Reduced functions and Jensen measures

Wolfhard Hansen and Ivan Netuka

Abstract

Let φ be a locally upper bounded Borel measurable function on a Greenian open set Ω in $\mathbb{R}^d$ and, for every $x \in \Omega$, let $v_\varphi(x)$ denote the infimum of the integrals of φ with respect to Jensen measures for x on Ω. Twenty years ago, B.J. Cole and T.J. Ransford proved that v_φ is the supremum of all subharmonic minorants of φ on X and that the sets $\{v_\varphi < t\}$, $t \in \mathbb{R}$, are analytic. In this paper, a different method leading to the inf-sup-result establishes at the same time that, in fact, v_φ is the minimum of φ and a subharmonic function, and hence Borel measurable. This is presented in the generality of harmonic spaces, where semipolar sets are polar, and the key are measurability results for reduced functions on balayage spaces which are of independent interest.

Keywords: Reduced function; Jensen measure; axiom of polarity.

MSC: 31B05, 31D05, 35J15, 60J45, 60J60, 60J75.

1 Introduction

The motivation for our considerations is a question in connection with Jensen measures which could not be answered in [5]. Let Ω be an open set in $\mathbb{R}^d$, $d \geq 2$ (such that, if $d = 2$, $\mathbb{R}^d \setminus \Omega$ is not polar). We recall that a (Radon) measure μ with compact support in Ω is a *Jensen measure for a point* $x \in \Omega$ if

$$(1.1) \qquad \int v\, d\mu \geq v(x) \quad \text{for every subharmonic function } v \text{ on } \Omega.$$

Let φ be a locally upper bounded Borel measurable function φ on Ω and

$$v_\varphi(x) := \inf\{\int \varphi\, d\mu \colon \mu \text{ Jensen measure for } x\}, \qquad x \in \Omega.$$

The results [5, Theorem 1.6 and Corollary 1.7] show that

$$(1.2) \qquad v_\varphi = \sup\{v \colon \; v \text{ subharmonic on } \Omega,\, v \leq \varphi\}$$

and that the sets $\{v_\varphi < t\}$, $t \in \mathbb{R}$, are analytic (which led the authors B.J. Cole and T.J. Ransford to a definition and the study of quasi-subharmonic functions; cf. also [1]). It remained an open question if the function v_φ is, in fact, Borel measurable (see the lines following [5, Theorem 1.6]).

In this short paper, we shall give a positive answer to this question (even in a much more general setting) using a different method which, at the same time, provides a simpler proof for (1.2).

Our essential tool are measurability properties which we shall prove for reduced functions on balayage spaces $(X, \mathcal{W})$ satisfying the axiom of polarity (Section 2) and which are of independent interest.

In our application to Jensen measures on harmonic spaces (Section 3) it is natural to consider superharmonic functions instead of subharmonic functions. Recalling that a function u is superharmonic if and only if $-u$ is subharmonic, this requires to look upside-down at the definitions, assumptions and statements above.

In both sections, the reader, who is not familiar with or not interested in general potential theory, may suppose that X is an open subset Ω of $\mathbb{R}^d$ and that $\mathcal{W}$ is the set of all functions $u \geq 0$ on Ω which are hyperharmonic on Ω (that is, which, for each connected component U of Ω, are either superharmonic on U or are identically $+\infty$ on U).

2 Measurability of reduced functions

Let $(X, \mathcal{W})$ be a balayage space (X a locally compact space with countable base and $\mathcal{W}$ the set of positive hyperharmonic functions on X, see [4] or [10]). For every open set U in X, let $\mathcal{B}(U)$ ($\mathcal{C}(U)$, respectively) denote the set of all numerical Borel measurable (real continuous, respectively) functions on U. As usual, given a set $\mathcal{F}$ of functions, let $\mathcal{F}^+$ be the set of all $f \in \mathcal{F}$ such that $f \geq 0$. In the following, let u_0 be any strictly positive function in $\mathcal{W} \cap \mathcal{C}(X)$.

We recall that, for every numerical function $\varphi \geq 0$, a reduced function R_φ is defined by

$$R_\varphi := \inf\{u \in \mathcal{W} \colon u \geq \varphi\}. \tag{2.1}$$

In particular, we have $R_v^A := R_{1_A v}$ for $A \subset X$ and $v \in \mathcal{W}$, which leads to reduced measures ε_x^A, $x \in X$, characterized by $\int v \, d\varepsilon_x^A = R_v^A(x)$, $v \in \mathcal{W}$.

