

# REGULARIZATION BY NOISE FOR STOCHASTIC HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We study regularizing effects of nonlinear stochastic perturbations for fully nonlinear PDE. More precisely, path-by-path  $L^\infty$  bounds for the second derivative of solutions to such PDE are shown. These bounds are expressed as solutions to reflected SDE and are shown to be optimal.

## 1. INTRODUCTION

The questions of regularizing effects and well-posedness by noise for (stochastic) partial differential equations have attracted much interest in recent years. The principle idea is that the inclusion of stochastic perturbations may lead to more regular solutions and in some cases even to uniqueness of solutions. Historically, possible regularizing effects of additive noise have been investigated, e.g. for (stochastic) reaction diffusion equations

$$dv = \Delta v dt + f(v) dt + dW_t$$

in [20] and for Navier-Stokes equations in [13, 14]. In [3, 10, 11], well-posedness and regularization by linear multiplicative noise for transport equations, that is for

$$dv = b(x)\nabla_x v dt + \nabla v \circ d\beta_t,$$

have been obtained. We refer to [12] for more details on the literature. Only very recently, regularizing effects of *non-linear* stochastic perturbations in the setting of (stochastic) scalar conservation laws have been discovered in [17]. In particular, in [17] it has been shown that quasi-solutions to

$$dv + \frac{1}{2}\partial_x v^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T} \tag{1.1}$$

where  $\mathbb{T}$  is the one-dimensional torus, enjoy fractional Sobolev regularity of the order

$$v(t) \in W^{\alpha,1}(\mathbb{T}) \quad \text{for all } \alpha < \frac{4}{5}, t > 0, \mathbb{P}\text{-a.s.} \tag{1.2}$$

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This is in contrast to the deterministic case, in which examples of quasi-solutions to

$$\partial_t v + \frac{1}{2} \partial_x v^2 = 0 \quad \text{on } \mathbb{T}$$

have been given in [8] such that, for all  $\alpha > \frac{1}{3}$ ,

$$v(t) \notin W^{\alpha,1}(\mathbb{T}) \quad \text{for all } t > 0.$$

In this sense, the stochastic perturbation introduced in (1.1) has a regularizing effect. In [17], the question of optimality of the estimate (1.2) remained open.

Subsequently, the results and techniques developed in [17] have been (partially) extended in [18] to a class of parabolic-hyperbolic SPDE, as a particular example including the SPDE

$$dv + \frac{1}{2} \partial_x v^2 \circ d\beta_t = \frac{1}{12} \partial_{xx} v^3 dt \quad \text{on } \mathbb{T}. \quad (1.3)$$

In [18], the regularity of solutions to (1.3) was analyzed. More precisely, it was shown that

$$v(t) \in W^{\alpha,1}(\mathbb{T}) \quad \text{for all } \alpha < \frac{2}{3}, \mathbb{P}\text{-a.s.} \quad (1.4)$$

However, neither optimality of these results nor regularization by noise could be observed in this case. That is, the regularity estimates for solutions to (1.3) proven in [18] did not exceed the known regularity for the solutions to the non-perturbed cases

$$\partial_t v + \frac{1}{2} \partial_x v^2 = \frac{1}{12} \partial_{xx} v^3 \quad \text{or} \quad \partial_t v = \frac{1}{12} \partial_{xx} v^3 \quad \text{on } \mathbb{T}. \quad (1.5)$$

In [17, 18] the estimation of the regularity of solutions to (1.1), (1.3) relied on properties of the law of Brownian motion. The question of the *pathwise* properties of  $\beta$  leading to regularization by noise could thus not be answered (cf. [6] for related questions in the case of linear transport equations).

The purpose of this paper is to provide sharp, pathwise regularity estimates to a class of SPDE, in particular including (1.1), (1.3) and to prove regularization by noise in this case. More precisely, sharp estimates are obtained for the  $L^\infty$  norm of the second derivative of solutions to SPDE of the type

$$du + \frac{1}{2} |Du|^2 \circ d\xi_t = F(x, u, Du, D^2u) dt \quad \text{on } \mathbb{R}^N, \quad (1.6)$$

for  $F$  satisfying appropriate assumptions detailed below and  $\xi$  being a continuous function.

Our proof is based on the regularizing effects of the semi-groups  $S_H$  and  $S_{-H}$  associated to the Hamiltonians  $H := p \mapsto \frac{1}{2} p^2$  and  $-H$ . It is well-known that  $S_H$  and  $S_{-H}$  allow to obtain one-sided bounds (of the opposite sign) on the second derivative (cf e.g. [27]), and the fact that one can combine these two bounds to

obtain  $C^{1,1}$  bounds goes back to Lasry and Lions [25]. Our main theorem is in a sense a generalization of their result.

Let us emphasize that while one-sided (i.e. semiconcavity or semiconvexity) bounds are typical for solutions of deterministic Hamilton-Jacobi-Bellman equations (cf. [5, 15]), two-sided (i.e.  $C^{1,1}$ ) bounds in general do not hold for degenerate parabolic equations<sup>1</sup>. The fact that we are able to obtain such two-sided bounds in our case depends crucially on the "stochastic" (or "rough") nature of the signal  $\xi$  in (1.6).

Before stating our theorem in detail let us first consider some concrete examples (cf. Section 3.2 below for details).

As a first example, as mentioned above, the results answer the question of optimal regularity and (pathwise) regularization by noise for (1.3). Indeed, let  $u$  be the unique viscosity solution to the SPDE

$$du + \frac{1}{2}(\partial_x u)^2 \circ d\beta_t = \frac{1}{12}\partial_x(\partial_x u)^3 dt, \quad \text{on } \mathbb{R}.$$

Then, informally,  $v = \partial_x u$  is a solution to (1.3). Our results (cf. Corollary 5.2 below) yield that if  $\beta = \sigma B$  where  $B$  is a standard Brownian motion, then

$$\sigma > 1 \Rightarrow v(t) \in W^{1,\infty} \quad \mathbb{P}\text{-a.s.},$$

whereas (at least for some choice of initial conditions)

$$\sigma \leq 1 \Rightarrow \mathbb{P}\text{-a.s.} \quad \exists T > 0, \forall t \geq T, v(t) \notin W^{1,\infty}.$$

More precisely, we obtain the sharp bound

$$\|\partial_x v(t)\|_{L^\infty} \leq \frac{1}{L^+(t) \wedge L^-(t)}, \quad (1.7)$$

where  $L^+$ ,  $L^-$  are the solutions to the reflected (at  $0^+$ ) SDE with dynamics on  $(0, \infty)$  given by

$$\begin{aligned} dL^+ &= -\frac{1}{2L^+(t)}dt + d\beta_t, & L^+(0) &= \frac{1}{\|(\partial_x v_0)_+\|_{L^\infty}} \\ dL^- &= -\frac{1}{2L^-(t)}dt - d\beta_t, & L^-(0) &= \frac{1}{\|(\partial_x v_0)_-\|_{L^\infty}}. \end{aligned}$$

This demonstrates that, when the noise coefficient is large enough, the stochastic perturbation in (1.3) has a regularizing effect as compared to the non-perturbed situation

$$\partial_t w = \frac{1}{12}\partial_{xx} w^3, \quad \text{on } \mathbb{R}$$

for which solutions are known to develop singularities in terms of a blow-up of  $\|\partial_x w\|_{L^\infty}$ . This dependence of a regularizing effect of noise on the strength of the noise  $\sigma$  seems to be observed here for the first time. Concerning the optimality of (1.7) and thus of the main result, we prove that for a certain class of initial

<sup>1</sup>See however the one-dimensional example in [22].

conditions (cf. Section 5 below) equality in (1.7) holds. The proof of optimality relies on a careful choice of approximations and on a monotonicity property with respect to the driving path  $\beta$ , which follows from results in [16].

As a second example, consider hyperbolic SPDE of the form

$$du + \frac{1}{2}|Du|^2 \circ d\beta_t^H = F(Du) dt, \quad \text{on } \mathbb{R}^N, \quad (1.8)$$

where  $\beta^H$  is a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$ . Typically, the solutions to the deterministic counterpart

$$\partial_t w + \frac{1}{2}|Dw|^2 = F(Dw) \quad \text{or} \quad \partial_t w = F(Dw) \quad \text{on } \mathbb{R}^N$$

develop singularities in terms of shocks of the derivative, that is,  $Dw$  will become discontinuous for large times, even if  $w_0$  is smooth. In contrast, our results yield that (cf. Example 3.5 below)

$$\mathbb{P}(\|D^2u(t, \cdot)\|_{L^\infty} < \infty) = 1 \quad \forall t > 0,$$

for  $u$  being a solution to (1.8).

