

# On algebraic integers in short intervals and near smooth curves

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## Abstract

In 1970 A. Baker and W. Schmidt introduced the concept of a regular system of numbers and vectors. They proved that the set of real algebraic numbers forms a regular system on any fixed interval. This fact has allowed to prove some important results in the metric theory of transcendental numbers. In this paper their approach is applied to the set of algebraic integers  $\alpha$  in short intervals of length depending on the height of  $\alpha$ .

## 1 Introduction

The solution of many problems in Diophantine approximation theory is connected with the question of the distribution of algebraic numbers and algebraic integers [15, 24]. In this paper we study the distribution of algebraic integers on the line and the distribution of the points with integer conjugate algebraic coordinates in the plane.

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Let  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ ,  $a_i \in \mathbb{Z}$  be a polynomial with integer coefficients of the degree  $\deg P = n$ . Given a polynomial  $P(x)$ , let  $H(P) = \max_{0 \leq j \leq n} |a_j|$  denotes the height of  $P(x)$ .

Consider an irreducible polynomial  $P(x)$  such that

$$\gcd(|a_n|, \dots, |a_0|) = 1.$$

The roots of this polynomial are the algebraic numbers  $\alpha$  of the degree  $n$  and of height  $H(\alpha) = H(P)$ . When  $a_n = 1$ , the roots of an irreducible polynomial  $P(x) = x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  are called algebraic integers  $\alpha$  of the degree  $n$  and of height  $H(\alpha) = H(P)$ . Denote by  $\#S$  the cardinality of a finite set  $S$  and by  $\mu D$  the Lebesgue measure of a measurable set  $D$ . We define the following class of polynomials

$$\mathcal{P}_n(Q) = \{P(x) \in \mathbb{Z}[x], \deg P \leq n, H(P) \leq Q\},$$

where  $Q > Q_0(n)$  is an integer. In the following denote by  $c_1 = c_1(n), c_2 = c_2(n) \dots$ , positive values which depend on  $n$  but don't depend on  $H(P)$  and  $Q$ .

During the last 20 years new insights about the distribution of algebraic numbers were obtained. Lower and upper estimates for a distance between conjugate algebraic numbers and for a distance between the roots of different integer polynomials were found in the papers [4, 14, 20, 11].

Consider an interval  $I \subset [-\frac{1}{2}; \frac{1}{2}]$  of length, say  $|I| = c_1 Q^{-1}$ . It would be interesting to know, whether an interval  $I$  of this length already contains algebraic numbers  $\alpha$  of the degree  $\deg \alpha \leq n$  and of height  $H(\alpha) \leq Q$ . In the case  $n = 3$  an answer was given in the paper of V. Bernik, N. Budarina and X. O'Donnell [10] in 2012 which has been extended for arbitrary  $n$  in the paper of V. Bernik and F. Götze [7]. Another result in [7] shows, that for any integer  $Q \geq 1$  there exists an interval  $I$  of length  $|I| = \frac{1}{2} Q^{-1}$ , which doesn't even contain algebraic numbers  $\alpha$  of any degree and height  $H(\alpha) \leq Q$ . On the other hand, for  $Q > Q_0(n)$  and for the sufficiently large value  $c_1$  any interval  $I$  of length  $|I| \geq c_1 Q^{-1}$  contains at least  $c_2 Q^{n+1} |I|$ ,  $c_2 > 0$  real algebraic numbers  $\alpha$  of the degree  $\deg \alpha \leq n$  and of the height  $H(\alpha) \leq Q$ . Furthermore, one can construct a regular system from this algebraic numbers [13].

In this paper we study the analogous question for the case of algebraic integers.

**Theorem 1.** *For any integer  $Q \geq 1$  there exists an interval  $I$  of length  $|I| = \frac{1}{2} Q^{-1}$ , which doesn't contain algebraic integers  $\alpha$  of height  $H(\alpha) \leq Q$  and arbitrary degree  $n \geq 2$ .*

It is easy to see, that Theorem 1 follows from the results [7], since algebraic integers are of course algebraic numbers.

**Theorem 2.** *For a sufficiently large value  $c_3 > 0$  and  $Q > Q_0(n)$  there exists a value  $c_4 > 0$  such that any interval  $I$  of length  $|I| \geq c_3 Q^{-1}$  contains at least  $c_4 Q^n |I|$  real algebraic integers  $\alpha$  of degree  $\deg \alpha = n$ ,  $n \geq 2$  and of height  $H(\alpha) \leq Q$ .*

Furthermore, we shall prove, that real algebraic integers of degree  $n$  form a regular system in a short intervals.

