LONG RANGE SCATTERING FOR THE CUBIC DIRAC EQUATION ON $\mathbb{R}^{1+1}$

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Abstract. We show that the cubic Dirac equation, also known as the Thirring model, scatters at infinity to a linear solution modulo a phase correction.

1. Introduction

We consider the cubic Dirac equation (also known as the Thirring model)

$$\begin{aligned}(\partial_t + \partial_x)u &= iv + i|v|^2 u \\ (\partial_t - \partial_x)v &= iu + i|u|^2 v\end{aligned} \tag{1.1}$$

with data $u(1) = f$, $v(1) = g$ where $u, v : \mathbb{R}^{1+1} \to \mathbb{C}$. This model was introduced by Thirring in [16] and describes the self interaction of a Dirac field.

It is known that solutions exist globally in time, provided that the data $f, g \in L^2$ [4]. With regards to regularity, this is sharp in the sense that the L^2_x norm is scale invariant (at least for no mass). Earlier local and global well-posedness results can be found in [13]. However the question of asymptotic behaviour is largely unknown. Some recent work in this direction has shown *orbital stability* of the solitons [5, 12], but this leaves open the question of pointwise behaviour. In higher dimensions, $n > 1$, the Thirring model is globally well-posed for small data and scatters to a linear solution in the scale invariant Sobolev space [1, 2, 3]. Thus in the small data regime, the asymptotic behaviour is understood provided $n \neq 1$.

In the current article, our goal is present a first step towards understanding the pointwise asymptotic stability of the Dirac equation (1.1). More precisely, we adapt the arguments of Lindblad-Soffer [9, 10, 11] (see also [14, 15]), and show via energy estimates, together with an ODE argument, that the cubic nonlinearity causes an additional phase correction in the scattering behaviour. Our main result is as follows.

Theorem 1.1. *Let $N \geqslant 1$. There exists $\epsilon > 0$ such that if the data satisfies*

$$\big\|\langle x\rangle^{3+\frac{N}{2}} f\big\|_{H^{N+4}} + \big\|\langle x\rangle^{3+\frac{N}{2}} g\big\|_{H^{N+4}} \leqslant \epsilon,$$

then in the exterior region $1 \leqslant t \leqslant \langle x\rangle$ we have

$$|u(t,x)| + |v(t,x)| \lesssim \langle x\rangle^{-\frac{N}{2}} \Big(\big\|\langle x\rangle^{3+\frac{N}{2}} f\big\|_{H^{N+4}} + \big\|\langle x\rangle^{3+\frac{N}{2}} g\big\|_{H^{N+4}}\Big).$$

On the other hand, when $t \geqslant \langle x\rangle$, there exists bounded continuous functions $f_\pm$, such that

$$\begin{aligned}u(t,x) &= \frac{1}{\sqrt{t-x}}\Big(e^{i\rho+2i|f_+(\frac{x}{t})|^2\ln(\rho)} f_+\big(\tfrac{x}{t}\big) + e^{-i\rho+2i|f_-(\frac{x}{t})|^2\ln(\rho)} f_-\big(\tfrac{x}{t}\big)\Big) + \mathcal{O}\Big(\frac{\rho^{-\frac12}}{\sqrt{t-x}}\Big),\\ v(t,x) &= \frac{1}{\sqrt{t+x}}\Big(e^{i\rho+2i|f_+(\frac{x}{t})|^2\ln(\rho)} f_+\big(\tfrac{x}{t}\big) - e^{-i\rho+2i|f_-(\frac{x}{t})|^2\ln(\rho)} f_-\big(\tfrac{x}{t}\big)\Big) + \mathcal{O}\Big(\frac{\rho^{-\frac12}}{\sqrt{t+x}}\Big)\end{aligned}$$

as $\rho = \sqrt{t^2 - x^2} \to \infty$.

We have made no attempt to optimise the decay or regularity assumptions on the data, and it is clear that the proof given below can be improved to somewhat sharpen the assumptions on the data. However, this would complicate the statements of our results, and thus we shall not pursue this further here.

