

# ENUMERATING GRADED IDEALS IN GRADED RINGS ASSOCIATED TO FREE NILPOTENT LIE RINGS

SEUNGJAI LEE AND CHRISTOPHER VOLL

ABSTRACT. We compute the zeta functions enumerating graded ideals in the graded Lie rings associated with the free  $d$ -generator Lie rings  $\mathfrak{f}_{c,d}$  of nilpotency class  $c$  for all  $c \leq 2$  and for  $(c, d) \in \{(3, 3), (3, 2), (4, 2)\}$ . We apply our computations to obtain information about  $\mathfrak{p}$ -adic, reduced, and topological zeta functions, in particular pertaining to their degrees and some special values.

## 1. INTRODUCTION

**1.1. Enumerating graded ideals in graded Lie rings.** Let  $R$  be the ring of integers of a number field or the completion of such a ring at a nonzero prime ideal. Let  $L$  be a nilpotent  $R$ -Lie algebra of nilpotency class  $c$ , free of finite rank over  $R$ , with lower central series  $(\gamma_i(L))_{i=1}^c$ . For  $i \in \{1, \dots, c\}$ , set  $L^{(i)} := \gamma_i(L)/\gamma_{i+1}(L)$ . The *associated graded  $R$ -Lie algebra* is  $\text{gr}L = \bigoplus_{i=1}^c L^{(i)}$ . An  $R$ -ideal  $I$  of  $\text{gr}L$  (of finite index in  $\text{gr}L$ ) is *graded* or *homogeneous* if it is generated by homogeneous elements or, equivalently, if  $I = \bigoplus_{i=1}^c (I \cap L^{(i)})$ . In this case we write  $I \triangleleft_{\text{gr}} \text{gr}L$ . We define the *graded ideal zeta function of  $L$*  as the Dirichlet generating series

$$(1.1) \quad \zeta_L^{\triangleleft_{\text{gr}}}(s) = \sum_{I \triangleleft_{\text{gr}} \text{gr}L} |\text{gr}L : I|^{-s},$$

enumerating graded ideals in  $\text{gr}L$  of finite index in  $\text{gr}L$ . Here  $s$  is a complex variable; our assumptions on  $L$  guarantee that  $\zeta_L^{\triangleleft_{\text{gr}}}(s)$  converges on a complex half-plane. (For a similar definition in a slightly more general setting, see [11, Section 3.1]).

Assume now that  $R = \mathcal{O}$  is the ring of integers of a number field. For a (nonzero) prime ideal  $\mathfrak{p} \in \text{Spec}(\mathcal{O})$  we write  $\mathcal{O}_{\mathfrak{p}}$  for the completion of  $\mathcal{O}$  at  $\mathfrak{p}$ , a complete discrete valuation ring of characteristic zero and residue field  $\mathcal{O}/\mathfrak{p}$  of cardinality  $q$ , say. Primary decomposition yields the Euler product

$$\zeta_L^{\triangleleft_{\text{gr}}}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft_{\text{gr}}}(s),$$

expressing the “global” zeta function  $\zeta_L^{\triangleleft_{\text{gr}}}(s)$  as an infinite product of “local”, or  $\mathfrak{p}$ -adic ones. By slight abuse of notation we denote here by  $\text{Spec}(\mathcal{O})$  the set of nonzero prime ideals of  $\mathcal{O}$ . Each individual Euler factor is a rational function in the parameter  $q^{-s}$ . In fact, (1.1) is an Euler product of *cone integrals* in the sense of [4] (cf. [11, Theorem 3.3]), and the far-reaching results of this paper – regarding both the factors and the product’s

---

*Date:* June 14, 2016.

*2000 Mathematics Subject Classification.* 17B70, 11M41, 11S40.

*Key words and phrases.* Graded ideal zeta functions, free nilpotent Lie rings, local functional equations.

analytic properties – apply. The fact that the analysis of [4] is restricted to the case  $R = \mathbb{Z}$  is insubstantial for this conclusion.

**1.2. Main results.** In the present paper we are concerned with graded ideal zeta functions of free nilpotent Lie rings of finite rank. Given  $c \in \mathbb{N}$  and  $d \in \mathbb{N}_{\geq 2}$ , let  $\mathfrak{f}_{c,d}$  be the free nilpotent Lie ring of nilpotency class  $c$  on  $d$  Lie generators. One may identify  $\mathfrak{f}_{c,d}$  with the quotient of the free  $\mathbb{Z}$ -Lie algebra  $\mathfrak{f}_d$  on  $d$  generators by the  $c + 1$ -th term  $\gamma_{c+1}(\mathfrak{f}_d)$  of its lower central series. For  $i \in \{1, \dots, c\}$ , the  $\mathbb{Z}$ -rank of the  $i$ -th lower-central-series quotient  $\gamma_i(\mathfrak{f}_{c,d})/\gamma_{i+1}(\mathfrak{f}_{c,d})$  is given by the Witt function

$$(1.2) \quad W_d(i) := \frac{1}{i} \sum_{j|i} \mu(j) d^{i/j},$$

where  $\mu$  denotes the Möbius function; cf. [18, Satz 3]. Hence  $\text{rk}_{\mathbb{Z}}(\mathfrak{f}_{c,d}) = \sum_{i=1}^c W_d(i)$ . Given a commutative ring  $R$  – in this paper always of the form  $\mathcal{O}$  or  $\mathcal{O}_{\mathfrak{p}}$  as above – we write  $\mathfrak{f}_{c,d}(R) = \mathfrak{f}_{c,d} \otimes_{\mathbb{Z}} R$ , considered as an  $R$ -Lie algebra.

For  $c = 1$ , the (graded ideal) zeta function of the free abelian Lie ring  $\mathfrak{f}_{1,d}(\mathcal{O})$ , enumerating all finite index  $\mathcal{O}$ -sublattices of  $\mathcal{O}^d$ , is well known to be equal to

$$(1.3) \quad \zeta_{\mathfrak{f}_{1,d}(\mathcal{O})}^{\triangleleft \text{gr}}(s) = \prod_{i=1}^d \zeta_K(s - i + 1),$$

where  $\zeta_K$  denotes the Dedekind zeta function of  $K$ ; cf. [7, Proposition 1.1]. The Euler product (1.1) reflects the well-known Euler product  $\zeta_K(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} (1 - |\mathcal{O}/\mathfrak{p}|^{-s})^{-1}$ :

$$(1.4) \quad \zeta_{\mathfrak{f}_{1,d}(\mathcal{O})}^{\triangleleft \text{gr}}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \prod_{i=1}^d \frac{1}{1 - |\mathcal{O}/\mathfrak{p}|^{i-1-s}} = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \zeta_{\mathcal{O}_{\mathfrak{p}}^d}(s), \text{ say.}$$

For  $c = 2$ , the second author computed in [15] the *ideal* zeta functions enumerating ( $\mathcal{O}$ -)ideals of finite index in the rings  $\mathfrak{f}_{2,d}(\mathcal{O})$ . (The paper only discusses the case  $R = \mathbb{Z}$ , but its computations carry over – *mutatis mutandis* – to the case of general number rings.) We compute the *graded ideal* zeta functions  $\zeta_{\mathfrak{f}_{2,d}(\mathcal{O})}^{\triangleleft \text{gr}}(s)$  for all  $d \geq 2$  in Section 4 and those for  $(c, d) \in \{(3, 2), (4, 2)\}$  in Section 5 of the current paper.

The paper's most involved result is the computation, in Section 3, of the graded ideal zeta function of  $\mathfrak{f}_{3,3}(\mathcal{O})$ . To this end we compute an explicit formula for  $\zeta_{\mathfrak{f}_{3,3}(\mathcal{O})}^{\triangleleft \text{gr}}(s)$ , valid for all finite extension  $\mathfrak{o}$  of the  $p$ -adic integers  $\mathbb{Z}_p$ , where  $p$  is a prime, viz. a local ring of the form  $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$  for a nonzero prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$  lying above  $p$ .

**Theorem 1.1.** *There exists an explicitly determined rational function  $W_{3,3}^{\triangleleft \text{gr}} \in \mathbb{Q}(X, Y)$  such that, for all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ , with residue cardinality  $q$ ,*

$$\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{3,3}^{\triangleleft \text{gr}}(q, q^{-s}).$$

*It may be written as  $W_{3,3}^{\triangleleft \text{gr}} = N_{3,3}/D_{3,3}$  where*

$$\begin{aligned} D_{3,3}(X, Y) = & (1 - Y)(1 - XY)(1 - X^2Y)(1 - X^3Y^4)(1 - XY^5)(1 - X^2Y^5)(1 - Y^6) \\ & (1 - X^7Y^7)(1 - X^8Y^7)(1 - X^8Y^8)(1 - X^{12}Y^8)(1 - X^6Y^9)(1 - X^{15}Y^9) \\ & (1 - X^{16}Y^{10})(1 - X^{15}Y^{11})(1 - X^{12}Y^{12})(1 - X^7Y^{13})(1 - Y^{14}), \end{aligned}$$

a polynomial of degree 115 in  $X$  and 131 in  $Y$ , and  $N_{3,3} \in \mathbb{Q}[X, Y]$  is a polynomial of degree 81 in  $X$  and 108 in  $Y$ .

The rational function  $W_{3,3}^{\triangleleft \text{gr}}$  satisfies the functional equation

$$(1.5) \quad W_{3,3}^{\triangleleft \text{gr}}(X^{-1}, Y^{-1}) = X^{34}Y^{23}W_{3,3}^{\triangleleft \text{gr}}(X, Y).$$

We note that the Witt function  $W_3$  (cf. (1.2)) takes the values  $(W_3(1), W_3(2), W_3(3)) = (3, 3, 8)$  and that  $115 - 81 = 34 = \binom{3}{2} + \binom{3}{2} + \binom{8}{2}$  and  $131 - 108 = 23 = 3 \cdot 3 + 2 \cdot 3 + 1 \cdot 8$ ; cf. Conjecture 6.2.

Our proof of Theorem 1.1 yields  $W_{3,3}^{\triangleleft \text{gr}}$  as a sum of 15 explicitly given summands, listed essentially in Section 3.2. We do not reproduce the “final” outcome of this summation here, as the numerator  $N_{3,3}$  of  $W_{3,3}^{\triangleleft \text{gr}}$  fills several pages. We do record, however, several corollaries of the explicit formula for  $W_{3,3}^{\triangleleft \text{gr}}$ .

The first corollary concerns analytic properties of the global zeta function  $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ .

**Corollary 1.2.** *The global graded ideal zeta function  $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$  converges on  $\{s \in \mathbb{C} \mid \text{Re}(s) > 3\}$  and may be continued meromorphically to  $\{s \in \mathbb{C} \mid \text{Re}(s) > 14/9\}$ .*

This follows from two observations. Firstly, the product  $\prod_{\mathfrak{p} \in \text{Spec}(\mathfrak{o})} D_{3,3}(q, q^{-s})^{-1}$  – a product of finitely many translates of the Dedekind zeta function  $\zeta_K(s)$  – has abscissa of convergence 3 and may be continued meromorphically to the whole complex plane. Secondly, the product  $\prod_{\mathfrak{p} \in \text{Spec}(\mathfrak{o})} N_{3,3}(q, q^{-s})$  may be continued meromorphically to  $\{s \in \mathbb{C} \mid \text{Re}(s) > 14/9\}$ . Indeed, if  $N_{3,3}(X, Y) = 1 + \sum_{i \in I} \alpha_i X^{a_i} Y^{b_i}$  for a finite index set  $I$  and  $\alpha_i \in \mathbb{Z} \setminus \{0\}$ ,  $a_i \in \mathbb{N}_0$ ,  $b_i \in \mathbb{N}$ , then  $\prod_{\mathfrak{p} \in \text{Spec}(\mathfrak{o})} N_{3,3}(q, q^{-s})$  may be continued meromorphically to  $\{s \in \mathbb{C} \mid \text{Re}(s) > \beta\}$ , where  $\beta := \max\{\frac{a_i}{b_i} \mid i \in I\}$ ; cf. [5, Lemma 5.5]. That  $\beta = 14/9$  follows from inspection of  $N_{3,3}$ .

The second corollary concerns the *reduced graded ideal zeta function*

$$\zeta_{\mathfrak{f}_{3,3,\text{red}}}^{\triangleleft \text{gr}}(Y) := W_{3,3}^{\triangleleft \text{gr}}(1, Y) \in \mathbb{Q}(Y);$$

cf. Section 2.4. This concept was introduced in [6], albeit not in the context of graded ideal zeta functions. We expect that our ad hoc definition will fit into a general definition of reduced graded ideal zeta functions along the lines of [6].

**Corollary 1.3.** *The reduced graded ideal zeta function  $\zeta_{\mathfrak{f}_{3,3,\text{red}}}^{\triangleleft \text{gr}}(Y)$  satisfies*

$$(1.6) \quad \zeta_{\mathfrak{f}_{3,3,\text{red}}}^{\triangleleft \text{gr}}(Y) = \frac{N_{3,3,\text{red}}(Y)}{(1-Y)^3(1-Y^3)(1-Y^4)^2(1-Y^5) \prod_{i=8}^{14} (1-Y^i)},$$

where  $N_{3,3,\text{red}}(Y) \in \mathbb{Z}[Y]$  is equal to

$$\begin{aligned} & 1 + 2Y^3 + 4Y^4 + 5Y^5 + 16Y^6 + 34Y^7 + 53Y^8 + 77Y^9 + 98Y^{10} + 121Y^{11} + 182Y^{12} + \\ & 302Y^{13} + 483Y^{14} + 712Y^{15} + 953Y^{16} + 1187Y^{17} + 1425Y^{18} + 1689Y^{19} + 2046Y^{20} + \\ & 2579Y^{21} + 3298Y^{22} + 4162Y^{23} + 5059Y^{24} + 5826Y^{25} + 6398Y^{26} + 6894Y^{27} + \\ & 7475Y^{28} + 8270Y^{29} + 9265Y^{30} + 10260Y^{31} + 11041Y^{32} + 11529Y^{33} + 11745Y^{34} + \\ & 11798Y^{35} + 11811Y^{36} + 11811Y^{37} + 11798Y^{38} + 11745Y^{39} + 11529Y^{40} + 11041Y^{41} + \\ & 10260Y^{42} + 9265Y^{43} + 8270Y^{44} + 7475Y^{45} + 6894Y^{46} + 6398Y^{47} + 5826Y^{48} + \\ & 5059Y^{49} + 4162Y^{50} + 3298Y^{51} + 2579Y^{52} + 2046Y^{53} + 1689Y^{54} + 1425Y^{55} + \\ & 1187Y^{56} + 953Y^{57} + 712Y^{58} + 483Y^{59} + 302Y^{60} + 182Y^{61} + 121Y^{62} + 98Y^{63} + \\ & 77Y^{64} + 53Y^{65} + 34Y^{66} + 16Y^{67} + 5Y^{68} + 4Y^{69} + 2Y^{70} + Y^{73}. \end{aligned}$$

It seems remarkable that  $W_{3,3}^{\triangleleft\text{gr}}(1, Y)$  has a simple pole at  $Y = 1$  of order  $14 = 3+3+8$ , the  $\mathbb{Z}$ -rank of  $\mathfrak{f}_{3,3}$ , and that  $N_{3,3,\text{red}}$  has nonnegative, unimodal coefficients. This is consistent with the speculation that  $W_{3,3}^{\triangleleft\text{gr}}(1, Y)$  is in fact the Hilbert-Poincaré series of a 14-dimensional graded algebra, associated to  $\mathfrak{f}_{3,3}$  in a natural way; cf. Remark 6.5. Note that the right hand side of (1.6) is not in lowest terms. The palindromic symmetry of the coefficients of  $N_{3,3,\text{red}}$  is implied by the functional equation (1.5).

