On points with algebraically conjugate coordinates close to smooth curves

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Abstract

Let $y = f(x)$ be a continuous differentiable function on an interval $J \subset \mathbb{R}$. In this paper we show that for any $n \in \mathbb{N}$, $n \geq 2$, sufficiently large integer Q and a real $0 < \lambda < \frac{3}{4}$ there exists a positive value $c(n, f, J)$ such that all strips $L_J(Q, \lambda) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2 - f(x_1)| \ll Q^{-\lambda}, x_1 \in J\}$ contain at least $c(n, f, J)Q^{n+1-\lambda}$ points $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with algebraically conjugate coordinates which minimal polynomial P satisfies $\deg P \leq n$, $H(P) \leq Q$. The proof is based on a metric theorem on the measure of the set of vectors (x_1, x_2) lying in a rectangle Π of dimensions $\asymp Q^{-s_1} \times Q^{-s_2}$ with $|P(x_1)|, |P(x_2)|$ bounded from above and $|P'(x_1)|, |P'(x_2)|$ bounded from below, where P is a polynomial of degree $\deg P \leq n$ and height $H(P) \leq Q$. This theorem is a generalization of a result obtained by V. Bernik, F. Götze and O. Kukso for $s_1 = s_2 = \frac{1}{2}$ and $\lambda = \frac{1}{2}$ [10] .

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1 Introduction

Let Q be a sufficiently large number. We denote by $\mathcal{P}_n(Q)$ the following class of polynomials:

$$\mathcal{P}_n(Q) = \{P \in \mathbb{Z}[t] : \deg P \leq n, H(P) \leq Q\},$$

where $H(P) = \max\limits_{0 \leq j \leq n} |a_j|$ denotes the height of an integer polynomial $P(t) = a_n t^n + \ldots + a_1 t + a_0$.

The point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is called an *algebraic point* if α_1 and α_2 are roots of the same polynomial $P \in \mathbb{Z}[t]$. The polynomial P of smallest degree such that $P(\alpha_1) = P(\alpha_2) = 0$ and $\gcd(|a_n|, \ldots, |a_0|) = 1$ is called the minimal polynomial of the algebraic point $\boldsymbol{\alpha}$. Denote by $\deg(\boldsymbol{\alpha}) = \deg P$ the degree of the algebraic point $\boldsymbol{\alpha}$, and by $H(\boldsymbol{\alpha}) = H(P)$ the height of the algebraic point $\boldsymbol{\alpha}$. Define the following sets: $\mathbb{A}_n^2(Q)$ is the set of algebraic points $\boldsymbol{\alpha}$ of degree at most n and of height at most Q; $\mathbb{A}_n^2(Q, D) = \mathbb{A}_n^2(Q) \cap D$ is the set of algebraic points $\boldsymbol{\alpha} \in \mathbb{A}_n^2(Q)$ lying in a domain $D \subset \mathbb{R}^2$. Denote by $\#S$ the cardinality of a finite set S, by $\mu_1 S$ the Lebesgue measure of a measurable set $S \subset \mathbb{R}$ and by $\mu_2 S$ the Lebesgue measure of a measurable set $S \subset \mathbb{R}^2$. Further, denote by $c_j > 0$, $j \in \mathbb{N}$, positive values which do not depend on $H(P)$ or Q. We are also going to use the Vinogradov symbol $A \ll B$, which means that there exists a value $c > 0$ such that $A \le c \cdot B$ and c doesn't depend on B.

An important and interesting topic in the theory of Diophantine approximation is the distribution of algebraic numbers [1, 7, 8, 12]. In this paper we consider problems related to the distribution of algebraic points in domains of small measure and the distribution of algebraic points near smooth curves.

Consider rectangles $\Pi = I_1 \times I_2$ where $\mu_1 I_1 = c_{1,1} \cdot Q^{-s_1}$ and $\mu_1 I_2 = c_{1,2} \cdot Q^{-s_2}$ under the conditions $0 < s_1 + s_2 \le 1$, $s_1, s_2 < 1$, $\Pi \cap \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1 - x_2| < \varepsilon\} = \emptyset$ and $c_{1,1}c_{1,2} \ge c_0$. The condition $|x_1 - x_2| > \varepsilon$ means that we exclude from consideration a strip F of small measure such that the coordinates $(x_1, x_2) \in F$ are well approximated by points of form (α, α).

We can prove the following theorem.

Theorem 1. *For any rectangle $\Pi = I_1 \times I_2$ satisfying the following conditions:*

1. $\mu_1 I_i = c_{1,i} Q^{-s_i}$ where $s_i < 1$ and $0 < s_1 + s_2 \le 1$, $i = 1, 2$;

2. $\Pi \cap \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1 - x_2| < \varepsilon\} = \emptyset$;

3. $c_{1,1}c_{1,2} > c_0(n, \varepsilon, \mathbf{d}) > 0$ for $s_1 + s_2 = 1$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of Π;

there exists a constant $c_2 = c_2(n, \varepsilon, \mathbf{d}) > 0$, such that

$$\#\mathbb{A}_n^2(Q, \Pi) \ge c_2 Q^{n+1} \mu_2 \Pi,$$

for $Q > Q_0(n, \varepsilon, \mathbf{d}, \mathbf{s})$.

For $s_1 + s_2 > 1$, we can find a rectangle Π such that the statement of Theorem 1 does not hold. The example of such rectangle is $\Pi = (0, 0.5Q^{-1}) \times (0, 0.5)$. It is easy to prove [9] that the interval $(0, 0.5Q^{-1})$ doesn't contain algebraic numbers of any degree and height $\le Q$. Let us introduce some restrictions on the domains to be used in the following proofs.

Consider a square $\overline{\Pi} = I_1 \times I_2$ of size $\mu_1 I_1 = \mu_1 I_2 = c_3 Q^{-s}$ such that $\frac{1}{2} < s < \frac{3}{4}$. Given positive u_1, u_2 under the condition $u_1 + u_2 = 1$ let us say that the square $\overline{\Pi}$ is (u_1, u_2)-*ordinary* square if it doesn't contain points $(x_1', x_2') \in \mathbb{R}^2$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ of the form $P(t) = b_2 t^2 + b_1 t + b_0$ satisfying the system of inequalities

$$\begin{cases} |P(x_i')| \ll Q^{-u_i}, \quad i = 1, 2, \\ |b_2| < Q^{s - \frac{1}{2}}. \end{cases} \tag{1}$$

Otherwise, the square $\overline{\Pi}$ is going to be called (u_1, u_2)- *special.*

For (u_1, u_2)-*ordinary* squares, the following result holds.

Theorem 2. *For any $\left(\frac{1}{2}, \frac{1}{2}\right)$-ordinary square $\overline{\Pi} = I_1 \times I_2$ under the following conditions:*

1. $\mu_1 I_i = c_3 Q^{-s}$, where $\frac{1}{2} < s < \frac{3}{4}$;

2. $\overline{\Pi} \cap \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1 - x_2| < \varepsilon\} = \emptyset$;

3. $c_3 > c_0(n, \varepsilon, \mathbf{d}) > 0$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of $\overline{\Pi}$;

there exists a constant $c_4 = c_4(n, \varepsilon, \mathbf{d}) > 0$, such that

$$\#\mathbb{A}_n^2(Q, \overline{\Pi}) \ge c_4 Q^{n+1} \mu_2 \overline{\Pi}$$

for $Q > Q_0(n, \varepsilon, \mathbf{d}, s)$.

Another interesting and important topic is the distribution of algebraic points near smooth curves. The result presented in this paper is a natural generalization of problems related to distribution of rational points near smooth curves [3, 4, 12, 15, 13, 14]. In 2014 a lower bound for the number of algebraic points lying at a distance of at most $Q^{-\lambda}$, $0 < \lambda < \frac{1}{2}$, from a smooth curve was obtained by V. Bernik, F. Götze and O. Kukso [10]. We improve on this result and obtain an identical estimate for $0 < \lambda < \frac{3}{4}$.

Theorem 3. *Let $y = f(x)$ be a continuous differentiable function on an interval $J = [a, b]$ such that $\sup_{x \in J} |f'(x)| := c_5 < \infty$ and $\# \{x \in \mathbb{R} : f(x) = x\} < \infty$. Denote by $L_J(Q, \lambda)$ the following set:*

$$L_J(Q, \lambda) = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2 - f(x_1)| < \left(\tfrac{1}{2} + c_5\right) \cdot c_3 Q^{-\lambda}, \quad x_1 \in J\right\},$$

for $0 < \lambda < \frac{3}{4}$. Then there exists a positive value $c_6(J, f, n) > 0$ such that

$$\# \left\{\mathbb{A}_n^2(Q) \cap L_J(Q, \lambda)\right\} \geq c_6 Q^{n+1-\lambda}$$

for $Q > Q_0(J, f, n, \lambda)$.

2 Auxiliary statements

For a polynomial P with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, let

$$S(\alpha_i) = \left\{x \in \mathbb{R} : |x - \alpha_i| = \min_{1 \leq j \leq n} |x - \alpha_j|\right\}.$$

From now on, we assume that the roots of the polynomial P are sorted by distance from $\alpha_i = \alpha_{i,1}$:

$$|\alpha_{i,1} - \alpha_{i,2}| \leq |\alpha_{i,1} - \alpha_{i,3}| \leq \ldots \leq |\alpha_{i,1} - \alpha_{i,n}|.$$

Lemma 1. *Let $x \in S(\alpha_i)$. Then*

$$|x - \alpha_i| \leq n \cdot \frac{|P(x)|}{|P'(x)|}, \quad |x - \alpha_i| \leq 2^{n-1} \cdot \frac{|P(x)|}{|P'(\alpha_i)|}, \tag{2}$$

$$|x - \alpha_i| \leq \min_{1 \leq j \leq n} \left(2^{n-j} \frac{|P(x)|}{|P'(\alpha_i)|} |\alpha_i - \alpha_{i,2}| \ldots |\alpha_i - \alpha_{i,j}|\right)^{1/j}. \tag{3}$$

The first inequality follows from the identity

$$|P'(x)||P(x)|^{-1} = \sum_{j=1}^{n} |x - \alpha_j|^{-1}.$$

For a proof of the second and the third inequalities see [1], [2].

Lemma 2. *Let I be an interval, and let $A \subset \mathbb{R}$ be a measurable set, $A \subset I$, $\mu_1 A \geq \frac{1}{2}\mu_1 I$. If for some $v > 0$ and all $x \in A$ the inequality $|P(x)| < c_7 Q^{-v}$, where $v > 0$, holds, then*

$$|P(x)| < 6^n (n+1)^{n+1} c_7 Q^{-v}$$

for all points $x \in I$, where $n = \deg P$.

The proof of this lemma can be found in [6].

Lemma 3. *Let δ, η_1, η_2 be real positive numbers, and let $P_1, P_2 \in \mathbb{Z}[t]$ be a co-prime polynomials of degrees at most n such that*

$$\max\left(H(P_1), H(P_2)\right) < K,$$

where $K > K_0(\delta)$. Let $J_1, J_2 \subset \mathbb{R}$ be intervals of sizes $\mu J_1 = K^{-\eta_1}$, $\mu J_2 = K^{-\eta_2}$. If for some $\tau_1, \tau_2 > 0$ and for all $(x_1, x_2) \in J_1 \times J_2$, the inequalities

$$\max\left(|P_1(x_i)|, |P_2(x_i)|\right) < K^{-\tau_i}, \quad i = 1, 2,$$

hold, then

$$\tau_1 + \tau_2 + 2 + 2\max(\tau_1 + 1 - \eta_1, 0) + 2\max(\tau_2 + 1 - \eta_2, 0) < 2n + \delta. \tag{4}$$

The proof of this lemma can be found in [17].

Lemma 4. *Let $P \in \mathbb{Z}[t]$ be a reducible polynomial, $P = P_1 \cdot P_2$, $\deg P = n \geq 2$. Then there exist $c_8, c_9 > 0$ such that*

$$c_8 H(P) < H(P_1)H(P_2) < c_9 H(P).$$

The proof of Lemma 4 can be found, for example, in [1].

3 Proof of Theorem 1

Before we start it should be noted that there exists a constant $h_n = h_n(\mathbf{d}) > 0$ such that for every point $(x_1, x_2) \in \Pi$ and every $\mathbf{v} = (v_1, v_2)$ with $v_1 + v_2 = n - 1$ there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the inequalities:

$$|P(x_i)| < h_n \cdot Q^{-v_i}, \quad i = 1, 2,$$

for $Q > Q_0$. This simple fact follows from Dirichlet's principle and estimates $\#\mathcal{P}_n(Q) > 2^n Q^{n+1}$ and $|P(x_i)| < \left((|d_i| + 1)^{n+1} - 1\right) |d_i|^{-1} \cdot Q$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of Π.

To prove Theorem 1, we are going to rely on the following Lemma 5.

Lemma 5. *For all rectangles $\Pi = I_1 \times I_2$ under the conditions:*

1. *$\mu_1 I_i = c_{1,i} Q^{-s_i}$ where $s_i < 1$ and $0 < s_1 + s_2 \leq 1$, $i = 1, 2$;*
2. *$\Pi \cap \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1 - x_2| < \varepsilon\} = \emptyset$;*
3. *$c_{1,1} c_{1,2} > c_0(n, \varepsilon, \mathbf{d}) > 0$ for $s_1 + s_2 = 1$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of Π;*

let $L = L(Q, \delta_n, \mathbf{v}, \Pi)$ be the set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, \quad v_i > 0, \\ \min\limits_i \{|P'(x_i)|\} < \delta_n \cdot Q, \\ v_1 + v_2 = n - 1, \quad i = 1, 2. \end{cases} \tag{5}$$

Then for $\delta_n \leq \delta_0(n, \varepsilon, \mathbf{d})$ and $Q > Q_0(n, \varepsilon, \mathbf{s}, \mathbf{v}, \mathbf{d})$, the estimate

$$\mu_2 L < \tfrac{1}{4} \mu_2 \Pi$$

holds.

Proof. Denote by L_1 the set of points $(x_1, x_2) \in \Pi$ such that the system of inequalities (5) has a solution in irreducible polynomials $P \in \mathcal{P}_n(Q)$ under condition $|P'(x_1)| < \delta_n \cdot Q$, by L_2 the set of points $(x_1, x_2) \in \Pi$ such that the system of inequalities (5) has a solution in irreducible polynomials $P \in \mathcal{P}_n(Q)$ under condition $|P'(x_2)| < \delta_n \cdot Q$ and by L_3 the set of points $(x_1, x_2) \in \Pi$ such that the system of inequalities (5) has a solution in reducible polynomials $P \in \mathcal{P}_n(Q)$. Thus, $L = L_1 \cup L_2 \cup L_3$.

Let us estimate the measure of L_1. The main idea is to split the range of the possible values of $|P'(x_i)|$, $|P'(\alpha_i)|$, where $x_i \in S(\alpha_i)$, $i = 1, 2$ into a total of $r = r(n) = (n-1)^2$ sub-ranges and consider them separately.