Our results may also be applied to some cases where, unlike in the previous examples, the deterministic part of the equation has a regularizing effect. For example, consider the equation

$$\partial_t w = \frac{1}{2}(1 - (\partial_x w)^2)\partial_{xx} w, \quad \text{on } \mathbb{R}, \quad w(0, \cdot) = w^0$$

with initial condition  $w^0$  such that  $\|\partial_x w^0\|_{L^\infty} < 1$ . Since this is preserved by the equation, that is  $\|\partial_x w(t, \cdot)\|_{L^\infty} < 1$  for all  $t \geq 0$ , the deterministic part is uniformly elliptic. In particular, the solutions are smooth at positive times. Our result yields that this is still true for the solution  $u$  to

$$du + \frac{1}{2}(\partial_x u)^2 \circ d\xi_t = \frac{1}{2}(1 - (\partial_x u)^2)(\partial_{xx} u)dt, \quad \text{on } \mathbb{R}, \quad u(0, \cdot) = w^0$$

if the intensity of the noise is small enough. More precisely, if  $\xi \in C^\alpha$ ,  $\alpha > \frac{1}{2}$  or  $\xi = \sigma B$  with  $B$  a Brownian motion and  $\sigma < 1$ , then (almost surely in the latter case)

$$\forall t > 0, \quad \|\partial_{xx} u(t, \cdot)\|_{L^\infty} < \infty.$$

Again this follows from properties of SDE, namely that the solutions to

$$dL^\pm = \frac{1}{L^\pm} dt \pm d\beta(t)$$

do not hit 0 at positive times.

Finally, let us mention that our regularity results imply some estimates for large time behavior. For instance, if  $u$  is a solution to the stochastic Hamilton-Jacobi equation

$$du + \frac{1}{2}(\partial_x u)^2 \circ d\beta_t = 0, \quad u(0, \cdot) = u^0,$$

then for all  $t \geq 0$ , (cf. Proposition 3.8 below)

$$\|Du(t, \cdot)\|_{L^\infty} \leq \sqrt{\frac{2 \|u^0\|_{L^\infty}}{\max_{0 \leq s \leq t} \beta(s) - \inf_{0 \leq s \leq t} \beta(s)}}.$$

Note that when  $\beta$  is a Brownian motion, we get a rate of decay in  $t^{-1/4}$  which is the same rate as obtained in [17].

**1.1. Organization of the paper.** In Section 2 we give the precise statement of the assumptions and the main theorem. Subsequently, we provide sufficient conditions for these assumptions as well as a series of applications of the main result to specific SPDE in Section 3. The proof of the main result is given in Section 4 while the proof of optimality is given in Section 5. In the Appendix A we recall the employed well-posedness and stability results for stochastic viscosity solutions.

**1.2. Notation.** We let  $\mathbb{R}_+ := [0, \infty)$  and  $S^N$  be the set of all symmetric  $N \times N$  matrices. We further define  $C_0^k([0, T]; \mathbb{R}) := \{\xi \in C^k([0, T]; \mathbb{R}) : \xi(0) = 0\}$ ,  $\text{Lip}_{loc}(\mathbb{R}^N)$  to be the space of all locally Lipschitz continuous functions on  $\mathbb{R}^N$  and  $\text{Lip}_b(\mathbb{R}_+)$  to be the space of all bounded Lipschitz continuous functions on  $\mathbb{R}_+$ . For a càdlàg path  $\xi$  we set  $\xi_{s,t} := \xi_t - \xi_{s-}$ .

Given continuous functions  $F, H$  we let  $S_F(t), S_H(t)$  be the semigroups, in the sense of viscosity solutions and in case they exist, for the PDE

$$\partial_t u = F(t, x, u, Du, D^2u)$$

and

$$\partial_t u + H(Du) = 0,$$

respectively. For a locally Lipschitz continuous function  $V : (0, \infty) \rightarrow \mathbb{R}$  we define  $\varphi^V(t) : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}_+$  as the solution flow to the ODE  $\dot{\ell}(t) = V(\ell)$  (stopped when reaching the boundaries 0 or  $+\infty$ ). For notational convenience, we set  $H(p) := \frac{1}{2}|p|^2$  and  $S_H(-\delta) := S_{-H}(\delta)$  for  $\delta \geq 0$ .

A modulus of continuity is a nondecreasing, subadditive function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{r \rightarrow 0} \omega(r) = \omega(0) = 0$ . We define  $UC(\mathbb{R}^N)$  to be the space of all uniformly continuous functions, that is,  $u \in UC(\mathbb{R}^N)$  if  $|u(x) - u(y)| \leq \omega(|x - y|)$  for some modulus of continuity  $\omega$ . If, in addition,  $u$  is bounded, we say  $u \in BUC(\mathbb{R}^N)$ . Furthermore,  $USC(\mathbb{R}^N)$  (resp.  $LSC(\mathbb{R}^N)$ ) denotes the set of all upper- (resp. lower) semicontinuous functions in  $\mathbb{R}^N$ , and  $BUSC(\mathbb{R}^N)$  (resp.  $BLSC(\mathbb{R}^N)$ ) is the set of all bounded functions in  $USC(\mathbb{R}^N)$  (resp.  $LSC(\mathbb{R}^N)$ ).

We say that a function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  is semiconvex (resp. semiconcave) of order  $C$  if  $x \mapsto u(x) + \frac{1}{2}C|x|^2$  is convex (resp.  $x \mapsto u(x) - \frac{1}{2}C|x|^2$  is concave).

For  $a, b \in \mathbb{R}$  we set  $a \wedge b := \min(a, b)$ ,  $a \vee b := \max(a, b)$ ,  $a_+ := \max(a, 0)$  and  $a_- := \max(-a, 0)$ . We let  $K, \tilde{K}$  be generic constants that may change value from line to line.

## 2. MAIN RESULT

We consider rough PDE of the form

$$\begin{aligned} du + \frac{1}{2}|Du|^2 \circ d\xi(t) &= F(t, x, u, Du, D^2u)dt \\ u(0) &= u_0, \end{aligned} \tag{2.1}$$

where  $u_0 \in BUC(\mathbb{R}^N)$ ,  $\xi$  is a continuous path and  $F$  satisfies the typical assumptions from the theory of viscosity solutions, that is,

**Assumption 2.1.** (1) *Degenerate ellipticity:* For all  $X, Y \in S^N$ ,  $X \leq Y$  and all  $(t, x, r, p) \in [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ ,

$$F(t, x, r, p, X) \leq F(t, x, r, p, Y).$$

(2) *Lipschitz continuity in  $r$ :* There exists an  $L > 0$  such that

$$|F(t, x, r, p, X) - F(t, x, s, p, X)| \leq L|r - s| \quad \forall (t, x, s, r, p, X) \in [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times S^N.$$

(3) *Boundedness in  $(t, x)$ :*

$$\sup_{[0, T] \times \mathbb{R}^N} |F(\cdot, \cdot, 0, 0, 0)| < \infty.$$

(4) *Uniform continuity in  $(t, x)$ :* For any  $R > 0$ ,

$F$  is uniformly continuous on  $[0, T] \times \mathbb{R}^N \times [-R, R] \times B_R \times B_R$ .

(5) *Joint continuity in  $(X, p, x)$ :* For each  $R > 0$  there exists a modulus of continuity  $\omega_{F,R}$  such that, for all  $\alpha, \varepsilon > 0$  and uniformly in  $t \in [0, T]$ ,  $r \in [-R, R]$

$$F(t, x, r, \alpha p, X) - F(t, x, r, \alpha p, Y) \leq \omega_{F,R}(\alpha|p|^2 + |p| + \alpha^{-1}),$$

for all  $p \in \mathbb{R}^N$  and  $X, Y \in S^N$  such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

We refer to the Appendix A for an according well-posedness result for (2.1). We will make the following assumption on  $F$ :

**Assumption 2.2.** *There exists  $V_F : (0, \infty) \rightarrow \mathbb{R}$ , locally Lipschitz and bounded from above on  $[1, \infty)$  such that for all  $g \in BUC(\mathbb{R}^n)$ ,  $t \geq 0$ , one has*

$$D^2g \leq \ell^{-1}Id \Rightarrow D^2(S_F(t, g)) \leq \frac{Id}{\varphi^{V_F}(t)(\ell)},$$

*the inequalities being understood in distribution sense.*

The above assumption yields a control on the rate of loss of semiconcavity for  $S_F$ . Note that  $\varphi^{V_F}$  may take the value 0 and thus no preservation of semiconcavity is assumed.

**Theorem 2.3.** *Let  $u_0 \in BUC(\mathbb{R}^N)$ ,  $\xi \in C(\mathbb{R}_+)$ , suppose that Assumptions 2.1, 2.2 are satisfied and let  $u$  be the solution to*

$$\begin{cases} du + \frac{1}{2}|Du|^2 \circ d\xi(t) = F(t, x, u, Du, D^2u)dt, \\ u(0, \cdot) = u_0. \end{cases} \quad (2.2)$$

Suppose that  $D^2u_0 \leq \frac{Id}{\ell_0}$  for some  $\ell_0 \in [0, \infty)$ , in the sense of distributions. Then, for each  $t \geq 0$ ,

$$D^2u(t, \cdot) \leq \frac{Id}{L(t)}, \quad (2.3)$$

in the sense of distributions, where  $L$  is the maximal continuous solution on  $[0, \infty)$  to

$$\begin{aligned} dL(t) &= V_F(L(t))dt + d\xi(t) \text{ on } \{t \geq 0 : L(t) > 0\}, \quad L \geq 0, \\ L(0) &= \ell_0. \end{aligned} \quad (2.4)$$

The proof of Theorem 2.3 is given in Section 4 below.

### 3. EXAMPLES

In this section we present applications of our main Theorem 2.3 to certain classes of PDE. To do so, in particular, Assumption 2.2 has to be verified. We first provide a general result on the preservation of semiconvexity for fully nonlinear PDE in Section 3.1, which is then applied to several PDE in Section 3.2.