Let $\mathcal{P}(X)$ denote the set of all continuous real potentials on X, that is, of all $p \in \mathcal{W} \cap \mathcal{C}(X)$ satisfying

$$\inf\{R_p^{X \setminus K} \colon K \text{ compact in } X\} = 0.$$

A real function φ on X is called $\mathcal{P}$-bounded, if $|\varphi| \leq p$ for some $p \in \mathcal{P}(X)$ (every bounded φ with compact support is $\mathcal{P}$-bounded).

The following properties of R_φ are well known (see [4, p. 58] and [10, Corollary 1.2.2]). If φ is lower semicontinuous, then $R_\varphi \in \mathcal{W}$. If φ is $\mathcal{P}$-bounded and continuous (upper semicontinuous, respectively), then R_φ is continuous (upper semicontinuous, respectively).

We now claim that, to some extent, we may replace the values of φ at points, where R_φ is strictly greater than φ, by the value 0 without changing the reduced function. The following would be sufficient for our purposes.

PROPOSITION 2.1. *Let $\varphi \geq 0$ be a numerical function on X and $x \in X$ such that $R_\varphi(x) > \varphi(x)$. Then $R_{1_{X \setminus \{x\}}\varphi} = R_\varphi$.*

Proof. Let $u \in \mathcal{W}$, $u \geq \varphi$ on $X \setminus \{x\}$. To see that also $u(x) \geq \varphi(x)$ and hence $u \geq R_\varphi$, we may assume that $u(x) < \infty$, take $\varepsilon > 0$ and consider

$$v := u + \varepsilon u_0 \in \mathcal{W}, \qquad \gamma := 1 \vee \frac{\varphi(x)}{v(x)}.$$

Then $\gamma v \ge \varphi$ on X, and hence $\gamma v \ge R_\varphi$. In particular,

$$v(x) \vee \varphi(x) = \gamma v(x) \ge R_\varphi(x) > \varphi(x).$$

This shows that $v(x) > \varphi(x)$, and hence $u(x) \ge \varphi(x)$, since $\varepsilon > 0$ is arbitrary. $\square$

In fact, we may carry out this replacement on sets exhausting $\{R_\varphi > \varphi\}$.

PROPOSITION 2.2. *Let $\varphi \ge 0$ be a numerical function on X, $\alpha \in (1,\infty)$, $M \in (0,\infty)$, and $A := \{R_\varphi > \alpha\varphi\} \cap \{\varphi < Mu_0\}$. Then $R_{1_{X\setminus A}\varphi} = R_\varphi$.*

Proof. Clearly, it suffices to consider the case, where φ is not identically zero on A. Let $u \in \mathcal{W}$, $u \ge \varphi$ on $X \setminus A$, $\varepsilon > 0$, and $v := u + \varepsilon u_0$. To show that $v \ge R_\varphi$, let

$$\beta := \inf\{b \in (0,\infty) \colon bv \ge \varphi \text{ on } A\} \quad \text{and} \quad \gamma := 1 \vee \beta. \tag{2.2}$$

Then $0 < \beta \le M/\varepsilon$ and $\gamma v \ge \varphi$ on X. Hence $(\beta/\alpha)v(x) < \varphi(x)$ for some $x \in A$ and $\gamma v \ge R_\varphi$, which leads to $\beta v(x) < \alpha\varphi(x) < R_\varphi(x) \le \gamma v(x)$. Thus $\gamma = 1$, $v \ge R_\varphi$. $\square$

COROLLARY 2.3. *Suppose that $\varphi \in \mathcal{C}^+(X)$ is $\mathcal{P}$-bounded and let $A := \{R_\varphi > \varphi\}$. Then $R_{1_{X\setminus A}\varphi} = R_\varphi$.*

Proof. The sets $A_n := \{R_\varphi > (1+n^{-1})\varphi\} \cap \{\varphi < nu_0\}$, $n \in \mathbb{N}$, are open, and hence the functions $\varphi_n := 1_{X\setminus A_n}\varphi$ are upper semicontinuous. Since $\varphi_n \downarrow 1_{X\setminus A}\varphi$ as $n \to \infty$, we conclude that $\lim_{n\to\infty} R_{\varphi_n} = R_{1_{X\setminus A}\varphi}$ (see [10, Corollary 1.2.2,3]). The proof is finished, since, by Proposition 2.2, $R_{\varphi_n} = R_\varphi$ for every $n \in \mathbb{N}$. $\square$

REMARK 2.4. Let us observe that neither the continuity nor the $\mathcal{P}$-boundedness of φ in Corollary 2.3 can be deleted without replacement. Indeed, if we consider classical potential theory on a connected open set X in $\mathbb{R}^d$, $d \ge 1$, containing the unit ball B (for example, $X = B$ or $\overline{B} \subset X$), and take $\varphi(x) := 1_B|\cdot|$, then $A = X$ and hence $R_{1_{X\setminus A}\varphi} = 0$.