**Definition 1.** Let  $\Gamma$  be a countable set of real numbers and  $N : \Gamma \rightarrow \mathbb{R}$  be a positive function. The pair  $(\Gamma, N)$  is called a **regular system** if there exists a value  $c_5 = c_5(\Gamma, N) > 0$ , such that for any interval  $I \subset \mathbb{R}$  there exists a sufficiently large number  $T_0 = T_0(\Gamma, N, I) > 0$  such that for any integer  $T > T_0$  there are  $\gamma_1, \gamma_2, \dots, \gamma_t \in \Gamma \cap I$  such that

- 1)  $N(\gamma_i) \leq T, \quad 1 \leq i \leq t,$
- 2)  $|\gamma_i - \gamma_j| > T^{-1}, \quad 1 \leq i < j \leq t,$
- 3)  $t > c_5 |I| T.$

Obviously the set of rational numbers  $p/q$  together with function  $N(p/q) := q^2$  is a regular system. Similarly, real algebraic numbers  $\alpha$  of degree  $n$  form a regular system with respect to  $N(\alpha) = H(\alpha)^{n+1}$ , see e.g. [2, 1].

Regular systems of algebraic numbers are used for obtaining lower bounds for Hausdorff dimension of the set of real algebraic numbers with a given measure of transcendence [1] and of the set of real numbers approximating by zeros of integer combinations of non-degenerate functions with given order [18], as well as for studying of Khinchin-type theorems in case of divergence [2, 6, 9].

Analogous questions of the distribution of points with algebraic conjugate coordinates in the plane were considered by V.Bernik, F.Götze and O.Kukso in the paper [8]. Consider the rectangle  $E = I_1 \times I_2 \subset [-\frac{1}{2}; \frac{1}{2}]^2$ , where  $I_1, I_2$  are the intervals of length  $|I_1| = Q^{-\kappa_1}, |I_2| = Q^{-\kappa_2}$  such that  $0 < \kappa_1, \kappa_2 < \frac{1}{2}$ . Furthermore, also demand that  $E \cap \{|x - y| \leq \epsilon\} = \emptyset$ , where  $\epsilon > 0$  is sufficiently small. This specific choice of a rectangle will simplify our calculations. The point  $(\alpha, \beta)$  is called an *algebraic point* if  $\alpha$  and  $\beta$  are algebraic conjugate numbers and an *algebraic integer point* if  $\alpha$  and  $\beta$  are algebraic conjugate integers. In the paper [8] it is shown, that for  $Q > Q_0(n)$  any rectangle  $E$  of size  $\mu E = |I_1| \cdot |I_2| = Q^{-\kappa_1 - \kappa_2}, 0 < \kappa_1, \kappa_2 < \frac{1}{2}$  contains at

least  $c_6 Q^{n+1} \mu E$ ,  $c_6 > 0$  algebraic points  $(\alpha, \beta)$  of degree  $\deg \alpha = \deg \beta \leq n$ ,  $n \geq 2$  and of height  $H(\alpha) = H(\beta) \leq Q$ .

We will prove that the same estimate holds for algebraic integer points.

**Theorem 3.** *For sufficiently large  $Q > Q_0(n)$  there exists a value  $c_7 > 0$  such that any rectangle  $E = I_1 \times I_2$  of size  $\mu E = |I_1| \cdot |I_2| = Q^{-\kappa_1 - \kappa_2}$ ,  $0 < \kappa_1, \kappa_2 < \frac{1}{2}$  contains at least  $c_7 Q^n \mu E$  algebraic integer points  $(\alpha, \beta)$  of degree  $\deg \alpha = \deg \beta = n$ ,  $n \geq 4$  and of height  $H(\alpha) = H(\beta) \leq Q$ .*

Interesting questions arise in the study of the distribution of algebraic points near smooth curves [22]. Recently new results in estimating of the quantity of rational points near smooth curves were obtained. V. Beresnevich, D. Dickinson and S. Velani [3] and R. Vaughan, S. Velani [25] obtained lower and upper estimates of the same order were. Recently, V. Bernik, F. Götze and O. Kukso [8] proved lower estimates for the quantity of algebraic points of arbitrary degree near smooth curves.

In this paper we shall derive lower estimates for the number of algebraic integer points of arbitrary degree near smooth curves.

**Theorem 4.** *Let  $f(x)$  be a continuous function on the interval  $J = [a, b]$  and let*

$$L(Q, \lambda) = \{(x, y) : x \in J, |y - f(x)| < Q^{-\lambda}\}, \quad 0 < \lambda < \frac{1}{2}.$$

*Then for  $Q > Q_0(n, J, f)$  there are at least  $c_8(n, J, f)Q^{n-\lambda}$ ,  $c_8(n, J, f) > 0$  algebraic integer points  $(\alpha, \beta)$  of degree  $\deg \alpha = \deg \beta = n$ ,  $n \geq 4$  and of height  $H(\alpha) = H(\beta) \leq Q$  such that  $(\alpha, \beta) \in L(Q, \lambda)$ .*