**3.1. Semiconvexity preservation.** In this section we provide sufficient conditions on  $F$  to satisfy Assumption 2.2. From [28] we recall

**Proposition 3.1.** *Let  $F = F(t, x, p, A) \in C([0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times S^N)$  be degenerate elliptic and such that, for all  $t \geq 0$ ,  $x, p \in \mathbb{R}^N$ ,  $\xi \neq 0 \in \mathbb{R}^N$ ,*

$$(y, A) \mapsto F(t, x + y, p, B) \text{ is convex on } (\mathbb{R}\xi)^\perp \times X_\xi, \quad (3.1)$$

where  $X_\xi = \{A \in S^N, A\xi = 0, A > 0 \text{ on } (\mathbb{R}\xi)^\perp\}$ ,  $B\xi = 0$ ,  $B = A^{-1}$  on  $(\mathbb{R}\xi)^\perp$ .

Let  $u$  be coercive in  $x$  i.e.

$$\lim_{|x| \rightarrow \infty} \inf_{t \in [0, T]} \frac{u(t, x)}{|x|} = +\infty$$

and a classical supersolution on  $[0, T] \times \mathbb{R}^N$  to

$$\partial_t u = F(t, x, Du, D^2u), \quad (3.2)$$

and let

$$u_{**}(t, x) := \inf \left\{ \sum_{i=1}^m \lambda_i u(t, x_i), \quad 0 \leq \lambda_i \leq 1, \sum_{i=1}^m \lambda_i = 1, \sum_{i=1}^m \lambda_i x_i = x \right\}$$

be the partial convex envelope of  $u$ . Then  $u_{**}$  is a viscosity supersolution to (3.2).

*Proof.* For the reader's convenience we provide a proof. First note that by continuity of  $F$ , it is straightforward to see that the assumption (3.1) is equivalent to the fact that for any subspace  $V \subset \mathbb{R}^n$  which is not reduced to  $\{0\}$ , the map

$$(y, A) \mapsto F(t, x + y, p, B) \text{ is convex on } V^\perp \times X_V, \quad (3.3)$$

where  $X_V = \{A \in S^N, A|_V = 0, A > 0 \text{ on } V^\perp\}$ ,  $B|_V = 0, B = A^{-1}$  on  $V^\perp$ .

Now consider  $(t, x) \in (0, T] \times \mathbb{R}^n$ , and let  $(q, p, A)$  be in the parabolic subset of  $u_{**}$  at  $(t, x)$  (we refer e.g. to [7] for definitions). Assume that  $u_{**}(t, x) < u(t, x)$  (otherwise there is nothing to prove), let  $\lambda_i, x_i, i = 1, \dots, m$  be such that  $u_{**}(t, x) = \lambda_1 u(t, x_1) + \dots + \lambda_m u(t, x_m)$ , and let  $V$  be the span of  $(x_1 - x, \dots, x_m - x)$ . Then by similar computations as in [1, pp.272-273], letting  $A_i = D^2 u(t, x_i)$ , it holds that

$$A_i \geq 0, \quad A \leq \left( \sum \lambda_i A_i^{-1} \right)^{-1}, \quad (3.4)$$

$$q = \sum_{i=1}^m \lambda_i \partial_t u(t, x_i), \quad (3.5)$$

$$p = Du(t, x_i), \quad i = 1, \dots, m. \quad (3.6)$$

Note that since  $u_{**}(t, \cdot)$  is affine in the directions spanned by  $V$  in a neighborhood of  $x$ , one has  $A \leq 0$  on  $V$ , so that by ellipticity

$$q - F(t, x, p, A) \geq q - F(t, x, p, B),$$

where  $B = \left( \sum \lambda_i A_i^{-1} \right)^{-1}$  on  $V^\perp$ ,  $B = 0$  on  $V$ , and by (3.3), we obtain

$$q - F(t, x, p, A) \geq \sum_{i=1}^m \lambda_i (\partial_t u(t, x_i) - F(t, x_i, Du(t, x_i), \tilde{A}_i))$$

where  $\tilde{A}_i = A_i$  on  $V^\perp$ ,  $\tilde{A}_i = 0$  on  $V$ , so that  $\tilde{A}_i \leq A_i$ , and by ellipticity of  $F$  and the fact that  $u$  is a supersolution to the equation we finally obtain

$$q - F(t, x, p, A) \geq 0.$$

□

We deduce the following

**Theorem 3.2.** *Let  $F = F(t, x, p, A) \in C([0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times S^N)$  be degenerate elliptic such that there exists a  $\Phi \in \text{Lip}_{loc}(\mathbb{R}_+; \mathbb{R})$  with  $\Phi(0+) \geq 0$  such that for all  $\lambda \in \mathbb{R}_+$ ,  $t \in [0, T]$ ,  $x, p \in \mathbb{R}^N$ ,  $\xi \neq 0 \in \mathbb{R}^N$ ,*

$$\begin{cases} (y, A) \mapsto F(t, x + y, p - \lambda(x + y), B - \lambda I) + \frac{1}{2}\Phi(\lambda)|x + y|^2 \\ \text{is convex on } (\mathbb{R}\xi) \times X_\xi, \end{cases} \quad (3.7)$$

where  $X_\xi = \{A \in S^N, A\xi = 0, A > 0 \text{ on } (\mathbb{R}\xi)^\perp\}$ ,  $B\xi = 0$ ,  $B = A^{-1}$  on  $(\mathbb{R}\xi)^\perp$ . Let  $u_0 \in C^2(\mathbb{R}^N)$  satisfy  $D^2u_0 \geq -\lambda_0 I$  for some  $\lambda_0 \geq 0$  and assume that  $u$  satisfies for some  $K > 0$ ,

$$|u(t, x)| \leq K(1 + |x|) \quad \forall x \in \mathbb{R}^N, t \in [0, T] \quad (3.8)$$

and is a classical solution to

$$\begin{cases} \partial_t u = F(t, x, Du, D^2u), \\ u(0, \cdot) = u_0, \end{cases} \quad (3.9)$$

then if  $\lambda(t)$  is the solution to

$$\begin{cases} \dot{\lambda}(t) = \Phi(\lambda(t)), \\ \lambda(0) = \lambda_0, \end{cases} \quad (3.10)$$

one has  $D^2u(t, \cdot) \geq -\lambda(t)I$  for all  $t \geq 0$ .

*Proof.* Let  $\varepsilon > 0$  arbitrary, fix and let  $\lambda^\varepsilon$  be the solution to (3.10) with initial condition  $\lambda^\varepsilon(0) = \lambda_0 + \varepsilon$ . We set  $v(t, x) := u(t, x) + \frac{1}{2}\lambda^\varepsilon(t)|x|^2$ . Since  $\lambda(t) > 0$ ,  $v(t)$  is coercive, in the sense that  $\inf_{t \in [0, T]} \frac{v(t, x)}{|x|} \rightarrow \infty$  for  $|x| \rightarrow \infty$ . Moreover,  $v$  is a classical solution to

$$\begin{aligned} \partial_t v &= F(t, x, Du, D^2u) + \frac{1}{2}\Phi(\lambda^\varepsilon(t))|x|^2 \\ &= F(t, x, Dv - \lambda^\varepsilon(t)x, D^2v - \lambda^\varepsilon(t)Id) + \frac{1}{2}\Phi(\lambda^\varepsilon(t))|x|^2 \\ &=: \tilde{F}(t, x, Dv, D^2v). \end{aligned} \quad (3.11)$$

By (3.7),  $\tilde{F}$  satisfies (3.1). Hence, by Proposition 3.1, the convex envelope  $v_{**}$  of  $v$  is a supersolution to (3.11). Equivalently,  $\hat{u} := v_{**} - \frac{1}{2}\lambda^\varepsilon(t)|x|^2$  is a supersolution to (3.9). By (3.8) we have that

$$v(t, x) \geq \frac{1}{2}\lambda^\varepsilon(t)|x|^2 - K - K|x|$$

for all  $x \in \mathbb{R}^d$  which implies that

$$v_{**}(t, x) \geq \frac{1}{2}\lambda^\varepsilon(t)|x|^2 - \tilde{K} - K|x|,$$

for some  $\tilde{K} > 0$  and all  $x \in \mathbb{R}^d$ . Hence,  $\hat{u} \geq -\tilde{K}(1 + |x|)$  and we may apply the comparison result [19, Theorem 4.2] to obtain

$$u \leq \hat{u}.$$

On the other hand, since  $v_{**} \leq v$  we have that

$$\hat{u} \leq v - \frac{1}{2}\lambda^\varepsilon(t)|x|^2 = u.$$

Hence,  $\hat{u} = u$  and, since  $v_{**}$  is convex, we conclude

$$D^2u = D^2\hat{u} = D^2v_{**} - \lambda^\varepsilon(t)Id \geq -\lambda^\varepsilon(t)Id.$$

Since this is true for all  $\varepsilon > 0$  the proof is finished.  $\square$

We next provide a series of abstract PDE for which condition (3.7) is satisfied.