Next let us recall the following. If $\mathcal{V} \subset \mathcal{W}$ and $v := \inf \mathcal{V}$, then the lower semicontinuous regularized function $\hat{v}$ (defined by $\hat{v}(x) := \liminf_{y\to x} v(y)$) is contained in $\mathcal{W}$. The set $\{\hat{v} < v\}$ is semipolar ([4, VI.5.11], for a converse see [4, VI.5.13]). If S is a semipolar set and $u, w \in \mathcal{W}$ such that $u \ge w$ outside S, then $u \ge w$ on X (see [4, II.4.2 and VI.5.9]).

Moreover, we note the following. If $u \in \mathcal{W}$ and $\varphi \ge 0$ is a numerical function on X such that $u \ge \varphi$ outside a set P which is polar (that is, satisfies $\hat{R}_{1_P u_0} = 0$), then

$$u \ge R_\varphi \quad \text{on } X \setminus P \quad \text{and} \quad u \ge \hat{R}_\varphi. \tag{2.3}$$

Indeed, for all $x \in X \setminus P$ and $\varepsilon > 0$, there exists $w \in \mathcal{W}$ such that $w = \infty$ on P and $w(x) < \varepsilon$ (see [4, VI.5.1 and VI.2.3]), and hence $v := u + w \in \mathcal{W}$, $v \ge \varphi$ on X. So $v \ge R_\varphi$, $u(x) + \varepsilon \ge v(x) \ge R_\varphi(x)$. Knowing that $u \ge R_\varphi$ on $X \setminus P$, we have a fortiori $u \ge \hat{R}_\varphi$ on $X \setminus P$ whence $u \ge \hat{R}_\varphi$, since the polar set P is semipolar.

From now on let us suppose that $(X, \mathcal{W})$ satisfies the axiom of polarity, that is, every semipolar set is polar.

The following is known (without assuming the axiom of polarity) for the special cases $\varphi = 1_A u$, where $A \subset X$ and $u \in \mathcal{W}$ (see [4, VI.2.4]).

THEOREM 2.5. *Let φ be a positive numerical function on X. Then*

$$R_\varphi = \varphi \vee \hat{R}_\varphi.$$

In particular, R_φ is Borel measurable if φ is Borel measurable.

Proof. Of course, $R_\varphi \geq \varphi \vee \hat{R}_\varphi$. To prove the reverse inequality we have to show that $\hat{R}_\varphi = R_\varphi$ on $\{R_\varphi > \varphi\}$. This will be achieved in two steps.

(a) Let $\psi \colon X \to [0, \infty]$. The semipolar set $P := \{\hat{R}_\psi < R_\psi\}$ is polar, by the axiom of polarity, and $\hat{R}_\psi \geq \psi$ outside the subset $P' := P \setminus \{\psi = 0\}$ of P. Hence, by (2.3), $\hat{R}_\psi \geq R_\psi$ on the set $X \setminus P'$ containing $\{\psi = 0\}$. Thus

$$\hat{R}_\psi = R_\psi \quad \text{on } \{\psi = 0\}.$$

(b) If $x \in \{R_\varphi > \varphi\}$ and $\psi := 1_{X \setminus \{x\}} \varphi$, then $R_\psi = R_\varphi$, by Proposition 2.1, and therefore, by (a),

$$\hat{R}_\varphi(x) = \hat{R}_\psi(x) = R_\psi(x) = R_\varphi(x)$$

□

COROLLARY 2.6. *If $\varphi \geq 0$ is a $\mathcal{P}$- bounded upper semicontinuous function on X and $A := \{R_\varphi > \varphi\}$, then $R_{1_{X \setminus A}\varphi} = R_\varphi$.*

Proof. By Theorem 2.5, $\{R_\varphi > \alpha\varphi\} = \{\hat{R}_\varphi > \alpha\varphi\}$ for every $\alpha > 1$. Hence the sets A_n in the proof of Corollary 2.3 are open, and we may conclude as before. □