*Proof.* Consider the graph of the function  $f(x)$  and the strip  $L(Q, \lambda)$  for fixed  $0 < \lambda < \frac{1}{2}$ . Divide the argument interval  $J$  into sub-intervals  $J_i = [x_{i-1}, x_i]$  of length  $|J_i| = Q^{-\lambda}$ ,  $i = \overline{1, t}$  where  $t > (a - b)Q^\lambda - 1 > c_9(J)Q^\lambda$  for  $Q > Q_0(n, J, f)$ . Let  $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$  denote their midpoints. Consider the rectangles

$$E_i = \{(x, y) : |x - \bar{x}_i| \leq c_{10}(n, f)Q^{-\lambda}; |y - f(\bar{x}_i)| \leq c_{11}(n, f)Q^{-\lambda}\},$$

where  $c_{10}(n, f)$  and  $c_{11}(n, f)$  are chosen such that rectangles  $E_i$  lie completely inside the domains  $T_i = \{(x, y) : |x - \bar{x}_i| \leq \frac{1}{2}Q^{-\lambda}; |y - f(x)| \leq Q^{-\lambda}\}$ ,  $i = \overline{1, t}$ .

It follows from Theorem 3 that every rectangle  $E_i$ ,  $i = \overline{1, t}$  contains at least  $c_{12}(n, f)Q^{n-2\lambda}$  algebraic integer points of degree  $n$  and height at most  $Q$ . Hence, as  $t > c_9(J)Q^\lambda$  than there are at least  $c_8(n, J, f)Q^{n-\lambda}$  algebraic integer points  $(\alpha, \beta) \in L(Q, \lambda)$ .  $\square$

## 2 Auxiliary statements

This section contains several lemmas which will be used in the proof of Theorems 2–3. Among other results we shall use some facts of the geometry of numbers, see [16]. The first work on approximation by algebraic integers is due to Davenport and Schmidt [17]. Recently their approach has been reworked by Y. Bugeaud [12] and we shall use ideas of this paper.

**Lemma 1** (Minkowski 2nd theorem on successive minima). *Let  $K$  be a bounded central symmetric convex body in  $\mathbb{R}^n$  with successive minima  $\tau_1, \dots, \tau_n$ . Then*

$$\frac{2^n}{n!} \leq \tau_1 \tau_2 \dots \tau_n V(K) \leq 2^n.$$

For a proof, see [16, pp. 203], [21, pp. 59].

**Lemma 2** (Bertrand postulate). *For any integer  $n \geq 2$  there exists a prime  $p$  such that  $n < p < 2n$ .*

Proved by P. Chebyshev in 1850, a proof can be found, for example, in [23, Theorem 2.4].

**Lemma 3** (Eisenstein criterion). *Let  $P(x)$  denote a polynomial with integer coefficients,*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

*If there exists a prime number  $p$  such that:*

$$(1) \quad \begin{cases} a_n \not\equiv 0 \pmod{p}, \\ a_i \equiv 0 \pmod{p}, \quad i = 0, \dots, n-1 \\ a_0 \not\equiv 0 \pmod{p^2}, \end{cases}$$

*then  $P(x)$  is irreducible over the rational numbers.*

For a proof see [19].

**Lemma 4.** *Consider a point  $x_1 \in \mathbb{R}$  and polynomial  $P(x)$  with zeros  $\alpha_1, \alpha_2, \dots, \alpha_n$  where  $|x_1 - \alpha_1| = \min_i |x_1 - \alpha_i|$ . Then*

$$|x_1 - \alpha_1| \leq n \frac{|P(x_1)|}{|P'(x_1)|}.$$

*Proof.* Consider the polynomial  $P(x)$  and its derivative  $P'(x)$  at point  $x_1$ . Since

$$\frac{|P'(x_1)|}{|P(x_1)|} \leq \sum_{i=1}^n \frac{1}{|x_1 - \alpha_i|} \leq \frac{n}{|x_1 - \alpha_1|},$$

then

$$|x_1 - \alpha_1| \leq n \frac{|P(x_1)|}{|P'(x_1)|}.$$

□

**Lemma 5** (see [7]). *Let  $\mathcal{L}_n = \mathcal{L}_n(Q, \delta_0, I)$  be the set of  $x \in I$ , such that the system*

$$\begin{cases} |P(x)| < Q^{-n}, \\ |P'(x)| < \delta_0 Q, \end{cases}$$

*has a solution in polynomials  $P(x) \in \mathcal{P}_n(Q)$ . For any sufficiently small value  $\delta_0 = \delta_0(n)$  and sufficiently large value  $c_{13}$  we have*

$$\mu \mathcal{L}_n < \frac{1}{4} |I|$$

*for all intervals  $I$  such that*

$$|I| > c_{13} Q^{-1}.$$

In [7] it is shown that it suffices to take  $\delta_0(n) = 2^{-n-8} n^{-2}$ .