**Example 3.3.** (1) *First-order PDE: Let*

$$F = F(t, x, p) \in C([0, T]; C_b^2(\mathbb{R}^N \times \mathbb{R}^N)).$$

*Then (3.7) is satisfied with*

$$\Phi(\lambda) = \|F_{xx}\|_\infty + 2|\lambda| \|F_{xp}\|_\infty + \lambda^2 \|F_{pp}\|_\infty.$$

*More generally, let  $F = F(t, x, p) \in C([0, T] \times \mathbb{R}^N \times \mathbb{R}^N)$  such that  $(x, p) \mapsto F(t, x, p)$  is semiconvex of order  $C_F$ . Then, (3.7) is satisfied with*

$$\Phi(\lambda) = C_F(1 + \lambda^2).$$

(2) *Quasilinear PDE: Let*

$$F(x, p, A) = \text{Tr}(a(x, p)A) \in C(\mathbb{R}^N \times \mathbb{R}^N \times S^N),$$

*where  $a(x, p) \in C^2(\mathbb{R}^N \times \mathbb{R}^N)$  is nonnegative, has bounded second derivative and  $(y, p) \mapsto \sqrt{a(y, p)}$  is convex. Then (3.7) is satisfied with*

$$\Phi(\lambda) = N\lambda \|a_{xx}\|_\infty + 2N\lambda^2 \|a_{xp}\|_\infty + N\lambda^3 \|a_{pp}\|_\infty.$$

(3) *Monotone, convex, fully nonlinear PDE: Let*

$$F = F(t, A) \in C([0, T] \times S^N)$$

*be convex and non-decreasing in  $A \in S^N$ . Then (3.7) is satisfied with  $\Phi = 0$ .*

(4) *One-dimensional, fully nonlinear PDE: Let  $F = F(t, x, p, A) \in C([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$  such that  $(x, p) \mapsto F(t, x, p, A)$  is semiconvex of order  $C_F(A)$ . Then, (3.7) is satisfied with*

$$\Phi(\lambda) = C_F(\lambda)(1 + \lambda^2).$$

*Proof.* (1): Immediate.

(2): For  $\lambda \in \mathbb{R}_+$ ,  $x, p \in \mathbb{R}^N$ ,  $\xi \neq 0 \in \mathbb{R}^N$  we aim to prove convexity of

$$\begin{aligned} (y, A) &\mapsto F(x + y, p - \lambda(x + y), B - \lambda I) + \frac{1}{2}\Phi(\lambda)|x + y|^2 \\ &= a(x + y, p - \lambda(x + y))\text{Tr}(B) \\ &\quad - a(x + y, p - \lambda(x + y))\lambda N + \frac{1}{2}\Phi(\lambda)|x + y|^2 \\ &=: F_1(x + y, p, B) + F_2(x + y, p). \end{aligned}$$

For the first part,  $F_1$ , we note that, by [28, Theorem 3.1, Remark (ii)], convexity of  $(y, A) \mapsto F_1(x + y, p, B)$  follows from convexity of  $\sqrt{a}$ . For the second part  $F_2$  we note that

$$\begin{aligned} D_{yy}F_2 &= -\lambda N D_{yy}a(x + y, p - \lambda(x + y)) + N\lambda^2 D_{yp}a(x + y, p - \lambda(x + y)) \\ &\quad + N\lambda^3 D_{pp}a(x + y, p - \lambda(x + y)) + \Phi(\lambda) \\ &\geq -\lambda N \|D_{yy}a\|_\infty - N\lambda^2 \|D_{yp}a\|_\infty - N\lambda^3 \|D_{pp}a\|_\infty + \Phi(\lambda) \\ &\geq 0. \end{aligned}$$

(3): Let  $\xi \neq 0 \in \mathbb{R}^N$ . By [1, Appendix] the map  $A \mapsto A^{-1}$  is convex on  $X_\xi$ , which implies (3.7) with  $\Phi = 0$ .

(4): Note that we have  $X_\xi = \{0\}$  in (3.7) and thus only convexity in  $y$  has to be checked, which easily follows from semiconvexity of  $F$ .  $\square$

**3.2. Examples.** In this section we provide a series of PDE for which regularization by noise can be observed based on our main result.

**Example 3.4.** *We consider the quasilinear PDE*

$$\begin{aligned} du + \frac{1}{2}|Du|^2 \circ d\xi(t) &= a(Du)\Delta u dt \quad \text{on } [0, T] \times \mathbb{R}^N, \\ u(0) &= u_0, \end{aligned} \tag{3.12}$$

where  $u_0 \in (BUC \cap W^{1,\infty})(\mathbb{R}^N)$ ,  $a \in C^2(\mathbb{R}^N)$  is nonnegative such that  $p \mapsto \sqrt{a(p)}$  is convex. Then,

$$\|D^2u(t, \cdot)\|_\infty \leq \frac{1}{L^+(t) \wedge L^-(t)},$$

where  $L^\pm$  are the maximal solutions on  $\mathbb{R}_+$  to

$$\begin{aligned} dL^+(t) &= -\frac{N\|a_{pp}\|_{L^\infty(B_R(0))}}{L^+(t)} + d\xi(t), \quad L^+(0) = \frac{1}{\|(D^2u_0)_+\|_\infty}, \\ dL^-(t) &= -\frac{N\|a_{pp}\|_{L^\infty(B_R(0))}}{L^-(t)} - d\xi(t), \quad L^-(0) = \frac{1}{\|(D^2u_0)_-\|_\infty}, \end{aligned}$$

with  $R := \|u_0\|_{W^{1,\infty}}$ .

In particular this includes the  $p$ -Laplace equation in one space dimension

$$du + \frac{1}{2} |\partial_x u|^2 \circ d\xi(t) = \partial_x (\partial_x u)^{[m]} dt,$$

with  $a(p) = m|p|^{m-1}$  and  $m \geq 3$ .

*Proof.* We aim to apply Theorem 2.3. Hence, we have to verify Assumption 2.2.

For  $\varepsilon > 0$  we consider

$$\begin{aligned} \partial_t u^\varepsilon &= (a^\varepsilon(Du^\varepsilon) + \varepsilon)\Delta u^\varepsilon, \\ u^\varepsilon(0) &= u_0^\varepsilon, \end{aligned} \tag{3.13}$$

where  $u_0^\varepsilon \in C_b^2(\mathbb{R}^N)$  converges to  $u_0$  uniformly with  $\|u_0^\varepsilon\|_{W^{1,\infty}(\mathbb{R}^N)} \leq \|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ ,  $a^\varepsilon \in C_b^2(\mathbb{R}^N)$ ,  $a^\varepsilon = a$  on  $B_R(0)$  and

$$\|a_{pp}^\varepsilon\|_\infty \leq \|a_{pp}\|_{L^\infty(B_{R+\varepsilon}(0))}.$$

By [24] there is a unique classical solution  $u^\varepsilon$  to (3.13).

The partial derivatives  $u_{x_i}^\varepsilon$  then satisfy

$$\partial_t u_{x_i}^\varepsilon = (a^\varepsilon(Du^\varepsilon) + \varepsilon)\Delta u_{x_i}^\varepsilon + (Da^\varepsilon(Du^\varepsilon), Du_{x_i}^\varepsilon)\Delta u^\varepsilon.$$

By the maximum principle we conclude that  $\|Du^\varepsilon(t, \cdot)\|_\infty \leq \|Du_0^\varepsilon\|_\infty$  and thus

$$\sup_{t \in [0, T]} \|u^\varepsilon(t)\|_{W^{1,\infty}} \leq \|u_0\|_{W^{1,\infty}}.$$

By Example 3.3, (2) we have that (3.7) is satisfied for (3.13) with

$$\Phi(\lambda) = N\lambda^3 \|a_{pp}^\varepsilon\|_{L^\infty} \leq N\lambda^3 \|a_{pp}\|_{L^\infty(B_{R+\varepsilon}(0))}.$$

Hence, by Theorem 3.2 applied to  $-u^\varepsilon$ , we have  $D^2u^\varepsilon(t, \cdot) \leq \lambda(t)Id$  for all  $t \geq 0$ , where  $\lambda$  is the (local) solution to (3.10). Setting  $l(t) := \frac{1}{\lambda(t)}$ , with the convention  $\frac{1}{\infty} = 0$ , we have

$$l'(t) = -\frac{\lambda'(t)}{\lambda^2(t)} = -\frac{\Phi(\lambda(t))}{\lambda^2(t)} = -\frac{N}{l(t)} \|a_{pp}\|_{L^\infty(B_{R+\varepsilon})}$$

and

$$D^2u^\varepsilon(t, \cdot) \leq \frac{1}{l(t)} Id.$$

By [2, 9] we have  $u^\varepsilon \rightarrow u$  locally uniformly and thus

$$D^2u(t, \cdot) \leq \frac{1}{l(t)} Id,$$

in the sense of distributions. In conclusion, since  $\varepsilon > 0$  was arbitrary, Assumption 2.1 is satisfied with

$$V_F(l) = -\frac{N \|a_{pp}\|_{L^\infty(B_R)}}{l}.$$

With  $l_0 := \|(D^2u_0)_+\|_\infty$  Theorem 2.3 implies

$$D^2u(t, \cdot) \leq \frac{1}{L^+(t)} \quad (3.14)$$

Since  $\tilde{u} := -u$  solves (3.12) with  $\xi$  replaced by  $\tilde{\xi} := -\xi$ , and  $a$  by  $\tilde{a} := a(-\cdot)$ , we also have

$$D^2u(t, \cdot) = -D^2\tilde{u}(t, \cdot) \geq -\frac{1}{L^-(t)}.$$

In conclusion,

$$\|D^2u(t, \cdot)\|_\infty \leq \frac{1}{L^+(t) \wedge L^-(t)}.$$

□

**Example 3.5.** We consider the quasilinear PDE

$$du + \frac{1}{2}|Du|^2 \circ d\xi(t) = F(Du)dt.$$

where  $u_0 \in (BUC \cap W^{1,\infty})(\mathbb{R}^N)$  and  $F \in C^2(\mathbb{R}^N)$ . Then,

$$\|D^2u(t, \cdot)\|_{L^\infty} \leq \frac{1}{L^+(t) \wedge L^-(t)}, \quad (3.15)$$

where  $L^\pm$  are the maximal continuous solutions on  $\mathbb{R}_+$  to

$$\begin{aligned} dL^+(t) &= -\|F_{pp}\|_{L^\infty(B_R(0))}dt + d\xi(t), & L^+(0) &= \frac{1}{\|(D^2u_0)_+\|_\infty}, \\ dL^-(t) &= -\|F_{pp}\|_{L^\infty(B_R(0))}dt - d\xi(t), & L^-(0) &= \frac{1}{\|(D^2u_0)_-\|_\infty}, \end{aligned}$$

where  $R = \|u_0\|_{W^{1,\infty}}$ .