The proof of Theorem 1.1 in the exterior region only exploits the additional decay of the Klein-Gordon equation when $t \leqslant \langle x\rangle$ by using an argument of Klainerman [8]. In particular, the argument introduced here can be used to remove the compact support assumptions from related works on the cubic Klein-Gordon

The first author acknowledges support from the German Research Foundation via Collaborative Research Center 701.

equation [9, 14, 6]. On the other hand, in the interior region, we follow the argument given in [9], and use energy estimates on the hyperboloids $\{(t,x)|t^2 - x^2 = \rho^2\}$, together with an ODE formulation which reveals the precise asymptotic correction to the linear flow.

It is worth noting that, if we square the system (1.1), we obtain a nonlinear Klein-Gordon equation of the (schematic) form

$$\Box\phi + \phi = \phi^3 + \phi^2 \partial\phi \tag{1.2}$$

thus it is tempting to try and deduce the asymptotic behaviour of (u, v) from the corresponding result on the cubic Klein-Gordon equation given in [14, 6]. However, the nonlinear terms in (1.2) *do not* satisfy the requirements needed to apply the previous results, and thus we have to work harder to obtain the asymptotic behaviour given in Theorem 1.1.

2. Exterior Region

Given $T \geqslant 1$, we consider the domain

$$\mathcal{D}_T = \{(t,x) \in \mathbb{R}^{1+1} \,|\, \langle x \rangle \geqslant t,\ 1 \leqslant t \leqslant T\}$$

with boundary $S_T = \{(T,x) \in \mathbb{R}^{1+1} \,|\, \langle x \rangle \geqslant T\} \cup \{(\langle x \rangle, x) \,|\, \langle x \rangle \leqslant T\}$. Define

$$E_{ext,T}(\phi, f) = \left(\int_{S_T} n^\alpha\, Q_{\alpha 0}[\phi]\, f\, dx \right)^{\frac{1}{2}}$$

with $n^\alpha \partial_\alpha = \partial_t$ on $\{\langle x \rangle \geqslant T, t = T\}$, $n^\alpha \partial_\alpha = \partial_t + \frac{x}{\langle x \rangle}\partial_x$ on $\{\langle x \rangle = t, 1 \leqslant t \leqslant T\}$, and $Q_{\alpha\beta}$ is the Klein-Gordon energy momentum tensor

$$Q_{\alpha\beta} = \Re\Big(\partial_\alpha \phi^\dagger \partial_\alpha \phi - \frac{1}{2} m_{\alpha\beta}\big(\partial^\mu \phi^\dagger \partial_\mu \phi - |\phi|^2\big)\Big)$$

with the metric $m = \mathrm{diag}(1, -1)$ and $\partial_0 = \partial_t$, $\partial_1 = \partial_x$. Note that

$$n^\alpha\, Q_{\alpha 0} = \begin{cases} \frac{1}{2}\big(|\partial_t \phi|^2 + |\partial_x \phi|^2 + |\phi|^2\big) & \langle x \rangle > T \text{ and } t = T, \\ \frac{1}{2}\big(|\partial_t \phi|^2 + |\partial_x \phi|^2 + |\phi|^2\big) + \frac{x}{t}\Re\big((\partial_t\phi)^\dagger \partial_x \phi\big) & \langle x \rangle = t \text{ and } t < T \end{cases}$$

and hence $n^\alpha\, Q_{\alpha 0} \geqslant 0$. In particular $E_{ext,T}$ is well defined for any positive weight $f \geqslant 0$. Our goal is to prove the following weighted energy estimate (cf. [8, Theorem 3]).