**Lemma 6** (see [8]). *Let  $\mathcal{M}_n = \mathcal{M}_n(Q, \delta_0, E)$  be the set of points  $(x, y) \in E$ , such that the system*

$$\begin{cases} |P(x)| < Q^{-v_1}, \quad |P(y)| < Q^{-v_2} \\ \min \{|P'(x)|, |P'(y)|\} < \delta_0 Q, \\ v_1 + v_2 = n - 1, \end{cases}$$

*has a solution in polynomials  $P(x) \in \mathcal{P}_n(Q)$ . Then for  $\delta_0 = \delta_0(n) < 2^{-n-40} n^{-4}$  we have*

$$\mu \mathcal{M}_n < \frac{1}{4} \mu E$$

*for all rectangles  $E$  such that*

$$\mu E = Q^{-\kappa_1 - \kappa_2}, \quad 0 < \kappa_1, \kappa_2 < \frac{1}{2}.$$

### 3 Proof of Theorem 2

From lemma 5 it follows that the measure of the set of  $x \in I$  such that the system

$$(2) \quad \begin{cases} |P(x)| < Q^{-n+1}, \\ |P'(x)| < \delta_0 Q, \end{cases}$$

has a solution in polynomials  $P \in \mathcal{P}_{n-1}(Q)$  can be estimated as

$$\mu L_{n-1} \leq \frac{1}{4}|I|,$$

where  $|I| > c_{13}Q^{-1}$  for  $Q > Q_0(n)$  and  $\delta_0 = 2^{-n-7}(n-1)^{-2}$ .

Consider the set  $B_1 = I \setminus L_{n-1}$ . Since the first inequality of the system (2) is solvable in polynomials  $P \in \mathcal{P}_{n-1}(Q)$  for any  $x \in I$ , then for any  $x_0 \in B_1$  the system of inequalities

$$\begin{cases} |P(x_0)| < Q^{-n+1}, \\ |P'(x_0)| \geq \delta_0 Q, \end{cases}$$

holds for any polynomial  $P \in \mathcal{P}_{n-1}(Q)$  and  $\mu B_1 \geq \frac{3}{4}|I|$ .

Consider an arbitrary point  $x_0 \in B_1$  for  $Q > Q_0(n)$  and examine the successive minima  $\tau_1, \dots, \tau_n$  of the compact convex defined by the inequalities

$$(3) \quad \begin{cases} |a_{n-1}x_0^{n-1} + \dots + a_1x_0 + a_0| \leq Q^{-n+1}, \\ |(n-1)a_{n-1}x_0^{n-2} + \dots + 2a_2x_0 + a_1| \leq Q, \\ |a_{n-1}|, \dots, |a_2| \leq Q. \end{cases}$$

Let  $\tau_1 \leq \delta_0$ . Then there exists a polynomial  $P_0 \in \mathcal{P}_{n-1}(Q)$  such that the inequalities

$$\begin{cases} |P_0(x_0)| \leq \delta_0 Q^{-n+1} < Q^{-n+1}, \\ |P_0'(x_0)| \leq \delta_0 Q, \\ H(P_0) \leq \delta_0 Q < Q, \end{cases}$$

hold. This leads to a contradiction, since  $x_0 \notin L_{n-1}$ . Then  $\tau_1 > \delta_0$ . Since the volume of the compact convex set defined by the inequalities (3) is not less than  $2^n$ , it follows from Lemma 1 that  $\tau_1 \dots \tau_n \leq 1$  and  $\tau_{n-1} \leq \delta_0^{-n+1}$ . Thus we can choose  $n$  linearly independent polynomials  $P_i(x) \in \mathcal{P}_{n-1}(Q)$ ,  $i = \overline{1, n}$  such that the system

$$(4) \quad \begin{cases} |P_i(x_0)| \leq \delta_0^{-n+1} Q^{-n+1}, \\ |P_i'(x_0)| \leq \delta_0^{-n+1} Q, \\ H(P_i) \leq \delta_0^{-n+1} Q. \end{cases}$$

holds. Using well-known estimates from the geometry of numbers, see [16, pp. 219], we obtain for the polynomials  $P_i(x)$ ,

$$\Delta = \det |(a_{i,j-1})_{i,j=1}^n| \leq n!,$$

where  $P_i(x) = a_{i,n-1}x^{n-1} + \dots + a_{i,1}x + a_{i,0}$ ,  $i = \overline{1, n}$ . It follows from Lemma 2 that there exists a prime  $p$  not dividing  $\Delta$  with

$$(5) \quad n! < p < 2n!.$$

Consider the following system of linear equations in  $n$  variables  $\theta_1, \dots, \theta_n$

$$(6) \quad \begin{cases} x_0^n + p \cdot \sum_{i=1}^n \theta_i P_i(x_0) = p(n+1) \cdot \delta_0^{-n+1} Q^{-n+1}, \\ nx_0^{n-1} + p \cdot \sum_{i=1}^n \theta_i P'_i(x_0) = pQ + p \cdot \sum_{i=1}^n |P'_i(x_0)|, \\ \sum_{i=1}^n \theta_i a_{i,j} = 0, \quad j = 2, \dots, n-1. \end{cases}$$