Without loss of generality, we will assume that $|d_1| < |d_2|$. Let us show that the inequality

$$|P'(x_i)| \geq 2c_{10} \cdot Q^{\frac{1}{2} - \frac{v_i}{2}} \tag{6}$$

yields the following bounds on $P'(\alpha_i)$:

$$\tfrac{1}{2}|P'(x_i)| \leq |P'(\alpha_i)| \leq 2|P'(x_i)|,$$

where $c_{10} = n(n-1) \cdot \max\{h_n, 1\} \cdot (3\max\{1, |d_2|\})^{n-1} \cdot (1 + |d_2|^{-1})$. Let us write a Taylor expansion of $P'(t)$:

$$P'(x_i) = P'(\alpha_i) + \tfrac{1}{2}P''(\alpha_i)(x_i - \alpha_i) + \ldots + \tfrac{1}{(n-1)!}P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1}. \tag{7}$$

Using Lemma 1 and the estimates (5) for $Q > Q_0$, we have:

$$|x_i - \alpha_i| \leq nh_n c_{10}^{-1} \cdot Q^{-\frac{v_i+1}{2}} < \max\{1, |d_2|\} \cdot Q^{-\frac{v_i+1}{2}}.$$

Then, for $s_i > 0$ and $Q > Q_0$ we get $|x_i - d_1| < 1/2$ and thus:

$$|\alpha_i| \leq |x_i| + \tfrac{1}{2} < |d_2| + 1.$$

From this estimates we obtain the following inequality for every term in (7):

$$\left|\tfrac{1}{(k-1)!}P^{(k)}(\alpha_i)(x_i - \alpha_i)^{k-1}\right| < C_{n-1}^{k-1} \cdot \tfrac{n(n+1-k)(|d_2|+1)^{n-k+1}}{|d_2|} \cdot \max\{1, |d_2|\}^{k-1} \cdot Q^{1 - \frac{(k-1)(1+v_i)}{2}} \leq$$
$$\leq C_{n-1}^{k-1} \cdot \tfrac{n(n-1)(|d_2|+1)^{n-k+1}}{|d_2|} \cdot \max\{1, |d_2|\}^{k-1} Q^{\frac{1}{2} - \frac{v_i}{2}},$$

for $k \geq 2$. Thus, the estimate

$$\left|\tfrac{1}{2}P''(\alpha_i)(x_i - \alpha_i) + \ldots + \tfrac{1}{(n-1)!}P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1}\right| <$$
$$< n(n-1)\left(3\max\{1, |d_2|\}\right)^{n-1} \cdot \left(1 + |d_2|^{-1}\right) \cdot Q^{\frac{1}{2} - \frac{v_i}{2}} < \tfrac{1}{2} \cdot |P'(x_i)|$$

holds. By substituting these inequality to (7) we get

$$\tfrac{1}{2} \cdot |P'(x_i)| \leq |P'(\alpha_i)| \leq 2|P'(x_i)|.$$

This means that $|P'(\alpha_i)| \in T_i$, where

$$T_1 = \left[c_{10} \cdot Q^{\frac{1}{2} - \frac{v_1}{2}}; 2\delta_n \cdot Q\right), \qquad T_2 = \left[c_{10} \cdot Q^{\frac{1}{2} - \frac{v_2}{2}}; n \cdot \tfrac{(|d_2|+1)^n - 1}{|d_2|} \cdot Q\right)$$

if the inequalities (6) hold. Let us divide the intervals T_i into sub-intervals $T_{i,j} = [d_{j,i}Q^{t_{j,i}}; d_{j-1,i}Q^{t_{j-1,i}})$, $2 \le j \le n$, where

$$t_{k,i} = \begin{cases} 1, & k = 1, \\ \frac{1}{2} - \frac{(k-1)v_i}{2(n-1)}, & 2 \le k \le n, \end{cases} \qquad d_{k,i} = \begin{cases} 2\delta_n, & k = 1, i = 1, \\ n \cdot \frac{(|d_2|+1)^n - 1}{|d_2|}, & k = 1, i = 2, \\ 1, & 2 \le k \le n-1, \\ c_{10}, & k = n, \end{cases}$$

Now we are going to consider the following cases:

- the case of polynomials of the second degree $n = 2$ (see Section 3.1);
- the case of irreducible polynomials:

  $|P'(\alpha_1)| \in T_{1,j_1}$, $|P'(\alpha_2)| \in T_{2,j_2}$, where $1 \le j_1, j_2 \le n-1$ (see Section 3.2);

  $|P'(\alpha_1)| \in T_{1,n}$, $|P'(\alpha_2)| \in T_{2,n}$ (see Section 3.3);

  $|P'(x_1)| \le 2c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}$, $|P'(x_2)| \le 2c_{10}Q^{\frac{1}{2}-\frac{v_2}{2}}$ (see Section 3.4);

  $|P'(\alpha_1)| \in T_{1,j_1}$, $|P'(\alpha_2)| \in T_{2,n}$ or $|P'(\alpha_1)| \in T_{1,n}$, $|P'(\alpha_2)| \in T_{2,j_2}$, where $1 \le j_1, j_2 \le n-1$ (see Section 3.5);

  $|P'(\alpha_1)| \in T_{1,j_1}$, $|P'(x_2)| \le 2c_{10}Q^{\frac{1}{2}-\frac{v_2}{2}}$ or $|P'(x_1)| \le 2c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}$, $|P'(\alpha_2)| \in T_{2,j_2}$, where $1 \le j_1, j_2 \le n$ (see Section 3.5);
- the case of reducible polynomials (see Section 3.6).

We are going to use induction on the degree n. Let us prove the following statement, which will serve as the base of induction.

3.1 The base of induction: polynomials of the second degree.

Statement 1. *For all rectangles Π under the conditions 1— 3 let $L_{2,2} = L_{2,2}(Q, \delta_2, \boldsymbol{\gamma}_2, \Pi)$ be the set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities*

$$\begin{cases} |P(x_i)| < h_2 \cdot Q^{-\gamma_{2,i}}, & \gamma_{2,i} > 0, \\ \min\limits_i \{|P'(x_i)|\} < \delta_2 \cdot Q, \\ \gamma_{2,1} + \gamma_{2,2} = 1, & i = 1, 2. \end{cases} \tag{8}$$

Then for any $r > 0$ and for $\delta_2 < \delta_0(r, \varepsilon, \mathbf{d})$ and $Q > Q_0(n, \varepsilon, \mathbf{s}, \boldsymbol{\gamma}_2, \mathbf{d})$, the estimate

$$\mu_2 L_{2,2} < \tfrac{1}{4r} \cdot \mu_2 \Pi$$

holds.

Proof. Let $P(t)$ be a polynomial of the form $b_2t^2 + b_1t + b_0$. Let us estimate the values $|P'(\alpha_1)|$ and $|P'(\alpha_2)|$. By the third inequality of Lemma 1, for every polynomial P satisfying the inequalities (8) at a point $(x_1, x_2) \in \Pi$, we have the following estimates:

$$|x_i - \alpha_i| < \left(|P(x_i)||b_2|^{-1}\right)^{1/2} < h_2^{1/2} Q^{-\frac{\gamma_{2,i}}{2}} < \tfrac{\varepsilon}{8}, \tag{9}$$

for $Q > Q_0$ and $x_i \in S(\alpha_i)$, $i = 1, 2$.

From (9) and condition 2 we obtain that

$$|\alpha_1 - \alpha_2| > |x_1 - x_2| - |x_1 - \alpha_1| - |x_2 - \alpha_2| > \tfrac{3}{4} \cdot \varepsilon$$

and

$$|\alpha_1 - \alpha_2| < |x_1| + |x_2| + |x_1 - \alpha_1| + |x_2 - \alpha_2| < |d_1| + |d_2| + 1 + \tfrac{\varepsilon}{4}.$$

This leads to the following lower bounds for $|P'(\alpha_i)|$:

$$\left(|d_1| + |d_2| + 1 + \tfrac{\varepsilon}{4}\right) \cdot |b_2| > |P'(\alpha_i)| = \sqrt{D} = |b_2| \cdot |\alpha_1 - \alpha_2| > \tfrac{3}{4} \cdot \varepsilon \cdot |b_2|, \qquad (10)$$

where D is the discriminant of the polynomial P. The inequalities (9) also yield upper bounds for $|P'(x_i)|$:

$$|P'(x_i)| \le |b_2| \cdot (|\alpha_1 - x_i| + |\alpha_2 - x_i|) \le \left(|d_2| + 1 + \tfrac{\varepsilon}{4}\right) \cdot |b_2|. \qquad (11)$$

Now upper bounds for $|P'(\alpha_i)|$ can be obtained from the Taylor expansion of the polynomial P':

$$|P'(\alpha_i)| \le |P'(x_i)| + |P''(x_i)| \cdot |x_i - \alpha_i| \le |P'(x_i)| + \tfrac{\varepsilon}{2} \cdot |b_2|. \qquad (12)$$

Then, the estimates (10), (12) mean that

$$|b_2| < 4\varepsilon^{-1} \cdot \min_i \{|P'(x_i)|\} < 4\delta_2 \varepsilon^{-1} Q. \qquad (13)$$

From Lemma 1 and the estimates (10) it follows that the set $L_{2,2}$ is contained in a union $\bigcup\limits_{P \in \mathcal{P}_2(Q)} \sigma_P$, where

$$\sigma_P = \left\{(x_1, x_2) \in \Pi : \quad |x_i - \alpha_i| < 2h_2 \varepsilon^{-1} Q^{-\gamma_{2,i}} |b_2|^{-1}, i = 1, 2\right\}.$$

Simple calculations show that the measure of the set σ_P is lower than the measure of the rectangle Π:

$$\mu_2 \sigma_P \le 2^4 h_2^2 \varepsilon^{-2} Q^{-1} |b_2|^{-2} < c_{1,1} c_{1,2} Q^{-1} = \mu_2 \Pi$$

for $c_{1,1} c_{1,2} > 2^4 h_2^2 \varepsilon^{-2}$.

Let us estimate the measure of $L_{2,2}$:

$$\mu_2 L_{2,2} \le \mu_2 \bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P \le \sum_{P \in \mathcal{P}_2(Q)} \mu_2 \sigma_P \le 2^4 h_2^2 \varepsilon^{-2} Q^{-1} \sum_{\substack{b_2, b_1, b_0 \le Q: \\ P(t) = b_2 t^2 + b_1 t + b_0, \\ \sigma_P \ne \emptyset}} |b_2|^{-2}.$$

To do this, we need to estimate the number of polynomials $P \in \mathcal{P}_2(Q)$ such that the system (8) holds for some point $(x_1, x_2) \in \Pi$, where b_2 is fixed.

Let the inequalities (8) hold for polynomial P and point $(x_{0,1}, x_{0,2}) \in \Pi$. Let us estimate the value of the polynomial P at d_i. From the Taylor expansion of P, we have

$$P(d_i) = P(x_{0,i}) + P'(x_{0,i})(x_{0,i} - d_i) + \tfrac{1}{2} P''(x_{0,i})(x_{0,i} - d_i)^2.$$

It means that $|P(d_i)| \le |P(x_{0,i})| + |P'(x_{0,i})|\mu_1 I_i + |b_2| (\mu_1 I_i)^2$. Thus, from (11) for $Q > Q_0$ we can obtain the estimate

$$|P(d_i)| < |P(x_{0,i})| + c_{11} \cdot |b_2|\mu_1 I_i \le 2c_{11} \cdot \max\{1, |b_2|\mu_1 I_i\}.$$

Without loss of generality, let us assume that $\mu_1 I_1 \le \mu_1 I_2$.

Consider the system of equations

$$\begin{cases} b_2 d_1^2 + b_1 d_1 + b_0 = l_1, \\ b_2 d_2^2 + b_1 d_2 + b_0 = l_2 \end{cases} \tag{14}$$

in three variables $b_2, b_1, b_0 \in \mathbb{Z}$, where $|l_i| \le 2c_{11} \cdot \max\{1, |b_2|\mu_1 I_i\}$, $i = 1, 2$.

Let us estimate the number of possible pairs (b_1, b_0) such that the system (14) is satisfied for a fixed b_2. To obtain this estimate, we consider the system of linear equations (14) for two different combinations $b_2, b_{0,1}, b_{0,0}$ and $b_2, b_{j,1}, b_{j,0}$:

$$\begin{cases} b_2 d_1^2 + b_{0,1} d_1 + b_{0,0} = l_{0,1}, \\ b_2 d_1^2 + b_{j,1} d_1 + b_{j,0} = l_{j,1}, \\ b_2 d_2^2 + b_{0,1} d_2 + b_{0,0} = l_{0,2}, \\ b_2 d_2^2 + b_{j,1} d_2 + b_{j,0} = l_{j,2}. \end{cases}$$

Subtracting the second equation from the first and the forth equation from the third leads to the following system in two variables $b_{0,1} - b_{j,1}$ and $b_{0,0} - b_{j,0}$:

$$\begin{cases} (b_{0,1} - b_{j,1}) d_1 + (b_{0,0} - b_{j,0}) = l_{0,1} - l_{j,1}, \\ (b_{0,1} - b_{j,1}) d_2 + (b_{0,0} - b_{j,0}) = l_{0,2} - l_{j,2}. \end{cases} \tag{15}$$

The determinant of the system (15) can be written as

$$|\Delta| = \begin{vmatrix} d_1 & 1 \\ d_2 & 1 \end{vmatrix} = |d_1 - d_2| > \varepsilon > 0.$$

Since the determinant does not vanish, we can use Cramer's rule to solve the system (15). Using the inequalities $|l_{0,i} - l_{j,i}| \le 4c_{11} \cdot \max\{1, |b_2|\mu_1 I_i\}$, $i = 1, 2$, we estimate the determinant Δ_1 as follows:

$$|\Delta_1| \le 8c_{11} \cdot \max\{1, |b_2|\mu_1 I_2\}.$$

Hence by Cramer's rule we have

$$|b_{0,1} - b_{j,1}| \le \frac{|\Delta_1|}{|\Delta|} \le 8\varepsilon^{-1} c_{11} \cdot \max\{1, |b_2|\mu_1 I_2\}.$$

This inequality means that all possible values of the coefficient b_1 lie in an interval J_1 of length $\mu_1 J_1 = 2^4 \varepsilon^{-1} c_{11} \cdot \max\{1, |b_2|\mu_1 I_2\}$ centered at $b_{0,1}$. Since the values of the coefficient b_1 are integers, the number of these values does not exceed the measure of the interval J_1.

In addition, let us fix the value of the coefficient b_1. Choose a value $b_1 \in J_1$ and consider two different combinations $(b_2, b_1, b_{0,0})$ and $(b_2, b_1, b_{j,0})$. In this case, the system (14) can be transformed as follows:

$$\begin{cases} |b_{0,0} - b_{j,0}| \le 4c_{11} \cdot \max\{1, |b_2|\mu_1 I_1\}, \\ |b_{0,0} - b_{j,0}| \le 4c_{11} \cdot \max\{1, |b_2|\mu_1 I_2\}. \end{cases}$$

Similarly, we have $b_0 \in J_0$, where J_0 is an interval of length $\mu_1 J_0 = 8c_{11} \cdot \max\{1, |b_2|\mu_1 I_1\}$ centered at $b_{0,0}$, and the number of possible values for b_0 does not exceed the measure of the interval J_0.