Lemma 2.1 (Exterior Energy Estimates)**.** *Let $1 \leqslant T < \infty$, $N \in \mathbb{N}$ and for $0 \leqslant j \leqslant N$ define the weights*

$$w_j = (t + |x|)^{N-j}(|x| - t + 1)^j.$$

Then we have

$$\sum_{|I|\leqslant N} E_{ext,T}(\partial^I \phi, w_{|I|}) \lesssim \sum_{|I|\leqslant N} E_{ext,1}(\partial^I \phi, w_{|I|}) + \sum_{|I|\leqslant N} \int_1^T \left(\int_{\langle x \rangle \geqslant t} \big|(\Box + 1)\partial^I \phi\big|^2 w_{|I|} dx \right)^{\frac{1}{2}} dt.$$

Proof. We follow the argument of Klainerman [8]. An application of the divergence theorem gives for every $T \geqslant 1$

$$\big(E_{ext,T}(\phi, f)\big)^2 = \big(E_{ext,1}(\phi, f)\big)^2 + \int_{\mathcal{D}_T} Q_{\alpha 0}[\phi] \partial^\alpha f dx dt + \int_{\mathcal{D}_T} \partial^\alpha Q_{\alpha 0}[\phi] f dx dt.$$

Since $\partial^\alpha Q_{\alpha 0} = \Re[\partial_t \phi^\dagger (\Box\phi + \phi)]$, the last integral can be estimated by

$$\int_{\mathcal{D}_T} \partial^\alpha Q_{\alpha 0}[\phi] f dx dt = \int_1^T \int_{\langle x \rangle \geqslant t} \Re[\partial_t \phi^\dagger (\Box\phi + \phi)] f dx dt \lesssim \int_1^T E_{ext,t}(\phi, f) \Big(\int_{\langle x \rangle \geqslant t} |\Box\phi + \phi|^2 f dx \Big)^{\frac{1}{2}} dt.$$

Consequently, the lemma will follow provided we can show that there exists constants $c_j > 0$ (depending only on N) such that

$$\sum_{0 \leqslant j \leqslant N} c_j \sum_{|I| \leqslant j} Q_{\alpha 0}[\partial^I \phi] \partial^\alpha w_j \leqslant 0.$$

To this end, we define the vector fields $e_\pm = \partial_t \pm \frac{x}{|x|}\partial_x$ and observe that a computation gives the identity

$$Q_{\alpha 0}[\phi] \partial^\alpha f = \frac{1}{4}\Big(e_+(f)\big(|e_-(\phi)|^2 + |\phi|^2\big) + e_-(f)\big(|e_-(\phi)|^2 + |\phi|^2\big)\Big).$$

Moreover, we can check that the weights w_j satisfy $e_+(w_N) = e_-(w_0) = 0$ and

$$e_+(w_{j-1}) = 2(N-j+1)\big(t+|x|\big)^{N-j}\big(|x|-t+1\big)^j = -\frac{N-j+1}{j}e_-(w_j) \qquad 1 \leqslant j \leqslant N.$$

In particular we have $e_+(w_j) \geqslant 0$ and $e_-(w_j) \leqslant 0$ for every $0 \leqslant j \leqslant N$. Therefore, since $\sum_{|I|\leqslant j}\big(|e_-(\partial^I\phi)|^2 + |\partial^I\phi|^2\big) \leqslant 2\sum_{|I|\leqslant j+1}|\partial^I\phi|^2$, we deduce that

$$\begin{aligned}
4\sum_{0\leqslant j\leqslant N}\sum_{|I|\leqslant j} c_j Q_{\alpha 0}[\partial^I\phi]\partial^\alpha w_j &= \sum_{0\leqslant j\leqslant N} c_j \sum_{|I|\leqslant j} e_+(f)\big(|e_-(\partial^I\phi)|^2 + |\partial^I\phi|^2\big) + e_-(f)\big(|e_-(\partial^I\phi)|^2 + |\partial^I\phi|^2\big)\\
&\leqslant \sum_{0\leqslant j\leqslant N} c_j\Big(2e_+(w_j)\sum_{|I|\leqslant j+1}|\partial^I\phi|^2 + e_-(w_j)\sum_{|I|\leqslant j}|\partial^I\phi|^2\Big)\\
&= \sum_{1\leqslant j\leqslant N}\big(2c_{j-1}e_+(w_{j-1}) + c_j e_-(w_j)\big)\sum_{|I|\leqslant j}|\partial^I\phi|^2
\end{aligned}$$