Since the polynomials  $P_i(x), i = \overline{1, n}$  are linearly independent and the determinant of the system is equal to  $\Delta$ , the system (6) has a unique solution  $(\theta_1, \dots, \theta_n)$ . We choose  $n$  integers  $t_1, \dots, t_n$  such that

$$(7) \quad |\theta_i - t_i| \leq 1, \quad i = 1, \dots, n$$

Consider the following polynomial of degree  $n$  with integer coefficients

$$P(x) = x^n + p \cdot \sum_{i=1}^n t_i P_i(x) = x^n + p \cdot (a_{n-1}x^{n-1} + \dots + a_1x + a_0),$$

where  $a_j = \sum_{i=1}^n t_i a_{i,j}$ ,  $j = 0, \dots, n-1$ . In order to show that the polynomial  $P(x)$  is irreducible we check the conditions of Lemma 3. Obviously the first and the second condition of (1) hold. It remains to show that  $a_0 = t_1 a_{1,0} + \dots + t_n a_{n,0}$  is not divisible by  $p$ . Since  $p$  doesn't divide  $\Delta$ , there exists number  $1 \leq i \leq n$  such that  $a_{i,0}$  is not divisible by  $p$ . Consider two possible choices for  $t_i$ , which we denote by  $t_i^1, t_i^2 = t_i^1 + 1$ . In this case either  $a_0^1 = t_1 a_{1,0} + \dots + a_{i,0} t_i^1 + \dots + a_{n,0} t_n$  or  $a_0^2 = t_1 a_{1,0} + \dots + a_{i,0} t_i^2 + \dots + a_{n,0} t_n$  is not divisible by  $p$ . Hence lemma  $P(x)$  is irreducible.

In the following we shall estimate  $|P(x_0)|, |P'(x_0)|$  and  $H(P)$ .

From the first equation of the system (6) and inequalities (4) and (7) it follows that

$$(8) \quad p\delta_0^{-n+1}Q^{-n+1} \leq |P(x_0)| \leq p(2n+1)\delta_0^{-n+1}Q^{-n+1}.$$

Similarly by the second equation of the system (6) and inequalities (4) and (7) we see that

$$(9) \quad pQ \leq |P'(x_0)| \leq (p + 2pn\delta_0^{-n+1})Q.$$

In view of the other equations of the system (6) and inequalities (4) and (7) we have

$$(10) \quad |a_j| \leq n\delta_0^{-n+1}Q, \quad j = \overline{2, n-1}.$$

Finally we need to estimate  $|a_0|$  and  $|a_1|$ . Here we use the estimates (8)-(10) and inequality  $|x_0| \leq \frac{1}{2}$ . The result is

$$(11) \quad |a_1| \leq |P'(x_0)| + \sum_{j=2}^n |a_j| \leq (p + 2pn\delta_0^{-n+1} + n^2\delta_0^{-n+1})Q,$$

$$(12) \quad |a_0| \leq |P(x_0)| + |a_1| + \sum_{j=2}^n |a_j| \leq (p + p(4n + 1)\delta_0^{-n+1} + n^2\delta_0^{-n+1})Q.$$

From the estimates (10)-(12) and inequality (5) we conclude, that

$$(13) \quad H(P) \leq 20(n + 1)!\delta_0^{-n+1}Q.$$

Consider the roots  $\alpha_1, \dots, \alpha_n$  of the polynomial  $P(x)$ , where  $|x_0 - \alpha_1| = \min_i |x_0 - \alpha_i|$ . In view of Lemma 4 the following estimate holds

$$|x_0 - \alpha_1| \leq n|P(x_0)||P'(x_0)|^{-1}.$$

By inequalities (8) and (9) we have

$$(14) \quad |x_0 - \alpha_1| \leq n(2n + 1)\delta_0^{-n+1}Q^{-n} = c_{14}Q^{-n},$$

where  $c_{14} = n(2n + 1)\delta_0^{-n+1}$ .

If  $\alpha_1$  is a complex root, then its conjugate is a root of  $P(x)$ . Hence, by (13), (14) and estimates  $|\alpha_i| \leq H(P) + 1$ ,  $i = \overline{3, n}$  we conclude that

$$|P(x_0)| = \prod_{i=1}^n |x_0 - \alpha_i| \leq c_{14}^2 Q^{-2n} \cdot (2 + 20(n + 1)!\delta_0^{-n+1}Q)^{n-2}.$$

This inequality contradict (8) for  $Q > Q_0(n)$ . Thus,  $\alpha_1$  is real.