The following estimate

$$\#(b_1, b_0) \le \mu_1 J_1 \cdot \mu_1 J_0 = \begin{cases} 2^7\varepsilon^{-1}c_{11}^2 \cdot |b_2|^2\mu_2\Pi, & |b_2| \ge (\mu_1 I_1)^{-1}, \\ 2^7\varepsilon^{-1}c_{11}^2 \cdot |b_2|\mu_1 I_2, & (\mu_1 I_2)^{-1} \le |b_2| \le (\mu_1 I_1)^{-1}, \\ 2^7\varepsilon^{-1}c_{11}^2, & |b_2| \le (\mu_1 I_2)^{-1}, \end{cases} \tag{16}$$

holds for a fixed value of the coefficient b_2.

Let us use the estimates (13) and (16) to consider the following three cases.

Case 1: $(\mu_1 I_1)^{-1} \le |b_2| \le 4\delta_2\varepsilon^{-1}Q$.

In this case, the first estimate of (16) holds, and we have

$$\mu_2 L_{2,2} \le 2^{11}\varepsilon^{-3}c_{11}^2 h_2^2 \cdot Q^{-1}\mu_2\Pi \cdot 4\delta_2\varepsilon^{-1}Q < \tfrac{1}{12r}\mu_2\Pi,$$

for $\delta_1 < 2^{-17}r^{-1}\varepsilon^4 c_{11}^{-2}h_2^{-2}$.

Case 2: $(\mu_1 I_2)^{-1} \le |b_2| \le (\mu_1 I_1)^{-1}$.

Then the second estimate of (16) holds, and we have

$$\mu_2 L_{2,2} \ll Q^{-1}\mu_1 I_2 \sum_{(\mu_1 I_2)^{-1} \le |b_2| \le (\mu_1 I_1)^{-1}} |b_2|^{-1} \ll Q^{-1}\ln Q \cdot \mu_1 I_2.$$

Consequently, for $\varepsilon_1 = \frac{1-s_1}{2}$ and $Q > Q_0$ we obtain

$$\mu_2 L_{2,2} \ll Q^{-1+\varepsilon_1}\mu_1 I_2 \ll Q^{-\varepsilon_1}\mu_2\Pi \le \tfrac{1}{12r}\mu_2\Pi.$$

Case 3: $1 \le |b_2| \le (\mu_1 I_2)^{-1}$.

In this case, the third estimate of (16) holds, leading to

$$\mu_2 L_{2,2} \le 2^{11}\varepsilon^{-3}c_{11}^2 h_2^2 \cdot Q^{-1} \sum_{1 \le |b_2| \le (\mu_1 I_2)^{-1}} |b_2|^{-2} \le \tfrac{1}{12r}\mu_2\Pi,$$

for $c_{1,1}c_{1,2} > 2^{12}r\pi^2 c_{11}^2\varepsilon^{-3}h_2^2$. □

3.2 The induction step: reducing the degree of the polynomial.

Let us return to the proof of Lemma 5. For $|P'(\alpha_1)| \in T_{1,j_1}$ and $|P'(\alpha_2)| \in T_{2,j_2}$, we have the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, & v_i > 0, \\ d_{j_i,i}Q^{t_{j_i,i}} \le |P'(\alpha_i)| < d_{j_i-1,i}Q^{t_{j_i-1,i}}, \\ v_1 + v_2 = n-1, & i = 1,2. \end{cases} \tag{17}$$

Without loss of generality, assume that $j_1 \le j_2$. Denote by L_{j_1,j_2} the set of points $(x_1, x_2) \in \Pi$ such that the system of inequalities (17) has a solution in polynomials $P \in \mathcal{P}_n(Q)$. By Lemma 1, it follows that L_{j_1,j_2} is contained in a union $\bigcup_{P \in \mathcal{P}_n(Q)} \sigma_P$, where

$$\sigma_P = \left\{(x_1, x_2) \in \Pi : \quad |x_i - \alpha_i| < 2^{n-1}h_n \cdot Q^{-v_i}|P'(\alpha_i)|^{-1}, i = 1,2\right\}. \tag{18}$$

It means that the following estimate for $\mu_2 L_{j_1,j_2}$ holds:

$$\mu_2 L_{j_1,j_2} \leq \mu_2 \bigcup_{P\in\mathcal{P}_n(Q)} \sigma_P \leq \sum_{P\in\mathcal{P}_n(Q)} \mu_2\sigma_P.$$

Together with the sets σ_P consider the following expanded sets

$$\sigma'_P = \sigma'_{P,1} \times \sigma'_{P,2} = \left\{(x_1,x_2)\in\Pi : \quad |x_i-\alpha_i| < c_{12}Q^{-\gamma_{j_2,i}}|P'(\alpha_i)|^{-1}, i=1,2\right\}. \qquad (19)$$

where $\gamma_{j_2,i} = \frac{(j_2-1)v_i}{n-1}$. Simple calculations show that the measure of the set σ'_P is smaller than the measure of the rectangle Π for $Q > Q_0$:

$$\mu_2\sigma'_P \leq 4c_{12}^2 \cdot Q^{1-j_2}Q^{-t_{1,j_1}-t_{2,j_2}} < 4c_{12}^2\cdot Q^{-\frac{j_2+1}{2}} < \mu_2\Pi.$$

Using (18) and (19), we find that the measures $\mu_2\sigma_P$ and $\mu_2\sigma'_P$ are connected as follows:

$$\mu_2\sigma_P \leq 2^{2n-2}h_n^2c_{12}^{-2}\cdot Q^{-n+j_2}\mu_2\sigma'_P. \qquad (20)$$

Fix the vector $\mathbf{b}_{j_2} = (a_n,\ldots,a_{j_2+1})$, where $a_n,\ldots,a_{j_2+1}$ are the coefficients of the polynomial $P\in\mathcal{P}_n(Q)$. Denote by $\mathcal{P}_n(\mathbf{b}_{j_2})\subset\mathcal{P}_n(Q)$ a subclass of polynomials with the same vector of coefficients $\mathbf{b}_{j_2}$. The number of subclasses $\mathcal{P}_n(\mathbf{b}_{j_2})$ is equal to the number of vectors $\mathbf{b}_{j_2}$ which can be estimated as follows:

$$\#\left\{\mathbf{b}_{j_2}\right\} = (2Q+1)^{n-j_2} < 2^{2n}Q^{n-j_2}. \qquad (21)$$

We are going to apply Sprindžuk's method of essential and non-essential sets [1]. A set σ'_{P_1}, $P_1\in\mathcal{P}_n(\mathbf{b}_{j_2})$ is called *essential* if for every σ'_{P_2}, $P_2\in\mathcal{P}_n(\mathbf{b}_{j_2})$, $P_2\neq P_1$, the inequality

$$\mu_2\left(\sigma'_{P_1}\cap\sigma'_{P_2}\right) < \tfrac{1}{2}\mu_2\sigma'_{P_1}, \qquad (22)$$

is satisfied. Otherwise, σ'_{P_1} is called *non-essential.*

The case of essential sets. For essential sets, we have the following estimate:

$$\sum_{\substack{P\in\mathcal{P}_n(\mathbf{b}_{j_2})\\ \sigma'_P-\text{essential}}} \mu_2\sigma'_P \leq 4\mu_2\Pi. \qquad (23)$$

Then from (20), (21) and (23) we can write

$$\sum_{\mathbf{b}_{j_2}}\sum_{\substack{P\in\mathcal{P}_n(\mathbf{b}_{j_2})\\ \sigma'_P-\text{essential}}} \mu_2\sigma_P \leq 2^{4n-2}h_n^2c_{12}^{-2}\sum_{\substack{P\in\mathcal{P}_n(\mathbf{b}_{j_2})\\ \sigma'_P-\text{essential}}}\mu_2\sigma'_P < \tfrac{1}{24r}\mu_2\Pi, \qquad (24)$$

for $c_{12} = 2^{2n+3}r^{1/2}h_n$.

The case of non-essential sets. If a set σ'_{P_1} is non-essential, then there exists a set σ'_{P_2} such that $\mu_2\left(\sigma'_{P_1}\cap\sigma'_{P_2}\right) > \frac{1}{2}\mu_2\sigma'_{P_1}$. Consider the polynomial $R = P_2 - P_1$, $\deg R\leq j_2$, $H(R)\leq 2Q$, on the set $\left(\sigma'_{P_1}\cap\sigma'_{P_2}\right)$. Let us estimate the values $|R(x_i)|$ and $|R'(x_i)|$, $i,j=1,2$.

Let us write Taylor expansions of the polynomials P_1 and P_2 in the interval $\sigma'_{P_1,i}\cap\sigma'_{P_2,i}$, $i=1,2$:

$$P_j(x_i) = P'_j(\alpha_{j,i})(x_i-\alpha_{j,i}) + \ldots + \tfrac{1}{n!}\cdot P_j^{(n)}(\alpha_{j,i})(x_i-\alpha_{j,i})^n,$$

where $\alpha_{j,i} \in \sigma'_{P_j,i}$. From the estimate (19), we have:

$$|P'_j(\alpha_{j,i})(x_i - \alpha_{j,i})| \le c_{12} Q^{-\gamma_{j_2,i}},$$

$$\left|\tfrac{1}{k!} P_j^{(k)}(\alpha_{j,i})(x_i - \alpha_{j,i})^k\right| \le c_{13,k} Q^{1-k\gamma_{j_2,i}-kt_{j_2,i}} \le c_{13,k} Q^{1-\frac{k}{2}+\frac{k}{2}\gamma_{j_2,i}-k\gamma_{j_2,i}} \le c_{13,k} Q^{-\gamma_{j_2,i}},$$

for $k \ge 2$ and $Q > Q_0$.

Thus, the estimate $|R(x_i)| < |P_1(x_i)| + |P_2(x_i)| < c_{13} \cdot Q^{-\gamma_{j_2,i}}$ holds. From Lemma 2 it follows that for every point $(x_1, x_2) \in \sigma'_{P_1}$, the inequalities

$$|R(x_i)| < c_{14} \cdot Q^{-\gamma_{j_2,i}}, \quad i = 1, 2,$$

are satisfied.

Now let us write Taylor expansions of the polynomials P'_1 and P'_2 in the interval $\sigma'_{P_1,i} \cap \sigma'_{P_2,i}$, $j, i = 1, 2$:

$$P'_j(x_i) = P'_j(\alpha_{j,i}) + \ldots + \tfrac{1}{(n-1)!} P_j^{(n)}(\alpha_{j,i})(x_i - \alpha_{j,i})^{n-1},$$

where $\alpha_{j,i} \in \sigma'_{P_j,i}$. From the estimate (19), we have:

$$\left|\tfrac{1}{(k-1)!} P_j^{(k)}(\alpha_i)(x_i - \alpha_i)^{k-1}\right| \le c_{15,k} Q^{1+(k-1)\left(\frac{\gamma_{j_2,i}}{2} - \gamma_{j_2,i} - \frac{1}{2}\right)} \le c_{15,k} |P'(\alpha_i)|$$

for $Q > Q_0$. Thus, we obtain $|R'(x_i)| \le |P'_1(x_i)| + |P'_2(x_i)| \le c_{15} |P'(\alpha_i)|$. From Lemma 2 it follows that for a sufficiently large $Q > Q_0$ the following inequalities hold:

$$\min_i \{|R'(x_i)|\} \le c_{16} \min_i \{|P'(\alpha_i)|\} \le \begin{cases} 2c_{16}\delta_n Q, & j_1 = j_2 = 2, \\ c_{16} Q^{\frac{1}{2}}, & j_1 \ne 2 \text{ or } j_2 \ne 2, \end{cases}$$

for every point $(x_1, x_2) \in \sigma'_{P_1}$. Thus, the measure of L_{j_1,j_2} for non-essential sets does not exceed the respective measure for the system

$$\begin{cases} |R(x_i)| < h_{j_2} Q_1^{-\gamma_{j_2,i}}, & \gamma_{j_2,i} > 0, \\ \min_i \{|R'(x_i)|\} < \delta_{j_2} Q_1, \\ \gamma_{1,j_2} + \gamma_{2,j_2} = j_2 - 1, & i = 1, 2, \end{cases} \tag{25}$$

where $Q_1 = \min_i \left\{(h_{j_2}/c_{14})^{1/\gamma_{j_2,i}}\right\} \cdot Q$ and $\delta_{j_2} = 2c_{16} \cdot \left(\min_i \left\{(h_{j_2}/c_{14})^{1/\gamma_{j_2,i}}\right\}\right)^{-1} \cdot \delta_n$.

It should be mentioned that if polynomial $R(t) = a_1 t - a_0$ is linear, then by Lemma 1 we obtain:

$$\left|x_i - \tfrac{a_0}{a_1}\right| \ll Q_1^{-\gamma_{j_2,i}} < \tfrac{\varepsilon}{4}, \quad i = 1, 2$$

for $Q_1 > Q_0$. Hence, we immediately have $|x_1 - x_2| < \varepsilon$ which contradicts to condition 2 for polynomial Π. Thus, $\deg R \ge 2$ and we can use induction. Since $j_2 < n$, by the induction hypothesis the measure of solutions of the system (25) is bounded from above by $\frac{1}{24r}\mu_2\Pi$ for $\delta_{j_2} \le \delta_0$ and $Q_1 > Q_0$. Thus,

$$\sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P - \text{non-essential}}} \mu_2 \sigma_P \le \sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P - \text{non-essential}}} \mu_2 \sigma'_P \le \tfrac{1}{24r} \mu_2 \Pi$$

and together with the estimate (24), this implies that

$$\mu_2 L_{j_1,j_2} \le \sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P - \text{essential}}} \mu_2 \sigma_P + \sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P - \text{non-essential}}} \mu_2 \sigma_P \le \tfrac{1}{12r} \mu_2 \Pi.$$

3.3 The case of sub-intervals $T_{1,n}$ and $T_{2,n}$

For $|P'(\alpha_1)| \in T_{1,n}$ and $|P'(\alpha_2)| \in T_{2,n}$ we have the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, \quad v_i > 0, \\ c_{10} Q^{\frac{1}{2}-\frac{v_i}{2}} \le |P'(\alpha_i)| < Q^{\frac{1}{2}-\frac{v_i}{2}+\frac{v_i}{2(n-1)}}, \\ v_1 + v_2 = n-1, \quad i = 1,2. \end{cases} \tag{26}$$

By Lemma 1, the set $L_{n,n}$ of solutions of the system (26) is contained in a union $\bigcup\limits_{P\in\mathcal{P}_n(Q)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : \quad |x_i - \alpha_i| \le 2^{n-1} h_n c_{10}^{-1} Q^{-\frac{v_i+1}{2}}, i = 1,2 \right\}. \tag{27}$$

This leads to the following estimate for $\mu_2 L_{n,n}$:

$$\mu_2 L_{n,n} \le \mu_2 \bigcup_{P\in\mathcal{P}_n(Q)} \sigma_P \le \sum_{P\in\mathcal{P}_n(Q)} \mu_2 \sigma_P.$$

In this case we can not apply induction since the degree of the polynomial can not be reduced. Let us use a different method to estimate the measure $\mu_2 L_{n,n}$.