Next let us recall the following consequence of [4, VI.1.9]: For every Borel set A in X and for every $u \in \mathcal{W} \cap \mathcal{C}(X)$, there exists an increasing sequence (K_n) of compact sets in A such that

$$\sup\nolimits_{n \in \mathbb{N}} \hat{R}_u^{K_n} = \hat{R}_u^A \tag{2.4}$$

(in fact, having (2.4) for some strict continuous potential, (2.4) holds for every $u \in \mathcal{W}$). Indeed, by [4, VI.1.9], we have

$$\sup\{\hat{R}_u^L \colon L \text{ compact in } A\} = \hat{R}_u^A.$$

Hence, by [4, I.1.7], there exists a sequence (L_n) of compact sets in A such that $\sup_{n \in \mathbb{N}} \hat{R}_u^{L_n} = \hat{R}_u^A$. To obtain (2.4) it now suffices to take $K_n := L_1 \cup \cdots \cup L_n$.

THEOREM 2.7. *For every $\varphi \in \mathcal{B}^+(X)$, there exist bounded upper semicontinuous functions ψ_n with compact support in $\{\varphi > 0\}$ such that $0 \leq \psi_n \leq \psi_{n+1} \leq \varphi$ for every $n \in \mathbb{N}$ and*

$$\sup\nolimits_{n \in \mathbb{N}} \hat{R}_{\psi_n} = \hat{R}_\varphi. \tag{2.5}$$

In particular,

$$R_\varphi = \varphi \vee \sup\nolimits_{n \in \mathbb{N}} \hat{R}_{\psi_n} = \varphi \vee \sup\nolimits_{n \in \mathbb{N}} R_{\psi_n} \tag{2.6}$$

and

$$R_\varphi = \sup\{R_\psi \colon 0 \leq \psi \leq \varphi, \psi \text{ u.s.c.}\}.$$

Proof. (a) Suppose first that φ/u_0 is simple, that is,

$$\varphi/u_0 = \sum_{j=1}^{m} \alpha_j 1_{A_j},$$

where $A_1, \dots, A_m$ are pairwise disjoint Borel sets in X and $\alpha_1, \dots, \alpha_m \in (0, \infty)$. By (2.4), for every $1 \le j \le m$, there exists an increasing sequence $(K_{j,n})_{n\in\mathbb{N}}$ of compact sets in A_j such that

$$\lim_{n\to\infty} \hat{R}_{u_0}^{K_{j,n}} = \hat{R}_{u_0}^{A_j}.$$

We define

$$P := \bigcup_{j=1}^{m} \{\hat{R}_{u_0}^{A_j} < R_{u_0}^{A_j}\}$$

and

$$\psi_n := \sum_{j=1}^{m} \alpha_j 1_{K_{j,n}} u_0, \quad n \in \mathbb{N}.$$

Then P is a polar set and the sequence $(\hat{R}_{\psi_n})$ is increasing.

For the moment, let us fix $1 \le j \le m$. Clearly, $R_{\psi_n} \ge \alpha_j R_{u_0}^{K_{j,n}}$ for every $n \in \mathbb{N}$, and hence

$$v := \sup_{n\in\mathbb{N}} \hat{R}_{\psi_n} \ge \alpha_j \sup_{n\in\mathbb{N}} \hat{R}_{u_0}^{K_{j,n}} = \alpha_j \hat{R}_{u_0}^{A_j},$$

$v \ge \alpha_j u_0$ on $A_j \setminus P$. Thus $v \ge \varphi$ on $X \setminus P$. By (2.3), we conclude that $v \ge \hat{R}_\varphi$. The reverse inequality holds trivially.

(b) Let us now consider the general case $\varphi \in \mathcal{B}^+(X)$. There exist simple functions $\tilde{\varphi}_k \in \mathcal{B}^+(X)$ such that $\tilde{\varphi}_k \uparrow \varphi/u_0$ as $k \to \infty$. Then $\varphi_k := \tilde{\varphi}_k u_0 \uparrow \varphi$ as $k \to \infty$, and hence $\sup_{k\in\mathbb{N}} \hat{R}_{\varphi_k} = \hat{R}_\varphi$.