which is less than zero provided we choose the constants c_j such that

$$c_j = \frac{2(N-j+1)}{j}c_{j-1}, \qquad c_0 = 1.$$

□

If we have a function that satisfies $E_{ext,T}(\phi, w_j) < \infty$, then an application of Sobolev embedding[1] gives for any $\langle x\rangle \geqslant T$ and $0 \leqslant j \leqslant N$

$$\big(|\phi|^2 w_j\big)(T,x) \lesssim \big(E_{ext,T}(\phi, w_j)\big)^2.$$

We can use this estimate together with Lemma 2.1 and the formulation (1.2) to deduce the exterior component of Theorem 1.1. Let $Z = t\partial_x + x\partial_t$. A computation shows that

$$\begin{aligned}
&\sum_{|I|\leqslant N}\sum_{k=0}^{3}\big|\partial^I Z^k\big(\phi^3 + \phi^2\partial\phi\big)\big|^2 w_{|I|})\\
&\qquad\lesssim \sum_{k_1,k_2,k_3=0}^{3}\sum_{|I|\leqslant N} w_{|I|}\sum_{J_1+J_2+J_3=I}|\partial^{J_1}Z^{k_1}\phi|^2|\partial^{J_2}Z^{k_2}\phi|^2\big(|\partial^{J_3}Z^{k_3}\phi|^2 + |\partial^{J_3}\partial Z^{k_3}\phi|^2\big)\\
&\qquad\lesssim \sum_{k_1,k_2,k_3=0}^{3}\sum_{|I|\leqslant N}\sum_{J_1+J_2+J_3=I}\frac{w_{|I|}}{w_{|J_1|}w_{|J_2|}w_{|J_3|}}\\
&\qquad\qquad\times\big(E_{ext,T}(\partial^{J_1}Z^{k_1}\phi, w_{|J_1|})E_{ext,T}(\partial^{J_2}Z^{k_2}\phi, w_{J_2})\big)^2\big(|\partial^{J_3}Z^{k_3}\phi|^2 + |\partial^{J_3}\partial Z^{k_3}\phi|^2\big)w_{J_3}\\
&\qquad\lesssim t^{-2N}\sup_{\substack{|I|\leqslant N\\0\leqslant k\leqslant 3}}\big(E_{ext,T}(\partial^I Z^k\phi, w_{|I|})\big)^4\sum_{\substack{|I|\leqslant N\\0\leqslant k\leqslant 3}}\big(|\partial^I Z^k\phi|^2 + |\partial\partial^I Z^k\phi|^2\big)w_{|I|}.
\end{aligned} \tag{2.1}$$

Let (u, v) denote a solution to (1.1) and define

$$\mathcal{E}_{ext}(T) = \sum_{\substack{|I|\leqslant N\\0\leqslant k\leqslant 3}}\big(E_{ext,T}(\partial^I Z^k u, w_{|I|}) + E_{ext,T}(\partial^I Z^k v, w_{|I|})\big).$$

Then an application of Lemma 2.1, together with the formulation (1.2), the previous computation (2.1), and the fact that Z commutes with $\Box$, gives

$$\mathcal{E}_{ext}(T) \lesssim \mathcal{E}_{ext}(1) + \int_1^T t^{-2N}\mathcal{E}_{ext}(t)dt.$$

Thus we obtain the following.

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[1]We use the observation that $\partial_x w_j \lesssim w_j$, together with the embedding $\|f\|_{L^\infty(x\geqslant 0)} \lesssim \|f\|_{H^1(x\geqslant 0)}$ which follows from the standard inequality on $\mathbb{R}$ by extending f by reflection.

Theorem 2.2. *Let $N \geqslant 1$ and (u, v) be a solution to (1.1). There exists a constant $\epsilon > 0$ such that if the data satisfies*

$$\mathcal{E}_{ext}(1) \leqslant \epsilon,$$

then for every $T \geqslant 1$ we have

$$\mathcal{E}_{ext}(T) \lesssim \mathcal{E}_{ext}(1).$$

It is easy to check that this theorem gives the claimed decay rate in Theorem 1.1 in the exterior region $\langle x \rangle \geqslant t$.