Cover the rectangle Π by a system of disjoint rectangles $\Pi_k = J_{1,k} \times J_{2,k}$, where $\mu_1 J_{i,k} = Q^{-\frac{v_i+1}{2}+\varepsilon_{2,i}}$, $i = 1,2$, such that $\Pi \subset \bigcup\limits_k \Pi_k$ and $\Pi_k \cap \Pi \neq \emptyset$. Thus, the number of rectangles Π_k can be estimated as follows:

$$2\max\left\{\frac{\mu_1 I_1}{\mu_1 J_{1,k}}, 1\right\} \cdot 2\max\left\{\frac{\mu_1 I_2}{\mu_1 J_{2,k}}, 1\right\} = \begin{cases} 4Q^{\frac{n+1}{2}-\varepsilon_{2,1}-\varepsilon_{2,2}} \mu_2 \Pi, & s_i < \frac{v_i+1}{2}, \\ 4Q^{\frac{v_1+1}{2}-\varepsilon_{2,1}} \mu_1 I_1, & s_1 < \frac{v_1+1}{2}, s_2 \ge \frac{v_2+1}{2}, \\ 4Q^{\frac{v_2+1}{2}-\varepsilon_{2,2}} \mu_1 I_2, & s_1 \ge \frac{v_1+1}{2}, s_2 < \frac{v_2+1}{2}. \end{cases} \tag{28}$$

We are going to say that a polynomial P belongs to Π_k if there is a point $(x_1, x_2) \in \Pi_k$ such that the inequalities (26) are satisfied.

Now let us prove that there is no rectangle Π_k containing two or more irreducible polynomials $P \in \mathcal{P}_n(Q)$. Assume the converse: let $P_1, P_2 \in \Pi_k$ be irreducible polynomials and let the inequalities (26) hold for each polynomial P_j at a point $(x_{j,1}, x_{j,2}) \in \Pi_k$, $j = 1,2$. Thus, for $Q > Q_0$ and for every point $(x_1, x_2) \in \Pi_k$, the estimates

$$|x_i - \alpha_{j,i}| \le |x_i - x_{j,i}| + |x_{j,i} - \alpha_{j,i}| \le 2Q^{-\frac{v_i+1}{2}+\varepsilon_{2,i}}, \tag{29}$$

are satisfied, where $x_{j,i} \in S(\alpha_{j,i})$.

Let us estimate the values $|P_j(x_i)|$, $i,j = 1,2$ where $(x_1, x_2) \in \Pi_k$. Let us write Taylor expansions of P_j in the interval $J_{i,k}$:

$$P_j(x_i) = P_j'(\alpha_{j,i})(x_i - \alpha_{j,i}) + \ldots + \tfrac{1}{n!} \cdot P_j^{(n)}(\alpha_{j,i})(x_i - \alpha_{j,i})^n.$$

From estimates (26) and (29) we obtain that

$$\left|P_j'(\alpha_{j,i})(x_i - \alpha_{j,i})\right| \ll Q^{-v_i + \frac{v_i}{2(n-1)} + \varepsilon_{2,i}},$$

$$\left|\tfrac{1}{k!} \cdot P_j^{(k)}(\alpha_{j,i})(x_i - \alpha_{j,i})^k\right| \ll Q^{1-\frac{k}{2}-\frac{kv_i}{2}+k\varepsilon_{2,i}} \ll Q^{-v_i + \frac{v_i}{2(n-1)} + \varepsilon_{2,i}}$$

for $\varepsilon_{2,i} < \frac{v_i}{2(n-1)(k-1)}$ and $Q > Q_0$.

Then we can write the following estimate:

$$|P_j(x_i)| \ll Q^{-v_i+\frac{v_i}{2(n-1)}+\varepsilon_{2,i}} < Q^{-v_i+\frac{v_i}{2(n-1)}+\varepsilon_{2,i}+\varepsilon_3}, \tag{30}$$

where $\varepsilon_{2,i} < \frac{v_i}{2(n-1)^2}$.

From Lemma 3 for $\eta_i = \frac{v_i+1}{2} - \varepsilon_{2,i}$ and $\tau_i = v_i - \frac{v_i}{2(n-1)} - \varepsilon_{2,i} - \varepsilon_3$, $i = 1, 2$, we have

$$\tau_1 + \tau_2 + 2 = (n-1) - \tfrac{1}{2} - \varepsilon_{2,1} - \varepsilon_{2,2} + 2 - 2\varepsilon_3 = n + \tfrac{1}{2} - \varepsilon_{2,1} - \varepsilon_{2,2} - 2\varepsilon_3,$$

$$2(\tau_i + 1 - \eta_i) = 2\left(v_i - \tfrac{v_i}{2(n-1)} - \varepsilon_{2,i} - \varepsilon_3 + 1 - \tfrac{v_i+1}{2} + \varepsilon_{2,i}\right) = v_i + 1 - \tfrac{v_i}{n-1} - 2\varepsilon_3.$$

Substitution of this expressions into (4) leads to the inequality

$$\tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) = 2n + \tfrac{1}{2} - \varepsilon_{2,1} - \varepsilon_{2,2} - 6\varepsilon_3 \geq 2n + \tfrac{1}{8}$$

for $\varepsilon_{2,i} = \frac{v_i}{4(n-1)^2}$, $\varepsilon_3 = \frac{1}{48}$. This contradict to Lemma 3 with $\delta = \frac{1}{8}$.

Hence, every rectangle Π_k contains at most one polynomial $P \in \mathcal{P}_n(Q)$. In this case, we have the following estimate for the measure of the set $L_{n,n}$:

$$\mu_2 L_{n,n} \leq \sum_{\Pi_k} \mu_2 \sigma_P,$$

and together with the estimates (27) and (28) this leads to

$$\mu_2 L_{n,n} \ll Q^{-\varepsilon_{2,1}-\varepsilon_{2,2}} \mu_2 \Pi < \tfrac{1}{12r} \mu_2 \Pi$$

for $Q > Q_0$ and $s_i < \frac{v_i+1}{2}$, $i = 1, 2$. If $s_i \geq \frac{v_i+1}{2}$, then we obtain the estimate

$$\mu_2 L_{n,n} \leq \sum_{P \in \mathcal{P}_n(Q)} \mu_2 \sigma_P \ll Q^{-\varepsilon_{2,i}} \mu_1 I_1 \mu_1 I_2 < \tfrac{1}{12r} \mu_2 \Pi$$

for $Q > Q_0$.

3.4 The case of a small derivative

Let us discuss a situation where $|P'(x_i)| \leq 2c_{10} Q^{\frac{1}{2}-\frac{v_i}{2}}$, $i = 1, 2$. In this case, we can show that $|P'(\alpha_i)| \leq 2^{n-1} c_{10} Q^{\frac{1}{2}-\frac{v_i}{2}}$, where $x_i \in S(\alpha_i)$.

Indeed, let $|P'(\alpha_i)| > 2^{n-1} c_{10} Q^{\frac{1}{2}-\frac{v_i}{2}}$. Let us write a Taylor expansions of the polynomial P':

$$P'(x_i) = P'(\alpha_i) + P''(\alpha_i)(x_i - \alpha_i) + \ldots + \tfrac{1}{(n-1)!} P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1}.$$

Using our assumption and repeating analogous computations to those from the beginning of the proof of Lemma 5 (see page 5) we have:

$$\left| P''(\alpha_i)(x_i - \alpha_i) + \ldots + \tfrac{1}{(n-1)!} P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1} \right| \leq c_{10} Q^{\frac{1}{2}-\frac{v_i}{2}}.$$

This leads to the following upper bound for $|P'(\alpha_i)|$:

$$|P'(\alpha_i)| \leq 3c_{10} Q^{\frac{1}{2}-\frac{v_i}{2}},$$

which contradicts our assumption for $n \geq 3$.

Now let $L_{n+1,n+1} \subset \Pi$ be the set of points satisfying the system

$$\begin{cases} |P(x_i)| < h_n Q^{-v_i}, \quad v_i > 0, \\ |P'(\alpha_i)| < 2^{n-1} c_{10} Q^{\frac{1}{2}-\frac{v_i}{2}}, \\ v_1 + v_2 = n-1, \quad i = 1, 2. \end{cases} \tag{31}$$

The polynomials $P \in \mathcal{P}_n(Q)$ satisfying (31) are going to be classified according to the distribution of their roots and the size of the leading coefficient $|a_m|$. This classification was introduced by Sprindžuk [1].

For every polynomial $P \in \mathcal{P}_n(Q)$ of degree $3 \leq m \leq n$ we define numbers $\rho_{1,j}$ and $\rho_{2,j}$, $2 \leq j \leq m$, as solutions of equations

$$|\alpha_{1,1} - \alpha_{1,j}| = Q^{-\rho_{1,j}}, \quad |\alpha_{2,1} - \alpha_{2,j}| = Q^{-\rho_{2,j}}.$$

Let us also define the vectors $\mathbf{k}_1 = (k_{1,2}, \ldots, k_{1,m})$ and $\mathbf{k}_2 = (k_{2,2}, \ldots, k_{2,m})$ with integer coefficients as solutions of the inequalities

$$k_{i,j}\varepsilon_4 - \varepsilon_4 \leq \rho_{i,j} < k_{i,j}\varepsilon_4, \quad i = 1, 2, j = \overline{2, m},$$

where $\varepsilon_4 > 0$ is some small constant.

Denote by $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u) \subset \mathcal{P}_n(Q)$ a subclass of polynomials with the same pair of vectors $(\mathbf{k}_1, \mathbf{k}_2)$ and the following bounds on leading coefficients: $Q^u \leq |a_m| < Q^{u+\varepsilon_4}$, where $u \in \mathbb{Z}\cdot\varepsilon_4$. Since $1 \leq |a_m| \leq Q$, the following estimate holds for u: $0 \leq u \leq 1-\varepsilon_4$. The roots of the polynomial P are bounded, and we can write $Q \gg |\alpha_{j_1} - \alpha_{j_2}| \gg H^{-m+1} \gg Q^{-m+1}$, which leads to the estimates $-\frac{1}{\varepsilon_4} \leq k_{i,j} \leq \frac{m-1}{\varepsilon_4} + 1$. Thus, an integer vector $\mathbf{k}_i = (k_{i,2}, \ldots, k_{i,m})$ can take at most $\left(\frac{m}{\varepsilon_4} + 1\right)^{m-1}$ values, the number of subclasses $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ can be estimated as follows:

$$\#\{m, \mathbf{k}_1, \mathbf{k}_2, l\} \leq n c_{16}^2 c_{17}, \tag{32}$$

where $c_{16} = \sum_{i=2}^{n} \left(\frac{i}{\varepsilon_4} + 1\right)^{i-1}$, $c_{17} = \varepsilon_4^{-1} + 1$.

Let $p_{i,j}$, $i = 1, 2$, $j = \overline{1, m}$ be defined as follows:

$$\begin{cases} p_{i,j} = (k_{i,j+1} + \ldots + k_{i,m}) \cdot \varepsilon_4, \qquad 1 \leq j \leq m-1, \\ p_{i,j} = 0, \quad j = m. \end{cases} \tag{33}$$

For a polynomial $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$, we can write the following estimates for its derivatives at the root α_i:

$$\begin{gathered} Q^{u-p_{i,1}} \leq |P'(\alpha_i)| = |a_m||\alpha_{i,1} - \alpha_{i,2}| \ldots |\alpha_{i,1} - \alpha_{i,m}| \leq Q^{u-p_{i,1}+(m+1)\varepsilon_4}, \\ |P^{(j)}(\alpha_i)| \ll |a_m||\alpha_{i,1} - \alpha_{i,j+1}| \ldots |\alpha_{i,1} - \alpha_{i,m}| \ll Q^{u-p_{i,j}+(m+1)\varepsilon_4}, \quad j = \overline{2, m}. \end{gathered} \tag{34}$$

Consider polynomials which solve the system (31). We can assume that the following inequalities hold:

$$Q^{u-p_{1,1}} \leq |P'(\alpha_1)| \ll Q^{\frac{1}{2}-\frac{v_1}{2}},$$
$$Q^{u-p_{2,1}} \leq |P'(\alpha_2)| \ll Q^{\frac{1}{2}-\frac{v_2}{2}},$$

which leads to the inequalities

$$p_{1,1} > u + \frac{v_1 - 1}{2}, \quad p_{2,1} > u + \frac{v_2 - 1}{2}. \tag{35}$$

Now let us obtain an estimate for the measure of the set $L_{n+1,n+1}$. From Lemma 1 it follows that this set is contained in a union $\bigcup\limits_{m,\mathbf{k}_1,\mathbf{k}_2,u}\ \bigcup\limits_{P\in\mathcal{P}_m(Q,\mathbf{k}_1,\mathbf{k}_2,u)} \sigma_P$, where

$$\sigma_P = \left\{(x_1,x_2)\in\Pi : \quad |x_i-\alpha_i| \le \min_{1\le j\le m}\left(2^{m-j}\frac{h_n\cdot Q^{-v_i}}{|P'(\alpha_{i,1})|}\cdot|\alpha_{i,1}-\alpha_{i,2}|\ldots|\alpha_{i,1}-\alpha_{i,j}|\right)^{1/j}\right\}.$$

This, together with the previous notation (33) and the estimates (34), yields the formula

$$\sigma_P = \left\{(x_1,x_2)\in\Pi : \quad |x_i-\alpha_i| \le \tfrac{1}{2}\cdot\min_{1\le j\le m}\left((2^m h_n)^{1/j}\cdot Q^{\frac{-u-v_i+p_{i,j}}{j}}\right), i=1,2\right\} \tag{36}$$

for $P\in\mathcal{P}_m(Q,\mathbf{k}_1,\mathbf{k}_2,u)$. If the inequalities

$$(2^m h_n)^{1/m_i}\cdot Q^{\frac{-u-v_i+p_{i,m_i}}{m_i}} \le (2^m h_n)^{1/k}\cdot Q^{\frac{-u-v_i+p_{i,k}}{k}}, \quad 1\le k\le m, i=1,2, \tag{37}$$

are satisfied, then the numbers $j=m_1$ and $j=m_2$ provide the best estimates for the roots α_1 and α_2 respectively, and the inequalities

$$\sigma_P = \left\{(x_1,x_2)\in\Pi : \quad |x_i-\alpha_i| \le \tfrac{1}{2}\cdot(2^m h_n)^{1/m_i}\cdot Q^{\frac{-u-v_i+p_{i,m_i}}{m_i}}, i=1,2\right\} \tag{38}$$

hold.

Let us cover the rectangle Π by a system of disjoint rectangles $\Pi_{m_1,m_2} = J_{m_1}\times J_{m_2}$, where $\mu_1 J_{m_i} = Q^{-\frac{u+v_i-p_{i,m_i}}{m_i}+\varepsilon_5}$, such that $\Pi\subset\bigcup\limits_k \Pi_{m_1,m_2}$ and $\Pi_{m_1,m_2}\cap\Pi\ne\emptyset$. The number of rectangles Π_{m_1,m_2} can be estimated as follows:

$$\#\Pi_{m_1,m_2} \le 4\cdot Q^{\frac{u+v_1-p_{1,m_1}}{m_1}+\frac{u+v_2-p_{2,m_2}}{m_2}-2\varepsilon_5}\mu_2\Pi. \tag{39}$$

Now let us show that there is no rectangle Π_{m_1,m_2} containing two or more irreducible polynomials. Let $P_1,P_2\in\Pi_{m_1,m_2}$ be irreducible polynomials, and let the inequalities (31) hold for polynomials P_j at points $(x_{j,1},x_{j,2})\in\Pi_{m_1,m_2}$, $j=1,2$. Thus, estimates

$$|x_i-\alpha_{j,i}| \le |x_i-x_{j,i}|+|x_{j,i}-\alpha_{j,i}| \le 2\cdot Q^{-\frac{u+v_i-p_{i,m_i}}{m_i}+\varepsilon_5} \tag{40}$$

are satisfied for every point $(x_1,x_2)\in\Pi_{m_1,m_2}$ and for $Q>Q_0$, where $x_{j,i}\in S(\alpha_{j,i})$.