By (a), there exist upper semicontinuous bounded functions $\psi_{k,n}$ with compact support in $\{\varphi_k > 0\}$, $k, n \in \mathbb{N}$, such that $0 \le \psi_{k,n} \le \psi_{k,n+1} \le \varphi_k$ and, for every $k \in \mathbb{N}$,

$$\sup_{n\in\mathbb{N}} \hat{R}_{\psi_{k,n}} = \hat{R}_{\varphi_k}.$$

For every $n \in \mathbb{N}$, let

$$\psi_n := \sup\{\psi_{k,n} \colon 1 \le k \le n\}.$$

Then, for every $k \in \mathbb{N}$,

$$\sup_{n\in\mathbb{N}} \hat{R}_{\psi_n} \ge \sup_{n\in\mathbb{N}} \hat{R}_{\psi_{k,n}} = \hat{R}_{\varphi_k},$$

and hence

$$\sup_{n\in\mathbb{N}} \hat{R}_{\psi_n} \ge \sup_{k\in\mathbb{N}} \hat{R}_{\varphi_k} = \hat{R}_\varphi.$$

The reverse inequality is trivial.

(c) The proof is finished by an application of Theorem 2.5, the trivial inequalities $\hat{R}_{\psi_n} \le R_{\psi_n} \le R_\varphi$, and the choice of $\psi := 1_{\{x\}}\varphi$ to have $R_\psi(x) \ge \psi(x) = \varphi(x)$. □

3 Application to Jensen measures

From now on, we suppose more restrictively that the balayage space $(X, \mathcal{W})$ satisfying the axiom of polarity is a harmonic space, that is, $\mathcal{W}$ has the following local truncation property: For all open sets U in X and all $u, v \in \mathcal{W}$ such that $u \ge v$ on the boundary ∂U of U, the function w defined by $w := u \wedge v$ on U and v on $X \setminus U$ is contained in $\mathcal{W}$ (see [4, Section III.8]). This means that the reduced measures $\varepsilon_x^{X\setminus V}$ (that is, the harmonic measures μ_x^V) for open sets V and $x \in V$ are supported by ∂V (instead of having supports which could be the entire complement of V).

In probabilistic terms, an associated process will be a diffusion (instead of a process possibly having many jumps). We recall that fairly general linear differential operators L of second order on open subsets X of $\mathbb{R}^d$ (L being the Laplacian in the classical case) lead to harmonic spaces (see, for example, [9, Section 7]).

Given an open set U in X, let ${}^*\mathcal{H}(U)$ denote the set of all hyperharmonic functions v on U, that is, of all lower semicontinuous $v\colon U \to]-\infty, \infty]$ such that $\int v\,d\mu_x^V \le v(x)$ for every open set V, which is relatively compact in U, and every $x \in V$. If, in addition, the functions $x \mapsto \int v\,d\mu_x^V$ are continuous and real on V, then such a function v is called superharmonic on U. The set of all superharmonic functions on U is denoted by $\mathcal{S}(U)$, and $\mathcal{H}(U) = \mathcal{S}(U) \cap (-\mathcal{S}(U))$ is the set of all harmonic functions on U.

We note that ${}^*\mathcal{H}^+(X) = \mathcal{W}$ and $\mathcal{S}^+(X) \cap \mathcal{C}(X) = \mathcal{W} \cap \mathcal{C}(X)$. In particular, it is compatible with (2.1) to define, for *every* numerical function φ on X,

$$R_\varphi := \inf\{v \in {}^*\mathcal{H}(X)\colon v \ge \varphi\}.$$

In our proofs we shall tacitly use that, for every (relatively compact) open set U in X, $(U, {}^*\mathcal{H}^+(U))$ is a harmonic space as well (see, for example, [4, Section VI.1]) and that sets $A \subset U$ which are polar (semipolar, respectively) with respect to $(U, {}^*\mathcal{H}^+(U))$ are polar (semipolar, respectively) with respect to $(X, \mathcal{W})$ (see [6, Sections 6.2 and 6.3]; the converse is trivial).

Given an open set U in X, we say that a locally lower bounded function v on U is nearly hyperharmonic if $\int^* v\,d\mu_x^V \le v(x)$ for every open set V, which is relatively compact in U, and every $x \in V$. As is well-known, $\hat{v} \in {}^*\mathcal{H}(U)$ for every nearly hyperharmonic function on U.

LEMMA 3.1. *Let v be a locally lower bounded numerical function on an open set U in X. The following statements are equivalent:*

(i) *v is nearly hyperharmonic on U and the set $\{\hat{v} < v\}$ is polar.*

(ii) *v is the infimum of its hyperharmonic majorants on U.*

Proof. If (i) holds, we may argue as in the proof of (1) ⇒ (2) in [1, Theorem 2]): Let $x \in U$ be such that $v(x) < \infty$, and let $\varepsilon > 0$. There exists $v_x \in {}^*\mathcal{H}^+(U)$ such that $v_x(x) = v(x) - \hat{v}(x) + \varepsilon$ and $v_x = \infty$ on the polar set $\{\hat{v} < v\} \setminus \{x\}$. Then $w := \hat{v} + v_x \in {}^*\mathcal{H}^+(U)$, $w \ge v$ and $w(x) = v(x) + \varepsilon$.