3. Hyperbolic Coordinates

We now turn to the more difficult inner region $t \geqslant \langle x \rangle$. As in the previous works [9, 14, 6], hyperbolic coordinates play a key role. Define the coordinates

$$t = \rho \cosh(y), \qquad x = \rho \sinh(y)$$

and let

$$u(t, x) = (\rho e^{-y})^{-\frac{1}{2}} U(\rho, y), \qquad v(t, x) = (\rho e^{y})^{-\frac{1}{2}} V(\rho, y).$$

To control the solution in the interior region, we define the energy

$$\mathcal{E}_{int}(\rho) = \sum_{0 \leqslant k \leqslant 3} \|\partial_y^k U(\rho)\|_{L^2_y} + \|\partial_y^k V(\rho)\|_{L^2_y}.$$

Arguing as in [7, Section 7.6], [10], the point wise identity

$$\sum_{0 \leqslant k \leqslant 3} |\partial_y^k U|^2 + |\partial_y^k V|^2 \lesssim \rho \cosh(y) \Big(\sum_{0 \leqslant k \leqslant 3} |Z^k u|^2 + |Z^k v|^2 \Big)$$

implies that

$$\mathcal{E}_{int}(1) \lesssim \lim_{T \to \infty} \mathcal{E}_{ext}(T) \lesssim \mathcal{E}_{ext}(1).$$

Consequently, in view of the results in the previous section, we may assume that $\mathcal{E}_{int}(1)$ is small. The next step is to derive the equations satisfied by (U, V). To this end, we note that since

$$(\partial_t + \partial_x) u = e^{-\frac{y}{2}} \rho^{-\frac{1}{2}} \Big(\partial_\rho U + \frac{1}{\rho} \partial_y U \Big)$$

and

$$(\partial_t - \partial_x) v = e^{\frac{y}{2}} \rho^{-\frac{1}{2}} \Big(\partial_\rho V - \frac{1}{\rho} \partial_y V \Big)$$

the system (1.1) becomes

$$\begin{aligned} \partial_\rho U + \frac{1}{\rho} \partial_y U &= iV + i \frac{1}{\rho} |V|^2 U \\ \partial_\rho V - \frac{1}{\rho} \partial_y V &= iU + i \frac{1}{\rho} |U|^2 V. \end{aligned}$$

We require another version of the equation (1.1) to exploit the oscillatory behaviour of the solution. Define

$$\phi_\pm = e^{\mp i\rho} (U \pm V)$$

and $\phi = (\phi_+, \phi_-)$. Observe that

$$e^{\pm i\rho} \partial_\rho \phi_\pm = \big(\partial_\rho U - iV \big) \pm \big(\partial_\rho V - iU \big) = \frac{1}{\rho} \Big(i|V|^2 U \pm i|U|^2 V - \partial_y \big(U \mp V \big) \Big).$$

Consequently we see that $\phi_\pm$ satisfies

$$\partial_\rho \phi_\pm + e^{\mp 2i\rho} \frac{1}{\rho} \partial_y \phi_\mp = \frac{1}{\rho} i F_\pm$$

with

$$F_\pm = e^{\mp i\rho} \big(|V|^2 U \pm |U|^2 V \big).$$

To compute $F_\pm$ in terms of $\phi_\pm$, we start by observing that

$$|V|^2U \pm |U|^2V = \pm\big(U \pm V\big)^\dagger UV = \pm\frac{1}{2}\big(U \pm V\big)^\dagger\Big((U+V)^2 - (U-V)^2\Big)$$

which implies that

$$\pm 2F_\pm = e^{\mp i\rho}\big(e^{\pm i\rho}\phi_\pm\big)^\dagger\Big(e^{2i\rho}\phi_+^2 - e^{-2i\rho}\phi_-^2\Big).$$

Rearranging this then gives

$$2F_\pm = |\phi_\pm|^2\phi_\pm - e^{\mp 4i\rho}\big(\phi_\pm^\dagger\phi_\mp\big)\phi_\mp.$$