Finally we shall construct a regular system of real algebraic integers. We choose a maximal system of real algebraic integers  $\Gamma = \beta_1, \dots, \beta_t$  such that  $|\beta_i - \beta_j| > c_{14}Q^{-n}$ ,  $i, j = \overline{1, t}$ ,  $i \neq j$ . There is a real algebraic integer  $\alpha_1 \in I$  for any point  $x_0 \in B_1$  such that  $|x_0 - \alpha_1| \leq c_{14}Q^{-n}$ . If  $\alpha_1 \notin \Gamma$ , then there exists  $\beta_i \in \Gamma$  such that

$$|\alpha_1 - \beta_i| \leq c_{14}Q^{-n}$$

and

$$|x_0 - \beta_i| \leq 2c_{14}Q^{-n}.$$

Hence

$$B_1 \subset \bigcup_{i=1}^t \{x_0 \in B_1 : |x_0 - \beta_i| \leq 2c_{14}Q^{-n}\}$$

and

$$4c_{14}Q^{-n} \cdot t > \frac{3}{4}|I|.$$

Thus, we have

$$t > \frac{3}{16}c_{14}^{-1}Q^n|I| = c_4Q^n|I|$$

and Theorem 2 is proved.

It follows from the proof of Theorem 2, that the set of algebraic integers of degree less or equal  $n$  forms a regular system with respect to the function  $N(\alpha) = H(\alpha)^n$  and  $T_0 = c_{15}|I|^{-n}$ .

## 4 Proof of Theorem 3

The proof of Theorem 3 is based on the same method as the proof of Theorem 2 but it contains some non-trivial moments that require attention.

In order to prove the Theorem 3 we use Lemma 6. Consider the system

$$(15) \quad \begin{cases} |P(x)| < Q^{-v_1}, & |P(y)| < Q^{-v_2}, \\ \min \{|P'(x)|, |P'(y)|\} < \delta_0 Q, \\ v_1 + v_2 = n - 2. \end{cases}$$

Lemma 6 implies, that the measure of the set of points  $(x, y) \in E$  such that the system (15) has a solution in polynomials  $P \in \mathcal{P}_{n-1}(Q)$  can be estimated as

$$\mu M_{n-1} \leq \frac{1}{4}\mu E$$

for  $Q > Q_0(n)$  and  $\delta_0 = 2^{-n-39}(n-1)^{-4}$ .

Thus for any point  $(x, y) \in K_1 = E \setminus M_{n-1}$  the system

$$\begin{cases} |P(x)| < Q^{-v_1}, & |P(y)| < Q^{-v_2}, \\ |P'(x)| \geq \delta_0 Q, & |P'(y)| \geq \delta_0 Q, \\ v_1 + v_2 = n - 2. \end{cases}$$

holds for any polynomial  $P \in \mathcal{P}_{n-1}(Q)$  and  $\mu K_1 \geq \frac{3}{4}\mu E$ .

Consider an arbitrary point  $(x_0, y_0) \in K_1$  and examine the successive minima  $\tau_1, \dots, \tau_n$  of the compact convex set defined by

$$\begin{cases} |a_{n-1}x_0^{n-1} + \dots + a_1x_0 + a_0| \leq Q^{-v_1}, \\ |a_{n-1}y_0^{n-1} + \dots + a_1y_0 + a_0| \leq Q^{-v_2}, \\ |(n-1)a_{n-1}x_0^{n-2} + \dots + 2a_2x_0 + a_1| \leq Q, \\ |(n-1)a_{n-1}y_0^{n-2} + \dots + 2a_2y_0 + a_1| \leq Q, \\ |a_{n-1}|, \dots, |a_2| \leq Q. \end{cases}$$

Assume  $\tau_1 \leq \delta_0$ . Then there exists a polynomial  $P_0 \in \mathcal{P}_{n-1}(Q)$  such that the inequalities

$$\begin{cases} |P_0(x_0)| < \delta_0 Q^{-v_1} < Q^{-v_1}, & |P_0(y_0)| < \delta_0 Q^{-v_2} < Q^{-v_2}, \\ |P'_0(x_0)| < \delta_0 Q, & |P'_0(y_0)| < \delta_0 Q, \\ H(P_0) \leq \delta_0 Q < Q, \\ v_1 + v_2 = n - 2. \end{cases}$$

hold, contradicting to  $(x_0, y_0) \in K_1$ . Thus  $\tau_1 > \delta_0$  and by  $\tau_1 \dots \tau_n \leq 1$  and by Lemma 1 we have  $\tau_{n-1} \leq \delta_0^{-n+1}$ . Hence, there exist  $n$  linearly independent polynomials  $P_i(x) \in \mathcal{P}_{n-1}(Q)$ ,  $i = \overline{1, n}$  with integer coefficients, satisfying

$$(16) \quad \begin{cases} |P_i(x_0)| \leq \delta_0^{-n+1} Q^{-v_1}, & |P_i(y_0)| \leq \delta_0^{-n+1} Q^{-v_2}, \\ |P'_i(x_0)| \leq \delta_0^{-n+1} Q, & |P'_i(y_0)| \leq \delta_0^{-n+1} Q, \\ H(P_i) \leq \delta_0^{-n+1} Q. \end{cases}$$