Let us estimate $|P_j(x_i)|$, where $(x_1,x_2)\in\Pi_{m_1,m_2}$. Let us write Taylor expansions of the polynomials P_j in the interval J_{m_i}:

$$P_j(x_i) = P_j'(\alpha_{j,i})(x_i-\alpha_{j,i})+\ldots+\tfrac{1}{m!}\cdot P_j^{(m)}(\alpha_{j,i})(x_i-\alpha_{j,i})^m.$$

By estimates (34), (37) and (40) we have

$$\left|\tfrac{1}{k!}\cdot P_j^{(k)}(\alpha_{j,i})(x_i-\alpha_{j,i})^k\right| \ll Q^{-v_i+(m+1)\varepsilon_4+k\varepsilon_5}.$$

This leads to the following estimates for $|P_j(x_i)|$:

$$|P_j(x_i)| \ll Q^{-v_i+(m+1)\varepsilon_4+m\varepsilon_5} < Q^{-v_i+(m+1)(\varepsilon_4+\varepsilon_5)}. \tag{41}$$

From Lemma 3 with $\eta_i = \frac{u+v_i-p_{i,m_i}}{m_i} - \varepsilon_5$ and $\tau_i = v_i - (m+1)(\varepsilon_4+\varepsilon_5)$, where $i = 1, 2$ and $\varepsilon_4 = \frac{1}{12(m+1)}$, $\varepsilon_5 = \frac{1}{4(3m+1)}$, we obtain

$$\tau_1 + \tau_2 + 2 = n + 1 - \tfrac{1}{6} - 2(m+1)\varepsilon_5,$$

$$2(\tau_i + 1 - \eta_i) = 2v_i + 2 - 2 \cdot \tfrac{u+v_i-p_{i,m_i}}{m_i} - \tfrac{1}{6} - 2m\varepsilon_5.$$

Let us estimate the expression $2(\tau_i + 1 - \eta_i)$ by applying the inequalities (35):

$$2(\tau_i + 1 - \eta_i) \geq \begin{cases} v_i + 2 - u + \frac{2p_{i,m_i}}{m} - \frac{1}{6} - 2m\varepsilon_5, & m_i \geq 2, \\ v_i + 1 - \frac{1}{6} - 2m\varepsilon_5, & m_i = 1, \end{cases} \geq v_i + 1 - \tfrac{1}{6} - 2m\varepsilon_5.$$

Substituting this expressions into (4) yields

$$\tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) = 2n + \tfrac{3}{2} - (6m+2)\varepsilon_5 > 2n + \tfrac{1}{2},$$

which contradicts to Lemma 3 with $\delta = \frac{1}{2}$.

This means that every rectangle Π_{m_1,m_2} contains at most one polynomial $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. Thus, the measure of solutions of the system (31) can be estimated as follows:

$$\mu_2 L_{n+1,n+1} \leq \sum_{m,\mathbf{k}_1,\mathbf{k}_2,u} \sum_{P \in \mathcal{P}_m(Q,\mathbf{k}_1,\mathbf{k}_2,u)} \mu_2\sigma_P \leq \sum_{m,\mathbf{k}_1,\mathbf{k}_2,u} \sum_{\Pi_{m_1,m_2}} \mu_2\sigma_P.$$

Thus, by estimates (32), (38) and (39), we can obtain the inequality

$$\mu_2 L_{n+1,n+1} \ll Q^{-2\varepsilon_5} \cdot \mu_2\Pi < \tfrac{1}{12r}\mu_2\Pi$$

for $Q > Q_0$.

3.5 Mixed cases

The case of sub-intervals $T_{1,n}$, $T_{2,j}$ ($T_{1,j}$, $T_{2,n}$), $j = \overline{2, n-1}$

Consider the system of inequalities

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, & v_i > 0, \\ c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}} \leq |P'(\alpha_1)| < Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{2(n-1)}}, \\ Q^{\frac{1}{2}-\frac{(j-1)v_2}{2(n-1)}} \leq |P'(\alpha_2)| < Q^{\frac{1}{2}-\frac{v_2(j-2)}{2(n-1)}}, \\ v_1 + v_2 = n - 1, & i = 1, 2. \end{cases} \tag{42}$$

Let $L_{n,j}$ be the set of solutions of the system (42). In this case we need to consider two different sets. Let $L^1_{n,j}$ and $L^2_{n,j}$ be the sets of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the system (42) under condition $c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}} \leq |P'(\alpha_1)| < Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{4(n-1)}}$ and $Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{4(n-1)}} \leq |P'(\alpha_1)| < Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{2(n-1)}}$ respectively.

As in the case of small derivatives, we classify polynomials $P \in \mathcal{P}_n(Q)$ according to the distribution of their roots and the size of their leading coefficients. We will consider the subclasses of polynomials $\mathcal{P}_m(Q, \mathbf{k}_2, u)$ with the same vector $\mathbf{k}_2$ and the following bounds on leading coefficient: $Q^u < |a_m| < Q^{u+\varepsilon_4}$, where $0 \le u \le 1-\varepsilon_4$, $0 < \varepsilon_4 < 1$ and $u \in \mathbb{Z} \cdot \varepsilon_4$. Then

$$\#\{m, \mathbf{k}_2, u\} \le nc_{17} \cdot c_{16}. \tag{43}$$

From Lemma 1, the set $L_{n,j}^g$, $g = 1, 2$ is contained in a union $\bigcup\limits_{m,\mathbf{k}_2,u} \bigcup\limits_{P \in \mathcal{P}_m(Q,\mathbf{k}_2,u)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : \quad \begin{array}{l} |x_1 - \alpha_1| \le 2^{m-1} h_n \max\{c_{10}^{-1}, 1\} \cdot Q^{-\frac{v_1}{2} - \frac{1}{2} - \frac{v_1(g-1)}{4(n-1)}}, \\ |x_2 - \alpha_2| \le 2^{m-1} h_n Q^{-v_2 + p_{2,1} - u} \end{array} \right\}. \tag{44}$$

Define the value $l = v_2 - p_{2,1} + u - k_{2,2}\varepsilon_4$ and let us write l as $l = [l] + \{l\}$, where $[l]$ is the integer part of l and $\{l\}$ is the fractional part. Now let us cover the rectangle Π by a system of disjoint rectangles $\Pi_k = J_{1,k} \times J_{2,k}$, where $\mu_1 J_{1,k} = Q^{-\frac{v_1}{2} - \frac{1}{2} - \frac{v_1(g-1)}{4(n-1)} + \varepsilon_6}$ and $\mu_1 J_{2,k} = Q^{-k_{2,2}\varepsilon_4 - \{l\}}$, such that $\Pi \subset \bigcup\limits_k \Pi_k$ and $\Pi_k \cap \Pi \ne \emptyset$. The number of rectangles $\Pi_k \in \Pi$ can be estimated as

$$\#\{\Pi_k\} \le 4Q^{\frac{v_1}{2} + \frac{1}{2} + \frac{v_1(g-1)}{4(n-1)} + k_{2,2}\varepsilon_4 - \varepsilon_6 + \{l\}} \mu_2 \Pi. \tag{45}$$

Assume that every rectangle Π_k contains no more than $2^m Q^{[l] + \frac{\varepsilon_6}{2}}$ points (α_1, α_2), where α_1, α_2 are the roots of polynomial $P \in \mathcal{P}_m(Q, \mathbf{k}_2, u)$. Then by inequalities (43), (44) and (45) it follows that the measure of the set $L_{n,j}^g$ can be estimated as:

$$\mu_2 L_{n,j}^g \le 2^{3m+4} n c_{10} c_{16} c_{17} \cdot Q^{-v_2 + p_{2,1} - u + k_{2,2}\varepsilon_4 - \frac{\varepsilon_6}{2} + [l] + \{l\}} \mu_2 \Pi \le Q^{-\frac{\varepsilon_6}{2}} \mu_2 \Pi \le \tfrac{1}{24r} \mu_2 \Pi, \tag{46}$$

where $Q > Q_0$.

Now assume that there exists a rectangle Π_k containing more than $2^m Q^{[l] + \frac{\varepsilon_6}{2}}$ polynomials $P_j \in \mathcal{P}_m(Q, \mathbf{k}_2, u)$. From the Taylor expansions of polynomials P_j in the interval $J_{2,k}$, the estimates (34) and condition $(\alpha_{j,1}, \alpha_{j,2}) \in \Pi_k$ it follows that

$$\left| \tfrac{1}{k!} \cdot P_j^{(k)}(\alpha_{j,2})(x_2 - \alpha_{j,2})^k \right| \ll Q^{u - p_{2,k} + (m+1)\varepsilon_4 - k \cdot k_{2,2}\varepsilon_4 - k\{l\}} < Q^{u - p_{2,1} - k_{2,2}\varepsilon_4 - \{l\} + (m+1)\varepsilon_4},$$

which allows us to write

$$|P_j(x_2)| < Q^{u - p_{2,1} - k_{2,2}\varepsilon_4 - \{l\} + (m+2)\varepsilon_4}. \tag{47}$$

Similarly, repeating the calculations by analogy with Section 3.3 (see inequality (30)), we have

$$|P_j(x_1)| < Q^{-v_1 + \frac{v_1}{4(n-1)} + 2\varepsilon_6} \tag{48}$$

for $\varepsilon_6 < \frac{v_1}{(n-1)^2}$.

By Dirichlet's principle we can find at least $\left[Q^{\frac{\varepsilon_6}{2}}\right] + 1$ polynomials from $\mathcal{P}_m(Q, \mathbf{k}_2, u)$ contained in Π_k such that their coefficients $a_m, \ldots, a_{m+1-[l]}$ coincide. Let us call them $P_1, \ldots, P_{\left[Q^{\frac{\varepsilon_6}{2}}\right]+1}$. If $[l] = 0$, then we can simply ignore this step. Let us consider the differences $R_{i,j} = P_i - P_j$, $1 \le i < j \le \left[Q^{\frac{\varepsilon_6}{2}}\right] + 1$.

From the inequalities (47) and (48), we obtain that at every point of the rectangle Π_k the polynomials $R_{i,j}$ satisfy

$$\begin{cases} |R_{i,j}(x_1)| < 2Q^{-v_1+\frac{v_1}{4(n-1)}+2\varepsilon_6}, \quad |R_{i,j}(x_2)| < 2Q^{u-p_{2,1}-k_{2,2}\varepsilon_4-\{l\}+(m+2)\varepsilon_4}, \\ \deg R_{i,j} \le m-[l] = m-v_2+p_{2,1}-u+k_{2,2}\varepsilon_4+\{l\}. \end{cases} \tag{49}$$

Assume that among polynomials $R_{i,j}$ we can find at least two polynomials without common roots. Then we can apply Lemma 3 with $\tau_1 = v_1 - \frac{v_1}{4(n-1)} - 2\varepsilon_6$, $\tau_2 = -u + p_{2,1} + k_{2,2}\varepsilon_4 + \{l\} - (m+2)\varepsilon_4$, $\eta_1 = \frac{v_1}{2} + \frac{1}{2} + \frac{v_1(g-1)}{4(n-1)} - \varepsilon_6$, $\eta_2 = k_{2,2}\varepsilon_4 + \{l\}$, so that we have

$$\tau_1 + 1 = v_1 + 1 - \tfrac{v_1}{4(n-1)} - 2\varepsilon_6, \quad \tau_2 + 1 = 1 - u + p_{2,1} + k_{2,2}\varepsilon_4 + \{l\} - (m+2)\varepsilon_4,$$

$$2(\tau_1+1-\eta_1) = v_1 + 1 - \tfrac{gv_1}{2(n-1)} - 2\varepsilon_6, \quad 2(\tau_2+1-\eta_2) = 2 - 2u + 2p_{2,1} - 2(m+2)\varepsilon_4.$$

Substituting these expressions into (4) for $\varepsilon_4 = \frac{1-\{l\}}{9(m+2)}$ and $\varepsilon_6 = \frac{1-\{l\}}{12}$ yields

$$\begin{aligned} \tau_1+\tau_2+2+2(\tau_1+1-\eta_1)+2(\tau_2+1-\eta_2) = 2v_1 + 5 - \tfrac{(2+g)v_1}{4(n-1)} + 3p_{2,1} + k_{2,2}\varepsilon_4 - 3u + \\ + \{l\} - 3(m+2)\varepsilon_4 - 4\varepsilon_6 \ge 2n - 2v_2 + 2p_{2,1} + 2k_{2,2}\varepsilon_4 - 2u + (p_{2,1} - k_{2,2}\varepsilon_4) + \\ + (1-u) + \{l\} + 2 - \tfrac{2(1-\{l\})}{3} - \tfrac{v_1}{n-1} \ge 2(m-[l]) - \tfrac{\{l\}}{3} + \tfrac{1}{3} \end{aligned}$$

This inequality contradict to Lemma 3 for $\delta = \frac{1-\{l\}}{3}$.

The case when among polynomials $R_{i,j}$, $1 \le i < j \le \left[Q^{\frac{\varepsilon_6}{2}}\right] + 1$ we can not find two polynomials without common roots is considered in [10].

Hence, we obtain

$$\mu_2 L_{n,j} \le \mu_2 L^1_{n,j} + \mu_2 L^2_{n,j} \le \tfrac{1}{12r}\mu_2 \Pi.$$

The case where one derivative is small and the other derivative lies in the sub-interval $T_{2,j}$, $j = \overline{2,n}$ ($T_{2,j}$)

Given the estimate for $|P'(\alpha_1)|$ obtained in Section 3.4 for $|P'(x_1)| \le 2c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}$ consider the system of inequalities

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, \quad v_i > 0, \\ |P'(\alpha_1)| < 2^{n-1}c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}, \\ Q^{\frac{1}{2}-\frac{(j-1)v_2}{2(n-1)}} \le |P'(\alpha_2)| < Q^{\frac{1}{2}-\frac{(j-2)v_2}{2(n-1)}}, \\ v_1 + v_2 = n-1, \quad i = 1,2. \end{cases} \tag{50}$$

Denote by $L_{n+1,j}$ the set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the system (50). Once again let us classify polynomials $P \in \mathcal{P}_n(Q)$ according to the distribution of their roots and the size of leading coefficients. We will consider the subclasses of polynomials $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ defined above.

From Lemma 1 by analogy with Section 3.4 (see inequality (36)) we conclude that the set $L_{n+1,j}$ is contained in a union $\bigcup\limits_{m,\mathbf{k}_1,\mathbf{k}_2,u} \bigcup\limits_{P \in \mathcal{P}_m(Q,\mathbf{k}_1,\mathbf{k}_2,u)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1,x_2) \in \Pi : \quad \begin{aligned} &|x_1 - \alpha_1| \le \tfrac{1}{2} \min_{1 \le j \le m} \left((2^m h_n)^{1/j} \cdot Q^{\frac{-u-v_1+p_{1,j}}{j}} \right), \\ &|x_2 - \alpha_2| \le 2^{m-1} h_n Q^{-u-v_2+p_{2,1}} \end{aligned} \right\}$$

for $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$.