The third corollary concerns another limiting object of the local graded ideal zeta functions  $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)$ , viz. the *topological graded ideal zeta function*  $\zeta_{\mathfrak{f}_{3,3,\text{top}}}^{\triangleleft\text{gr}}(s)$ ; cf. Section 2.5. In [9], Rossmann introduced and studied so-called topological zeta functions associated to a range of Dirichlet generating series. Informally speaking, topological zeta functions may be viewed as suitably defined limits of local zeta functions. Whilst topological versions of zeta functions such as  $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)$  have not yet been studied specifically, one may define  $\zeta_{\mathfrak{f}_{3,3,\text{top}}}^{\triangleleft\text{gr}}(s)$  as the coefficient of  $(q-1)^{-14}$  in the expansion of  $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)$  in  $q-1$ ; cf. [9, Definition 5.13].

**Corollary 1.4.** *The topological graded ideal zeta function  $\zeta_{\mathfrak{f}_{3,3,\text{top}}}^{\triangleleft\text{gr}}(s)$  satisfies*

$$\zeta_{\mathfrak{f}_{3,3,\text{top}}}^{\triangleleft\text{gr}}(s) = \frac{(33250s^4 - 81537s^3 + 66573s^2 - 20800s + 1920)/56}{D_{3,3,\text{top}}(s)},$$

where  $D_{3,3,\text{top}}(s) \in \mathbb{Z}[s]$  is defined to be

$$s^3(s-1)^4(s-2)(3s-2)(4s-3)(5s-1)(5s-2)(7s-8)(2s-3)(3s-5)(5s-8)(11s-15)(13s-7)$$

A number of further features of the  $\mathfrak{p}$ -adic, topological, and reduced graded ideal zeta functions associated to  $\mathfrak{f}_{3,3}$  considered above seem remarkable. Some of the numerical values in the following corollary are presented so as to illustrate the general conjectures in Section 6 which we extracted from the paper's explicit computations.

By the *degree* of a rational function  $f = P/Q \in \mathbb{Q}(Y)$  in  $Y$  we mean  $\deg_Y f = \deg_Y P - \deg_Y Q$ .

**Corollary 1.5.** (1)

$$\deg_s \left( \zeta_{\mathfrak{f}_{3,3,\text{top}}}^{\triangleleft\text{gr}}(s) \right) = -14 = -(3+3+8) = -\text{rk}_{\mathbb{Z}}(\mathfrak{f}_{3,3}),$$

(2)

$$s^{-14} \zeta_{f_{3,3}, \text{top}}^{\triangleleft \text{gr}}(s^{-1})|_{s=0} = \frac{19}{288, 288} = (1 - Y)^{14} \zeta_{f_{3,3}, \text{red}}^{\triangleleft \text{gr}}(Y)|_{Y=1} \in \mathbb{Q}_{>0},$$

(3)

$$s^3 \zeta_{f_{3,3}, \text{top}}^{\triangleleft \text{gr}}(s)|_{s=0} = \frac{-1}{70, 560} = \frac{(-1)^{2+2+7} 3 \cdot 3 \cdot 8}{3 \cdot 6 \cdot 14 \cdot 2! \cdot 2! \cdot 7!},$$

(4)

$$\left. \frac{\zeta_{f_{3,3}(\mathcal{O})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathcal{O}^3}(s) \zeta_{\mathcal{O}^3}(s) \zeta_{\mathcal{O}^8}(s)} \right|_{s=0} = \frac{2}{7} = \frac{3 \cdot 3 \cdot 8}{3 \cdot 6 \cdot 14}.$$

If the above-mentioned connection between the reduced graded ideal zeta function and Hilbert-Poincaré series of graded algebras were to hold, the rational number in part (2) – for which we currently do not have an interpretation – were to be interpreted in terms of the multiplicity of the associated graded algebra; cf., for instance, [1, Section 4].

**1.3. Background, motivation, and methodology.** Recall that  $L$  is a nilpotent Lie algebra over the ring of integers  $\mathcal{O}$  of a number field. The *ideal zeta function* of  $L$  is the Dirichlet generating series enumerating the  $\mathcal{O}$ -ideals of  $L$  of finite index, viz.

$$\zeta_L^{\triangleleft}(s) = \sum_{I \triangleleft L} |L : I|^{-s} = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s),$$

where  $s$  is a complex variable and, for a (nonzero) prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$ ,

$$\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s) = \sum_{I \triangleleft L(\mathcal{O}_{\mathfrak{p}})} |L(\mathcal{O}_{\mathfrak{p}}) : I|^{-s}$$

enumerates  $\mathcal{O}_{\mathfrak{p}}$ -ideals of  $L(\mathcal{O}_{\mathfrak{p}})$ . Ideal zeta functions of nilpotent Lie rings are well-studied relatives of the graded ideal zeta functions studied in the present paper. One of the main results of [7], which introduced the former zeta functions, establishes the rationality of each of the Euler factors  $\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s)$  in  $q^{-s}$ , where  $q = |\mathcal{O} : \mathfrak{p}|$ ; cf. [7, Theorem 3.5]. For numerous examples of ideal zeta functions of nilpotent Lie rings, see [5].

By the Mal'cev correspondence, ideal zeta functions of nilpotent Lie rings are closely related to the *normal subgroup zeta functions* enumerating finite index normal subgroups of finitely generated nilpotent groups. In particular, given  $c, d \in \mathbb{N}$ , almost all (i.e. all but finitely many) of the Euler factors of  $\zeta_{f_{c,d}}^{\triangleleft}(s)$  coincide with those of the normal subgroup zeta function of the free class- $c$ -nilpotent  $d$ -generator group  $F_{c,d}$ ; cf. [7, Section 4]. The study of the normal subgroup growth of free nilpotent groups is connected with the enumeration of finite  $p$ -groups up to isomorphism; cf. [3].

Informally speaking, graded ideal zeta functions of nilpotent Lie rings may be seen as “approximations” of their ideal zeta functions. Indeed, almost all Euler factors  $\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft \text{gr}}(s)$  actually enumerate a sublattice of the lattice of ideals enumerated by  $\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s)$ . In general, this approximation is quite coarse. In nilpotency class  $c \leq 2$ , however, the problems of computing ideal zeta functions and *graded* ideal zeta functions are closely related, as the following example shows.

*Example 1.6.* Assume that  $L$  is nilpotent of class 2, with isolated commutator ideal  $L'$  such that  $\text{rk}_{\mathcal{O}}(L/L') = d$ . Then, essentially by [7, Lemma 6.1],

$$(1.7) \quad \zeta_L^{\triangleleft}(s) = \sum_{\Lambda_1 \leq L/L'} |L/L' : \Lambda_1|^{-s} \sum_{[\Lambda_1, L] \leq \Lambda_2 \leq L'} |L' : \Lambda_2|^{d-s},$$

whereas

$$(1.8) \quad \zeta_L^{\triangleleft, \text{gr}}(s) = \sum_{\Lambda_1 \leq L/L'} |L/L' : \Lambda_1|^{-s} \sum_{[\Lambda_1, L] \leq \Lambda_2 \leq L'} |L' : \Lambda_2|^{-s}.$$

In the special cases  $L = \mathfrak{f}_{2,d}$ , the proximity between (1.7) and (1.8) explains the proximity between the explicit formulae recorded in Theorems 4.1 and 4.2 of the current paper.

In higher nilpotency classes, we are not aware of any such simple parallels. In the realm of free nilpotent Lie rings of class greater than two, explicit computations of ideal zeta functions seem all but unfeasible. In particular, we do not know of formulae for the ideal zeta functions of the Lie rings  $\mathfrak{f}_{3,3}(\mathcal{O})$  and  $\mathfrak{f}_{4,2}(\mathcal{O})$ .

Numerous questions regarding ideal zeta functions have analogues regarding the “approximating” *graded* ideal zeta functions, and one may speculate that the latter are easier to answer than the former. On p. 188 of [7], Grunewald, Segal, and Smith formulate, for example, a conjecture which would imply that, for any  $c, d \in \mathbb{N}$  there exists a rational function  $W_{c,d}^{\triangleleft}(X, Y) \in \mathbb{Q}(X, Y)$  such that, for almost all  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ ,  $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft}(s) = W_{c,d}^{\triangleleft}(q, q^{-s})$ . This consequence is known to hold for  $c \leq 2$  and  $(c, d) = (3, 2)$  but wide open in general, including the cases  $(c, d) \in \{(3, 3), (4, 2)\}$ . Our Conjecture 6.1 – which is verified by the explicit computations in the current paper in particular for  $c = 2$  and  $(c, d) \in \{(3, 3), (3, 2), (4, 2)\}$  – may be viewed as a “graded” version of this conjecture.

In [9], Rossmann formulates a number of conjectures on certain special values of  $\mathfrak{p}$ -adic and topological zeta functions, pertaining in particular to ideal zeta functions of nilpotent Lie rings. Our Conjectures 6.7, 6.9 and 6.11 are “graded” counterparts.

In [17, Theorem 4.4], the second author proved a local functional equation for the generic Euler factors of the ideal zeta functions  $\zeta_{\mathfrak{f}_{c,d}(\mathcal{O})}^{\triangleleft}(s)$  upon inversion of the prime for all  $c \in \mathbb{N}$  and  $d \in \mathbb{N}_{\geq 2}$ . Our results suggest that this phenomenon also appears for graded ideal zeta functions of free nilpotent Lie rings; cf. Conjecture 6.2.

All our computations owe their feasibility to the fact that, for the parameter values considered, viz.  $c = 2$  and  $(c, d) \in \{(3, 3), (3, 2), (4, 2)\}$ , the enumeration of graded ideals is equivalent to the enumeration of various flags of lattices in free  $\mathfrak{o}$ -modules which depend only, and in a linear fashion, on the lattices’ *elementary divisor types*. Our computations rely on a simple polynomial formula, due to Birkhoff, for the numbers of  $\mathfrak{o}$ -submodules of given type in a finite  $\mathfrak{o}$ -module of given type; cf. Proposition 2.2. To obtain closed formulae for the relevant graded ideal zeta functions we need to organize the enumeration of infinitely many values of Birkhoff’s formula in a manageable way. We meet this challenge by organizing pairs of partitions, encoding two lattices’ elementary divisor types, by their *overlap type*, formally one of finitely many multiset permutations, viz. words in two letters (each with multiplicity); cf. Section 3.1. For  $(c, d) = (3, 3)$ , for instance, we are led to consider 15 specific words of length 11 in the alphabet  $\{\mathbf{1}, \mathbf{2}\}$ ; cf.

Table 3.1. Restricting the relevant enumerations to a fixed word yields formulae which are products of  $q$ -multinomial coefficients and so-called *Igusa functions*; cf. Section 2.2. In Theorem 3.6 we compute such formulae for each of the 15 words in question. Similar methodology has been applied before to compute ideal zeta functions of Heisenberg Lie rings over number rings ([12]) and free nilpotent Lie rings of class 2 ([15]).

That we are able to reduce our computations to combinatorial considerations with partitions is essentially owed to the fact that, for the parameters  $(c, d)$  considered, the automorphism groups of the free Lie rings  $\mathfrak{f}_{c,d}$  act transitively on the lattices of given elementary divisor types in relevant sections of  $\mathfrak{f}_{c,d}$ . This allows us to assume that the relevant lattices are generated by multiples of elements of *Hall bases*; cf. Section 2.3. That a lattice may always be generated by multiples of *some* linear basis follows, of course, from the elementary divisor theorem; that (subsets of) Hall bases may be used is a lucky consequence of the existence of “many” automorphisms, as we now explain. The automorphism group of  $\mathfrak{f}_{c,d}(\mathfrak{o})$  contains a copy of  $\mathrm{GL}_d(\mathfrak{o})$  which allows us to perform arbitrary invertible  $\mathfrak{o}$ -linear transformations of Lie generators  $x_1, \dots, x_d$  of  $\mathfrak{f}_{c,d}(\mathfrak{o})$ . For  $c = 2$ , this observation – which was already exploited in the proof of [7, Theorem 2] – in conjunction with Birkhoff’s formula is sufficient to compute the (graded) ideal zeta function of  $\mathfrak{f}_{c,d}(\mathfrak{o})$ ; cf. Section 4. On the Lie commutators  $[x_i, x_j]$ ,  $1 \leq i < j \leq n$ , these linear transformations act via the exterior square representation. In general, the image of the natural map  $\bigwedge^2 \mathrm{GL}_d(\mathfrak{o}) \rightarrow \mathrm{GL}_{\binom{d}{2}}(\mathfrak{o})$  is rather small. The map is surjective, however, if  $d \leq 3$ , and an isomorphism if  $d = 3$ . This facilitates our computations for  $(c, d) \in \{(3, 3), (3, 2)\}$ . If  $d = 2$ , then the above-mentioned copy of  $\mathrm{GL}_2(\mathfrak{o})$  even induces the full automorphism group of the  $\mathfrak{o}$ -module generated by the weight-3-commutators  $[[x_1, x_2], x_1]$  and  $[[x_1, x_2], x_2]$ , which we exploit for  $(c, d) = (4, 2)$ .

**1.4. Notation.** Given  $n \in \mathbb{N} = \{1, 2, \dots\}$ , we write  $[n]$  for  $\{1, 2, \dots, n\}$ . We write  $\mathbb{N}_{\geq 2}$  for  $\{2, 3, \dots\} \subseteq \mathbb{N}$ . Given a subset  $I \subseteq \mathbb{N}$ , we write  $I_0$  for  $I \cup \{0\}$ . If  $I \subseteq [n-1]$ , we denote  $n - I = \{n - i \mid i \in I\}$ . Given  $a, b \in \mathbb{N}_0$ , we denote the interval  $\{a + 1, \dots, b\}$  by  $]a, b]$  and the interval  $\{a + 1, \dots, b - 1\}$  by  $]a, b[$ . We write  $2^S$  for the power set of a set  $S$ . The notation  $I = \{i_1, \dots, i_h\}_<$  for a subset of  $I \subset \mathbb{N}_0$  indicates that  $i_1 < i_2 < \dots < i_h$ . Similarly,  $(\lambda_1, \dots, \lambda_n)_{\geq} \in \mathbb{N}_0^n$  denotes a partition with non-ascending parts  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ .

We write  $\mathfrak{o}$  for a compact discrete valuation ring of characteristic zero, i.e. a finite extension of the ring  $\mathbb{Z}_p$  of  $p$ -adic integers or, equivalently, a ring of the form  $\mathcal{O}_{\mathfrak{p}}$ , the completion of  $\mathcal{O}$ , the ring of integers of a number field  $K$ , at a nonzero prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$ . We write  $q$  for the cardinality of the residue field of  $\mathfrak{o}$  and  $p$  for its residue characteristic. We set  $t = q^{-s}$ , where  $s$  is a complex variable.  $\zeta_K$  is the Dedekind zeta function of  $K$ .

## 2. GENERAL PRELIMINARIES

Let  $c \in \mathbb{N}$ ,  $d \in \mathbb{N}_{\geq 2}$ , and set  $\mathfrak{f} = \mathfrak{f}_{c,d}$  be as above. The following is analogous to [7, Lemma 6.1].

**Lemma 2.1.** *Let  $p$  be a prime and  $\mathfrak{o}$  be a finite extension of  $\mathbb{Z}_p$ . For each  $\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}$  let*

$$(2.1) \quad (X(\Lambda_2) \oplus \mathfrak{f}(\mathfrak{o})^{(2)})/\Lambda_2 = Z((\mathfrak{f}(\mathfrak{o})^{(1)} \oplus \mathfrak{f}(\mathfrak{o})^{(2)})/\Lambda_2).$$

Then

$$\begin{aligned}
\zeta_{\mathfrak{f}(\mathfrak{o})}^{\triangleleft_{\text{gr}}} (s) &= \sum_{\substack{\Lambda_1 \leq \mathfrak{f}(\mathfrak{o})^{(1)}, \dots, \Lambda_c \leq \mathfrak{f}(\mathfrak{o})^{(c)} \\ \forall i \in ]1, c]: [\Lambda_{i-1}, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_i}} \prod_{i=1}^c |\mathfrak{f}(\mathfrak{o})^{(i)} : \Lambda_i|^{-s} \\
(2.2) \quad &= \zeta_{\mathfrak{o}^d}(s) \sum_{\substack{\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}, \dots, \Lambda_c \leq \mathfrak{f}(\mathfrak{o})^{(c)} \\ \forall i \in ]2, c]: [\Lambda_{i-1}, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_i}} |\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)|^{-s} \prod_{i=2}^c |\mathfrak{f}(\mathfrak{o})^{(i)} : \Lambda_i|^{-s}.
\end{aligned}$$

*Proof.* A graded additive sublattice  $\Lambda \leq \text{gr } \mathfrak{f}(\mathfrak{o})$  determines and is determined by the sequence  $(\Lambda_1, \dots, \Lambda_c)$ , where  $\Lambda_i := \Lambda \cap \mathfrak{f}(\mathfrak{o})^{(i)}$  for all  $i \in [c]$ . We have  $\Lambda \triangleleft_{\text{gr}} \mathfrak{f}(\mathfrak{o})$  if and only if  $[\Lambda_{i-1}, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_i$  for all  $i \in ]1, c]$ . This proves the first equality. The second follows from the definition of  $X(\Lambda_2)$  given in (2.1), noting that  $[\Lambda_1, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_2$  if and only if  $\Lambda_1 \leq X(\Lambda_2)$  and that, for each  $\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}$ ,

$$\sum_{\Lambda_1 \leq X(\Lambda_2)} |\mathfrak{f}(\mathfrak{o})^{(1)} : \Lambda_1|^{-s} = \zeta_{\mathfrak{o}^d}(s) |\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)|^{-s}. \quad \square$$

**2.1. Birkhoff's formula.** Given a pair of partitions  $(\sigma, \tau)$ , let  $\alpha(\tau, \sigma; q)$  denote the number of torsion  $\mathfrak{o}$ -modules of type  $\tau$  contained in a fixed torsion  $\mathfrak{o}$ -module of type  $\sigma$ . Clearly  $\alpha(\sigma, \tau; q) = 0$  unless  $\tau \leq \sigma$ . Notice that, for  $\tau = (\tau_1, \dots, \tau_n)_{\geq}$ ,

$$\alpha(\tau_1^{(n)}, \tau; q) = \#\{\Lambda \leq \mathfrak{o}^n \mid \mathfrak{o}^n / \Lambda \cong \bigoplus_{j=1}^n \mathfrak{o} / \mathfrak{p}^{\tau_j}\}.$$

The following explicit general formula for  $\alpha(\sigma, \tau; q)$  is attributed to Birkhoff in [2].

**Proposition 2.2** (Birkhoff). *Let  $\tau \leq \sigma$  be partitions, with dual partitions  $\tau' \leq \sigma'$ . Then*

$$\alpha(\sigma, \tau; q) = \prod_{k \geq 1} q^{\tau'_k(\sigma'_k - \tau'_k)} \binom{\sigma'_k - \tau'_{k+1}}{\sigma'_k - \tau'_k}_{q^{-1}} \in \mathbb{Z}[q].$$

## 2.2. Igusa functions and their functional equations.

*Definition 2.3.* ([12, Definition 2.5]) Let  $h \in \mathbb{N}$ . Given variables  $\mathbf{X} = (X_1, \dots, X_h)$  and  $Y$ , we set

$$\begin{aligned}
I_h(Y; \mathbf{X}) &= \frac{1}{1 - X_h} \sum_{I \subseteq [h-1]} \binom{h}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_h), \\
I_h^\circ(Y; \mathbf{X}) &= \frac{X_h}{1 - X_h} \sum_{I \subseteq [h-1]} \binom{h}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_h).
\end{aligned}$$

We set  $I_0(Y) = I_0^\circ(Y) = 1 \in \mathbb{Q}(Y)$ .

Note that  $I_1(Y; X) = \frac{1}{1-X}$  and  $I_1^\circ(Y; X) = \frac{X}{1-X}$ . We will make repeated use of the functional equations ([12, Proposition 4.2])

$$\begin{aligned}
(2.3) \quad I_h(Y^{-1}; \mathbf{X}^{-1}) &= (-1)^h X_h Y^{-\binom{h}{2}} I_h(Y; \mathbf{X}), \\
I_h^\circ(Y^{-1}; \mathbf{X}^{-1}) &= (-1)^h X_h^{-1} Y^{-\binom{h}{2}} I_h^\circ(Y; \mathbf{X}).
\end{aligned}$$

*Remark 2.4.* The ‘‘symmetry centres’’  $(-1)^h X_h Y^{-\binom{h}{2}}$  resp.  $(-1)^h X_h^{-1} Y^{-\binom{h}{2}}$  in the functional equations (2.3) depend on  $Y$  and  $X_h$ , but not on  $X_1, \dots, X_{h-1}$ .

Throughout this paper, we always substitute  $q^{-1}$  for  $Y$  and hence write  $I_h(\mathbf{X})$  instead of  $I_h(q^{-1}; \mathbf{X})$  and  $I_h^\circ(\mathbf{X})$  instead of  $I_h^\circ(q^{-1}; \mathbf{X})$ .

Igusa functions are ubiquitous in the theory of zeta functions of groups and rings. Note, for instance, that

$$\zeta_{\mathfrak{o}^d}(s) = I_d((q^{(d-j)t} t^j)_{j \in [d]}) = \frac{1}{\prod_{j=0}^{d-1} (1 - q^j t)};$$

cf. (1.4).

**2.3. Hall bases for free nilpotent Lie rings.** Let  $x_1, \dots, x_d$  be Lie generators for  $\mathfrak{f} = \mathfrak{f}_{c,d}$ . A *Hall basis* for  $\mathfrak{f}_{c,d}$  (on  $x_1, \dots, x_d$ ) is a  $\mathbb{Z}$ -basis  $\mathcal{H}_{c,d}$  for  $\mathfrak{f}_{c,d}$  which may be constructed by selecting inductively certain *basic commutators* in the  $x_i$ ; see [8] for details. Note that  $|\mathcal{H}_{c,d}| = \sum_{i=1}^c W_d(i)$ , where  $W_d$  denotes the Witt function (1.2).

In Table 2.1 we record some Hall bases which are relevant for the current paper. Here we adopt the standard abbreviation  $x_{i_1} x_{i_2} x_{i_3} \dots x_{i_r}$  for left-normed commutators  $[\dots [[x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_r}]$ . In the case that  $d = 2$  (resp.  $d = 3$ ), we write  $x, y$  (and  $z$ ) instead of  $x_1, x_2$  (and  $x_3$ ).

$(c, d)$	Hall basis $\mathcal{H}_{c,d}$	$(W_d(i))_{i \in [c]}$
$(1, d)$	$\{x_1, \dots, x_d\}$	$(d)$
$(2, d)$	$\{x_1, \dots, x_d, x_i x_j \mid 1 \leq i < j \leq d\}$	$(d, \binom{d}{2})$
$(3, 3)$	$\{x, y, z, xy, xz, yz, xyx, xzx, yzx, xyy, xzy, yzy, xzz, yzz\}$	$(3, 3, 8)$
$(3, 2)$	$\{x, y, xy, xyx, xyy\}$	$(2, 1, 2)$
$(4, 2)$	$\{x, y, xy, xyx, xyy, xyxx, xyxy, xyyy\}$	$(2, 1, 2, 3)$

TABLE 2.1. Some Hall bases

**2.4. Reduced zeta functions.** In [6], Evseev introduced a certain “limit as  $\mathfrak{p} \rightarrow 1$ ” of various local zeta functions. Informally speaking, the idea is to exploit the fact that the coefficients of these generating functions enumerate  $\mathbb{F}_q$ -rational points of constructible sets. The desired limit is obtained by replacing these constructible sets by their Euler-Poincaré characteristics. The resulting rational functions are called *reduced zeta functions*; cf. [6, Section 3]. This paper does not define reduced *graded* ideal zeta functions explicitly, but provides a theoretical framework that allows this quite readily. If  $c, d$  are such that  $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s})$  for a rational function  $W_{c,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ , for almost all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ , then  $\zeta_{\mathfrak{f}_{c,d,\text{red}}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(1, Y) \in \mathbb{Q}(Y)$ . Evseev proved in [6, Section 4] that reduced ideal zeta functions of Lie rings with so-called *nice* and *simple* bases are Hilbert-Poincaré series enumerating integral points in rational polyhedral cones. The Hall bases  $\mathcal{H}_{2,d}$  given in Table 2.1, for instance, have this property. Comparing (1.7) and (1.8) one sees that, for nilpotent Lie rings  $L$  of nilpotency class 2,  $\zeta_{L,\text{red}}^{\triangleleft \text{gr}}(Y) = \zeta_{L,\text{red}}^{\triangleleft}(Y)$ ; cf. also Remark 5.2.

**2.5. Topological zeta functions.** Another means of defining limits of  $\mathfrak{p}$ -adic zeta functions are topological zeta functions. If  $c, d$  are such that  $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s})$  for

a rational function  $W_{c,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ , for almost all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ , then  $\zeta_{f_{c,d}, \text{top}}^{\triangleleft \text{gr}}(s)$  is simply the coefficient of  $(q-1)^{-r}$  in the expansion of  $W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s})$  in  $q-1$ , where  $r = \text{rk}_{\mathbb{Z}}(f_{c,d})$ .

*Example 2.5.* For  $a \in \mathbb{N}_0$  and  $b \in \mathbb{N}$ ,  $\frac{1}{1-q^{a-bs}} = \frac{1}{bs-a}(q-1)^{-1} + O(1)$ . Thus

$$\zeta_{f_{1,d}}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^d}(s) = \frac{1}{s(s-1)\dots(s-d+1)}(q-1)^{-d} + O((q-1)^{-d+1}),$$

whence

$$\zeta_{f_{1,d}, \text{top}}^{\triangleleft \text{gr}}(s) = \frac{1}{s(s-1)\dots(s-d+1)}.$$

More generally, [9, Definition 5.13] applies to any system of local zeta functions of Denef type over a number field, in the sense of [9, Definition 5.7]. Examples of such systems are families of  $\mathfrak{p}$ -adic zeta functions arising from families of the form  $(W(q, q^{-s}))_{\mathfrak{p} \in \text{Spec}(\mathfrak{o})}$ , for suitable  $W(X, Y) \in \mathbb{Q}(X, Y)$  (and, as usual,  $q = |\mathfrak{o}/\mathfrak{p}|$  for  $\mathfrak{p} \in \text{Spec}(\mathfrak{o})$ ); cf. also [10]. All the local graded ideal zeta functions considered in the current paper fit into such “uniform” families. We expect this phenomenon to be universal in the context of free nilpotent Lie rings; cf. Conjecture 6.1.

For a formal and far more general definition of topological zeta functions we refer to [9, Section 5]. In [9, Section 8], Rossmann collects a number of intriguing conjectures about analytic properties of topological zeta functions associated to various group- and ring-theoretic counting problems. We expect most of these conjectures to have analogues in the realm of topological graded ideal zeta functions. Motivated by the computation of various topological graded ideal zeta functions of free nilpotent Lie rings made throughout the current paper, we make a number of such conjectures in Section 6.