Next suppose that (ii) holds. Then v is obviously nearly hyperharmonic on U. Moreover, the set $\{\hat{v} < v\}$ is semipolar (see [6, Theorem 6.3.2]), and hence polar by the axiom of polarity. □

We shall need the following consequence.

LEMMA 3.2. *Let U_n, $n \in \mathbb{N}$, be relatively compact open sets in X such that $\overline{U}_n \subset U_{n+1}$ and $\bigcup_{n\in\mathbb{N}} U_n = X$. Moreover, let (v_n) be an increasing sequence of locally lower bounded numerical functions on X such that, for every $n \in \mathbb{N}$,*

$$v_n|_{U_n} = \inf\{w \in {}^*\mathcal{H}(U_n)\colon w \ge v_n|_{U_n}\},$$

and let $v := \lim_{n\to\infty} v_n$. *Then* $\hat v = \lim_{n\to\infty} \hat v_n$ *and*

$$v = \inf\{w \in {}^*\mathcal{H}(X) \colon w \ge v\}. \tag{3.1}$$

Proof. For every $n \in \mathbb{N}$, v_n is nearly hyperharmonic on U_n and $P_n := \{\hat v_n < v_n\}$ is polar, by Lemma 3.1. Therefore v is nearly hyperharmonic on X. Moreover, $\hat v = \lim_{n\to\infty} \hat v_n$ (see [2, p. 48]). Hence the set $P := \{\hat v < v\}$ is contained in the union of all P_n, $n \in \mathbb{N}$. So P is polar, and (3.1) holds, by Lemma 3.1. □

For every open set U in X, let $\mathcal{M}_c(U)$ denote the set of all measures with compact compact support in U. For every $x \in U$, let $J_x(U)$ denote the set of all *Jensen measures for* x *with respect to* U, that is,

$$J_x(U) := \{\mu \in \mathcal{M}_c(U) \colon \int v\, d\mu \le v(x) \text{ for every } v \in \mathcal{S}(U)\}.$$

Of course, $J_x(U)$ is a convex set containing the Dirac measure ε_x at x and the harmonic measures μ_x^V, V relatively compact open in U and $x \in V$ (see [11] for a detailed discussion).

If $h \in \mathcal{H}(U)$, then $\pm h \in \mathcal{S}(U)$, and hence

$$\int h\, d\mu = h(x) \qquad \text{for all } x \in U \text{ and } \mu \in J_x(U).$$

Since every function in ${}^*\mathcal{H}(U)$ is an increasing limit of functions in $\mathcal{S}(U) \cap \mathcal{C}(U)$ (see [6, Corollary 2.3.1]), a measure $\mu \in \mathcal{M}_c(U)$ is a Jensen measure for x with respect to U provided $\int u\, d\mu \le u(x)$ for every $u \in \mathcal{S}(U) \cap \mathcal{C}(U)$, and then $\int w\, d\mu \le w(x)$ for every $w \in {}^*\mathcal{H}(U)$.

Given a function $\varphi \in \mathcal{B}(X)$ which is locally bounded below, we define

$$u_\varphi(x) := \sup\{\int \varphi\, d\mu \colon \mu \in J_x(X)\}, \qquad x \in X. \tag{3.2}$$

Trivially,

$$u_\varphi \le \inf\{w \in {}^*\mathcal{H}(X) \colon w \ge \varphi\} = R_\varphi. \tag{3.3}$$

To prove that the reverse inequality holds as well, and hence $u_\varphi \in \mathcal{B}(X)$, let us assume that, for every relatively compact open set U in X, there exists a strictly positive harmonic function on U. This is a rather weak assumption; it is equivalent to $R_{u_0}^{X\setminus U} > 0$.

A first step is the following.

PROPOSITION 3.3. *Let* ψ *be a* $\mathcal{P}$*-bounded upper semicontinuous function on* X. *Then* $u_\psi = R_\psi$. *In particular,* u_ψ *is upper semicontinuous.*

Proof. Let us fix an exhaustion of X by relatively compact open sets U_n, $n \in \mathbb{N}$, such that $\overline{U}_n \subset U_{n+1}$. For $n \in \mathbb{N}$, we define a function $v_n \ge \psi$ on X by

$$v_n(x) := \inf\{s(x) \colon s \in \mathcal{S}(X) \cap \mathcal{C}(X),\, s \ge \psi \text{ on } \overline{U}_n\}, \qquad x \in U_n,$$

and $v_n := \psi$ on $X \setminus U_n$. Of course, $v_n|_{U_n} = \inf\{w \in {}^*\mathcal{H}(U_n) \colon w \geq v_n|_{U_n}\}$, $n \in \mathbb{N}$, and the sequence (v_n) is increasing. By Lemma 3.2, $v := \lim_{n\to\infty} v_n$ satisfies $v = R_v$. Since $v \geq \psi$, we see that $v \geq R_\psi$.