If the inequalities (37) hold for $i = 1$, then the estimate numbered as $j = m_1$ is optimal for the root α_1, and we have

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : \quad \begin{array}{l} |x_1 - \alpha_1| \leq \frac{1}{2} \cdot (2^m h_n)^{1/m_1} \cdot Q^{\frac{-u-v_1+p_{1,m_1}}{m_1}}, \\ |x_2 - \alpha_2| \leq 2^{m-1} h_n Q^{-u-v_2+p_{2,1}} \end{array} \right\}. \tag{51}$$

Define the value $l = v_2 - p_{2,1} + u - k_{2,2}\varepsilon_4$ as in the previous case and let us cover the rectangle Π by a system of disjoint rectangles $\Pi_k = J_{1,k} \times J_{2,k}$, where $\mu_1 J_{1,k} = Q^{-\frac{u+v_1-p_{1,m_1}}{m_1}+\varepsilon_7}$ and $\mu_1 J_{2,k} = Q^{-k_{2,2}\varepsilon_4 - \{l\}}$, such that $\Pi \subset \bigcup\limits_k \Pi_k$ and $\Pi_k \cap \Pi \neq \emptyset$. The number of rectangles $\Pi_k \in \Pi$ can be estimated as

$$\#\{\Pi_k\} \leq 4Q^{\frac{u+v_1-p_{1,m_1}}{m_1}+k_{2,2}\varepsilon_4+\{l\}-\varepsilon_7}\mu_2\Pi. \tag{52}$$

Let every rectangle Π_k contain no more than $2^m Q^{[l]+\frac{\varepsilon_7}{2}}$ polynomials $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. Then by inequalities (50), (32) and (52) it follows that the measure of the set $L_{n+1,j}$ can be estimated as:

$$\mu_2 L_{n+1,j} \ll Q^{-u-v_2+p_{2,1}+k_{2,2}\varepsilon_4-\frac{\varepsilon_7}{2}+[l]+\{l\}}\mu_2\Pi \ll Q^{-\frac{\varepsilon_7}{2}}\mu_2\Pi \leq \tfrac{1}{12r}\mu_2\Pi,$$

where $Q > Q_0$.

Now assume that there exists a rectangle Π_k containing more than $2^m Q^{[l]+\frac{\varepsilon_7}{2}}$ points (α_1, α_2), where α_1, α_2 are the roots of polynomial $P_j \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. Using the calculations described in the previous case (see estimate (47)) and in Section 3.4 (see estimate (41)) for every point $(x_1, x_2) \in \Pi_k$ we have:

$$|P_j(x_1)| < Q^{-v_1+(m+1)(\varepsilon_4+\varepsilon_7)}, \quad |P_j(x_2)| < Q^{u-p_{2,1}-k_{2,2}\varepsilon_4-\{l\}+(m+2)\varepsilon_4}. \tag{53}$$

By Dirichlet's principle we can find at least $\left[Q^{\frac{\varepsilon_7}{2}}\right]+1$ from $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ contained in Π_k such that their coefficients $a_m, \ldots, a_{m+1-[l]}$ coincide. Let us call them $P_1, \ldots, P_{\left[Q^{\frac{\varepsilon_7}{2}}\right]+1}$. Thus, let us consider the differences $R_{i,j} = P_i - P_j$, where $1 \leq i < j \leq \left[Q^{\frac{\varepsilon_7}{2}}\right]+1$.

From the inequalities (53), we obtain that at every point of the rectangle Π_k the polynomials $R_{i,j}$ satisfy

$$\begin{cases} |R_{i,j}(x_1)| < 2Q^{-v_1+(m+1)(\varepsilon_4+\varepsilon_7)}, \quad |R_{i,j}(x_2)| < 2Q^{u-p_{2,1}-k_{2,2}\varepsilon_4-\{l\}+(m+2)\varepsilon_4}, \\ \deg R_{i,j} \leq m - [l] = m - v_2 + p_{2,1} - u + k_{2,2}\varepsilon_4 + \{l\}. \end{cases}$$

Assume that among polynomials $R_{i,j}$ we can find at least two polynomials without common roots and apply Lemma 3 with $\tau_1 = v_1 - (m+1)(\varepsilon_4 + \varepsilon_7)$, $\tau_2 = -u + p_{2,1} + k_{2,2} \cdot \varepsilon_4 + \{l\} - (m+2)\varepsilon_4$, $\eta_1 = \frac{u+v_1-p_{1,m_1}}{m_1} - \varepsilon_7$, $\eta_2 = k_{2,2}\varepsilon_4 + \{l\}$, so that we have

$$\tau_1 + 1 = v_1 + 1 - (m+1)(\varepsilon_4 + \varepsilon_7), \quad \tau_2 + 1 = 1 - u + p_{2,1} + k_{2,2}\varepsilon_4 + \{l\} - (m+2)\varepsilon_4,$$

and repeating the arguments from the end of Section 3.4 we obtain

$$2(\tau_1 + 1 - \eta_1) \geq v_1 + 1 - 2(m+1)\varepsilon_4 - 2m\varepsilon_7, \quad 2(\tau_2 + 1 - \eta_2) = 2 - 2u + 2p_{2,1} - 2(m+2)\varepsilon_4.$$

Substituting these expressions into (4) for $\varepsilon_4 = \frac{1}{48(m+2)}$ and $\varepsilon_7 = \frac{1}{8(3m+1)}$ yields

$$\tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) \geq 2v_1 + 5 + 3p_{2,1} + k_{2,2}\varepsilon_4 - 3u + \{l\} - \tfrac{1}{4} \geq$$
$$\geq 2n - 2v_2 + 2p_{2,1} + 2k_{2,2}\varepsilon_4 - 2u + \{l\} + \tfrac{7}{4} \geq 2(m - [l]) - \{l\} + 1 + \tfrac{3}{4} \geq 2(m - [l]) + \tfrac{3}{4}.$$

This inequality contradicts to Lemma 3 with $\delta = \frac{3}{4}$.

If among polynomials $R_{i,j}$, $1 \leq i < j \leq \left[Q^{\frac{\varepsilon_7}{2}}\right] + 1$ we can not find two polynomials without common roots then we use the arguments described in [10].

This section concludes the proof of Lemma in case of irreducible polynomials. We have

$$\mu_2 L_1 \leq \sum_{2 \leq i,j \leq n+1} \mu_2 L_{i,j} \leq (n-1)^2 \cdot \tfrac{1}{12r} \cdot \mu_2 \Pi \leq \tfrac{1}{12} \cdot \mu_2 \Pi.$$

Similarly we obtain $\mu_2 L_2 \leq \frac{1}{12} \cdot \mu_2 \Pi$.

3.6 The case of reducible polynomials

Let us estimate the measure of the set L_3. Let a polynomial P of degree n be a product of several (not necessarily different) irreducible polynomials $P_1, P_2, \ldots, P_s$, $s \geq 2$, where $\deg P_i = n_i \geq 2$ and $n_1 + \ldots + n_s = n$. Then by Lemma 4 we have:

$$H(P_1) \cdot H(P_2) \cdot \ldots \cdot H(P_s) \leq c_9 H(P) \leq c_9 Q.$$

On the other hand, by the definition of height, we have $H(P_i) \geq 1$, and thus $H(P_i) \leq c_9 Q = Q_1$, $i = 1, \ldots, s$.

Denote by $L_3(k)$ a set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $R \in \mathcal{P}_k(Q_1)$ satisfying the inequality:

$$|R(x_1)R(x_2)| < h_n^2 Q_1^{-k+\frac{1}{2}}. \tag{54}$$

If a polynomial $P \in \mathcal{P}_n(Q)$ satisfies the inequalities (5) at a point $(x_1, x_2) \in \Pi$, we can write

$$|P(x_1)P(x_2)| = |P_1(x_1)P_1(x_2)| \cdot \ldots \cdot |P_s(x_1)P_s(x_2)| \leq h_n^2 Q^{-n+1}.$$

Since $n = n_1 + \ldots + n_s$ and $s \geq 2$, it is easy to see that at least one of the inequalities

$$|P_i(x_1)P_i(x_2)| \leq h_n^2 Q^{-n_i+\frac{1}{2}}, \quad i = 1, \ldots, s,$$

is satisfied at the point (x_1, x_2). Hence, $(x_1, x_2) \in L_3(n_j)$ and we have

$$L_3 \subset \bigcup_{k=2}^{n-2} L_3(k).$$

Let us estimate the measure of the set $L_3(k)$, $2 \leq k \leq n-2$. Denote by $L_3^1(k,t)$ a set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_k(Q_1)$ satisfying the inequalities:

$$\begin{cases} |P(x_1)| < h_n^2 Q_1^t, \quad |P(x_2)| < h_n^2 Q_1^{-k+1-t}, \\ \min\limits_i \{|P'(\alpha_i)|\} < \delta_k Q_1, \quad x_i \in S(\alpha_i), i = 1, 2. \end{cases} \tag{55}$$

and by $L_3^2(k,t)$ a set of points $(x_1,x_2)\in\Pi$ such that there exists a polynomial $P\in\mathcal{P}_k(Q_1)$ satisfying the inequality:

$$\begin{cases}|P(x_1)|<h_n^2Q_1^t, & |P(x_2)|<h_n^2Q_1^{-k+\frac{3}{4}-t},\\ |P'(\alpha_i)|>\delta_kQ_1, & x_i\in S(\alpha_i),\quad i=1,2.\end{cases}\tag{56}$$

By the definition of the set $L_3(k)$ it is easy to see that:

$$L_3(k)\subset\left(\bigcup_{i=0}^{2k}L_3^1(k,1-i/2)\right)\cup\left(\bigcup_{i=0}^{4k+1}L_3^2(k,1-i/4)\right).$$

The system (55) is a system of the form (5). Hence, as the polynomials $P\in\mathcal{P}_k(Q_1)$ are irreducible and $k<n$, we can apply the induction hypothesis to obtain the following estimate:

$$\mu_2L_3^1(k,t)<\tfrac{1}{2^6n^2}\cdot\mu_2\Pi\tag{57}$$

for $Q_1>Q_0$ and sufficiently small δ_k.

Now let us estimate the measure of the set $L_3^2(k,t)$. From Lemma 1 it follows that $L_3^2(k,t)$ is contained in a union $\bigcup\limits_{P\in\mathcal{P}_k(Q)}\sigma_P(t)$, where

$$\sigma_P(t):=\left\{(x_1,x_2)\in\Pi:\begin{array}{l}|x_1-\alpha_1|\le 2^{k-1}h_n^2\cdot Q^t\cdot|P'(\alpha_1)|^{-1},\\ |x_2-\alpha_2|\le 2^{k-1}h_n^2\cdot Q^{-k+\frac{3}{4}-t}\cdot|P'(\alpha_2)|^{-1}.\end{array}\right\}$$

Let us estimate the value of the polynomial P at a central point $\mathbf{d}$ of the square Π. A Taylor expansion of the polynomial P can be written as follows:

$$P(d_i)=P'(\alpha_i)(d_i-\alpha_i)+\tfrac{1}{2}P''(\alpha_i)(d_i-\alpha_i)^2+\ldots+\tfrac{1}{k!}\cdot P^{(k)}(\alpha_i)(d_i-\alpha_i)^k.\tag{58}$$

If polynomial P satisfy (56) at point $(x_{0,1},x_{0,2})\in\Pi$ then:

$$\begin{aligned}|d_1-\alpha_1|&\le|d_1-x_{0,1}|+|x_{0,1}-\alpha_1|\le\mu_1I_1+2^{k-1}h_n^2\delta_k^{-1}\cdot Q_1^{t-1},\\ |d_2-\alpha_2|&\le|d_2-x_{0,2}|+|x_{0,2}-\alpha_2|\le\mu_1I_2+2^{k-1}h_n^2\delta_k^{-1}\cdot Q_1^{-k+\frac{3}{4}-t-1}.\end{aligned}\tag{59}$$

Without loss of generality, let us assume that $t\ge -k+\frac{3}{4}-t$. Then we can rewrite the estimates (59) as follows:

$$|d_1-\alpha_1|\le\begin{cases}c_{18}\cdot\mu_1I_1, & t<1-s_1,\\ c_{18}\cdot Q_1^{t-1}, & 1-s_1\le t\le 1,\end{cases}\qquad |d_2-\alpha_2|\le\mu_1I_2.$$

where $c_{18}=2^{k-1}h_n^2\delta_k^{-1}+c_{1,1}$. We mention that $\Pi=I_1\times I_2$, $\mu_1I_i=c_{1,i}Q^{-s_i}$, $i=1,2$ and $s_1\le s_2$.

Using these inequalities and expression (58) allows us to write

$$|P(d_1)|<\begin{cases}c_{19}\cdot Q_1\cdot\mu_1I_1, & t<1-s_1,\\ c_{19}\cdot Q_1^t, & 1-s_1\le t<1,\end{cases}\qquad |P(d_2)|<c_{19}\cdot Q_1\cdot\mu_1I_2.\tag{60}$$

Fix a vector $\mathbf{A}_1 = (a_k, \dots, a_2)$, where $a_k, \dots, a_2$ will denote the coefficients of the polynomial $P \in \mathcal{P}_k(Q_1)$. Consider a subclass $\mathcal{P}_k(\mathbf{A}_1)$ of polynomials P which satisfy (56) and have the same vector of coefficients $\mathbf{A}_1$. For $Q_1 > Q_0$, the number of such classes can be estimated as follows

$$\#\{\mathbf{A}_1\} = (2Q_1+1)^{k-1} < 2^k Q_1^{k-1}. \tag{61}$$

Let us estimate the value $\#\mathcal{P}_k(\mathbf{A}_1)$. Take a polynomial $P_0 \in \mathcal{P}_k(\mathbf{A}_1)$ and consider the difference between the polynomials P_0 and $P_j \in \mathcal{P}_k(\mathbf{A}_1)$ at points d_i, $i = 1, 2$. By (60), we have that:

$$|P_0(d_1) - P_j(d_1)| = |(a_{0,1} - a_{j,1})d_1 + (a_{0,0} - a_{j,0})| \le \begin{cases} 2c_{19} \cdot Q_1 \mu_1 I_1, & t < 1 - s_1, \\ 2c_{19} \cdot Q_1^t, & 1 - s_1 \le t \le 1, \end{cases}$$

$$|P_0(d_2) - P_j(d_2)| = |(a_{0,1} - a_{j,1})d_2 + (a_{0,0} - a_{j,0})| \le 2c_{19} \cdot Q_1 \mu_1 I_2.$$

This implies that the number of different polynomials $P_j \in \mathcal{P}_k(\mathbf{A}_1)$ does not exceed the number of integer solutions of the system

$$|b_1 d_i + b_0| \le K_i, \quad i = 1, 2, \tag{62}$$

where $K_2 = 2c_{19} \cdot Q_1 \mu_1 I_2$ and $K_1 = 2c_{19} \cdot Q_1 \mu_1 I_1$ if $t < 1 - s_1$ and $K_1 = 2c_{19} \cdot Q_1^t$ if $1 - s_1 \le t \le 1$. It is easy to see that $K_i \ge 2c_{19} \cdot Q_1^{1-s_1} > Q_1^{\varepsilon} > 1$ for $Q_1 > Q_0$. Thus, using the scheme described in Section 3.1 to solve the system (62) we have

$$\#\mathcal{P}_k(\mathbf{A}_1) \le \begin{cases} 32\varepsilon^{-1} \cdot Q_1^2 \cdot \mu_2 \Pi, & t < 1 - s_1, \\ 32\varepsilon^{-1} \cdot Q_1^{t+1} \cdot \mu_1 I_2, & 1 - s_1 \le t \le 1. \end{cases}$$