### 3. PROOF OF THEOREM 1.1 ((c, d) = (3, 3))

Let  $\mathfrak{f} = \mathfrak{f}_{3,3}$  be the free nilpotent Lie ring on 3 generators of nilpotency class 3. Let  $p$  be a prime and  $\mathfrak{o}$  be a finite extension of  $\mathbb{Z}_p$ , with uniformizer  $\pi$ . Note that  $\mathfrak{f}(\mathfrak{o})^{(1)} \cong \mathfrak{o}^3 \cong \mathfrak{f}(\mathfrak{o})^{(2)}$  and  $\mathfrak{f}(\mathfrak{o})^{(3)} \cong \mathfrak{o}^8$ . In order to parameterize lattices in  $\mathfrak{f}(\mathfrak{o})^{(2)}$  and  $\mathfrak{f}(\mathfrak{o})^{(3)}$  we denote by

$$\mu = (\mu_1, \mu_2, \mu_3)_{\geq} \quad \text{and} \quad \nu = (\nu_1, \dots, \nu_8)_{\geq}$$

integer partitions of at most 3 and 8 parts respectively, and set

$$\bar{\mu} = (\mu_1^{(2)}, \mu_2^{(3)}, \mu_3^{(3)}) = (\mu_1, \mu_1, \mu_2, \mu_2, \mu_2, \mu_3, \mu_3, \mu_3)_{\geq}.$$

#### Proposition 3.1.

$$(3.1) \quad \zeta_{f_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^3}(s) \sum_{\mu} \alpha(\mu_1^{(3)}, \mu; q) q^{-s(3\mu_1 + 2\mu_2 + \mu_3)} \sum_{\nu \leq \bar{\mu}} \alpha(\bar{\mu}, \nu; q) q^{-s \sum_{k=1}^8 \nu_k}.$$

*Proof.* Our starting point is (2.2) in Lemma 2.1, which in this case reads

$$\zeta_{f_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^3}(s) \sum_{\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}} |\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)|^{-s} |\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2|^{-s} \sum_{\substack{\Lambda_3 \leq \mathfrak{f}(\mathfrak{o})^{(3)}, \\ [\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_3}} |\mathfrak{f}(\mathfrak{o})^{(3)} : \Lambda_3|^{-s}.$$

Let  $\mu = (\mu_1, \mu_2, \mu_3)_{\geq}$  be a partition. There are  $\alpha(\mu_1^{(3)}, \mu; q)$  lattices  $\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}$  whose elementary divisor type with respect to  $\mathfrak{f}(\mathfrak{o})^{(2)}$  is given by  $\mu$ . Clearly  $|\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2| = q^{\sum_{j=1}^3 \mu_j}$ . We claim that  $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu_1 + \mu_2}$ . Indeed, lattices such as  $\Lambda_2$  may be parametrized by their elementary divisor type and a coset  $\alpha\Gamma_\mu \in \mathrm{SL}_3(\mathfrak{o})/\Gamma_\mu$  of a certain stabilizer subgroup  $\Gamma_\mu \leq \mathrm{SL}_3(\mathfrak{o})$ . Then  $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)|$  is the index of the lattice of solutions to the simultaneous congruences

$$(x_1, x_2, x_3) \begin{pmatrix} 0 & \alpha_{1j} & \alpha_{2j} \\ -\alpha_{1j} & 0 & \alpha_{3j} \\ -\alpha_{2j} & -\alpha_{3j} & 0 \end{pmatrix} \equiv 0 \pmod{\mathfrak{p}^{\mu_j}}$$

for  $j = 1, 2, 3$ ; cf. [14, Section 2.2] for details. This index clearly is  $q^{2\mu_1 + \mu_2}$ .

After a change of generators for  $\mathfrak{f}(\mathfrak{o})$  if necessary, we may assume that

$$\Lambda_2 = \mathfrak{p}^{\mu_1}xy \oplus \mathfrak{p}^{\mu_2}xz \oplus \mathfrak{p}^{\mu_3}yz.$$

At this point we crucially use the fact that  $d = 3$ , as  $\mathrm{GL}_d(\mathfrak{o}) \cong \bigwedge^2 \mathrm{GL}_d(\mathfrak{o})$  if and only if  $d = 3$ . We obtain

$$(3.2) \quad [\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] = \langle \pi^{\mu_1}xyx, \pi^{\mu_1}xyy, \pi^{\mu_1}xyz, \pi^{\mu_2}xzx, \pi^{\mu_2}xzy, \pi^{\mu_2}xzz, \pi^{\mu_3}yzx, \pi^{\mu_3}yzy, \pi^{\mu_3}yzz \rangle_{\mathfrak{o}}.$$

The Jacobi identity involving the three underlined terms is the only nontrivial relation between these terms. Indeed, the relation  $xyz + zxy + yzx = 0$  implies that

$$\langle \pi^{\mu_1}xyz, \pi^{\mu_2}xzy, \pi^{\mu_3}yzx \rangle_{\mathfrak{o}} = \langle \pi^{\mu_1}(xzy - yzx), \pi^{\mu_2}xzy, \pi^{\mu_3}yzx \rangle_{\mathfrak{o}} = \mathfrak{p}^{\mu_2}xzy \oplus \mathfrak{p}^{\mu_3}yzx.$$

Hence (3.2) implies that

$$[\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] = \mathfrak{p}^{\mu_1}xyx \oplus \mathfrak{p}^{\mu_1}xyy \oplus \mathfrak{p}^{\mu_2}xzx \oplus \mathfrak{p}^{\mu_2}xzy \oplus \mathfrak{p}^{\mu_2}xzz \oplus \mathfrak{p}^{\mu_3}yzx \oplus \mathfrak{p}^{\mu_3}yzy \oplus \mathfrak{p}^{\mu_3}yzz,$$

whence  $[\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}]$  has type  $\bar{\mu}$  with respect to  $\mathfrak{f}(\mathfrak{o})^{(3)}$ . This shows that the number of lattices  $\Lambda_3 \leq \mathfrak{f}(\mathfrak{o})^{(3)}$  whose elementary divisor type with respect to  $\mathfrak{f}(\mathfrak{o})^{(3)}$  is given by a partition  $\nu = (\nu_1, \dots, \nu_8)_{\geq}$  and which satisfy  $[\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_3$  is equal to  $\alpha(\bar{\mu}, \nu; q)$ . Each such lattice satisfies  $|\mathfrak{f}(\mathfrak{o})^{(3)} : \Lambda_3| = q^{-s \sum_{k=1}^8 \nu_k}$ .  $\square$

**3.1. Overlap types and 2-dimensional words.** Our approach to computing the right hand side of (3.1) is similar to the one taken in [12] to compute local factors of the  $(\mathbb{Z}-)$  ideal zeta functions of Lie rings of the form  $\mathfrak{f}_{2,2}(\mathcal{O})$ .

To compute the right hand side of (3.1) we carry out a case distinction with respect to the finitely many ways in which the partitions  $\bar{\mu}$  and  $\nu$  may “overlap”. To be precise, let  $\mu$  and  $\nu$  be partitions of 3 and 8 parts, respectively, satisfying  $\nu \leq \bar{\mu}$ . There are uniquely determined numbers  $r \in \mathbb{N}_0$  and  $M_i, N_i \in \mathbb{N}$  ( $i \in [r-1]$ ), such that

$$\begin{aligned} \mu_1 &\geq \dots \geq \mu_{M_1} \geq \nu_1 \geq \dots \geq \nu_{N_1} > \\ \mu_{M_1+1} &\geq \dots \geq \mu_{M_2} \geq \nu_{N_1+1} \geq \dots \geq \nu_{N_2} > \dots \\ \mu_{M_{r-1}+1} &\geq \dots \geq \mu_3 \geq \nu_{N_{r-1}+1} \geq \dots \geq \nu_8. \end{aligned}$$

Define  $M_r = 3$ ,  $N_r = 8$ , and  $M_0 = N_0 = 0$ . We call the integer sequence  $(M_i, N_i)_{i \in [r-1]}$  arising the *overlap type* of the pair  $(\mu, \nu)$ . Set

$$\widehat{\cdot} : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad n \mapsto n \cdot (2, 3, 3) := 2\delta_{n \geq 1} + 3\delta_{n \geq 2} + 3\delta_{n \geq 3},$$

An overlap type determines and is determined by the *2D-word*

$$v = \mathbf{1}^{M_1} \mathbf{2}^{N_1} \mathbf{1}^{M_2 - M_1} \mathbf{2}^{N_2 - N_1} \dots \mathbf{1}^{3 - M_{r-1}} \mathbf{2}^{8 - N_{r-1}},$$

i.e. a word on the alphabet  $\{\mathbf{1}, \mathbf{2}\}$  of length eleven such that  $\mathbf{1}$  occurs three times and  $\mathbf{2}$  eight times, and  $\widehat{M}_i \geq N_i$  for  $i \in [r-1]$ . Observe that the latter condition is equivalent to  $\nu \leq \bar{\mu}$ . In this case we write  $v(\mu, \nu) = v$ . We denote by  $\mathcal{D}^{(2)}$  the set of 15 2D-words arising in this way.

*Remark 3.2.* Our notion of 2D-word is related to the classical one of (2-dimensional) *Dyck word of length  $2n$* . The latter are words in  $\{\mathbf{1}, \mathbf{2}\}$ , featuring  $n$  occurrences of each letter. They may be used to model the overlap types of two partitions, each of at most  $n$  parts; cf. [12, Section 2.4]. 2D-words, in contrast, model the overlaps of two partitions  $\nu$  and  $\bar{\mu}$ ; whilst  $\nu$  has genuinely at most 8 parts,  $\bar{\mu}$  has at most 3 distinct parts, with ties in blocks of respective sizes 2, 3, and 3.

We chose to phrase our results and the supporting notation as similar to the relevant material in [12] as possible, leaving the reader free to concentrate on the crucial technical differences. Our Lemmata 3.4 and 3.5, for instance, are similar to, but subtly different from Lemmata 2.16 and 2.17 in [12]. Theorem 3.6 is analogous to [12, Theorem 3.1].

For  $v \in \mathcal{D}^{(2)}$  we set

$$(3.3) \quad D_v(q, t) := \sum_{\substack{(\mu, \nu) \\ v(\mu, \nu) = v}} \alpha(\mu_1^{(3)}, \mu; q) \alpha(\bar{\mu}, \nu; q) q^{-s(3\mu_1 + 2\mu_2 + \mu_3 + \sum_{k=1}^8 \nu_k)}.$$

Proposition 3.1 allows us to write

$$(3.4) \quad \zeta_{f_{3,3}^{\text{gr}}(s)}(s) = \zeta_{\mathfrak{o}^3}(s) \sum_{v \in \mathcal{D}^{(2)}} D_v(q, t).$$

This section's main result is Theorem 3.6, giving a general formula for the functions  $D_v$ . Table 3.1 lists the 15 words in  $\mathcal{D}^{(2)}$  together with their overlap types and an indication where in Section 3.2 an explicit formula for  $D_v$  may be found.

In order to obtain such formulae, it is useful for us to have notation for the successive differences of the parts of  $\mu$  and  $\nu$ . For  $j \in [8]$  we set

$$(3.5) \quad r_j = \begin{cases} \nu_j - \nu_{j+1} & \text{if } j \notin \{N_1, \dots, N_r\}, \\ \nu_{N_i} - \mu_{M_i+1} & \text{if } j = N_i, \end{cases}$$

where we define  $\mu_4 = 0$ . Similarly, for  $j \in [3]$  we set

$$(3.6) \quad s_j = \begin{cases} \mu_j - \mu_{j+1} & \text{if } j \notin \{M_1, \dots, M_r\}, \\ \mu_{M_i} - \nu_{N_{i-1}+1} & \text{if } j = M_i. \end{cases}$$

By definition,  $r_j, s_j \geq 0$  for all  $j$ . Note also that  $r_j > 0$  if  $j \in \{N_1, \dots, N_{r-1}\}$  and observe that  $\mu_{M_i} > \mu_{M_i+1}$  and  $\nu_{N_i} > \nu_{N_i+1}$  for each  $i \in [r-1]$ . Set

$$\widetilde{\cdot} : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad n \mapsto n \cdot (3, 2, 1) := 3\delta_{n \geq 1} + 2\delta_{n \geq 2} + 1\delta_{n \geq 3}.$$

TABLE 3.1. Overlap types and 2D-words

$r$	$(M_i, N_i)_{i \in [r-1]}$	$v \in \mathcal{D}^{(2)}$	Section
1	–	$\mathbf{1}^3\mathbf{2}^8$	3.2.1
2	(2, 1)	$\mathbf{1}^2\mathbf{2}\mathbf{1}\mathbf{2}^7$	3.2.2
2	(2, 2)	$\mathbf{1}^2\mathbf{2}^2\mathbf{1}\mathbf{2}^6$	
2	(2, 3)	$\mathbf{1}^2\mathbf{2}^3\mathbf{1}\mathbf{2}^5$	
2	(2, 4)	$\mathbf{1}^2\mathbf{2}^4\mathbf{1}\mathbf{2}^4$	
2	(2, 5)	$\mathbf{1}^2\mathbf{2}^5\mathbf{1}\mathbf{2}^3$	
2	(1, 1)	$\mathbf{1}\mathbf{2}\mathbf{1}^2\mathbf{2}^7$	3.2.3
3	(1, 1; 2, 2)	$\mathbf{1}\mathbf{2}\mathbf{1}\mathbf{2}\mathbf{1}\mathbf{2}^6$	
3	(1, 1; 2, 3)	$\mathbf{1}\mathbf{2}\mathbf{1}\mathbf{2}^2\mathbf{1}\mathbf{2}^5$	
3	(1, 1; 2, 4)	$\mathbf{1}\mathbf{2}\mathbf{1}\mathbf{2}^3\mathbf{1}\mathbf{2}^4$	
3	(1, 1; 2, 5)	$\mathbf{1}\mathbf{2}\mathbf{1}\mathbf{2}^4\mathbf{1}\mathbf{2}^3$	
2	(1, 2)	$\mathbf{1}\mathbf{2}^2\mathbf{1}^2\mathbf{2}^6$	3.2.4
3	(1, 2; 2, 3)	$\mathbf{1}\mathbf{2}^2\mathbf{1}\mathbf{2}\mathbf{1}\mathbf{2}^5$	
3	(1, 2; 2, 4)	$\mathbf{1}\mathbf{2}^2\mathbf{1}\mathbf{2}^2\mathbf{1}\mathbf{2}^4$	
3	(1, 2; 2, 5)	$\mathbf{1}\mathbf{2}^2\mathbf{1}\mathbf{2}^3\mathbf{1}\mathbf{2}^3$	

Then

$$3\mu_1 + 2\mu_2 + \mu_3 = \sum_{j=1}^3 \check{j}s_j + \sum_{i=1}^r \check{M}_i (r_{N_{i-1}+1} + \cdots + r_{N_i}),$$

$$\nu_1 + \cdots + \nu_8 = \sum_{j=1}^8 j r_j + \sum_{i=1}^{r-1} N_i (s_{M_{i+1}} + \cdots + s_{M_{i+1}}).$$

Finally, for each  $i \in [r]$  we define

$$(3.7) \quad J_i^\mu = \{j \in [M_i - M_{i-1} - 1] \mid \mu_{M_i-j} > \mu_{M_i-j+1}\},$$

$$(3.8) \quad J_i^\nu = \{j \in [N_i - N_{i-1} - 1] \mid \nu_{N_i-j} > \nu_{N_i-j+1}\}.$$

We start with an immediate consequence of Proposition 2.2.

**Lemma 3.3.** *Let  $\mu$  and  $\nu$  be partitions of 3 and 8 parts, respectively, satisfying  $\nu \leq \bar{\mu}$ .*

*Then*

$$\sum_{\mu} \alpha(\mu_1^{(3)}, \mu; q) \alpha(\bar{\mu}, \nu; q) = \prod_{k \geq 1} q^{\mu'_k(3-\mu'_k) + \nu'_k(\widehat{\mu'_k} - \nu'_k)} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}}.$$

Let  $\mu$  and  $\nu$  be partitions as in Lemma 3.3. Let  $(M_i, N_i)_{i \in [r-1]}$  be their overlap type and  $\mu'$  resp.  $\nu'$  be their dual partitions. For  $k \in \mathbb{N}$ , set

$$d_k := \mu'_k(3 - \mu'_k) + \nu'_k(\widehat{\mu'_k} - \nu'_k).$$

Also, set

$$a_j := M_i(3 - M_i) + j(\widehat{M}_i - j), \quad j \in ]N_{i-1}, N_i],$$

$$b_j := j(3 - j) + N_{i-1}(\widehat{j} - N_{i-1}), \quad j \in ]M_{i-1}, M_i].$$

Note that  $j$  determines a unique  $i$ .