Well-known bounds from the geometry of numbers, see [16, pp. 219], yield the following bound for the polynomials  $P_i(x)$

$$\Delta = \det |(a_{i,j-1})_{i,j=1}^n| \leq n!,$$

where  $P_i(x) = a_{i,n-1}x^{n-1} + \dots + a_{i,1}x + a_{i,0}$ ,  $i = \overline{1, n}$ . For a prime  $p$  not dividing  $\Delta$ , Lemma 2 yields

$$(17) \quad p < 2n!.$$

Consider the system of linear equations for the  $n$  variables  $\theta_1, \dots, \theta_n$

$$(18) \quad \begin{cases} x_0^n + p \cdot \sum_{i=1}^n \theta_i P_i(x_0) = p(n+1) \cdot \delta_0^{-n+1} Q^{-v_1}, \\ y_0^n + p \cdot \sum_{i=1}^n \theta_i P_i(y_0) = p(n+1) \cdot \delta_0^{-n+1} Q^{-v_2}, \\ nx_0^{n-1} + p \cdot \sum_{i=1}^n \theta_i P'_i(x_0) = pQ + p \cdot \sum_{i=1}^n |P'_i(x_0)|, \\ ny_0^{n-1} + p \cdot \sum_{i=1}^n \theta_i P'_i(y_0) = pQ + p \cdot \sum_{i=1}^n |P'_i(y_0)|, \\ \sum_{i=1}^n \theta_i a_{i,j} = 0, \quad j = 4, \dots, n-1. \end{cases}$$

In order to determine the determinant of this system, we transform it as follows. We multiply the equation with number  $k = 5, 6, \dots, n$  by  $p \cdot x_0^{k-1}$  ( resp. by  $p \cdot y_0^{k-1}$ ) and subtract it from the first ( respectively the second) equation of the system (18). Similarly we multiply the equation with number  $k = 5, 6, \dots, n$  by  $p \cdot (k-1)x_0^{k-2}$  (respectively by  $p \cdot (k-1)y_0^{k-2}$ ) and

subtract it from the third (the respectively the fourth) equation. After these transformations the determinant of system (18) is given by

$$\hat{\Delta}(x_0, y_0) = p^4 \cdot \begin{vmatrix} \sum_{i=0}^3 a_{1,i}x_0^i & \dots & \sum_{i=0}^3 a_{n,i}x_0^i \\ \sum_{i=0}^3 a_{1,i}y_0^i & \dots & \sum_{i=0}^3 a_{n,i}y_0^i \\ \sum_{i=1}^3 i \cdot a_{1,i}x_0^{i-1} & \dots & \sum_{i=1}^3 i \cdot a_{n,i}x_0^{i-1} \\ \sum_{i=1}^3 i \cdot a_{1,i}y_0^{i-1} & \dots & \sum_{i=1}^3 i \cdot a_{n,i}y_0^{i-1} \\ a_{1,4} & \dots & a_{n,4} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix}$$

By simple transformations of the first four rows of the matrix we have

$$(19) \quad \hat{\Delta}(x_0, y_0) = 9p^4(y_0 - x_0)^4 \cdot \begin{vmatrix} a_{1,0} & \dots & a_{n,0} \\ \vdots & \ddots & \vdots \\ a_{1,n-1} & \dots & a_{n,n-1} \end{vmatrix} = 9p^4(y_0 - x_0)^4 \Delta > 0,$$

since the polynomials  $P_i(x)$ ,  $i = \overline{1, n}$  are linearly independent and  $|y_0 - x_0| > \epsilon > 0$ . By (19) the system (18) has a unique solution  $(\theta_1, \dots, \theta_n)$ . We take  $n$  integers  $t_1, \dots, t_n$  such that

$$(20) \quad |\theta_i - t_i| \leq 1, \quad i = 1, \dots, n$$

and construct the following polynomial with integer coefficients

$$P(x) = x^n + p \cdot \sum_{i=1}^n t_i P_i(x) = x^n + p \cdot (a_{n-1}x^{n-1} + \dots + a_1x + a_0),$$

where  $a_j = \sum_{i=1}^n t_i a_{i,j}$ ,  $j = 0, \dots, n-1$ . By Lemma 3 the polynomial  $P(x)$  is irreducible.

We finally derive bounds for  $|P(x_0)|$ ,  $|P(y_0)|$ ,  $|P'(x_0)|$  and  $|P'(y_0)|$ .