This has the important implication that we may write our equation as

$$\begin{aligned}\partial_\rho\phi_\pm &= \frac{i}{2\rho}|\phi_\pm|^2\phi_\pm - \frac{1}{\rho}\Big(e^{\mp 2i\rho}\partial_y\phi_\mp + \frac{i}{2}e^{\mp 4i\rho}(\phi_\pm^\dagger\phi_\mp)\phi_\mp\Big)\\ &= \frac{i}{2\rho}|\phi_\pm|^2\phi_\pm + \partial_\rho S_\pm + R_\pm \end{aligned}\tag{3.1}$$

where

$$S_\pm = \frac{-1}{\mp i\rho}e^{\mp i\rho}\partial_y\phi_\mp + \frac{-1}{\mp 8\rho}e^{\mp 4i\rho}(\phi_\pm^\dagger\phi_\mp)\phi_\mp$$

and

$$R_\pm = \frac{1}{\mp i}e^{\mp i\rho}\partial_\rho\Big(\frac{1}{\rho}\partial_y\phi_\mp\Big) + \frac{1}{\mp 8}e^{\mp i\rho}\partial_\rho\Big(\frac{1}{\rho}(\phi_\pm^\dagger\phi_\mp)\phi_\mp\Big).$$

The idea being that $R_\pm$ should be integrable, and thus can be considered a remainder term. On the other hand, the $\partial_\rho S_\pm$ is not (absolutely) integrable, but can be absorbed into the left hand side. The remaining non-resonant term $|\phi_\pm|^2\phi_\pm$ cannot be handled in this manner, and thus leads to the phase correction in the asymptotic behaviour.

4. Interior Region

Define

$$M(\rho) = \sup_y\big(|U|^2 + |V|^2\big)^{\frac{1}{2}} = 2\sup_y\big(|\phi_+|^2 + |\phi_-|^2\big)^{\frac{1}{2}}.$$

Our goal is to prove the following.

Lemma 4.1. *There exists $\epsilon > 0$ such that, if $\mathcal{E}_{int}(1) \leqslant \epsilon$, then we have the global bound*

$$\sup_{\rho\geqslant 1} M(\rho) \lesssim \mathcal{E}_{int}(1).$$

Proof. Fix $T > 0$. A continuity argument shows that it is enough to prove that, provided we take $\epsilon > 0$ sufficiently small, there exists a constant $C^* > 0$ such that

$$\sup_{1\leqslant\rho\leqslant T} M \leqslant 2C^*\mathcal{E}_{int}(1) \qquad \Longrightarrow \qquad \sup_{1\leqslant\rho\leqslant T} M \leqslant C^*\mathcal{E}_{int}(1).$$

If we take the derivative of the energy $\mathcal{E}_{int}$, we obtain

$$\begin{aligned}\frac{1}{2}\partial_\rho\mathcal{E}_{int}^2 = &\sum_{0\leqslant k\leqslant 3}\int_{\mathbb{R}}\Re\Big[(\partial_y^kU)^\dagger\Big(-\frac{1}{\rho}\partial_y^{k+1}U + i\partial_y^kV\Big) + (\partial_y^kV)^\dagger\Big(\frac{1}{\rho}\partial_y^{k+1}V + i\partial_y^kU\Big)\Big]dy\\ &\qquad + \frac{1}{\rho}\int_{\mathbb{R}}\Re\Big[i(\partial_y^kU)^\dagger\partial_y^k\big(|U|^2V\big) + i(\partial_y^kV)^\dagger\partial_y^k\big(|V|^2U\big)\Big]dy\\ = &\, L + \frac{1}{\rho}N.\end{aligned}$$

To control the linear component L, we simply observe that

$$\int_{\mathbb{R}}\Re\big[(\partial_y^kU)^\dagger\partial_y^{k+1}U + (\partial_y^kV)\partial_y^{k+1}V\big]dy = \int_{\mathbb{R}}\partial_y\big(|\partial_y^kU|^2 + |\partial_y^kV|^2\big)dy = 0$$

and

$$(\partial_y^kU)^\dagger\partial_y^kV + (\partial_y^kV)^\dagger\partial_y^kU = 2\Re\big[(\partial_y^kU)^\dagger\partial_y^kV\big]$$