Next, let us for the moment fix $n \in \mathbb{N}$ and choose $h \in \mathcal{H}(U_{n+1})$, $h > 0$. We introduce the set $\mathcal{R}$ of all functions $x \mapsto s(x)/h(x)$ on $\overline{U}_n$, where $s \in \mathcal{S}(X) \cap \mathcal{C}(X)$, and define (identifying measures on X not charging $X \setminus \overline{U}_n$ with measures on $\overline{U}_n$)

$$\mathcal{N}_x := \{\nu \in \mathcal{M}_c(X) \colon \nu(X \setminus \overline{U}_n) = 0, \int f\,d\nu \leq f(x) \text{ for all } f \in \mathcal{R}\}, \quad x \in X.$$

Then $\mathcal{R}$ is a convex cone in $\mathcal{C}(\overline{U}_n)$ separating the points of $\overline{U}_n$. Moreover, $\pm 1 \in \mathcal{R}$. So $\mathcal{N}_x$ consists of probability measures. By Edwards' theorem (see [5, Section 2]),

$$\frac{v_n(x)}{h(x)} = \sup\{\int \frac{\psi}{h}\,d\nu \colon \nu \in \mathcal{N}_x\}, \qquad x \in \overline{U}_n.$$

Obviously, for every $x \in \overline{U}_n$, a measure $\mu \in \mathcal{M}_c(X)$ satisfying $\mu(X \setminus \overline{U}_n) = 0$ is contained in $J_x(X)$ if and only if $(h/h(x))\mu \in \mathcal{N}_x$. Hence

$$v_n(x) = \sup\{\int \psi\,d\mu \colon \mu \in J_x(X),\, \mu(X \setminus \overline{U}_n) = 0\}, \qquad x \in \overline{U}_n.$$

Letting $n \to \infty$ we see that $v = u_\psi$, where $u_\psi \leq R_\psi$, by (3.3). Thus $u_\psi = R_\psi$. The proof is finished, since R_ψ is upper semicontinuous (see [10, Corollary 1.2.2]). □

Having Theorem 2.7, an immediate consequence is the following (we recall that $\hat{R}_\varphi$ is a hyperharmonic function on X and $\{\hat{R}_\varphi < R_\varphi\}$ is a polar subset of $\{R_\varphi = \varphi\}$).

COROLLARY 3.4. *Let $\varphi \in \mathcal{B}(X)$ and $\varphi + h \geq 0$ for some $h \in \mathcal{H}(X)$. Then*

$$u_\varphi = R_\varphi = \varphi \vee \hat{R}_\varphi.$$

In particular, $u_\varphi \in \mathcal{B}(X)$.

Proof. (a) Let us suppose first that $\varphi \geq 0$. By (3.3), $u_\varphi \leq R_\varphi$. On the other hand, by Theorem 2.7, there exist bounded upper semicontinuous functions ψ_n with compact support which satisfy $0 \leq \psi_n \leq \psi_{n+1} \leq \varphi$, $n \in \mathbb{N}$, and

$$R_\varphi = \varphi \vee \sup\nolimits_{n\in\mathbb{N}} R_{\psi_n}.$$

Since $\varepsilon_x \in J_x(X)$ for every $x \in X$, we know that $\varphi \leq u_\varphi$. By Proposition 3.3, $R_{\psi_n} = u_{\psi_n} \leq u_\varphi$ for all $n \in \mathbb{N}$. Thus also $R_\varphi \leq u_\varphi$. By Theorem 2.5, $R_\varphi = \varphi \vee \hat{R}_\varphi$.