*Proof.* In order to verify Assumption 2.2 we first consider

$$\begin{aligned} du^\varepsilon &= F^\varepsilon(Du^\varepsilon)dt + \varepsilon\Delta u^\varepsilon dt \\ u^\varepsilon(0) &= u_0^\varepsilon, \end{aligned} \quad (3.16)$$

with  $u_0^\varepsilon \in C_b^2(\mathbb{R}^N)$ ,  $u_0^\varepsilon \rightarrow u_0$  uniformly with  $\|u_0^\varepsilon\|_{W^{1,\infty}(\mathbb{R}^N)} \leq \|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ ,  $F^\varepsilon \in C_b^2$ ,  $F^\varepsilon = F$  on  $B_R$ ,  $\|F_{pp}^\varepsilon\|_\infty \leq \|F_{pp}\|_{L^\infty(B_{R+\varepsilon}(0))}$ . By [24] there is a unique, classical solution  $u^\varepsilon$  to (3.16). As in Example 3.4, we have the uniform estimate  $\|u^\varepsilon(t)\|_{W^{1,\infty}(\mathbb{R}^N)} \leq \|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ . By Example 3.3, (1) and arguing as in Example 3.4 we obtain that (3.7) is satisfied with  $\Phi(\lambda) = -\lambda^2\|F_{pp}\|_{L^\infty(B_R(0))}$ . Thus, for

$$l'(t) = -\|F_{pp}\|_{L^\infty(B_R)}$$

we have

$$D^2u(t, \cdot) \leq \frac{1}{l(t)}Id,$$

i.e. Assumption 2.2 is satisfied with  $V_F(l) = -\|F_{pp}\|_{L^\infty(B_R(0))}$ , which implies the claim as in Example 3.4.  $\square$

**Example 3.6.** *We consider the quasilinear, one-dimensional PDE*

$$\begin{aligned} \partial_t u + \frac{1}{2} |\partial_x u|^2 \circ d\xi(t) &= F(\partial_{xx} u) dt, \\ u(0) &= u_0 \in BUC(\mathbb{R}), \end{aligned} \quad (3.17)$$

where  $F \in C^0(\mathbb{R})$  is non-decreasing. Then,

$$\|\partial_{xx} u(t, \cdot)\|_{L^\infty} \leq \frac{1}{L^+(t) \wedge L^-(t)}, \quad (3.18)$$

where

$$L^+(t) = \xi(t) - \min_{s \in [0, t]} \xi(s), \quad L^-(t) = \max_{s \in [0, t]} \xi(s) - \xi(t).$$

*Proof.* We consider a smooth approximation  $F^\varepsilon$  of  $F$  such that  $F^\varepsilon \rightarrow F$  locally uniformly. Let  $u_0^\varepsilon \in C^\infty(\mathbb{R})$  such that  $u_0^\varepsilon \rightarrow u_0$  locally uniformly. By [26, Theorem 14.24] there is a (unique) classical solution  $u^\varepsilon$  to

$$\partial_t u^\varepsilon = F^\varepsilon(\partial_{xx} u^\varepsilon) + \varepsilon \partial_{xx} u^\varepsilon.$$

The second derivative  $v^\varepsilon := \partial_{xx} u^\varepsilon$  satisfies

$$\partial_t v^\varepsilon = \partial_{xx} F^\varepsilon(v^\varepsilon) + \varepsilon \partial_{xx} v^\varepsilon.$$

By the maximum principle we obtain that

$$\partial_{xx} u^\varepsilon(t) \leq \sup \partial_{xx} u_0^\varepsilon.$$

Since  $u^\varepsilon \rightarrow u$  locally uniformly, we conclude that Assumption 2.1 is satisfied with  $V_F = 0$ .  $\square$

**Remark 3.7.** *We emphasize that the estimate (3.18) is uniform in  $F$  and  $u_0$ . For example, consider  $F^m(r) := r^{[m]} = |r|^{m-1} r \rightarrow \text{sgn}(r)$  for all  $r \in \mathbb{R}$  for  $m \rightarrow 0$  and let  $u_0^m \in (BUC \cap W^{1,1})(\mathbb{R})$  with  $u_0^m \rightarrow u_0$  in  $W^{1,1}(\mathbb{R})$ . Then, at least formally, (3.18) continues to hold for the limit*

$$du + \frac{1}{2} |\partial_x u|^2 \circ d\xi(t) = \text{sgn}(\partial_{xx} u) dt$$

implying Lipschitz bounds for the stochastic total variation flow

$$dv + \frac{1}{2} \partial_x v^2 \circ d\xi(t) = \partial_x \text{sgn}(\partial_x v) dt.$$

These bounds improve the deterministic case. Indeed, in [4, Section 2.5] it has been shown that the solution  $v(t, \cdot)$  to the total variation flow in one spatial dimension

$$\partial_t v = \partial_x \text{sgn}(\partial_x v) \quad (3.19)$$

is a step-function if  $v_0$  is. In particular, for  $v_0 \in BV(\mathbb{R})$  one only has  $v(t) \in BV(\mathbb{R})$  in general.

**Proposition 3.8.** *Let  $u$  be the solution to*

$$\begin{aligned} du + \frac{1}{2}|Du|^2 \circ d\xi(t) &= F(Du, D^2u)dt, \\ u(0) &= u_0 \in BUC(\mathbb{R}^N), \end{aligned} \tag{3.20}$$

where  $F$  satisfies the assumptions of Theorem 2.3. Then for all  $t \geq 0$

$$\|Du(t, \cdot)\|_{L^\infty} \leq \inf_{0 \leq s \leq t} \sqrt{\frac{2\|u_0\|_\infty}{L^+(s) \vee L^-(s)}}$$

where  $L^\pm$  are the bounds on  $D^2u$  from Theorem 2.3.

*Proof.* This is an immediate consequence of Theorem 2.3, noting that if  $u$  is semi-concave (or semiconvex) of order  $C$  then  $\|Du\|_\infty \leq \sqrt{2C\|u\|_\infty}$  (e.g. [27, p.240]), and the fact that since the coefficients in (3.20) only depend on  $Du$  and  $D^2u$ ,  $\|u(t, \cdot)\|_\infty$  and  $\|Du(t, \cdot)\|_\infty$  are nonincreasing in  $t$ .  $\square$

#### 4. PROOF OF THEOREM 2.3

The proof of Theorem 2.3 is based on a Trotter-Kato splitting scheme for (2.1). The estimate (2.3) is then proven for the corresponding approximating solutions  $u^n$  with respect to a discretization  $L^n$  of  $L$ , based on semiconvexity estimates for  $S_H$ , with  $H(p) = \frac{1}{2}|p|^2$ . The corresponding estimates are derived in Section 4.1 below. The rest of the proof then consists in proving the convergence of the approximations  $L^n$  (cf. Section 4.2 below) and  $u^n$  (cf. Section 4.3 below). Finally, the proof of Theorem 2.3 is given in Section 4.

**4.1. Inf- and sup-convolution estimates.** In this section we provide Lipschitz and semiconvexity estimates for  $S_H$  with  $H(p) = \frac{1}{2}|p|^2$ . We refer to [25, 27] for related arguments.

Recall that  $S_H(\delta)$  can be written as

$$S_H(\delta, \phi)(x) = \begin{cases} \sup_{y \in \mathbb{R}^N} \left( \phi(y) - \frac{|x-y|^2}{2\delta} \right), & \text{if } \delta \geq 0 \\ \inf_{y \in \mathbb{R}^N} \left( \phi(y) + \frac{|x-y|^2}{2|\delta|} \right), & \text{if } \delta \leq 0. \end{cases}$$

**Lemma 4.1.** *If  $\phi \in BUC(\mathbb{R}^N)$  is convex (resp. concave), then so is  $S_H(\delta, \phi)$ , for all  $\delta \in \mathbb{R}$ .*

*Proof.* We will prove the claim only for  $\delta > 0$ , the case  $\delta < 0$  then follows noting that  $S_H(\delta, -\phi) = -S_H(-\delta, \phi)$ .