**Lemma 3.4.** *For  $i \in [r]$ ,*

$$\begin{aligned} \prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}+1}} q^{d_k} \binom{3-\mu'_{k+1}}{3-\mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{M_i - M_{i-1}} q^{b_{M_{i-1}+j} s_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}}. \end{aligned}$$

*Proof.* Since all the indices  $k$  appearing in the product on the left hand side satisfy  $\nu_{N_{i-1}} > \mu_{M_{i-1}+1} \geq k \geq \mu_{M_i} \geq \nu_{N_{i-1}+1}$ , we have  $\nu'_k = \nu'_{k+1} = N_{i-1}$  for all  $k$  in the interval  $[\mu_{M_i}, \mu_{M_{i-1}+1}]$ . Thus

$$\prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}+1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} = 1.$$

For  $j \in [M_i - M_{i-1}]$  we have  $\mu'_k = M_{i-1} + j$  when  $\mu_{M_{i-1}+j+1} < k \leq \mu_{M_{i-1}+j}$ ; observe that it may be the case for some  $j$  that no index  $k$  satisfies this condition. As a result, we see that for each  $j \in [M_i - M_{i-1}]$ , there are exactly  $s_{M_{i-1}+j}$  elements  $k$  of the segment  $[\mu_{M_i}, \mu_{M_{i-1}+1}]$  for which  $\mu'_k = M_{i-1} + j$ . Now, note that

$$\binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \neq 1$$

if and only if  $\mu'_{k+1} \neq \mu'_k$ . This is the case exactly when there exists an  $i$  such that  $\mu_i = k$ . It follows that if  $J_i^\mu = \{j_{i,1}, \dots, j_{i,\gamma_i}\}$ , with  $j_{i,1} < \dots < j_{i,\gamma_i}$ ,  $j_{i,0} := 0$ ,  $j_{i,\gamma_i+1} := M_i - M_{i-1}$ , then

$$(3.9) \quad \prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}+1}} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} = \prod_{m=0}^{\gamma_i} \binom{3 - M_i + j_{i,m+1}}{3 - M_i + j_{i,m}}_{q^{-1}}.$$

We make use of the well-known identity

$$\binom{\alpha}{\beta}_X = \frac{1 - X^\alpha}{1 - X^{\alpha-\beta}} \binom{\alpha-1}{\beta}_X$$

for Gaussian binomial coefficients. Applying it inductively, we see that for all  $m \in [\gamma_i - 1]$ ,

$$\binom{3 - M_i + j_{i,m+1}}{3 - M_i + j_{i,m}}_{q^{-1}} = \binom{j_{i,m+1}}{j_{i,m}}_{q^{-1}} \frac{\binom{3 - M_i + j_{i,m+1}}{3 - M_i}_{q^{-1}}}{\binom{3 - M_i + j_{i,m}}{3 - M_i}_{q^{-1}}}.$$

Hence the right-hand side of (3.9) is equal to

$$\binom{M_i - M_{i-1}}{J_i^\lambda}_{q^{-1}} \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}}.$$

Thus

$$\begin{aligned} \prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}+1}} q^{d_k} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{M_i - M_{i-1}} q^{b_{M_{i-1}+j} s_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}} \end{aligned}$$

as required.  $\square$

**Lemma 3.5.** *For  $i \in [r]$ ,*

$$\begin{aligned} \prod_{k=\mu_{M_i+1}+1}^{\mu_{M_i}-1} q^{d_k} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{N_i - N_{i-1}} q^{a_{N_{i-1}+j} r_{N_{i-1}+j}} \binom{N_i - N_{i-1}}{J_i^\nu}_{q^{-1}} \binom{\widehat{M}_i - N_{i-1}}{\widehat{M}_i - N_i}_{q^{-1}}. \end{aligned}$$

*Proof.* Note that the product on the left hand side may be empty; this happens in the case  $\mu_{M_i} = \nu_{N_{i-1}+1} = \dots = \nu_{N_i} = \mu_{M_{i+1}} + 1$ . Since  $\mu_{M_i} > k > \mu_{M_{i+1}}$ , we have  $\mu'_k = \mu'_{k+1} = M_i$  for all  $k$  in the interval  $]\mu_{M_{i+1}}, \mu_{M_{i-1}}]$ . Finally, observe that for  $j \in [N_i - N_{i-1}]$  we have  $\nu'_k = N_{i-1} + j$  when  $\nu_{N_{i-1}+j+1} < k \leq \nu_{N_{i-1}+j}$ . The claim follows as in the proof of the previous lemma.  $\square$

With these preparations in place we now give a formula for the functions  $D_v$ .

**Theorem 3.6.** *Let  $v = \prod_{i=1}^r (\mathbf{1}^{M_i - M_{i-1}} \mathbf{2}^{N_i - N_{i-1}}) \in \mathcal{D}^{(2)}$ . Then*

$$\begin{aligned} D_v(q, t) = \prod_{i=1}^r \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}} \binom{\widehat{M}_i - N_{i-1}}{\widehat{M}_i - N_i}_{q^{-1}} \prod_{i=1}^r I_{M_i - M_{i-1}}(Y_{M_{i-1}+1}, \dots, Y_{M_i}) \\ \cdot \left( \prod_{i=1}^{r-1} I_{N_i - N_{i-1}}^{\circ}(X_{N_{i-1}+1}, \dots, X_{N_i}) \right) \cdot I_{8 - N_{r-1}}(X_{N_{r-1}+1}, \dots, X_8), \end{aligned}$$

with numerical data

$$(3.10) \quad X_j = q^{a_j} t^{\widetilde{M}_i + j} \quad j \in ]N_{i-1}, N_i],$$

$$(3.11) \quad Y_j = q^{b_j} t^{\check{j} + N_{i-1}} \quad j \in ]M_{i-1}, M_i].$$

*Proof.* Given  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{N}_0^3$  and  $\mathbf{v}' = (v'_1, \dots, v'_8) \in \mathbb{N}_0^8$ , we set, for each  $i \in [r]$ ,

$$\text{supp}_i^M(\mathbf{v}) = \{j \in [M_i - M_{i-1} - 1] \mid v_{M_{i-1}+j} > 0\},$$

$$\text{supp}_i^N(\mathbf{v}') = \{j \in [N_i - N_{i-1} - 1] \mid v'_{N_{i-1}+j} > 0\}.$$

In practice,  $\mathbf{v}$  will be one of the vectors of successive differences  $\mathbf{s} = (s_1, s_2, s_3)$  (cf. (3.6)) and  $\mathbf{v}'$  will be one of the vectors of successive differences  $\mathbf{r} = (r_1, \dots, r_8)$  (cf. (3.5)). Given a pair  $(\mu, \nu)$  of partitions satisfying  $\nu \leq \bar{\mu}$ , recall our definitions (3.7) and (3.8). It is easy to see that, for every  $i \in [r]$ ,

$$\text{supp}_i^M(\mathbf{v}) = M_i - M_{i-1} - J_i^\mu \quad \text{and} \quad \text{supp}_i^N(\mathbf{v}') = N_i - N_{i-1} - J_i^\nu.$$

Thus it follows that

$$\binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} = \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}} \quad \text{and} \quad \binom{N_i - N_{i-1}}{J_i^{\nu'}}_{q^{-1}} = \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v}')}_{q^{-1}}.$$

Let  $\delta_{ij}$  be the usual Kronecker delta function. Substituting Lemma 3.4 and 3.5, rewriting the expressions in terms of  $r_j$  and  $s_j$  and using (3.1), we find that formula (3.3) for  $D_v(q, t)$  splits into a product as follows:

$$D_v(q, t) = \prod_{i=1}^r \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}} \binom{\widehat{M}_i - N_{i-1}}{\widehat{M}_i - N_i}_{q^{-1}} \prod_{i=1}^r A_i B_i,$$

where, for  $i \in [r]$ ,

$$A_i = \sum_{r_{N_{i-1}+1}=0}^{\infty} \cdots \sum_{r_{N_{i-1}=0}}^{\infty} \sum_{r_{N_i=1-\delta_{ir}}}^{\infty} \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v}')}_{q^{-1}} \prod_{j=N_{i-1}+1}^{N_i} \left( q^{a_j} t^{\widetilde{M}_i+j} \right)^{r_j},$$

$$B_i = \sum_{s_{M_{i-1}+1}=0}^{\infty} \cdots \sum_{s_{M_{i-1}=0}}^{\infty} \sum_{s_{M_i}=0}^{\infty} \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}} \prod_{j=M_{i-1}+1}^{M_i} \left( q^{b_j} t^{N_{i-1}+\check{j}} \right)^{s_j}.$$

We now show that all of the factors  $A_i$  and  $B_i$  are products of Igusa functions and Gaussian binomial coefficients. Given  $i \in [r]$  and  $I \subseteq [M_i - M_{i-1} - 1]$ , we define  $\mathbf{S}^i(I)$  to be the set of vectors  $\mathbf{s}^i = (s_{M_{i-1}+1}, \dots, s_{M_i}) \in \mathbb{N}_0^{M_i - M_{i-1}}$  such that  $s_j = 0$  unless  $j \in \{M_{i-1} + k \mid k \in I\} \cup \{M_i\}$ . With the numerical data defined in (3.11), we have

$$\begin{aligned} B_i &= \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I}_{q^{-1}} \sum_{\mathbf{s}^i \in \mathbf{S}^i(I)} \prod_{j \in (I + M_{i-1}) \cup \{M_i\}} \left( q^{b_j} t^{N_{i-1}+\check{j}} \right)^{s_j} \\ &= \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I}_{q^{-1}} \left( \prod_{\iota \in I} \left( \sum_{s_{M_{i-1}+\iota}=1}^{\infty} (Y_{M_{i-1}+\iota})^{s_{M_{i-1}+\iota}} \right) \right) \sum_{s_{M_i}=0}^{\infty} (Y_{M_i})^{s_{M_i}} \\ &= \frac{1}{1 - Y_{M_i}} \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I}_{q^{-1}} \prod_{\iota \in I} \frac{Y_{M_{i-1}+\iota}}{1 - Y_{M_{i-1}+\iota}} \\ &= I_{M_i - M_{i-1}}(Y_{M_{i-1}+1}, \dots, Y_{M_i}). \end{aligned}$$

Analogously one shows that, with the numerical data defined in (3.10),

$$A_i = \begin{cases} I_{N_i - N_{i-1}}^\circ(X_{N_{i-1}+1}, \dots, X_{N_i}) & \text{for } i < r, \\ I_{8 - N_{r-1}}(X_{N_{r-1}+1}, \dots, X_8) & \text{for } i = r. \end{cases}$$

This completes the proof of Theorem 3.6.  $\square$

## 3.2. Explicit formulae for the functions $D_v$ .

### 3.2.1. $v = \mathbf{1}^3 \mathbf{2}^8$ .

$$D_{\mathbf{1}^3 \mathbf{2}^8}(q, t) = I_3(q^2 t^3, q^2 t^5, t^6) I_8((q^{i(8-i)} t^{6+i})_{i=1, \dots, 8}).$$

*Remark 3.7.* Note that  $\zeta_{\mathfrak{f}_{2,3}(0)}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^3}(s) I_3(q^2 t^3, q^2 t^5, t^6)$ ; cf. Proposition 4.10.

3.2.2.  $v = \mathbf{1}^2\mathbf{2}^j\mathbf{12}^{8-j}$ ,  $j \in \{1, \dots, 5\}$ .

$$D_{\mathbf{1}^2\mathbf{2}^j\mathbf{12}^{8-j}}(q, t) = \binom{5}{j}_{q^{-1}} \binom{3}{1}_{q^{-1}} I_2(q^2t^3, q^2t^5) \cdot I_j((q^{2+i(5-i)}t^{5+i})_{i=1, \dots, j}) I_1(q^{j(8-j)}t^{6+j}) I_{8-j}((q^{i(8-i)}t^{6+i})_{i=j+1, \dots, 8}).$$

3.2.3.  $v = \mathbf{1212}^{j-1}\mathbf{12}^{8-j}$ ,  $j \in \{1, \dots, 5\}$ . For  $j = 1$ ,

$$D_{\mathbf{1212}^{2^7}}(q, t) = \binom{3}{1}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_1(q^2t^3) I_1^\circ(q^3t^4) I_2(q^6t^6, q^7t^7) I_7((q^{i(8-i)}t^{6+i})_{i=2, \dots, 8}).$$

For  $j \in \{2, \dots, 5\}$ ,

$$D_{\mathbf{1212}^{j-1}\mathbf{12}^{8-j}}(q, t) = \binom{4}{j-1}_{q^{-1}} \binom{3}{1}_{q^{-1}} \binom{2}{1}_{q^{-1}}^2 I_1(q^2t^3) I_1^\circ(q^3t^4) I_1(q^6t^6) \cdot I_{j-1}^\circ((q^{2+i(5-i)}t^{5+i})_{i=2, \dots, j}) I_1(q^{j(8-j)}t^{6+j}) I_{8-j}((q^{i(8-i)}t^{6+i})_{i=j+1, \dots, 8}).$$

3.2.4.  $v = \mathbf{12}^2\mathbf{12}^{j-1}\mathbf{12}^{7-j}$ ,  $j \in \{1, \dots, 4\}$ . For  $j = 1$ ,

$$D_{\mathbf{12}^2\mathbf{12}^{2^6}}(q, t) = \binom{3}{1}_{q^{-1}} I_1(q^2t^3) I_2^\circ(q^3t^4, q^2t^5) I_2(q^8t^7, q^{12}t^8) I_6((q^{i(8-i)}t^{6+i})_{i=3, \dots, 8}).$$

For  $j \in \{2, 3, 4\}$ ,

$$D_{\mathbf{12}^2\mathbf{12}^{j-1}\mathbf{12}^{7-j}}(q, t) = \binom{3}{j-1}_{q^{-1}} \binom{3}{1}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_1(q^2t^3) I_2^\circ(q^3t^4, q^2t^5) I_1(q^8t^7) \cdot I_{j-1}^\circ((q^{2+i(5-i)}t^{5+i})_{i=3, \dots, j+1}) I_1(q^{(j+1)(8-(j+1))}t^{7+j}) I_{8-(j+1)}((q^{i(8-i)}t^{6+i})_{i=j+2, \dots, 8}).$$

**3.3. Completion of the proof.** The first two claims of Theorem 1.1 follows from (3.4) and the explicit formulae for  $D_v$ ,  $v \in \mathcal{D}^{(2)}$ , given in Section 3.2. To deduce the local functional equation (1.5), one checks, repeatedly using the functional equations (2.3) for the Igusa functions  $I_h$  resp.  $I_h^\circ$  and the well-known fact that, for  $a, b \in \mathbb{N}_0$ ,  $a \geq b$ ,  $\binom{a}{b}_q = q^{b(a-b)} \binom{a}{b}_{q^{-1}}$ , that each of the functions  $D_v$  satisfies

$$D_v(q^{-1}t^{-1}) = -q^{31}t^{20}D_v(q, t).$$

The functional equation (1.5) follows from (3.4) as

$$\zeta_{\mathfrak{o}^3}(s)|_{q \rightarrow q^{-1}} = -q^3t^3\zeta_{\mathfrak{o}^3}(s).$$

#### 4. NILPOTENCY CLASS TWO ( $c = 2$ )

Let  $d \in \mathbb{N}_{\geq 2}$ . In this section we compute the local graded ideal zeta functions  $\zeta_{\mathfrak{f}_{d,2}^{\langle \text{gr} \rangle}}(s)$ . We prove functional equations for these functions and establish their behaviour at  $s = 0$ . We use them to determine the abscissae of convergence of the global graded ideal zeta functions  $\zeta_{\mathfrak{f}_{d,2}^{\langle \text{gr} \rangle}}(s)$  and some properties of the associated topological and reduced graded ideal zeta functions. Throughout we write  $d'$  for  $\binom{d}{2} = W_2(d)$ ; cf. (1.2).