which implies that $L = 0$. On the other hand, an application of Hölder together with the product inequality for Sobolev spaces gives $N(\rho) \lesssim M^2(\rho)\mathcal{E}_{int}(\rho)^2$. The assumed bound on $M(\rho)$ and $\mathcal{E}_{int}(1)$ then implies that

$$\partial_\rho \mathcal{E}_{int}^2(\rho) \leqslant \frac{C}{\rho} M^2(\rho)\mathcal{E}_{int}(\rho)^2 \leqslant \frac{C(2C^*\epsilon)^2}{\rho}\mathcal{E}_{int}(\rho)^2$$

for some constant $C > 0$. Therefore, letting $\delta = \frac{1}{2}C(2C^*\epsilon)^2$ denote half the constant in the above inequality, we deduce that

$$\mathcal{E}_{int}(\rho) \leqslant \mathcal{E}_{int}(1) e^{\frac{1}{2}C(2C^*\epsilon)^2 \ln \rho} = \mathcal{E}_{int}(1)\rho^\delta. \tag{4.1}$$

Thus the energy $\mathcal{E}_{int}(\rho)$ is slowly growing. This bound is not enough on its own to control the solution, and we need to use the precise structure of the nonlinear terms to deduce the bound on $M(\rho)$. More precisely, a computation using (3.1) shows that

$$\partial_\rho\big(|\phi_\pm|^2 - 2\Re(\phi_\pm^\dagger S_\pm) - |S_\pm|^2\big) = 2\Re\Big(- \frac{i}{2\rho}|\phi_\pm|^2\phi_\pm^\dagger S_\pm + R_\pm^\dagger S_\pm + \phi_\pm^\dagger R_\pm\Big). \tag{4.2}$$

The definitions of $S_\pm$ and $R_\pm$ implies that

$$|S_\pm| \lesssim \frac{1}{\rho}(|\partial_y \phi| + |\phi|^3) \lesssim \frac{1}{\rho}\mathcal{E}_{int}(1 + \mathcal{E}_{int}^2)$$

and

$$\begin{aligned} |R_\pm| &\lesssim \frac{1}{\rho^2}\big(|\partial_y\phi| + |\phi|^2\big) + \frac{1}{\rho}\big(|\partial_y\partial_\rho\phi| + |\phi|^2|\partial_\rho\phi|\big) \\ &\lesssim \frac{1}{\rho^2}\big(|\partial_y\phi| + |\phi|^2\big) + \frac{1}{\rho^2}\big(|\partial_y^2\phi| + |\phi|^2|\partial_y\phi| + |\phi|^2|\partial_y\phi| + |\phi|^5\big) \\ &\lesssim \frac{1}{\rho^2}\mathcal{E}_{int}(1 + \mathcal{E}_{int})^5. \end{aligned}$$

Thus an application of the bound (4.1) gives

$$|S_\pm| \lesssim \mathcal{E}_{int}(1)\rho^{-1+3\delta}, \qquad |R_\pm| \lesssim \mathcal{E}_{int}(1)\rho^{-2+5\delta}$$

(here we assumed that $\mathcal{E}_{int}(1) \lesssim 1$, and $\rho \geqslant 1$). Therefore, provided we assume that $0 < \delta < \frac{1}{10}$, we may integrate the equation (4.2) to deduce that there exists a constant C (independent of C^*, ϵ, and ϕ) such that

$$M(\rho) \leqslant M(1) + C\mathcal{E}_{int}(1).$$

Consequently, assuming that $C^* > C + 1$ and choosing $\epsilon \ll \frac{1}{C^*}$, we obtain

$$M(\rho) \leqslant C^*\mathcal{E}_{int}(1)$$

as required. □

Remark 4.2. The proof of the above lemma shows something more. Namely, that there exists functions $a_\pm(y) \geqslant 0$ such that

$$\lim_{\rho\to\infty} |\phi_\pm|^2 = a_\pm$$

and moreover,

$$\big||\phi_\pm|^2 - a_\pm\big| \lesssim \rho^{-1+5\delta} \leqslant \rho^{-\frac{1}{2}}$$

(by perhaps choosing ϵ slightly smaller).