This estimate and the inequality (61) mean that the number N of polynomials $P \in \mathcal{P}_k(Q_1)$ satisfying the conditions (56) can be estimated as follows:

$$N \le \begin{cases} 2^{k+5}\varepsilon^{-1} \cdot Q_1^{k+1} \cdot \mu_2 \Pi, & t < 1 - s_1, \\ 2^{k+5}\varepsilon^{-1} \cdot Q_1^{k+t} \cdot \mu_1 I_2, & 1 - s_1 \le t \le 1. \end{cases} \tag{63}$$

On the other hand, the measure of the set $\sigma_P(t)$ satisfies the inequality

$$\mu_2 \sigma_P(t) \le \begin{cases} 2^{2k} h_n^4 \delta_k^{-2} \cdot Q_1^{-k-\frac{5}{4}}, & t < 1 - s_1, \\ 2^{2k} h_n^4 \delta_k^{-2} \cdot Q_1^{-k-\frac{1}{4}-t} \cdot \mu_1 I_1, & 1 - s_1 \le t \le 1. \end{cases} \tag{64}$$

Then, by estimates (63) and (64), for $Q_1 > Q_0$ we can write

$$\mu_2 L_3^2(k,t) \le 2^{3k+5} \delta_k^{-2} h_n^4 \varepsilon^{-1} Q_1^{-\frac{1}{4}} \mu_2 \Pi < \tfrac{1}{2^7 n^2} \cdot \mu_2 \Pi. \tag{65}$$

The inequalities (57) and (65) lead to the following estimate of the measure of the set $L_3(k)$:

$$\mu_2 L_3(k) \le \sum_{i=0}^{2k} \mu_2 L_3^1(k, 1 - i/2) + \sum_{i=0}^{4k+1} \mu_2 L_3^2(k, 1 - i/4) \le \tfrac{5+6k}{2^7 n^2} \cdot \mu_2 \Pi \le \tfrac{1}{12n} \cdot \mu_2 \Pi.$$

Therefore

$$\mu_2 L_3 \le \sum_{k=2}^{n-2} \mu_2 L_3(k) \le \tfrac{n-3}{12n} \cdot \mu_2 \Pi \le \tfrac{1}{12} \cdot \mu_2 \Pi.$$

This proves Lemma 5 in the case of reducible polynomials.

Thus, we have

$$\mu_2 L \le \mu_2 L_1 + \mu_2 L_2 + \mu_2 L_3 \le \tfrac{1}{4}\mu_2 \Pi.$$

□

3.7 The final part of the proof

The proof of Theorem 1 is going to be based on Lemma 5. Consider a set $B = \Pi \setminus L$. From Lemma 5 it follows that

$$\mu_2 B \ge \tfrac{3}{4}\mu_2 \Pi \tag{66}$$

for $Q > Q_0$.

It should be recalled that the value h_n is defined in the beginning of the section 3 such that for every point $\mathbf{x} \in \Pi$ there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying

$$|P(x_i)| \le h_n Q^{-\frac{n-1}{2}}, \quad i = 1, 2.$$

Then, for every point $(x_{1,1}, x_{1,2}) \in B$ there exists an irreducible polynomial $P_1 \in \mathcal{P}_n(Q)$ satisfying the system of inequalities

$$\begin{cases} |P_1(x_{1,i})| < h_n Q^{-\frac{n-1}{2}}, \\ |P_1'(x_{1,i})| > \delta_n Q, \quad i = 1, 2. \end{cases}$$

Let α_i, $x_{1,i} \in S(\alpha_i)$, $i = 1, 2$ be roots of the polynomial P_1. By Lemma 1, we have

$$|x_{1,i} - \alpha_i| \le n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, \quad i = 1, 2. \tag{67}$$

We are going to choose a maximal system of points $\Gamma = (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_t)$ satisfying the following conditions

1. $H(\boldsymbol{\gamma}_j) \le Q, \deg(\boldsymbol{\gamma}_j) \le n$;
2. Rectangles

$$\sigma(\boldsymbol{\gamma}_j) = \left\{ |x_i - \gamma_{j,i}| < n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, i = 1, 2 \right\}, \quad j = \overline{1, t},$$

do not intersect.

Let us introduce an expanded rectangles

$$\sigma_1(\boldsymbol{\gamma}_j) = \left\{ |x_i - \gamma_{j,i}| < 2n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, i = 1, 2 \right\}, \quad j = \overline{1, t}, \tag{68}$$

and show that

$$B \subset \bigcup_{j=1}^{t} \sigma_1(\boldsymbol{\gamma}_j). \tag{69}$$

We obtain this by proving that for every point $(x_{1,1}, x_{1,2}) \in B$ there exists a point $\boldsymbol{\gamma}_j \in \Gamma$ such that $(x_{1,1}, x_{1,2}) \in \sigma_1(\boldsymbol{\gamma}_j)$. Since $(x_{1,1}, x_{1,2}) \in B$, there is a point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ such that the inequalities (67) are true. Thus, either $\boldsymbol{\alpha} \in \Gamma$ and $(x_{1,1}, x_{1,2}) \in \sigma_1(\boldsymbol{\alpha})$, or there exists a point $\boldsymbol{\gamma}_j \in \Gamma$ satisfying

$$|\alpha_i - \gamma_{j,i}| \le n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, \quad i = 1, 2.$$

Hence, $(x_{1,1}, x_{1,2}) \in \sigma_1(\boldsymbol{\gamma}_j)$.

In this case, by (66),(68) and (69) we have:

$$\tfrac{3}{4}\mu_2\Pi \le \mu_2 B \le \sum_{j=1}^{t} \mu_2\sigma_1(\boldsymbol{\gamma}_j) \le t \cdot 2^4 n^2 h_n^2 \delta_n^{-2} Q^{-n-1},$$

which yields the estimate

$$t \ge c_2 Q^{n+1} \mu_2 \Pi.$$

4 Proof of Theorem 2

The proof of Theorem 2 is based on the following Lemma.

Lemma 6. *For all $\left(\frac{v_1}{n-1}, \frac{v_2}{n-1}\right)$- ordinary rectangles $\overline{\Pi} = I_1 \times I_2$ such that:*

1. $\mu_1 I_1 = \mu_1 I_2 = c_3 Q^{-s}$, where $\frac{1}{2} < s < \frac{3}{4}$;

2. $\overline{\Pi} \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$;

3. $c_3 > c_0(n, \varepsilon, \mathbf{d})$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of $\overline{\Pi}$;

let $L = L(Q, \delta_n, \mathbf{v}, \overline{\Pi})$ be the set of points $(x_1, x_2) \in \overline{\Pi}$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the following system of inequalities

$$\begin{cases} |P(x_i)| < h_n Q^{-v_i}, \quad v_i > 0, \\ \min\limits_i \{|P'(x_i)|\} < \delta_n Q, \\ v_1 + v_2 = n - 1, \quad i = 1, 2. \end{cases} \tag{70}$$

Then for a sufficiently small constant $\delta_n < \delta_0(n, \varepsilon, \mathbf{d})$ and a sufficiently large $Q > Q_0(n, \varepsilon, \mathbf{v}, s, \mathbf{d})$, the estimate

$$\mu_2 L < \tfrac{1}{4}\mu_2\overline{\Pi}$$

holds.

Proof. Lemma 6 can be proved in the same way as Lemma 5, we only need to replace the base of induction.

Statement 2. *For all $(\gamma_{2,1}, \gamma_{2,2})$- ordinary squares $\overline{\Pi} = I_1 \times I_2$ under conditions 1 — 3 let $L_{2,2} = L_{2,2}(Q, \delta_2, \boldsymbol{\gamma}_2, \overline{\Pi})$ be the set of points $(x_1, x_2) \in \overline{\Pi}$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities*

$$\begin{cases} |P(x_i)| < h_2 Q^{-\gamma_{2,i}}, \quad \gamma_{2,i} > 0, \\ \min\limits_i \{|P'(x_i)|\} < \delta_2 Q, \quad i = 1, 2 \\ \gamma_{2,i} + \gamma_{2,i} = 1, \quad |b_2| > Q^{s-\frac{1}{2}}. \end{cases} \tag{71}$$

Then for any $r > 0$, $\delta_2 \le \delta_0(\varepsilon, r, \mathbf{d})$ and $Q > Q_0(n, \varepsilon, s, \boldsymbol{\gamma}_2, \mathbf{d})$, the estimate

$$\mu_2 L_{2,2} < \tfrac{1}{4r}\mu_2 \overline{\Pi}$$

holds.

Proof. Let P be a polynomial of the form $P(t) = b_2 t^2 + b_1 t + b_0$. Applying the same argument that we used to prove the Statement 1, we obtain upper and lower bounds for the absolute value of the derivative P' at roots α_1, α_2 and at points x_1, x_2, where $x_i \in S(\alpha_i)$, $i = 1, 2$:

$$|P'(\alpha_i)| > \tfrac{3}{4} \cdot \varepsilon \cdot |b_2|, \quad |P'(x_i)| \le \left(|d_1| + |d_2| + 1 + \tfrac{\varepsilon}{4}\right) \cdot |b_2|. \tag{72}$$

These estimates lead to the following inequality:

$$|b_2| < 4\delta_2 \varepsilon^{-1} Q.$$

From Lemma 1 and the estimates (71), (72) it follows that the set $L_{2,2}$ is contained in a union $\bigcup\limits_{P \in \mathcal{P}_2(Q)} \sigma_P$, where

$$\sigma_P = \left\{(x_1, x_2) \in \overline{\Pi} : \quad |x_i - \alpha_i| < 2h_2 \varepsilon^{-1} Q^{-\gamma_{2,i}} |b_2|^{-1}, i = 1, 2\right\}. \tag{73}$$

Since the square $\overline{\Pi}$ is $(\gamma_{2,1}, \gamma_{2,2})$- *ordinary* we have $|b_2| \ge Q^{s - \frac{1}{2}}$ and

$$\mu_2 \sigma_P \le 2^4 h_2^2 \varepsilon^{-2} Q^{-1} |b_2|^{-2} \le c_3^2 Q^{-2s} \le \mu_2 \overline{\Pi}$$

with $c_3 > 4h_2 \varepsilon^{-1}$ and $s > \frac{1}{2}$.

Then we can write the following estimate for the measure of the set $L_{2,2}$:

$$\mu_2 L_{2,2} \le \mu_2 \bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P \le \sum_{P \in \mathcal{P}_2(Q)} \mu_2 \sigma_P \le 2^4 h_2^2 \varepsilon^{-2} Q^{-1} \sum_{\substack{b_2, b_1, b_0 \le Q: \\ P(t) = b_2 t^2 + b_1 t + b_0 \\ \sigma_P \ne \emptyset}} |b_2|^{-2}.$$

As in the proof of Statement 1 of Lemma 5, we estimate the number of polynomials $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities (71) at some point $(x_1, x_2) \in \overline{\Pi}$ for a fixed value of b_2.

Let us estimate the polynomial P at the points d_1, d_2. From Taylor expansions and estimates (72) we have

$$|P(d_i)| \le |P(x_i)| + c_{20} \cdot |b_2| \mu_1 I_i, \tag{74}$$

for a sufficiently large $Q > Q_0$. Consider a system of equations

$$\begin{cases} b_2 d_1^2 + b_1 d_1 + b_0 = l_1, \\ b_2 d_2^2 + b_1 d_2 + b_0 = l_2, \end{cases} \tag{75}$$

in three variables $b_2, b_1, b_0 \in \mathbb{Z}$, where $|l_i| \le 2c_{20} \cdot \max\{1, |b_2| \mu_1 I_i\}$, $i = 1, 2$.

Let us estimate the number of possible pairs (b_1, b_0) such that the system (75) is satisfied for a fixed b_2. To obtain this estimate, we consider the system (75) for two different combinations $b_2, b_{0,1}, b_{0,0}$ and $b_2, b_{j,1}, b_{j,0}$. Simple transformations lead to the following system of linear equations in two variables $b_{0,1} - b_{j,1}$ and $b_{0,0} - b_{j,0}$:

$$\begin{cases} (b_{0,1} - b_{j,1}) d_1 + (b_{0,0} - b_{j,0}) = l_{0,1} - l_{j,1}, \\ (b_{0,1} - b_{j,1}) d_2 + (b_{0,0} - b_{j,0}) = l_{0,2} - l_{j,2}. \end{cases} \tag{76}$$

Since the determinant of this system does not vanish, we can use Cramer's rule to solve it. Using inequalities $|l_{0,i} - l_{j,i}| \le 4c_{20} \cdot \max\{1, |b_2|\mu_1 I_i\}$ we estimate the determinants Δ_i, $i = 1, 2$ as follows:

$$|\Delta_i| \le 8c_{20} \cdot \max\{1, |b_2|\mu_1 I_i\}.$$

Thus

$$|b_{0,i} - b_{j,i}| \le \frac{|\Delta_i|}{|\Delta|} \le 8c_{20}\varepsilon^{-1} \cdot \max\{1, |b_2|\mu_1 I_i\},$$

and for a fixed b_2 the following estimate holds:

$$\#(b_1, b_0) \le \begin{cases} 2^6 c_{20}^2 \varepsilon^{-2} |b_2|^2 \mu_2 \overline{\Pi}, & |b_2| > c_3^{-1} Q^s, \\ 2^6 c_{20}^2 \varepsilon^{-2}, & Q^{s-\frac{1}{2}} < |b_2| < c_3^{-1} Q^s. \end{cases} \tag{77}$$

Depending on the absolute value $|b_2|$, let us consider the following two sets:

$$L_{2,2}^1 = \bigcup_{\substack{P \in \mathcal{P}_2(Q), \\ c_3^{-1} Q^s < |b_2| < 4\delta_2 \varepsilon^{-1} Q}} \sigma_P, \qquad L_{2,2}^2 = \bigcup_{\substack{P \in \mathcal{P}_2(Q), \\ Q^{s-\frac{1}{2}} < |b_2| < c_3^{-1} Q^s}} \sigma_P.$$

The set $L_{2,2}^1$: In this case for $\delta_2 < 2^{-15} r^{-1} c_{20}^{-2} h_2^{-2} \varepsilon^5$ can be estimated as:

$$\mu_2 L_{2,2} \le 2^{10} c_{20}^2 h_2^2 \varepsilon^{-4} Q^{-1} \cdot 4\delta_2 \varepsilon^{-1} Q \mu_2 \overline{\Pi} < \tfrac{1}{8r} \cdot \mu_2 \overline{\Pi}.$$

The set $L_{2,2}^2$: Consider the polynomials P under condition $Q^{s-\frac{1}{2}} < |b_2| < c_3^{-1} Q^s$. For every set σ_P we define the expanded set:

$$\sigma'_P = \left\{ (x_1, x_2) \in \overline{\Pi} : \quad |x_i - \alpha_i| < 2^5 h_2 \varepsilon^{-1} \sqrt{r} \cdot Q^{-\gamma_{2,i}} |b_2|^{-1}, i = 1, 2 \right\}. \tag{78}$$

Let us prove that for $|b_2| < c_{21} \cdot Q^{\frac{1}{2}}$, where $c_{21} = \varepsilon \left(2^5 h_2 \sqrt{r}\right)^{-1} \cdot (|d_1| + |d_2| + 2)^{-1}$ this sets do not intersect.