**4.1. p-Adic formulae.** The paper [15] determines the *normal subgroup zeta functions* of the free class-2-nilpotent  $d$ -generator groups  $F_{2,d}$ , enumerating these groups' normal subgroups of finite index. By the Mal'cev correspondence, these are the ideal zeta functions of the free nilpotent  $\mathbb{Z}$ -Lie rings  $\mathfrak{f}_{2,d}$ . The computations generalize to the case of general number rings in a straightforward manner. To recall the paper's main result define the function

$$\varphi : [d-1]_0 \rightarrow [d'], \quad i \mapsto id - \binom{i+1}{2}.$$

Given  $(I, J) \in 2^{[d-1]_0} \times 2^{[d']}$ , the paper [15, p. 71] defines a total order  $<_{\varphi(I),J}$  on the disjoint union  $I \cup J$ . Without loss of generality, we may assume that the sets  $I$  and  $J$  have the same cardinality  $h$ . For  $i \in I = \{i_1, \dots, i_h\}_{<}$  and  $j \in J = \{j_1, \dots, j_h\}_{<}$ , set

$$\begin{aligned} j(i) &:= \min \{j \in J \cup \{d'\} \mid \varphi(i) <_{\varphi(I),J} j\}, \\ i(j) &:= \max \{i \in I \cup \{0\} \mid \varphi(i) <_{\varphi(I),J} j\}, \end{aligned}$$

and  $j_0 := i_0 := 0 <_{\varphi(I),J} \varphi([d-2]) \uplus [d'-1] <_{\varphi(I),J} j_{h+1} := d'$ . The following is essentially the main result of [15], in notation compatible with the current paper. (The underlining of terms of the form  $d+$  in (4.3) is meant to facilitate comparison with the “graded” numerical data (4.5) in Theorem 4.2, and may be ignored.)

**Theorem 4.1.** [15, Theorem 4] *For all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ ,*

$$(4.1) \quad \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s) = \zeta_{\mathfrak{o}^d}(s) \sum_{\substack{I \subseteq [d-2], J \subseteq [d'-1] \\ |I|=|J|, \varphi(I) \leq J}} \mathcal{A}_{I,J}^{\triangleleft}(q, t),$$

where

$$(4.2) \quad \begin{aligned} \mathcal{A}_{I,J}^{\triangleleft}(q, t) &= \zeta_{\mathfrak{o}^{i_h}}(s)^{-1} I_{j_1}((X_\alpha)_{\alpha \in [j_1-1]}, X_0) \cdot \prod_{r=1}^h \binom{j_{r+1} - \varphi(i_r)}{j_r - \varphi(i_r)}_{q^{-1}} \binom{d - i_{r-1}}{d - i_r}_{q^{-1}} \\ &\quad \cdot I_{i_r - i_{r-1}}^{\circ}((Y_\beta)_{\beta \in [i_r-1, i_r]}, Y'_{i_r}) I_{j_{r+1} - j_r}((X_\alpha)_{\alpha \in [j_r, j_{r+1}]}, X_{j_r}), \end{aligned}$$

with numerical data

$$(4.3) \quad \begin{aligned} X_j &= q^{i(j)(d-i(j))+(d'-j)(\underline{d+j}-\varphi(i(j)))} t^{d-i(j)+d'-j}, & j \in [d'-1]_0, \\ Y_i &= q^{i(d-i)+(d'-j(i))(\underline{d+j(i)}-\varphi(i))} t^{d-i+d'-j(i)}, & i \in [d-2], \\ Y'_{i_r} &= q^{i_{r-1}(d-i_{r-1})+(d'-j(i_r))(\underline{d+j(i_r)}-\varphi(i_{r-1}))} t^{d-i_{r-1}+d'-j(i_r)}, & r \in [h]. \end{aligned}$$

The following is a graded analogue of Theorem 4.1.

**Theorem 4.2.** *For all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ ,*

$$(4.4) \quad \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft, \text{gr}}(s) = \zeta_{\mathfrak{o}^d}(s) \sum_{\substack{I \subseteq [d-2], J \subseteq [d'-1] \\ |I|=|J|, \varphi(I) \leq J}} \mathcal{A}_{I,J}^{\triangleleft, \text{gr}}(q, t),$$

where  $\mathcal{A}_{I,J}^{\triangleleft \text{gr}}(q, t)$  is defined as  $\mathcal{A}_{I,J}^{\triangleleft}(q, t)$  in (4.2), but with numerical data

$$(4.5) \quad \begin{aligned} X_j &= q^{i(j)(d-i(j))+(d'-j)(j-\varphi(i(j)))} t^{d-i(j)+d'-j}, & j \in [d'-1]_0, \\ Y_i &= q^{i(d-i)+(d'-j(i))(j(i)-\varphi(i))} t^{d-i+d'-j(i)}, & i \in [d-2], \\ Y'_{i_r} &= q^{i_{r-1}(d-i_{r-1})+(d'-j(i_r))(j(i_r)-\varphi(i_{r-1}))} t^{d-i_{r-1}+d'-j(i_r)}, & r \in [h]. \end{aligned}$$

*Proof.* The difference between (4.3) and (4.5) reflects the difference between (1.7) and (1.8).  $\square$

**Corollary 4.3.**

$$(4.6) \quad \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{d+d'} q^{\binom{d+d'}{2}} t^{2d+d'} \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s),$$

$$(4.7) \quad \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{d+d'} q^{\binom{d}{2} + \binom{d'}{2}} t^{2d+d'} \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s).$$

*Proof.* Eq. (4.6) is [15, Theorem 3(a)]. The proof proceeds by establishing the symmetry in question for each of the summands in (4.1), using (2.3). The formulae (4.1) and (4.4) only differ in their numerical data. By Remark 2.4, it suffices to control the value of  $X_0 \cdot \prod_{r=1}^h \frac{X_{j_r}}{Y'_{i_r}}$  in both cases. Inspection of (4.3) and (4.5), respectively, reveals that in the ideal case this term is by a factor  $q^{dd'}$  larger than in the graded ideal case. Thus

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{d+d'} q^{\binom{d+d'}{2} - dd'} t^{2d+d'} \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s).$$

As  $\binom{d+d'}{2} - dd' = \binom{d}{2} + \binom{d'}{2}$ , this establishes (4.7).  $\square$

**4.2. p-Adic behaviour at zero.** We describe the behaviours of  $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)$  and  $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$  at  $s = 0$ .

**Theorem 4.4.** *For all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ ,*

$$(4.8) \quad \frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)}{\zeta_{\mathfrak{o}^{d+d'}}(s)} \Big|_{s=0} = 1,$$

$$(4.9) \quad \frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^d}(s) \zeta_{\mathfrak{o}^{d'}}(s)} \Big|_{s=0} = \frac{d \cdot d'}{d \cdot (d + d')} = \frac{d-1}{d+1}.$$

*Proof.* To prove (4.8) we note that, by (4.1),

$$\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)}{\zeta_{\mathfrak{o}^{d+d'}}(s)} = \prod_{j=d}^{d+d'-1} (1 - q^j t) \left( \mathcal{A}_{\emptyset, \emptyset}^{\triangleleft}(q, t) + \sum_{(I,J) \neq (\emptyset, \emptyset)} \mathcal{A}_{I,J}^{\triangleleft}(q, t) \right).$$

By inspection of (4.3) we see that  $\mathcal{A}_{\emptyset, \emptyset}^{\triangleleft}(q, t) = I_{d'}((q^{(d+(d'-j))j} t^{d+j})_{j \in [d']})$  and that

$$\mathcal{A}_{I,J}^{\triangleleft}(q, 1) = \begin{cases} 0 & \text{if } (I, J) \neq (\emptyset, \emptyset), \\ I_{d'}((q^{(d+(d'-j))j})_{j \in [d']}) = \prod_{j=d}^{d+d'-1} (1 - q^j) & \text{if } (I, J) = (\emptyset, \emptyset). \end{cases}$$

To prove (4.9) we note that

$$\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^d}(s) \zeta_{\mathfrak{o}^{d'}}(s)} = \prod_{j=0}^{d'-1} (1 - q^j t) \left( \mathcal{A}_{\emptyset, \emptyset}^{\triangleleft \text{gr}}(q, t) + \sum_{(I,J) \neq (\emptyset, \emptyset)} \mathcal{A}_{I,J}^{\triangleleft \text{gr}}(q, t) \right).$$

By inspection of (4.5) we see that  $\mathcal{A}_{\emptyset, \emptyset}^{\triangleleft \text{gr}}(q, t) = I_{d'}((q^{(d'-j)j} t^{d+j})_{j \in [d']})$ , which has a simple pole at  $t = 1$ . In contrast,  $\mathcal{A}_{I, J}^{\triangleleft \text{gr}}(q, t)$  has no pole at  $t = 1$  when  $(I, J) \neq (\emptyset, \emptyset)$ . Hence

$$\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^{d'}}(s)\zeta_{\mathfrak{o}^{d'}}(s)} \Big|_{s=0} = \frac{I_{d'}((q^{(d'-j)j} t^{d+j})_{j \in [d']})}{I_{d'}((q^{(d'-j)j} t^j)_{j \in [d']})} \Big|_{t=1} = \frac{1-t^{d'}}{1-t^{d+d'}} \Big|_{t=1} = \frac{d'}{d+d'}. \quad \square$$

*Remark 4.5.* In the pertinent special cases, (4.8) confirms [9, Conjecture IV ( $\mathfrak{P}$ -adic form)].

**4.3. Global analytic properties.** The following result compares some of the known analytic properties of  $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}$  (cf. [15, Theorem 3]) with those of  $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}$  (s).

**Theorem 4.6.** *The abscissae of convergence of  $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)$  resp.  $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$  are*

$$(4.10) \quad \alpha^{\triangleleft}(\mathfrak{f}_{2,d}) = \max \left\{ d, \frac{((\binom{d}{2}) - j)(d+j) + 1}{\binom{d+1}{2} - j} \mid j \in [(\binom{d}{2}) - 1] \right\} \text{ resp. ,}$$

$$(4.11) \quad \alpha^{\triangleleft \text{gr}}(\mathfrak{f}_{2,d}) = \max \left\{ d, \frac{((\binom{d}{2}) - j)j + 1}{\binom{d+1}{2} - j} \mid j \in [(\binom{d}{2}) - 1] \right\}.$$

*The respective meromorphic continuations of both zeta functions beyond their abscissae of convergence have simple poles at  $s = \alpha^{\triangleleft}(\mathfrak{f}_{2,d})$  resp.  $s = \alpha^{\triangleleft \text{gr}}(\mathfrak{f}_{2,d})$ .*

*Proof.* Eq. (4.10) and the ensuing claim about the meromorphic continuation is essentially [15, Theorem 3]. Eq. (4.11) and the analogous claim are proved analogously.  $\square$

#### 4.4. Topological zeta functions – degree and behaviour at zero.

**Theorem 4.7.**

$$\deg_s \left( \zeta_{\mathfrak{f}_{2,d}, \text{top}}^{\triangleleft}(s) \right) = -(d+d') = \deg_s \left( \zeta_{\mathfrak{f}_{2,d}, \text{top}}^{\triangleleft \text{gr}}(s) \right).$$

*Moreover,*

$$(4.12) \quad s \zeta_{\mathfrak{f}_{2,d}, \text{top}}^{\triangleleft}(s) \Big|_{s=0} = \frac{(-1)^{d+d'-1}}{(d+d'-1)!},$$

$$(4.13) \quad s^2 \zeta_{\mathfrak{f}_{2,d}, \text{top}}^{\triangleleft \text{gr}}(s) \Big|_{s=0} = \frac{(-1)^{(d-1)+(d'-1)} d'}{(d+d')(d-1)!(d'-1)!}.$$

*Proof.* Note that, given  $h \in \mathbb{N}$  and, for  $i \in [h]$ ,  $X_i = q^{a_i} t^{b_i}$  for  $a_i \in \mathbb{N}_0$ ,  $b_i \in \mathbb{N}$ ,

$$I_h(X_1, \dots, X_h) = \frac{h!}{\prod_{i=1}^h (b_i s - a_i)} (q-1)^{-h} + O((q-1)^{-h+1}).$$

The summands in the formula (4.1) for  $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)$  are all products of Igusa functions and Gaussian binomial coefficients. Hence there exist  $a_{I,J,i}^{\triangleleft} \in \mathbb{N}_0$  and  $b_{I,J,i}^{\triangleleft}, c_{I,J}^{\triangleleft} \in \mathbb{N}$  such that

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft} s - a_{I,J,i}^{\triangleleft})} (q-1)^{-(d+d')} + O((q-1)^{-(d+d'+1)}).$$

(Here and in the sequel, the sums are over pairs  $(I, J)$  as in (4.1).) Hence

$$(4.14) \quad \zeta_{\mathfrak{f}_{2,d}, \text{top}}^{\triangleleft}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft} s - a_{I,J,i}^{\triangleleft})}$$

is a rational function in  $s$  of degree  $-(d + d')$ , confirming the first claim.