We use the inequalities (16), (18) and the inequality (20) and obtain the following estimates:

$$(21) \quad p\delta_0^{-n+1}Q^{-v_1} \leq |P(x_0)| \leq p(2n+1)\delta_0^{-n+1}Q^{-v_1}.$$

$$(22) \quad p\delta_0^{-n+1}Q^{-v_2} \leq |P(y_0)| \leq p(2n+1)\delta_0^{-n+1}Q^{-v_2}.$$

$$(23) \quad pQ \leq |P'(x_0)| \leq (p + 2pn\delta_0^{-n+1})Q.$$

$$(24) \quad pQ \leq |P'(y_0)| \leq (p + 2pn\delta_0^{-n+1})Q.$$

We now estimate the height  $H(P)$ . By equation 4 to  $n$  of the system (18) and inequalities (16) and (20) we have

$$(25) \quad |a_j| \leq n\delta_0^{-n+1}Q, \quad j = \overline{4, n-1}.$$

It remains to estimate  $|a_0|$ ,  $|a_1|$ ,  $|a_2|$  and  $|a_3|$ . By (21) – (25) and inequalities  $|x_0| \leq \frac{1}{2}$ ,  $|y_0| \leq \frac{1}{2}$ , we have

$$(26) \quad \begin{cases} |a_3x_0^3 + a_2x_0^2 + a_1x_0 + a_0| \leq |P(x_0)| + \sum_{j=4}^n |a_j| < 2pn\delta_0^{-n+1}Q, \\ |a_3y_0^3 + a_2y_0^2 + a_1y_0 + a_0| \leq |P(y_0)| + \sum_{j=4}^n |a_j| < 2pn\delta_0^{-n+1}Q, \\ |3a_3x_0^2 + 2a_2x_0 + a_1| \leq |P'(x_0)| + \sum_{j=4}^n j \cdot |a_j| < 2pn^3\delta_0^{-n+1}Q, \\ |3a_3y_0^2 + 2a_2y_0 + a_1| \leq |P'(y_0)| + \sum_{j=4}^n j \cdot |a_j| < 2pn^3\delta_0^{-n+1}Q. \end{cases}$$

Consider the system of linear equations for  $a_0$ ,  $a_1$ ,  $a_2$  and  $a_3$

$$(27) \quad \begin{cases} a_3x_0^3 + a_2x_0^2 + a_1x_0 + a_0 = l_1, \\ a_3y_0^3 + a_2y_0^2 + a_1y_0 + a_0 = l_2, \\ 3a_3x_0^2 + 2a_2x_0 + a_1 = l_3, \\ 3a_3y_0^2 + 2a_2y_0 + a_1 = l_4. \end{cases}$$

Since the determinant of the system (27) does not vanish, there exists a unique solution. We solve the system (27) subject to the estimates (26) and inequalities  $|x_0| \leq \frac{1}{2}$ ,  $|y_0| \leq \frac{1}{2}$ . Thus, we have

$$|a_i| < 10^4pn^3\delta_0^{-n+1}Q, \quad i = 0, 1, 2, 3.$$

Hence, by (17) and (25), we obtain

$$H(P) < 2 \cdot 10^4(n+4)!\delta_0^{-n+1}Q.$$

Consider the roots  $\alpha_1, \dots, \alpha_n$  of the polynomial  $P(x)$ , where  $|x_0 - \alpha_1| = \min_i |x_0 - \alpha_i|$  and consider a permutation  $\beta_1, \dots, \beta_n$  of these roots such that  $|y_0 - \beta_1| = \min_i |y_0 - \beta_i|$ . By Lemma 4 the following estimates hold

$$\begin{cases} |x_0 - \alpha_1| \leq n|P(x_0)||P'(x_0)|^{-1}, \\ |y_0 - \beta_1| \leq n|P(y_0)||P'(y_0)|^{-1}. \end{cases}$$

By (21) – (24) we have

$$\begin{cases} |x_0 - \alpha_1| < n(2n+1)\delta_0^{-n+1}Q^{-v_1-1} < c_{16}Q^{-v_1-1}, \\ |y_0 - \beta_1| < n(2n+1)\delta_0^{-n+1}Q^{-v_2-1} < c_{16}Q^{-v_2-1}, \end{cases}$$

where  $c_{16} = n(2n + 1)\delta_0^{-n+1}$ . For  $Q > Q_0(n)$  the roots  $\alpha_1$  and  $\beta_1$  are real.

We choose a maximal set of real algebraic integer points  $\Gamma = (\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t)$  such that

$$|\alpha_i - \alpha_j| > c_{16}Q^{-v_1-1},$$

or

$$|\beta_i - \beta_j| > c_{16}Q^{-v_2-1},$$

$$i, j = \overline{1, t}, i \neq j.$$

For any point  $(x_0, y_0) \in K_1$  there exists  $(\alpha_i, \beta_i) \in \Gamma$  such that

$$\begin{cases} |x_0 - \alpha_i| < 2c_{16}Q^{-v_1-1}, \\ |y_0 - \beta_i| < 2c_{16}Q^{-v_2-1}. \end{cases}$$

Hence

$$K_1 \subset \bigcup_{i=1}^t \{(x_0, y_0) \in K_1 : |x_0 - \alpha_i| < 2c_{16}Q^{-v_1-1}, |y_0 - \beta_i| < 2c_{16}Q^{-v_2-1}\}$$

and

$$t > \frac{3}{64}c_{16}^{-2}Q^n \mu E = c_7 Q^n \mu E$$

and hence Theorem 3 is proved.

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