(b) In the general case $\varphi + h \geq 0$ it suffices to observe that $u_{\varphi+h} = R_{\varphi+h}$, by (a), and that obviously $u_\varphi = u_{\varphi+h} - h$ and $R_\varphi = R_{\varphi+h} - h$. □

We finally apply Corollary 3.4 to relatively compact open subsets U of X in order to obtain the same result for functions $\varphi \in \mathcal{B}(X)$ which are only supposed to be locally bounded below. In this process, it will be natural to work with the following subset $J_x'(X)$ of $J_x(X)$, $x \in X$, defined by

$$J_x'(X) := \{\mu \in \mathcal{M}_c(X) \colon \mu \in J_x(U) \text{ for some relatively compact open } U \text{ in } X\},$$

and to consider also functions u'_φ defined by

$$u'_\varphi(x) := \sup\{\int \varphi\, d\mu \colon \mu \in J'_x(X)\}.$$

For the sake of completeness, we recall from [11] that fairly weak assumptions on $(X, \mathcal{W})$ imply that $J'_x(X) = J_x(X)$ for every $x \in X$ (see Remark 3.6).

Here is the main result in this Section.

THEOREM 3.5. *Let $\varphi \in \mathcal{B}(X)$ be locally bounded below. Then*

$$u_\varphi = u'_\varphi = R_\varphi = \varphi \vee \hat{R}_\varphi.$$

In particular, $u_\varphi \in \mathcal{B}(X)$.

Proof. Since $J'_x(X) \subset J_x(X)$, $x \in X$, and (3.3) holds, we have the inequalities

$$R_\varphi \ge u_\varphi \ge u'_\varphi.$$

To prove that $u'_\varphi \ge R_\varphi$ let us choose again relatively compact open sets U_n exhausting X such that $\overline{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$. For the moment, let us fix $n \in \mathbb{N}$. By assumption, there is a strictly positive function $h_{n+1} \in \mathcal{H}(U_{n+1})$, and there exists $a_n > 0$ such that the function $h_n := a_n h_{n+1}|_{U_n} \in \mathcal{H}^+(U_n)$ satisfies $\varphi + h_n > 0$ on U_n. By Corollary 3.4 (applied to U_n instead of X),

$$v_n := \inf\{w \in {}^*\mathcal{H}(U_n) \colon w \ge \varphi \text{ on } U_n\} = (\varphi|_{U_n}) \vee \hat{v}_n \tag{3.4}$$

and, for every $x \in U_n$,

$$v_n(x) = \sup\{\int \varphi\, d\mu \colon \mu \in J_x(U_n)\}. \tag{3.5}$$

Extending the functions v_n to functions on X by $v_n(x) := \varphi(x)$, $x \in X \setminus U_n$, (3.5) implies that the sequence (v_n) is increasing to $v := u'_\varphi$. By Lemma 3.2, we conclude that $v = R_v$ and $\hat{v} = \lim_{n\to\infty} \hat{v}_n$. Since $v \ge \varphi$, we obtain that $u'_\varphi = v \ge R_\varphi$.

Thus $u_\varphi = u'_\varphi = R_\varphi$, and we finally see that $R_\varphi = \varphi \vee \hat{R}_\varphi$, by (3.4). □

REMARK 3.6. The detailed description of Jensen measures in [11] led to various simple properties implying that (without assuming the axiom of polarity)

$$J'_x(X) = J_x(X) \qquad \text{for every } x \in X. \tag{3.6}$$

For example, (3.6) holds if $(X, \mathcal{W})$ has the following approximation property (AP): For every compact K in X, there exists a relatively compact open neighborhood U of K such that, for all $u \in \mathcal{S}(U) \cap \mathcal{C}(U)\}$ and $\varepsilon > 0$, there exists a function $v \in \mathcal{S}(X) \cap \mathcal{C}(X)$ satisfying $|u - v| < \varepsilon$ on K.

If $(X, \mathcal{W})$ is elliptic, that is, if every positive superharmonic function $s \ne 0$ on a domain U in X is strictly positive, (AP) follows from [3, Theorem 6.1 and Remark 6.2.1] (cf. also [7, Theorem 6.9] for the classical case and [8, Theorem 1] for the case of a Brelot space satisfying the axiom of domination).

An approach to (3.6), which is much less involved and, by [11, Proposition 3.2], covers the classical case as well, assumes that $(X, \mathcal{W})$ is h_0-transient for some strictly positive $h_0 \in \mathcal{H}(X)$, that is, for every compact K in X, the (closed) set $\{R^K_{h_0} = h_0\}$ is compact ([11, Theorem 3.3], see also [11, Corollary 4.4] for several characterizations of 1-transient bounded open sets in the classical case).

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Wolfhard Hansen, Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany, e-mail: hansen@math.uni-bielefeld.de

Ivan Netuka, Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, 186 75 Praha 8, Czech Republic, email: netuka@karlin.mff.cuni.cz