The second claim, on the degree of  $\zeta_{f_{2,d},\text{top}}^{\triangleleft\text{gr}}(s)$ , follows from analogous considerations based on (4.4). Indeed, there exist  $a_{I,J,i}^{\triangleleft\text{gr}} \in \mathbb{N}_0$  and  $b_{I,J,i}^{\triangleleft\text{gr}}, c_{I,J}^{\triangleleft\text{gr}} \in \mathbb{N}$  such that

$$\zeta_{f_{2,d}(\emptyset)}^{\triangleleft\text{gr}}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft\text{gr}}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft\text{gr}} s - a_{I,J,i}^{\triangleleft\text{gr}})} (q-1)^{-(d+d')} + O((q-1)^{-(d+d'+1)}).$$

Hence

$$(4.15) \quad \zeta_{f_{2,d},\text{top}}^{\triangleleft\text{gr}}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft\text{gr}}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft\text{gr}} s - a_{I,J,i}^{\triangleleft\text{gr}})}$$

is a rational function in  $s$  of degree  $-(d + d')$ , too.

Turning to the behaviour at  $s = 0$ , we start with the observation that  $\zeta_{\sigma^d}(s) \mathcal{A}_{I,J}^{\triangleleft}(q, t)$  has no pole at  $t = 1$  unless  $(I, J) = (\emptyset, \emptyset)$ , in which case it is simple. Thus the summand  $\frac{c_{I,J}^{\triangleleft}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft} s - a_{I,J,i}^{\triangleleft})}$  in (4.14) has no pole at  $s = 0$  unless  $(I, J) = (\emptyset, \emptyset)$ , in which case it is simple. Specifically,

$$\zeta_{\sigma^d}(s) \mathcal{A}_{\emptyset,\emptyset}^{\triangleleft}(q, t) = I_d((q^{i(d-i)} t^i)_{i \in [d]}) I_{d'}((q^{(d+d'-j)j} t^{d+j})_{j \in [d']}),$$

whence

$$\frac{c_{\emptyset,\emptyset}^{\triangleleft}}{\prod_{i=1}^{d+d'} (b_{\emptyset,\emptyset,i}^{\triangleleft} s - a_{\emptyset,\emptyset,i}^{\triangleleft})} = \frac{1}{\prod_{i=0}^{d-1} (s - i)} \cdot \frac{(d')!}{\prod_{j=1}^{d'} ((d+j)s - (d+d'-j)j)}$$

and thus

$$s \zeta_{f_{2,d},\text{top}}^{\triangleleft}(s) \Big|_{s=0} = \frac{(d')!}{\prod_{i=1}^{d-1} (-i) \prod_{j=1}^{d'} (-(d+d'-j)j)} = \frac{(-1)^{d+d'-1}}{(d+d'-1)!},$$

which establishes (4.12).

The proof of (4.13) goes along similar lines. We observe that  $\zeta_{\sigma^d}(s) \mathcal{A}_{I,J}^{\triangleleft\text{gr}}(q, t)$  has a simple pole at  $t = 1$  unless  $(I, J) = (\emptyset, \emptyset)$ , in which case it is double. Thus the summand  $\frac{c_{I,J}^{\triangleleft\text{gr}}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft\text{gr}} s - a_{I,J,i}^{\triangleleft\text{gr}})}$  in (4.15) has a simple pole at  $s = 0$  unless  $(I, J) = (\emptyset, \emptyset)$ , in which case it is double. Specifically,

$$\zeta_{\sigma^d}(s) \mathcal{A}_{\emptyset,\emptyset}^{\triangleleft\text{gr}}(q, t) = I_d((q^{i(d-i)} t^i)_{i \in [d]}) I_{d'}((q^{(d'-j)j} t^{d+j})_{j \in [d']}),$$

whence

$$\frac{c_{\emptyset,\emptyset}^{\triangleleft\text{gr}}}{\prod_{i=1}^{d+d'} (b_{\emptyset,\emptyset,i}^{\triangleleft\text{gr}} s - a_{\emptyset,\emptyset,i}^{\triangleleft\text{gr}})} = \frac{1}{\prod_{i=0}^{d-1} (s - i)} \cdot \frac{(d')!}{\prod_{j=1}^{d'} ((d+j)s - (d'-j)j)}$$

and thus

$$s^2 \zeta_{f_{2,d},\text{top}}^{\triangleleft\text{gr}}(s) \Big|_{s=0} = \frac{(d')!}{(d+d') \prod_{i=1}^{d-1} (-i) \prod_{j=1}^{d'-1} (-(d'-j)j)} = \frac{(-1)^{(d-1)+(d'-1)} d'}{(d+d')(d-1)!(d'-1)!},$$

which proves (4.13).  $\square$

*Remark 4.8.* In the pertinent special cases, the theorem's first statement confirms [9, Conjecture 1], whereas (4.12) confirms [9, Conjecture IV (topological form)].

**4.5. Explicit examples.** We record the following consequences of Theorem 4.2.

**Proposition 4.9**  $((c, d) = (2, 2) - \text{Heisenberg})$ .

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \zeta_{\mathfrak{o}^2}(s)I_1(t^3) =: W_{2,2}^{\triangleleft \text{gr}}(q, t), \\ \zeta_{\mathfrak{f}_{2,2,\text{red}}}^{\triangleleft \text{gr}}(Y) &= W_{2,2}^{\triangleleft \text{gr}}(1, Y) = \frac{1}{(1-Y)^2(1-Y^3)}, \\ \zeta_{\mathfrak{f}_{2,2,\text{top}}}^{\triangleleft \text{gr}}(s) &= \frac{1}{3s^2(s-1)}.\end{aligned}$$

The global graded ideal zeta function  $\zeta_{\mathfrak{f}_{2,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_K(s)\zeta_K(s-1)\zeta_K(3s)$  has abscissa of convergence 2 and meromorphic continuation to the whole complex plane.

**Proposition 4.10**  $((c, d) = (2, 3))$ .

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \zeta_{\mathfrak{o}^3}(s)I_3(q^2t^3, q^2t^5, t^6) =: W_{2,3}^{\triangleleft \text{gr}}(q, t), \\ \zeta_{\mathfrak{f}_{2,3,\text{red}}}^{\triangleleft \text{gr}}(Y) &= W_{2,3}^{\triangleleft \text{gr}}(1, Y) = \frac{1 + 2Y^3 + 2Y^5 + Y^8}{(1-Y)^3(1-Y^3)(1-Y^4)(1-Y^5)(1-Y^6)}, \\ \zeta_{\mathfrak{f}_{2,3,\text{top}}}^{\triangleleft \text{gr}}(s) &= \frac{1}{s^2(s-1)(s-2)(3s-2)(5s-2)}.\end{aligned}$$

The global graded ideal zeta function  $\zeta_{\mathfrak{f}_{2,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$  has abscissa of convergence 3 and may be continued meromorphically to  $\{s \in \mathbb{C} \mid \text{Re}(s) > 1/3\}$ .

*Remark 4.11.* The formulae for  $\zeta_{\mathfrak{f}_{2,2}(\mathfrak{o})}^{\triangleleft \text{gr}}$  resp.  $\zeta_{\mathfrak{f}_{2,3}(\mathfrak{o})}^{\triangleleft \text{gr}}$  (for almost all  $p$ ) are also given in [11, Table 2] under the labels m3.2 resp. m6.2.1.

We omit the (largish) formula for  $\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ , but do note the following consequences.

**Proposition 4.12**  $((c, d) = (2, 4))$ . *There exists a rational function  $W_{2,4}(X, Y) \in \mathbb{Q}[X, Y]$  such that*

$$\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{2,4}^{\triangleleft \text{gr}}(q, q^{-s}).$$

*Setting*

$$\begin{aligned}N_{2,4}(Y) &= Y^{33} + 5Y^{33} + 12Y^{28} + 6Y^{27} + 15Y^{26} + 26Y^{25} + 11Y^{24} + 39Y^{23} + 40Y^{22} + 43Y^{21} + \\ &\quad 62Y^{20} + 45Y^{19} + 66Y^{18} + 61(Y^{17} + Y^{16}) + 66Y^{15} + 45Y^{14} + 62Y^{13} + \\ &\quad 43Y^{12} + 40Y^{11} + 39Y^{10} + 11Y^9 + 26Y^8 + 15Y^7 + 6Y^6 + 12Y^5 + 5Y^3 + 1 \in \mathbb{Q}[Y]\end{aligned}$$

*we have*

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,4,\text{red}}}^{\triangleleft \text{gr}}(Y) &= W_{2,4}^{\triangleleft \text{gr}}(1, Y) = \frac{N_{2,4}(Y)}{(1-Y)^4(1-Y^3)\prod_{i=6}^{10}(1-Y^i)}, \\ \zeta_{\mathfrak{f}_{2,4,\text{top}}}^{\triangleleft \text{gr}}(s) &= \frac{3}{10} \frac{36s^2 - 43s + 12}{s^2(s-1)^3(s-2)(s-3)(3s-4)^2(2s-1)(7s-9)(9s-5)}.\end{aligned}$$

The global graded ideal zeta function  $\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$  has abscissa of convergence 4 and may be continued meromorphically to  $\{s \in \mathbb{C} \mid \text{Re}(s) > 7/6\}$ .

5. TWO GENERATORS ( $d = 2$ )

We compute the graded ideal zeta functions  $\zeta_{\mathfrak{f}_{c,2}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)$  for  $c \in \{3, 4\}$  as well as their reduced and topological counterparts. (For  $c = 1$  see (1.3), for  $c = 2$  see Proposition 4.9.)

**Proposition 5.1.**  $((c, d) = (3, 2))$

$$\begin{aligned}\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) &= \zeta_{\mathfrak{o}^2}(s) \frac{1+t^4}{(1-t^3)(1-qt^4)(1-t^5)} =: W_{3,2}^{\triangleleft\text{gr}}(q, t), \\ \zeta_{\mathfrak{f}_{3,2,\text{red}}(\mathfrak{o})}^{\triangleleft\text{gr}}(Y) &= W_{3,2}^{\triangleleft\text{gr}}(1, Y) = \frac{1+Y^4}{(1-Y)^2(1-Y^3)(1-Y^4)(1-Y^5)}, \\ \zeta_{\mathfrak{f}_{3,2,\text{top}}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) &= \frac{2}{15s^3(s-1)(4s-1)}.\end{aligned}$$

The global graded ideal zeta function

$$\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) = \frac{\zeta_K(s)\zeta_K(s-1)\zeta_K(3s)\zeta_K(4s)\zeta_K(4s-1)\zeta_K(5s)}{\zeta_K(8s)}$$

has abscissa of convergence 2 and meromorphic continuation to the whole complex plane.

*Proof.* (sketch) Let  $\mathfrak{f} = \mathfrak{f}_{3,2}$  and recall the Hall basis  $\mathcal{H}_{3,2}$  for  $\mathfrak{f}$  from Table 2.1. Recall formula (2.2) for the zeta function of  $\mathfrak{f}(\mathfrak{o})$  in terms of pairs  $(\Lambda_2, \Lambda_3)$  of sublattices of  $\mathfrak{f}(\mathfrak{o})^{(2)} \cong \mathfrak{o}$  and  $\mathfrak{f}(\mathfrak{o})^{(3)} \cong \mathfrak{o}^2$ , respectively. Note that  $|\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2| = q^\mu$  for some  $\mu \in \mathbb{N}_0$ ,  $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu}$ , and  $(\Lambda_2, \Lambda_3)$  satisfies the condition in (2.2) if and only if

$$(5.1) \quad \mathfrak{p}^\mu \mathfrak{f}(\mathfrak{o})^{(3)} \leq \Lambda_3.$$

This proves the first claim; the others are trivial consequences.  $\square$

*Remark 5.2.* Comparison with  $\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}^{\triangleleft}(s)$  (cf. [5, Theorem 2.35]) yields  $\zeta_{\mathfrak{f}_{3,2,\text{red}}(\mathfrak{o})}^{\triangleleft\text{gr}}(Y) = \zeta_{\mathfrak{f}_{3,2,\text{red}}(\mathfrak{o})}^{\triangleleft}(Y)$ . Note that the Hall basis  $\mathcal{H}_{3,2}$  is nice and simple in the sense of [6]. The formula for  $\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}^{\triangleleft\text{gr}}$  (for almost all  $p$ ) is also given in [11, Table 2] under the label m5\_3-1.

**Proposition 5.3.**  $((c, d) = (4, 2))$  Set

$$\begin{aligned}N_{4,2}(X, Y) &= -X^2(Y^{22} + Y^{18} + Y^{17} + Y^{16} + Y^{15} + Y^{11}) \\ &\quad - X(Y^{16} + Y^{15} - Y^{13} + Y^9 - Y^7 - Y^6) + Y^{11} + Y^7 + Y^6 + Y^5 + Y^4 + 1 \in \mathbb{Q}[X, Y].\end{aligned}$$

Then

$$\begin{aligned}\zeta_{\mathfrak{f}_{4,2}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) &= \zeta_{\mathfrak{o}^2}(s) \frac{N_{4,2}(q, t)}{(1-t^3)(1-qt^4)(1-t^5)(1-qt^5)(1-q^2t^6)(1-q^2t^7)(1-t^8)} \\ \zeta_{\mathfrak{f}_{4,2,\text{red}}(\mathfrak{o})}^{\triangleleft\text{gr}}(Y) &= \frac{Y^{17} + Y^{13} + 2(Y^{12} + Y^{11} + Y^{10} + Y^7 + Y^6 + Y^5) + Y^4 + 1}{(1-Y)^2 \prod_{i=3}^8 (1-Y^i)}, \\ \zeta_{\mathfrak{f}_{4,2,\text{top}}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) &= \frac{1}{60 s^4 (s-1)(5s-1)(4s-1)(7s-2)(3s-1)}.\end{aligned}$$

The global graded ideal zeta function  $\zeta_{\mathfrak{f}_{4,2}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)$  has abscissa of convergence 2 and may be continued meromorphically to  $\{s \in \mathbb{C} \mid \text{Re}(s) > 2/11\}$ .