We begin by the case when  $\phi$  is concave. Then for any  $x_1, x_2 \in \mathbb{R}^N$  and  $\lambda \in [0, 1]$ ,

$$\begin{aligned}
& S_H(\delta, \phi)(\lambda x_1 + (1 - \lambda)x_2) \\
&= \sup_{y \in \mathbb{R}^N} \left\{ \phi(y) - \frac{1}{2\delta} |y - (\lambda x_1 + (1 - \lambda)x_2)|^2 \right\} \\
&= \sup_{y_1, y_2 \in \mathbb{R}^N} \left\{ \phi(\lambda y_1 + (1 - \lambda)y_2) - \frac{1}{2\delta} |\lambda(y_1 - x_1) + (1 - \lambda)(y_2 - x_2)|^2 \right\} \\
&\geq \lambda \sup_{y_1 \in \mathbb{R}^N} \left\{ \phi(y_1) - \frac{1}{2\delta} |y_1 - x_1|^2 \right\} + (1 - \lambda) \sup_{y_2 \in \mathbb{R}^N} \left\{ \phi(y_2) - \frac{1}{2\delta} |y_2 - x_2|^2 \right\} \\
&= \lambda S_H(\delta, \phi)(x_1) + (1 - \lambda) S_H(\delta, \phi)(x_2),
\end{aligned}$$

where in the third inequality we have used the concavity of  $\phi$  and of  $-1/(2\delta)|\cdot|^2$ .

We now assume that  $\phi$  is convex. Then for  $x_1, x_2 \in \mathbb{R}^N$  and  $\lambda \in [0, 1]$ ,

$$\begin{aligned}
& S_H(\delta, \phi)(\lambda x_1 + (1 - \lambda)x_2) \\
&= \sup_{z \in \mathbb{R}^N} \left\{ \phi(\lambda x_1 + (1 - \lambda)x_2 - z) - \frac{1}{2\delta} |z|^2 \right\} \\
&\leq \sup_{z \in \mathbb{R}^N} \left\{ \lambda(\phi(x_1 - z) - \frac{1}{2\delta} |z|^2) + (1 - \lambda)(\phi(x_2 - z) - \frac{1}{2\delta} |z|^2) \right\} \\
&\leq \lambda \sup_{z \in \mathbb{R}^N} \left\{ \phi(x_1 - z) - \frac{1}{2\delta} |z|^2 \right\} + (1 - \lambda) \sup_{z \in \mathbb{R}^N} \left\{ \phi(x_2 - z) - \frac{1}{2\delta} |z|^2 \right\} \\
&= \lambda S_H(\delta, \phi)(x_1) + (1 - \lambda) S_H(\delta, \phi)(x_2).
\end{aligned}$$

□

**Proposition 4.2.** *Let  $\phi \in BUC(\mathbb{R}^N)$ ,  $\psi = S_H(\phi, \delta)$  for some  $\delta \in \mathbb{R}$  and  $\lambda \in [0, \infty)$ . Then*

$$D^2\phi \leq \lambda^{-1}Id \Rightarrow D^2\psi \leq (\lambda - \delta)_+^{-1}Id, \quad (4.1)$$

$$D^2\phi \geq -\lambda^{-1}Id \Rightarrow D^2\psi \geq -(\lambda + \delta)_+^{-1}Id. \quad (4.2)$$

*Proof.* To prove (4.1), (4.2), we again may assume without loss of generality that  $\delta > 0$ . We focus on (4.2) namely we prove that if  $\psi = S_H(\delta, \phi)$ ,

$$\phi + \frac{1}{2\lambda}|\cdot|^2 \text{ convex} \Rightarrow \psi + \frac{1}{2(\lambda + \delta)}|\cdot|^2 \text{ convex.}$$

Indeed,

$$\begin{aligned}
\psi(x) + \frac{1}{2(\lambda + \delta)}|x|^2 &= \sup_{y \in \mathbb{R}^N} \left\{ \phi(y) - \frac{1}{2\delta}|x - y|^2 + \frac{1}{2(\lambda + \delta)}|x|^2 \right\} \\
&= \sup_{y \in \mathbb{R}^N} \left\{ \phi(y) + \frac{1}{2\lambda}|y|^2 - \frac{1}{2\lambda}|y|^2 - \frac{1}{2\delta}|x - y|^2 + \frac{1}{2(\lambda + \delta)}|x|^2 \right\}.
\end{aligned}$$

By a direct computation,  $\frac{1}{2\lambda}|y|^2 + \frac{1}{2\delta}|x-y|^2 - \frac{1}{2(\lambda+\delta)}|x|^2$  can be written as  $\alpha|x - \beta y|^2$  for some  $\alpha, \beta \geq 0$ , so that (after an affine change of coordinates) one can apply Lemma 4.1 to obtain convexity of  $\psi + \frac{1}{2(\lambda+\delta)}|\cdot|^2$ .

The proof of (4.1) is similar (using the preservation of concavity from Lemma 4.1).  $\square$

**4.2. Reflected SDE.** In this section we first study stability properties of solutions to reflected SDE and then their boundary behavior.

Let  $V$  be locally Lipschitz on  $(0, +\infty)$ , bounded from above on  $[1, \infty)$ , and  $\xi$  be a continuous path. In this section we study the maximal solution on  $[0, T]$  to

$$\begin{aligned} dX(t) &= V(X(t))dt + d\xi(t) \text{ on } \{X > 0\}, \quad X \geq 0, \quad X \text{ continuous} \\ X(0) &= x \in \mathbb{R}_+. \end{aligned} \tag{4.3}$$

More precisely, a function  $X \in C([0, T]; \mathbb{R}_+)$  is said to be a solution to (4.3) if, for all  $s \leq t \in [0, T]$ ,

$$X > 0 \text{ on } [s, t] \Rightarrow X(t) = X(s) + \int_s^t V(X(u))du + \xi_{s,t}.$$

Let  $\mathcal{S}(V, \xi, x)$  be the set of solutions. Note that by the assumptions on  $V$  there exists a unique solution  $X$  to (4.3) until  $\tau = \inf\{t \geq 0 : \lim_{s \uparrow t} X(s) = 0\}$ , and a particular element of  $\mathcal{S}(V, \xi, x)$  is given by letting  $X(t) \equiv 0$  for  $t \geq \tau$ .

**Proposition 4.3.** *Let  $V$  be locally Lipschitz on  $(0, +\infty)$ , bounded from above on  $[1, \infty)$ , and  $\xi$  be a continuous path. Let*

$$\hat{X}(t) := \sup\{Y(t) : Y \in \mathcal{S}(V, \xi, x)\}.$$

*Then,  $\hat{X} \in \mathcal{S}(V, \xi, x)$ .*

*Proof.* We first show that elements of  $\mathcal{S}(V, \xi, x)$  are equibounded and equicontinuous. Indeed, it is easy to see that

$$M := x + 1 + T \|V_+\|_{L^\infty([1, +\infty))} + 2 \|\xi_{0,\cdot}\|_{L^\infty([0, T])}$$

is an upper bound for  $\hat{X}$ . Then letting for  $\varepsilon > 0$

$$\omega_\varepsilon(r) := r \|V\|_{L^\infty([\varepsilon, M])} + \omega^\xi(r)$$

where  $\omega^\xi$  is a modulus of continuity for  $\xi$  on  $[0, T]$ , one sees that each element  $X$  of  $\mathcal{S}(V, \xi, x)$  admits  $\omega_\varepsilon$  as a modulus of continuity on (connected subsets of)  $\{X \geq \varepsilon\}$ . This implies that

$$\omega(r) := \inf_{\varepsilon > 0} (2\varepsilon + 2\omega_\varepsilon(r))$$

is a modulus of continuity for  $X$ . Indeed, given  $s < t$  in  $[0, T]$ , either  $X \geq \varepsilon$  on  $[s, t]$ , or there exist  $s_1 \leq t_1 \in [s, t]$  with  $X(s_1), X(t_1) \leq \varepsilon$ , with  $X \geq \varepsilon$  on  $(s, s_1)$  and  $(t_1, t)$  (these intervals might be empty if  $X \leq \varepsilon$  in  $t$  or  $s$ ). Then one has

$$\begin{aligned} |X(t) - X(s)| &\leq |X(t) - X(t_1)| + |X(t_1)| + |X(s_1)| + |X(s) - X(s_1)| \\ &\leq 2\varepsilon + \omega_\varepsilon(t_1 - t) + \omega_\varepsilon(s - s_1). \end{aligned}$$

It follows that  $\hat{X}$  is non-negative, finite and continuous on  $[0, T]$ . Note that since  $\mathcal{S}(V, \xi, x)$  is stable under the maximum operation, one can find an increasing sequence  $X^n$  in  $\mathcal{S}(V, \xi, x)$  converging to  $\hat{X}$  uniformly. One then simply passes to the limit to check that

$$\hat{X} > 0 \text{ on } [s, t] \Rightarrow \hat{X}(t) = \hat{X}(s) + \int_s^t V(\hat{X}(u))du + \xi_{s,t}.$$

□

**Proposition 4.4.** *Let  $V$  admit a Lipschitz continuous extension to  $[0, \infty)$ , then  $\hat{X}$  as defined in Proposition 4.3 is the (unique continuous) solution to*

$$\begin{aligned} dX(t) &= V(X(t))dt + d\xi(t) + dK(t), \quad X \geq 0, \quad dK \geq 0, \quad dK(t)1_{\{X(t)>0\}} = 0, \\ X(0) &= x \end{aligned} \tag{4.4}$$

