\varphi) \approx 1 + |\log(1 - \cos \varphi)|$  and  $\int_0^{\pi/4} h(\varphi) \sin^{d-2} \varphi \, d\varphi < \infty$ , as needed. If  $\beta > d$ , then  $h(\varphi) \approx (1 - \cos \varphi)^{d-\beta}$ , and the integral is finite, too. If  $\beta < d$ , then  $h(\varphi) \approx 1$  and  $\int_0^{\pi/4} \frac{\sin^{d-2} \varphi}{(1-\cos \varphi)^{d-\beta}} \, d\varphi \approx \int_0^{\pi/4} \varphi^{(d-2)-2(d-\beta)} \, d\varphi$ , which converges if and only if  $\beta > (d+1)/2$ .  $\square$

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