*Proof.* (sketch) Let  $\mathfrak{f} = \mathfrak{f}_{4,2}$  and recall the Hall basis  $\mathcal{H}_{4,2}$  for  $\mathfrak{f}$  from Table 2.1. Recall formula (2.2) for the zeta function of  $\mathfrak{f}(\mathfrak{o})$  in terms of triples  $(\Lambda_2, \Lambda_3, \Lambda_4)$  of sublattices of  $\mathfrak{f}(\mathfrak{o})^{(2)} \cong \mathfrak{o}$ ,  $\mathfrak{f}(\mathfrak{o})^{(3)} \cong \mathfrak{o}^2$ , and  $\mathfrak{f}(\mathfrak{o})^{(4)} \cong \mathfrak{o}^3$ , respectively. Note that  $|\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2| = q^\mu$  for some  $\mu \in \mathbb{N}_0$ ,  $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu}$ , and  $(\Lambda_2, \Lambda_3, \Lambda_4)$  satisfies the condition in (2.2) if and only if  $(\Lambda_2, \Lambda_3)$  satisfies (5.1) and  $[\Lambda_3, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_4$ . To investigate the latter condition, we may assume that  $\Lambda_3 = \langle \pi^{\nu_1} xyx, \pi^{\nu_2} xyy \rangle_{\mathfrak{o}}$  for a partition  $\nu = (\nu_1, \nu_2)_{\geq}$ . Then

$$[\Lambda_3, \mathfrak{f}(\mathfrak{o})^{(1)}] = \langle \pi^{\nu_1} xyxx, \pi^{\nu_1} xyxy, \pi^{\nu_2} xyyx, \pi^{\nu_2} xyyy \rangle_{\mathfrak{o}}$$

As  $xyxy = xyyx$  by the Jacobi identity, the underlined term may be omitted, whence

$$[\Lambda_3, \mathfrak{f}(\mathfrak{o})^{(1)}] = \pi^{\nu_1} xyxx \oplus \pi^{\nu_2} xyyx \oplus \pi^{\nu_2} xyyy.$$

To enumerate the lattices  $\Lambda_4 \leq \mathfrak{f}(\mathfrak{o})^{(4)}$  satisfying  $[\Lambda_3, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_4$ , we distinguish according to the ‘‘overlap type’’ such a lattice may have with  $[\Lambda_3, \mathfrak{f}(\mathfrak{o})^{(1)}]$ . Indeed, if  $\Lambda_4$  has elementary divisor type given by a partition  $(\xi_1, \xi_2, \xi_3)_{\geq}$ , then either

$$\nu_1 \geq \nu_2 \geq \xi_1 \geq \xi_2 \geq \xi_3 \quad \text{or} \quad \nu_1 \geq \xi_1 > \nu_2 \geq \xi_2 \geq \xi_3.$$

Each of the two cases is covered by a formula similar to the one established in Theorem 3.6. Adding them yields a formula for the zeta function, viz.

$$\zeta_{\mathfrak{f}_{4,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^2}(s) I_1(t^3) \cdot \left( I_2(qt^4, t^5) I_3(q^2t^6, q^2t^7, t^8) + \binom{2}{1}_{q^{-1}} I_1(qt^4) I_1^{\circ}(qt^5) I_1(q^2t^6) I_2(q^2t^7, t^8) \right).$$

This proves the first claim; the others are trivial consequences. The claim about the meromorphic continuation follows from [5, Lemma 5.5].  $\square$

## 6. GENERAL CONJECTURES

Let  $c \in \mathbb{N}$  and  $d \in \mathbb{N}_{\geq 2}$ . We record a number of general conjectures regarding the graded ideal zeta functions  $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$  as well as their topological and reduced counterparts. We write  $W$  for the Witt function  $W_d$  (cf. (1.2)) and set  $r = \text{rk}_{\mathbb{Z}}(\mathfrak{f}_{c,d}) = \sum_{i=1}^c W(i)$ . With the exception of Conjecture 6.7 for  $c = 2$  and  $d \geq 5$ , all conjectures in this section are confirmed, in the relevant special cases, by the results in this paper.

**Conjecture 6.1** (Uniformity). *There exists  $W_{c,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$  such that, for almost all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ ,*

$$\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s}).$$

**Conjecture 6.2** (Local functional equations). *For almost all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ ,*

$$\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^r q^{\sum_{i=1}^c \binom{W(i)}{2} - \sum_{i=1}^c (c+1-i)W(i)s} \zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s).$$

*Remark 6.3.* Local functional equations are a universal feature for quite general subring zeta functions ([16, Theorem A]) and for ideal zeta functions of nilpotent Lie rings of nilpotency class 2 ([16, Theorem C]). In fact, we expect that the proof of the latter result should carry over to the graded setting, making the functional equations (4.7) an

instance of a general phenomenon in class 2; cf. Example 1.6. If Conjecture 6.1 holds, then Conjecture 6.2 states that

$$W_{c,d}^{\triangleleft \text{gr}}(X^{-1}, Y^{-1}) = (-1)^r X^{\sum_{i=1}^c} \binom{W(i)}{2} Y^{\sum_{i=1}^c (c+1-i)W(i)} W_{c,d}^{\triangleleft \text{gr}}(X, Y).$$

In general, the operation  $q \rightarrow q^{-1}$  is defined rather more delicately in terms of inversion of Frobenius eigenvalues.

In nilpotency class greater than 2 local functional equations are not to be expected in general for ideal zeta functions of nilpotent Lie rings: see [5, Section 1.2.3] for counterexamples in the ungraded setting and [11, Table 2] in the graded setting.

A sufficient criterion for local functional equations in an enumerative setup generalizing ideal zeta functions of nilpotent Lie rings is given in [17, Theorem 1.2]. It applies to the ideal zeta functions of the free nilpotent Lie rings  $\mathfrak{f}_{c,d}$ ; [17, Theorem 4.4]. Note that, in the notation of this result,  $N_0 = r$  and  $\sum_{i=0}^{c-1} N_i = \sum_{i=1}^c (c+1-i)W(i)$ .

For the following conjecture, assume that Conjecture 6.1 holds. In this case, we may define the *reduced graded ideal zeta function*

$$\zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y) = W_{c,d}^{\triangleleft \text{gr}}(1, Y) \in \mathbb{Q}(Y).$$

We expect that the technology in [6] may be adapted to define the reduced graded ideal zeta function even without the assumption of Conjecture 6.1.

**Conjecture 6.4** (Reduced zeta function). *The reduced graded ideal zeta function  $\zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y)$  has a pole of order  $r$  at 1 and there exist  $e_i \in \mathbb{N}$ ,  $i \in [r]$ , as well as nonnegative integers  $a_j \in \mathbb{N}_0$ ,  $j \in [N]_0$ , where  $N := (\sum_{i=1}^r e_i) - \sum_{j=1}^c (c+1-j)W_d(j)$ , with  $a_0 = 1$  and*

$$(6.1) \quad a_j = a_{N-j} \text{ for all } j \in [N]_0 \text{ and}$$

$$\zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y) = \frac{\sum_{j=0}^N a_j Y^j}{\prod_{i=1}^r (1 - Y^{e_i})}.$$

*Remark 6.5.* The palindromic property (6.1) holds, of course, in particular if Conjecture 6.2 holds. What Conjecture 6.4 suggests is that the reduced graded ideal zeta functions  $\zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y)$  share key properties with the Hilbert-Poincaré series of graded Cohen-Macaulay algebras. If there were such an algebra for each  $c, d$ , its Cohen-Macaulayness would explain the nonnegativity of the coefficients  $a_j$ , whereas their palindromy would reflect its Gorensteinness (assuming that it is a domain; cf. [13]).

For  $c = 2$ , the validity of Conjecture 6.4 may be seen from the facts that  $\zeta_{\mathfrak{f}_{2,d}, \text{red}}^{\triangleleft \text{gr}}(Y) = \zeta_{\mathfrak{f}_{2,d}, \text{red}}^{\triangleleft}(Y)$  and the Hall basis  $\mathcal{H}_{2,d}$  for  $\mathfrak{f}_{2,d}$  in Table 2.1 is nice and simple in the sense of [6]. Hence [6, Proposition 4.1] is applicable.

The following is a graded analogue of [9, Conjecture I].

**Conjecture 6.6** (Degree of topological zeta function).

$$\deg_s \left( \zeta_{\mathfrak{f}_{c,d}, \text{top}}^{\triangleleft \text{gr}}(s) \right) = -r.$$

**Conjecture 6.7** (Behaviour of topological zeta function at  $\infty$ ).

$$s^{-r} \zeta_{\mathfrak{f}_{c,d}, \text{top}}^{\triangleleft \text{gr}}(s^{-1}) \Big|_{s=0} = (1 - Y)^r \zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y) \Big|_{Y=1} \in \mathbb{Q}_{>0}.$$

*Remark 6.8.* We have no interpretation in terms of  $c$  and  $d$  for the numbers featuring in Conjecture 6.7. If Conjecture 6.4 were to hold for the reasons we explained in Remark 6.5, they were related to the “multiplicity” of the associated graded algebra.

We close with two conjectures on the behaviour of  $\mathfrak{p}$ -adic and topological graded ideal zeta functions at  $s = 0$ .

**Conjecture 6.9** (Behaviour of topological zeta function at zero). *The topological graded ideal zeta function  $\zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft\text{gr}}(s)$  has a pole of order  $c$  at  $s = 0$  with leading coefficient*

$$s^c \zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft\text{gr}}(s) \Big|_{s=0} = \frac{(-1)^{\sum_{i=1}^c (W(i)-1)} \prod_{i=1}^c W(i)}{\prod_{i=1}^c \left( \sum_{j=1}^i W(j) \right) (W(i) - 1)!}.$$

*Remark 6.10.* [9, Conjecture IV (topological form)] asserts that  $\zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft}(s)$  has a simple pole at  $s = 0$  with residue

$$s \zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft}(s) \Big|_{s=0} = \frac{(-1)^{r-1}}{(r-1)!}.$$

**Conjecture 6.11** (Behaviour of  $\mathfrak{p}$ -adic zeta function at zero).

$$(6.2) \quad \frac{\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)}{\prod_{i=1}^c \zeta_{\mathfrak{o}^{W(i)}}(s)} \Big|_{s=0} = \frac{\prod_{i=1}^c W(i)}{\prod_{i=1}^c \left( \sum_{j=1}^i W(j) \right)}.$$

*Remark 6.12.* [9, Conjecture IV ( $\mathfrak{P}$ -adic form)] asserts that

$$(6.3) \quad \frac{\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft}(s)}{\zeta_{\mathfrak{o}^r}(s)} \Big|_{s=0} = 1.$$

It seems remarkable that the right hand sides of both (6.2) and (6.3) are independent of  $\mathfrak{o}$ . Note that  $s = 0$  is outside the domain of convergence of the series defining  $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft}(s)$  resp.  $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)$ .

*Remark 6.13.* Simply replacing the numbers  $W(i)$  by the ranks of the successive quotients of the upper central series does not extend Conjectures 6.9 and 6.11 to graded ideal zeta functions of general (non-free) nilpotent Lie rings. Consider, for instance, the direct product  $L := \mathfrak{f}_{2,2} \times \mathfrak{f}_{2,2}$ . Then, for all primes  $p$  and all finite extensions  $\mathfrak{o}$  of  $\mathbb{Z}_p$ ,

$$(6.4) \quad \zeta_{L(\mathfrak{o})}^{\triangleleft\text{gr}}(s) = \zeta_{\mathfrak{o}^4}(s) \frac{1 - t^5}{(1 - t^3)^2 (1 - qt^5)}.$$

Indeed,  $L$  is the “fundamental graded Lie ring” m6.2.3, whose local graded ideal zeta functions are recorded (for almost all  $p$ ) in [11, Section 10, Table 2]. Formula (6.4) may also easily be derived directly from formulae for the ideal zeta functions of  $L$  (cf. [5, Theorem 2.4]), using the comparison identities recorded in Example 1.6. In any case,

$$s^2 \zeta_{L,\text{top}}^{\triangleleft\text{gr}}(s) \Big|_{s=0} = \frac{5}{54} \neq \frac{(-1)^{3+1} 4 \cdot 2}{4 \cdot 6 \cdot 3! \cdot 1!} = \frac{1}{18}.$$

Likewise,

$$\frac{\zeta_{L(\mathfrak{o})}^{\triangleleft\text{gr}}(s)}{\zeta_{\mathfrak{o}^4}(s) \zeta_{\mathfrak{o}^2}(s)} \Big|_{s=0} = \frac{5}{9} \neq \frac{4 \cdot 2}{4 \cdot 6} = \frac{1}{3}.$$

That this value, however, is a nonnegative rational number independent of  $\mathfrak{o}$ , as well as the coincidence

$$s^{-6} \zeta_{L, \text{top}}^{\triangleleft \text{gr}}(s^{-1}) \Big|_{s=0} = \frac{1}{9} = (1 - Y)^6 \zeta_{L, \text{red}}^{\triangleleft \text{gr}}(Y) \Big|_{Y=1} \in \mathbb{Q}_{>0},$$

seem to indicate that some of the conjectures made in this section may have generalizations to more general graded ideal zeta functions.

#### ACKNOWLEDGMENTS

Our work was supported by DFG Sonderforschungsbereich 701 ‘‘Spectral Structures and Topological Methods in Mathematics’’ at Bielefeld University. The first author was also supported by A23200000 fund from the National Institute for Mathematical Sciences. We acknowledge numerous helpful conversations with Tobias Rossmann.

#### REFERENCES

- [1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.
- [2] L. M. Butler, *A unimodality result in the enumeration of subgroups of a finite abelian group*, Proc. Amer. Math. Soc. **101** (1987), no. 4, 771–775.
- [3] M. P. F. du Sautoy, *Counting  $p$ -groups and nilpotent groups*, Publ. Math. I.H.E.S. **92** (2000), 63–112.
- [4] M. P. F. du Sautoy and F. J. Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. of Math. (2) **152** (2000), 793–833.
- [5] M. P. F. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, Lecture Notes in Mathematics, vol. 1925, Springer-Verlag, Berlin, 2008.
- [6] A. Evseev, *Reduced zeta functions of Lie algebras*, J. Reine Angew. Math. **633** (2009), 197–211.
- [7] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), 185–223.
- [8] M. Hall Jr., *A basis for free Lie rings and higher commutators in free groups*, Proc. Amer. Math. Soc. **1** (1950), 575–581.
- [9] T. Rossmann, *Computing topological zeta functions of groups, algebras, and modules, I*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 5, 1099–1134.
- [10] ———, *Stability results for local zeta functions of groups and related structures*, arXiv:1504.04164, 2015.
- [11] ———, *Computing local zeta functions of groups, algebras, and modules*, preprint, arXiv:1602.00919, 2016.
- [12] M. M. Schein and C. Voll, *Normal zeta functions of the Heisenberg groups over number rings I – the unramified case*, J. Lond. Math. Soc. (2) **91** (2015), no. 1, 19–46.
- [13] R. P. Stanley, *Hilbert functions of graded algebras*, Adv. Math. **28** (1978), 57–83.
- [14] C. Voll, *Zeta functions of groups and enumeration in Bruhat-Tits buildings*, Amer. J. Math. **126** (2004), 1005–1032.
- [15] ———, *Normal subgroup growth in free class-2-nilpotent groups*, Math. Ann. **332** (2005), 67–79.
- [16] ———, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.
- [17] ———, *Local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms*, preprint, arxiv.org/abs/1602.07025, 2016.
- [18] E. Witt, *Treue Darstellung Liescher Ringe*, J. Reine Angew. Math. **177** (1937), 152–160.

MATHEMATICAL INSTITUTE, OXFORD UNIVERSITY, OX2 6GG, UNITED KINGDOM

*Current address:* National Institute for Mathematical Sciences, Daejeon 305-811, South Korea

*E-mail address:* [seung.lee@mansfield.ox.ac.uk](mailto:seung.lee@mansfield.ox.ac.uk)

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, D-33501 BIELEFELD, GERMANY

*E-mail address:* [C.Voll.98@cantab.net](mailto:C.Voll.98@cantab.net)