*In particular,  $\xi \mapsto \hat{X}$  is continuous in supremum norm.*

*Proof.* Let  $\tilde{X}$  solve (4.4). Since  $\tilde{X} \in \mathcal{S}(V, \xi, x)$ , clearly  $\tilde{X} \leq \hat{X}$ . Then if  $\hat{X} > \tilde{X}$  on  $[s, t]$ , clearly  $\hat{X} > 0$  on this interval, so that

$$\begin{aligned} \hat{X}(t) - \tilde{X}(t) &= (\hat{X}(s) - \tilde{X}(s)) + \int_s^t (V(\hat{X}(u)) - V(\tilde{X}(u)))du - \int_s^t dKu \\ &\leq (\hat{X}(s) - \tilde{X}(s)) + \int_s^t C_V |\hat{X}(u) - \tilde{X}(u)| du, \end{aligned}$$

so that by Gronwall's lemma

$$\hat{X}(t) - \tilde{X}(t) \leq (\hat{X}(s) - \tilde{X}(s))e^{C_V(s-t)}.$$

Letting  $s \downarrow \inf\{r \in [0, t] : \hat{X} > \tilde{X} \text{ on } [r, t]\}$  we obtain that  $\hat{X}(t) - \tilde{X}(t) \leq 0$ , a contradiction. □

**Proposition 4.5.** *Let  $\xi \in C([0, T])$ ,  $V \in Lip(\mathbb{R}_+)$  with associated flow  $\varphi^V$ . Let  $\{t_i^n\}_{n \geq 0}$  be a sequence of partitions of  $[0, T]$  with step size  $\pi^n := \sup_i |t_{i+1}^n - t_i^n| \rightarrow 0$  as  $n \rightarrow \infty$ . For  $n \geq 0$ , define  $L^n$  by*

$$\begin{aligned} L^n(t_{i+1}^n) &= \left( \varphi^V(t_{i+1}^n - t_i^n, L_{t_i^n}^n) + \xi_{t_i^n, t_{i+1}^n} \right)_+ \\ L_0^n &= \ell_0. \end{aligned} \tag{4.5}$$

Then,  $L^n$  converges uniformly to  $L$  on  $[0, T]$ , where  $L$  is the (unique continuous) solution to the reflected SDE

$$\begin{aligned} dL(t) &= V(L(t))dt + d\xi(t) + dK(t), \quad L(t) \geq 0, \quad dK(t) \geq 0, \quad 1_{\{L(t) > 0\}}dK(t) = 0 \\ L(0) &= \ell_0. \end{aligned}$$

*Proof.* We first note that the  $L^n$  have a common modulus of continuity, uniformly in  $n \geq 0$ . Indeed, taking  $t_i^n < t_j^n$ , we distinguish two cases :

(1) If  $L^n(t_k^n) > 0$ , for each  $i < k < j$ , we then have

$$|L^n(t_i^n) - L^n(t_j^n)| \leq \|V\|_\infty(t_j^n - t_i^n) + \omega(t_j^n - t_i^n),$$

where  $\omega$  is the modulus of continuity of  $\xi$ .

(2) Otherwise considering the first last times where  $L^n = 0$  between  $t_i^n$  and  $t_j^n$  and applying the above bound, we obtain

$$|L^n(t_i^n) - L^n(t_j^n)| \leq 2(\|V\|_\infty(t_j^n - t_i^n) + \omega(t_j^n - t_i^n)).$$

This implies that, passing to a subsequence if necessary,  $L^n \rightarrow \hat{L}$  (locally uniformly), and it is enough to show that  $\hat{L} = L$ .

We can define  $L^n(s)$  for all  $s \geq 0$  by

$$L^n(s) = L^n(t_i^n) + \int_{t_i^n}^s V(L^n(u))du, \quad t_i^n \leq s < t_{i+1}^n, \quad L^n(t_{i+1}^n) = (L^n(t_{i+1}^n-) + \xi_{t_i^n, t_{i+1}^n})_+.$$

Letting

$$K^n(s) := \sum_{t_{i+1}^n \leq s} (L^n(t_{i+1}^n-) + \xi_{t_i^n, t_{i+1}^n})_-,$$

one has

$$L^n(t_i^n) = \int_0^{t_i^n} V(L^n(s))ds + \xi_{0, t_i^n} + K^n(t_i^n),$$

and it follows that  $K^n$  converges to some  $\hat{K}$ , which is continuous and nondecreasing, and such that

$$\hat{L}(t) = \int_0^t V(\hat{L}(s))ds + \xi_{0, t} + \hat{K}(t).$$

Therefore, it only remains to prove that  $\hat{L}(t)d\hat{K}(t) = 0$ . Assume that  $\hat{L}(s) \geq \varepsilon > 0$ . Then for  $n$  large enough, one has  $L^n(s) \geq \varepsilon/2$ , and then taking  $h$  such that for instance  $\|V\|_\infty h + \omega(h) \leq \varepsilon/4$ , one has  $L^n > 0$  on  $[s-h, s+h]$ . In particular,  $dK^n([s-h, s+h]) = 0$ , and passing to the limit,  $d\hat{K}([s-h, s+h]) = 0$ , and we have proven that  $1_{\{\hat{L}(t) \geq \varepsilon\}}d\hat{K}(t) = 0$ , for all  $\varepsilon > 0$ .  $\square$

**Proposition 4.6.** *Let  $V^1, V^2$  be locally Lipschitz on  $(0, +\infty)$ , bounded from above on  $[1, \infty)$ ,  $\xi$  be a continuous path,  $x \in \mathbb{R}_+$ , and let  $\hat{X}^1, \hat{X}^2$  be the associated maximal solutions. Then*

$$V^1 \geq V^2 \text{ on } (0, +\infty) \Rightarrow \hat{X}^1 \geq \hat{X}^2 \text{ on } \mathbb{R}_+.$$

*Proof.* Fix  $x \geq \varepsilon > 0$ , let  $V^{1,\varepsilon} = V^1 + \varepsilon$  and  $\hat{X}^{1,\varepsilon}$  be the corresponding solution reflected at  $\varepsilon$  (i.e.  $\hat{X}^{1,\varepsilon} = \hat{X}(x - \varepsilon, V^{1,\varepsilon}(\cdot + \varepsilon), \xi) + \varepsilon$ ). We first prove that  $\hat{X}^{1,\varepsilon} > \hat{X}^2$ . We proceed by contradiction, and let  $t = \inf\{s > 0, \hat{X}^{1,\varepsilon}(s) < \hat{X}^2(s)\}$ . By continuity of  $\hat{X}^{1,\varepsilon}, \hat{X}^2$  it holds that for some  $\delta > 0$ ,  $V^{1,\varepsilon}(\hat{X}^{1,\varepsilon}(s)) > V^2(\hat{X}^2(s))$  for  $s \in [t, t + \delta)$ . Then using Proposition 4.4 we obtain for  $s \in [t, t + \delta)$

$$\hat{X}^{1,\varepsilon}(s) - \hat{X}^2(s) = \int_t^s (V^{1,\varepsilon}(\hat{X}^{1,\varepsilon}(u)) - V^2(\hat{X}^2(u)))du + \int_t^s dK^{1,\varepsilon}(u) > 0,$$

which is a contradiction.

By the same argument, we see that  $\hat{X}^{1,\varepsilon}$  decreases as  $\varepsilon \downarrow 0$ , and as in the proof of Proposition 4.3 we can show that the limit  $\tilde{X}^1$  is in  $\mathcal{S}(V, x; \xi)$ . This yields  $\hat{X}^2 \leq \tilde{X}^1 \leq \hat{X}^1$  which finishes the proof.  $\square$

We next analyze the boundary behavior of the solutions to (4.3). The first result, Proposition 4.7 below, shows that if the signal  $\xi$  is too regular compared to the singularity of  $V$  at zero, then zero is absorbing or repelling depending on the sign of  $V$ . In contrast, in the case that  $\xi$  is given by Brownian motion, Proposition 4.8 below shows that zero may be either absorbing, reflecting or repelling, depending on the singularity of  $V$  at zero.

**Proposition 4.7.** *Assume that  $\xi \in C^\alpha$ ,  $\alpha \in (0, 1]$ . Then :*

(1) *If  $V$  is nonincreasing and satisfies  $\limsup_{T \rightarrow 0} T^{-\alpha} \int_0^T V(s^\alpha) ds = +\infty$ , then*

$$\forall t > 0, \hat{X}(t) > 0.$$

(2) *If  $V$  is nondecreasing and satisfies  $\limsup_{T \rightarrow 0} T^{-\alpha} \int_0^T V(s^\alpha) ds = -\infty$ , then*

$$\hat{X}(t) = 0 \Rightarrow \forall s \geq t, \hat{X}(s) = 0.$$

*Proof.* (1) The case where  $X(0) > 0$  is treated in [29, Prop. 2.2], and we only need to prove the case where  $X(0) = 0$ .