4.1. **Asymptotic Behaviour.** Our goal is to determine what happens to (U, V) for large ρ. Recall that we have the equation

$$\partial_\rho\phi_\pm = \frac{i}{2\rho}|\phi_\pm|^2\phi_\pm + \partial_\rho S_\pm + R_\pm$$

as well as the bounds

$$|S_\pm| \lesssim \rho^{-\frac{1}{2}}, \qquad |R_\pm| \lesssim \rho^{-\frac{3}{2}}, \qquad \big||\phi_\pm|^2 - a_\pm\big| \lesssim \rho^{-\frac{1}{2}}.$$

If we multiply the equation for $\phi_\pm$ with the integrating factor $e^{\frac{i}{2}a_\pm \ln(\rho)}$, we deduce that

$$\partial_\rho\big(e^{-\frac{i}{2}a_\pm\ln(\rho)}\phi_\pm - e^{-\frac{i}{2}a_\pm\ln(\rho)}S_\pm\big) = \frac{i}{2\rho}\big(|\phi_\pm|^2 - a_\pm\big)\phi_\pm + \frac{i}{2\rho}a_\pm S_\pm + e^{-\frac{i}{2}a_\pm\ln(\rho)}R_\pm.$$

The previous bounds imply that the right hand side is integrable, and hence

$$\lim_{\rho \to \infty} e^{-\frac{i}{2} a_{\pm} \ln(\rho)} \phi_{\pm} = \sigma_{\pm}(y)$$

exists, uniformly in $y \in \mathbb{R}$. Clearly we must have $|\sigma_{\pm}|^2 = a_{\pm}$, and consequently we can write

$$\phi_{\pm}(\rho, y) = e^{\frac{i}{2} |\sigma_{\pm}(y)|^2 \ln(\rho)} \sigma_{\pm}(y) + \mathcal{O}(\rho^{-\frac{1}{2}}).$$

In terms of (U, V), this becomes

$$U(\rho, y) = \frac{1}{2}\Big(e^{i\rho + \frac{i}{2} |\sigma_+(y)|^2 \ln(\rho)} \sigma_+(y) + e^{-i\rho + \frac{i}{2} |\sigma_-(y)|^2 \ln(\rho)} \sigma_-(y) \Big) + \mathcal{O}(\rho^{-\frac{1}{2}}),$$
$$V(\rho, y) = \frac{1}{2}\Big(e^{i\rho + \frac{i}{2} |\sigma_+(y)|^2 \ln(\rho)} \sigma_+(y) - e^{-i\rho + \frac{i}{2} |\sigma_-(y)|^2 \ln(\rho)} \sigma_-(y) \Big) + \mathcal{O}(\rho^{-\frac{1}{2}}).$$

If we return back to our original functions (u, v), this is

$$u(t, x) = \frac{1}{2\sqrt{t - x}}\Big(e^{i\rho + \frac{i}{2} |\sigma_+(y)|^2 \ln(\rho)} \sigma_+(y) + e^{-i\rho + \frac{i}{2} |\sigma_-(y)|^2 \ln(\rho)} \sigma_-(y) \Big) + \mathcal{O}\Big(\frac{\rho^{-\frac{1}{2}}}{\sqrt{(t - x)}} \Big),$$
$$v(t, x) = \frac{1}{2\sqrt{t + x}}\Big(e^{i\rho + \frac{i}{2} |\sigma_+(y)|^2 \ln(\rho)} \sigma_+(y) - e^{-i\rho + \frac{i}{2} |\sigma_-(y)|^2 \ln(\rho)} \sigma_-(y) \Big) + \mathcal{O}\Big(\frac{\rho^{-\frac{1}{2}}}{\sqrt{(t + x)}} \Big).$$

Defining the functions $f_{\pm}(s) = \frac{1}{2}\sigma_{\pm}\big(\frac{1}{2}\ln(1+s) - \frac{1}{2}\ln(1-s)\big)$ (which implies that $2f_{\pm}(\frac{x}{t}) = \sigma_{\pm}(y)$) we then obtain Theorem 1.1.

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