Consider polynomials P_j, $j = 1, 2$ with roots $\alpha_{j,1}, \alpha_{j,2}$ and leading coefficients $|b_{j,2}| < c_{21} \cdot Q^{\frac{1}{2}}$. Without loss of generality we will assume $|b_{1,2}| < |b_{2,2}|$. Let there exists a point $(x_{0,1}, x_{0,2}) \in \sigma'_{P_1} \cap \sigma'_{P_2}$. Since P_1 and P_2 have no common roots, the resultant $R(P_1, P_2)$ doesn't vanish, and the following estimate holds:

$$1 \le |R(P_1, P_2)| = |b_{1,2}|^2 |b_{2,2}|^2 |\alpha_{1,1} - \alpha_{2,1}| |\alpha_{1,1} - \alpha_{2,2}| |\alpha_{1,2} - \alpha_{2,1}| |\alpha_{1,2} - \alpha_{2,2}|. \tag{79}$$

By the estimates (78) we have

$$|\alpha_{1,i} - \alpha_{2,i}| \le |\alpha_{1,i} - x_{0,i}| + |\alpha_{2,i} - x_{0,i}| < 2^6 h_2 \varepsilon^{-1} \sqrt{r} \cdot Q^{-\gamma_{2,i}} |b_{1,2}|^{-1}.$$

On the other hand for $Q > Q_0$ we get

$$|\alpha_{1,1} - \alpha_{2,2}| \le |\alpha_{1,1}| + |\alpha_{2,2}| \le |d_1| + |d_2| + 2,$$
$$|\alpha_{1,2} - \alpha_{2,1}| \le |\alpha_{1,2}| + |\alpha_{2,1}| \le |d_1| + |d_2| + 2.$$

By substituting these inequalities to (79) we obtain

$$1 \le |R(P_1, P_2)| < 2^{12} h_2^2 \varepsilon^{-2} r \cdot (|d_1| + |d_2| + 2)^2 \cdot |b_{2,2}|^2 \cdot Q^{-1} < \tfrac{1}{4}.$$

This contradiction yields the following estimate

$$\sum_{\substack{P\in\mathcal{P}_2(Q),\\ Q^{s-\frac12}<|b_2|<c_{21}Q^{\frac12}}} \mu_2\sigma_P \le \tfrac{1}{16r}\cdot \sum_{\substack{P\in\mathcal{P}_2(Q),\\ Q^{s-\frac12}<|b_2|<c_{21}Q^{\frac12}}} \mu_2\sigma'_P \le \tfrac{1}{16r}\cdot\mu_2\overline{\Pi}.$$

Consider the case $|b_2| > c_{21}Q^{\frac12}$. Let $\mathcal{P}_2(Q,k)\subset\mathcal{P}_2(Q)$, $1\le k\le K = \left[\ln_2\left(\frac{2-2s}{3-4s}\right)\right]+1$ be a subclass of polynomials defined as follows:

$$\mathcal{P}_2(Q,k) := \left\{P\in\mathcal{P}_2(Q) : l_{k+1}\cdot Q^{\lambda_{k+1}} \le |b_2| \le l_k\cdot Q^{\lambda_k}\right\},$$

where

$$\begin{array}{lll} \lambda_1 = s, & l_1 = c_3^{-1}, & \\ \lambda_k = \lambda_{k-1}-(1-s)\cdot 2^{1-k}, & l_k = \frac{2^6 c_{20}h_2\cdot\sqrt{rK\cdot l_{k-1}}}{\varepsilon^2 c_3} & \text{for } 2\le k\le K, \\ \lambda_{K+1} = \frac12, & l_{K+1} = c_{21}. & \end{array}$$

This equations give $\lambda_k = s-(1-s)\cdot\left(1-\frac{1}{2^{k-1}}\right)$ for $2\le k\le K$.

Let us consider the following sets $L(k) = \bigcup\limits_{P\in\mathcal{P}_2(Q,k)}\sigma_P$ and estimate the measure of every one of them in the following way:

$$\mu_2L(k) = \sum_{P\in\mathcal{P}_2(Q,k)}\mu_2\sigma_P \le \frac{2^{10}h_2^2c_{20}^2}{\varepsilon^4}\cdot Q^{-1}\sum_{l_{k+1}Q^{\lambda_{k+1}}\le|b_2|\le l_kQ^{\lambda_k}}|b_2|^{-2} \le \frac{2^{10}h_2^2c_{20}^2l_k}{\varepsilon^4 l_{k+1}^2}\cdot Q^{-1-2\lambda_{k+1}+\lambda_k}.$$

Then for $k=1$ we obtain

$$\mu_2L(1) \le \tfrac{c_3^2}{16rK}\cdot Q^{-1-2s+1-s+s} \le \tfrac{1}{16rK}\cdot c_3^2Q^{-2s} < \tfrac{1}{16Kr}\cdot\mu_2\overline{\Pi};$$

for $1\le k\le K-1$ we have

$$\mu_2L(k) \le \tfrac{c_3^2}{16rK}\cdot Q^{-1+s-(1-s)\cdot\left(1-\frac{1}{2^{k-1}}\right)-2s+(1-s)\cdot\left(2-\frac{1}{2^{k-1}}\right)} \le \tfrac{1}{16rK}\cdot c_3^2Q^{-2s} = \tfrac{1}{16rK}\cdot\mu_2\overline{\Pi};$$

and for $k=K$, $s<\frac34$ and $Q>Q_0$ we get

$$\mu_2L(K) \le \tfrac{2^6h_2^2\cdot l_K}{\varepsilon^2c_{20}^2}\cdot Q^{-2+s-(1-s)\cdot\left(1-\frac{1}{2^{K-1}}\right)} \le \tfrac{2^4h_2^2\cdot l_K}{\varepsilon^2c_{21}^2}\cdot Q^{-3+2s+(1-s)\cdot\frac{3-4s}{2-2s}} \le \tfrac{2^4h_2^2\cdot l_K}{\varepsilon^2c_{21}^2}\cdot Q^{-\frac32} < \tfrac{1}{16rK}\cdot\mu_2\overline{\Pi}.$$

Then, we obtain following estimate for the measure of the set $L_{2,2}^2$

$$\mu_2L_{2,2}^2 \le \sum_{\substack{P\in\mathcal{P}_2(Q),\\ Q^{s-\frac12}<|b_2|<c_{21}Q^{\frac12}}}\mu_2\sigma_P + \sum_{1\le k\le K}\mu_2L(k) \le \tfrac{1}{8r}\cdot\mu_2\overline{\Pi},$$

and thus

$$\mu_2L_{2,2} \le \mu_2L_{2,2}^1 + \mu_2L_{2,2}^2 \le \tfrac{1}{4r}\cdot\mu_2\overline{\Pi}.$$

□

Now Lemma 6 can be proved by repeating the proof of Lemma 5. □

Theorem 2 is proved by applying the results of Lemma 6 to the proof of Theorem 1.

5 Proof of Theorem 3

To prove Theorem 3 we are going to use the results of Theorem 1 and Theorem 2. For this purpose we need to consider the set $J \backslash D = \bigcup_k J_k$, where $D := \{x \in J : \quad |f(x) - x| < \frac{1}{2}\varepsilon\}$. It is easy to see, that for sufficiently small ε the following estimate for the measure of the set $J \backslash D$ holds:

$$\mu_1(J \backslash D) \geq \tfrac{3}{4}\mu_1 J.$$

Now for every strip $L_{J_k}(Q, \lambda)$ we have $L_{J_k}(Q, \lambda) \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$.

Let us consider an interval $J_k = [a_k, b_k]$ and the strip $L_{J_k}(Q, \lambda)$ for a fixed $0 < \lambda < \frac{3}{4}$. Divide the strip $L_{J_k}(Q, \lambda)$ into segments

$$E_j := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in J_{k,j}, |x_2 - f(x_1)| < \left(\tfrac{1}{2} + c_5\right) \cdot c_3 Q^{-\lambda}\right\},$$

where $J_{k,j} = [x_j, x_{j+1}]$, $x_j = x_{j-1} + c_3 Q^{-\lambda}$, $x_0 = a_k$ and $1 \leq j \leq t_k$. The number of segments E_j can be estimated as follows:

$$t_k > \frac{\mu_1 J_k}{\mu_1 J_{k,j}} - 1 > \tfrac{1}{2} c_3^{-1} \mu_1 J_k \cdot Q^{\lambda}$$

for $Q > Q_0$.

Let $\overline{f}_j = \frac{1}{2} \cdot \left(\max\limits_{x \in J_{k,j}} f(x) + \min\limits_{x \in J_{k,j}} f(x)\right)$. Consider the rectangles

$$\Pi_j = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in J_{k,j}, \left|x_2 - \overline{f}_j\right| \leq \tfrac{1}{2} c_5 c_3 Q^{-\lambda}\right\}.$$

By mean value theorem, since f is continuous and differentiable function on every interval $J_{k,j}$ and $\sup\limits_{x \in J_{k,j}} |f'(x)| \leq \sup\limits_{x \in J} |f'(x)| := c_5$, we obtain:

$$\left|\max_{x \in J_{k,j}} f(x) - \min_{x \in J_{k,j}} f(x)\right| \leq |f'(\xi)| \cdot \mu_1 J_{k,j} < c_5 c_3 \cdot Q^{-\lambda}.$$

It means that $\Pi_j \subset E_j$ for every $1 \leq j \leq t_k$.

Case 1: $0 < \lambda \leq \frac{1}{2}$.

In this case, we apply the result of Theorem 1. From Theorem 1 it follows that every rectangle Π_j, $j = \overline{1, t_k}$, contains at least $c_2 Q^{n+1-2\lambda}$ algebraic points of degree at most n and height at most Q. Since we have $t_k > \frac{1}{2} c_3^{-1} \mu_1 J_k Q^{\lambda}$ and $\sum\limits_k \mu_1 J_k \geq \mu_1 J \backslash D > \frac{3}{4}\mu_1 J$, there must be at least $c_6 Q^{n+1-\lambda}$ algebraic points $\alpha \in L_J(Q, \lambda) \cap \mathbb{A}_n^2(Q)$.

Case 2: $\frac{1}{2} < \lambda < \frac{3}{4}$.

Theorem 2 will be used in that case. Let us count the number of $\left(\frac{1}{2}, \frac{1}{2}\right)$- *special* squares Π_i. By definition, a $\left(\frac{1}{2}, \frac{1}{2}\right)$- *special* square contains the points $(x_{0,1}, x_{0,2})$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities

$$\begin{cases} |P(x_{0,i})| < h_2 Q^{-\frac{1}{2}}, \quad i = 1, 2, \\ |b_2| \leq Q^{\lambda - \frac{1}{2}}. \end{cases} \tag{80}$$

Repeating the steps of the proof of Statement 1 from the beginning till inequality (13), we obtain the following estimates:

$$|P'(\alpha_1)| = |P'(\alpha_2)| > \tfrac{3}{4} \cdot \varepsilon \cdot |b_2|.$$

Thus, by Lemma 1 the set of points (x_1, x_2) satisfying the system (80) for a fixed polynomial P is a subset of the following square:

$$\sigma_P = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_i - \alpha_i| \le 2h_2\varepsilon^{-1}Q^{-\frac{1}{2}}|b_2|^{-1}, i = 1, 2\right\}.$$

Let us estimate the number of squares Π_j, such that $\Pi_j \cap \sigma_P \neq \emptyset$. It is easy to see that the width of the strip $L_{J_k}(Q, \lambda)$ is smaller than the heights of the squares σ_P for sufficiently large c_3. Hence, every square σ_P intersects with at most $4h_2\varepsilon^{-1}c_3^{-1}Q^{\lambda-\frac{1}{2}}|b_2|^{-1}$ squares Π_j. Therefore, the number m_1 of $\left(\frac{1}{2}, \frac{1}{2}\right)$- *special* squares Π_i can be estimated as

$$m_1 \le \sum_{P \in \mathcal{P}_2(Q)} 4h_2\varepsilon^{-1}c_3^{-1}Q^{\lambda-\frac{1}{2}}|b_2|^{-1} \le 4h_2\varepsilon^{-1}c_3^{-1}Q^{\lambda-\frac{1}{2}} \sum_{b_2, b_1, b_0} |b_2|^{-1}$$

Now we need to estimate the number of polynomials $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities (80) at some point $(x_1, x_2) \in L_{J_k}(Q, \lambda)$ for a fixed value of b_2. Since the function f is continuously differentiable on the interval J, and $\sup_{x \in J_k} |f'(x)| < c_5$, we get by the mean value theorem that

$$\left|\max_{x \in J_k} f(x) - \min_{x \in J_k} f(x)\right| < c_5 \cdot \mu_1 J_k,$$

which implies that the set $L_{J_k}(Q, \lambda)$ is contained in a rectangle $\Pi = I_1 \times I_2$, where $\mu_1 I_2 = c_5\mu_1 I_1 = c_5\mu_1 J_k$.

Let us estimate the polynomial P at the midpoint (d_1, d_2) of the rectangle Π. Using the steps of the proof of Statement 1 we obtain

$$|P(d_1)| \le c_{22} \cdot |b_2|\mu_1 J_k, \quad |P(d_2)| \le c_{22}c_5 \cdot |b_2|\mu_1 J_k.$$

and, hence, for a fixed value of b_2 the number of polynomials $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities (80) at some point $(x_1, x_2) \in \Pi$ can be estimated as follows:

$$\#(b_1, b_0) \le 2^5 c_5 c_{22}^2 \varepsilon^{-2} |b_2|^2 \left(\mu_1 J_k\right)^2.$$

Using this inequality we have:

$$m_1 \le \frac{2^7 h_2 c_5 c_{22}^2 \cdot (\mu_1 J_k)^2}{\varepsilon^3 c_3} \cdot Q^{\lambda-\frac{1}{2}} \sum_{|b_2| < Q^{\lambda-\frac{1}{2}}} |b_2| \le \frac{2^7 h_2 c_5 c_{22}^2 \cdot (\mu_1 J_k)^2}{\varepsilon^3 c_3} \cdot Q^{3\lambda-\frac{3}{2}} < \tfrac{1}{4}c_3^{-1}\mu_1 J_k \cdot Q^{\lambda} < \tfrac{t_k}{2}. \quad (81)$$

for $\lambda < \frac{3}{4}$ and $Q > Q_0$. By (81), it follows that the number of $\left(\frac{1}{2}, \frac{1}{2}\right)$-*ordinary* squares Π_j doesn't exceed

$$m_2 \ge t_k - \tfrac{1}{2}t_k > \tfrac{1}{2}t_k. \quad (82)$$

From Theorem 2 and the estimate (82) it now follows that in the case $\frac{1}{2} < \lambda < \frac{3}{4}$, the strip $L_J(Q, \lambda)$ contains at least $c_6 Q^{n+1-\lambda}$ algebraic points of degree at most n and height at most Q.

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