We fix  $\delta > 0$ , and take  $V^\delta \leq V$  with  $V^\delta$  bounded and Lipschitz on  $\mathbb{R}_+$ , and such that

$$V^\delta(0^+) > \inf_{\delta \geq t \geq s \geq 0} \frac{\xi_{s,t}}{(t-s)}. \quad (4.6)$$

Let  $X^\delta := \hat{X}(V^\delta, \xi, x)$ . Then by Proposition 4.6 one has  $\hat{X} \geq X^\delta$ , and by Proposition 4.4, for all  $s \leq t$ ,

$$X^\delta(t) \geq X^\delta(s) + \int_s^t V^\delta(X^\delta(s)) ds + \xi_{s,t}.$$

By (4.6),  $X^\delta$  is not identically 0 on  $[0, \delta]$ , and neither is  $\hat{X}$ . Hence there is a sequence  $t_\delta \rightarrow 0$  with  $\hat{X}_{t_\delta} > 0$ , and by the case  $\hat{X}_0 > 0$  we conclude that  $\hat{X} > 0$  on  $(0, \infty)$ .

(2) is a consequence of (1) by time-reversal: If for some  $s \leq t$ , one has  $\hat{X}(s) = 0$  and  $\hat{X} > 0$  on  $(s, t)$ , then letting  $Y(u) = \hat{X}(t - u)$ ,  $Y$  satisfies the assumptions of (1) (with  $V$  replaced by  $-V$ ,  $\xi$  by  $\xi_{t-\cdot}$ ), and  $Y(t - s) = 0$  which is a contradiction.  $\square$

When  $\xi$  is a standard Brownian motion, one has a complete classification of the boundary behavior at 0.

**Proposition 4.8.** *Let  $V$  be locally Lipschitz on  $(0, +\infty)$ , bounded from above on  $[1, \infty)$ ,  $x \in \mathbb{R}_+$ ,  $B$  be a linear Brownian motion, and let  $\hat{X} = \hat{X}(V, x, B)$ . Define*

$$I^+ = \int_0^1 \int_x^1 e^{2 \int_x^y V(u) du} dy dx, \quad I^- = \int_0^1 \int_x^1 e^{-2 \int_x^y V(u) du} dy dx.$$

Then one has the following four possible cases :

(1) (Regular boundary) If  $I^+ < \infty, I^- < \infty$ , then :

$$\forall t > 0, \mathbb{P}(\hat{X}(t) = 0) = 0, \quad \mathbb{P}(\exists s \leq t, \hat{X}(s) = 0) > 0.$$

(2) (Exit boundary) If  $I^- = \infty, I^+ < \infty$  :

$$\mathbb{P}(\exists s \leq t, \hat{X}(s) = 0) > 0, \quad \mathbb{P}(\exists s < t, \hat{X}(s) = 0, \hat{X}(t) > 0) = 0.$$

(3) (Entrance boundary) If  $I^+ = \infty, I^- < \infty$  :

$$\mathbb{P}(\forall t > 0, \hat{X}(t) > 0) = 1,$$

(4) (Natural boundary) If  $I^+ = I^- = \infty$  :

$$\text{If } x > 0, \text{ then } \mathbb{P}(\forall t > 0, \hat{X}(t) > 0) = 1, \text{ if } x = 0 \text{ then } \mathbb{P}(\forall t, \hat{X}(t) = 0) = 1.$$

*Proof.* This is mostly standard (cf. e.g. [21,23,31]), noting that  $I^+ = \int_0^1 dm(x) \int_x^1 ds(y)$ ,  $I^- = \int_0^1 ds(x) \int_x^1 dm(y)$  where  $s$  is the scale function and  $m$  is the speed measure associated to (4.3).

In case (1) the diffusion admits several possible boundary behaviors (so that  $\mathcal{S}(V, \xi, x)$  is in general infinite), but it is known that there exists a process  $X \in \mathcal{S}(V, \xi, x)$  which is instantaneously reflected i.e. such that  $\mathbb{P}(X(t) = 0) = 0$  for all  $t > 0$ . Since  $\hat{X} \geq X$  this implies that  $\mathbb{P}(\hat{X}(t) = 0) = 0$ .  $\square$

**4.3. A Trotter-Kato formula.** In this section we establish a Trotter-Kato formula for viscosity solutions to (2.1).

From Theorem A.1 recall that for  $u_0 \in BUC(\mathbb{R}^N)$ ,  $\xi, \zeta \in C([0, T]; \mathbb{R})$  we have

$$\|S^\xi(u_0) - S^\zeta(u_0)\|_\infty \leq \Phi(\|\xi_{0,\cdot} - \zeta_{0,\cdot}\|_\infty), \quad (4.7)$$

for some function  $\Phi$  as in Theorem A.1.

We now show that, as a consequence of this estimate, it is possible to define  $S^\xi(u_0)$  for paths  $\xi$  admitting jumps, in such a way that the estimate (4.7) remains true.

To this end, let  $\xi$  be a piecewise continuous path on  $[0, T]$  with jumps  $\Delta\xi(t_i) := \xi(t_i+) - \xi(t_i-)$  for  $i = 1, \dots, m-1$  along a partition  $(t_i)_{0 \leq i \leq m}$  of  $[0, T]$ . We then define  $u = S^\xi(u_0)$  as the solution to

$$\begin{aligned} u(0, \cdot) &= u_0, \\ u(t) &= \left( S^{\xi|_{[t_i, t_{i+1}]}} u(t_i) \right) (t) \text{ on } [t_i, t_{i+1}), \forall 0 \leq i \leq m, \\ u(t_{i+1}) &= S_H(\Delta\xi(t_{i+1}))(u(t_{i+1}-)), \quad 0 \leq i \leq m-1. \end{aligned}$$

We then have :

**Proposition 4.9.** *Let  $u_0 \in BUC(\mathbb{R}^N)$  and  $\xi, \zeta$  be piecewise-continuous paths. Then, (4.7) holds.*

*Proof.* The idea is to change the parametrization of  $\xi, \zeta$  in order to replace the piecewise-continuous paths by continuous paths.

We replace  $[0, T]$  by  $[0, \tilde{T}]$ , obtained from  $[0, T]$  by adding an interval for each jump of  $\xi$  and  $\zeta$ . For instance, say that  $\xi$  and  $\zeta$  have jumps at the points  $(t_i)_{i=0, \dots, m}$ . We then take  $\tilde{T} = T + m$ , and let

$$I = \cup_{i=0}^m [t_i + i, t_i + i + 1), \quad J = [0, \tilde{T}] \setminus I.$$

Then  $s(t) := t - i$  for  $t \in [t_i + i + 1, t_{i+1} + 1)$  defines a bijection from  $J$  to  $[0, T]$ . We define  $\tilde{\xi}$  such that  $\tilde{\xi} = \xi \circ s$  on  $J$  and  $\tilde{\xi}$  is affine linear on each interval of  $I$  and analogously for  $\tilde{\zeta}$ . Then,

$$\|\tilde{\xi}_{0,\cdot} - \tilde{\zeta}_{0,\cdot}\|_\infty = \|\xi_{0,\cdot} - \zeta_{0,\cdot}\|_\infty.$$

We further define

$$\tilde{F}(t, \cdot) = \begin{cases} 0, & t \in I, \\ F(s(t), \cdot), & t \in J, \end{cases}$$

Let  $\tilde{u}^\xi$  be the solution to

$$\begin{aligned} \partial_t u &= \tilde{F}(t, x, u, Du, D^2u) - H(Du)\tilde{\xi} \\ u(0) &= u_0, \end{aligned}$$

and  $\tilde{u}^\zeta$  analogously. Then  $S^\xi(u_0)(t, \cdot) = \tilde{u}^\xi(s^{-1}(t), \cdot)$ ,  $S^\zeta(u_0)(t, \cdot) = \tilde{u}^\zeta(s^{-1}(t), \cdot)$ , so that

$$\|S^\xi(u_0) - S^\zeta(u_0)\|_\infty \leq \|\tilde{u}^\xi - \tilde{u}^\zeta\|_\infty \leq \tilde{\Phi}(\|\tilde{\xi}_{0,\cdot} - \tilde{\zeta}_{0,\cdot}\|_\infty) = \tilde{\Phi}(\|\xi_{0,\cdot} - \zeta_{0,\cdot}\|_\infty),$$

where  $\tilde{\Phi}$  is given by Theorem A.1 applied to  $\tilde{F}, \tilde{T}$ . Now since  $\tilde{F}$  satisfies Assumption 2.1 with the same quantities as  $F$ , and  $\tilde{T}$  may be taken as close to  $T$  as one wishes, it follows that the estimate above also holds with  $\tilde{\Phi}$  replaced by  $\Phi$ .  $\square$

**Corollary 4.10** (Trotter-Kato formula). *Let  $\xi \in C([0, T])$ ,  $u_0 \in BUC(\mathbb{R}^N)$  and let  $u$  be the corresponding viscosity solution to (2.1). Further let  $(t_i^n)$  be a sequence of partitions of  $[0, T]$  with step-size going to 0. Define  $u^n$  by*

$$u^n(t, \cdot) := S_F(t - t_j^n) \circ S_H(\xi_{t_{j-1}^n, t_j^n}) \circ S_F(t_j^n - t_{j-1}^n) \circ \cdots \circ S_H(\xi_{0, t_1^n}) \circ S_F(t_1^n, u_0),$$

for  $t \in [t_j^n, t_{j+1}^n)$ . Then

$$\|u^n - u\|_{C([0, T] \times \mathbb{R}^N)} \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

*Proof.* We have  $u^n = S^{\xi^n}(u_0)$ , where  $\xi^n$  is the piecewise constant path equal to  $\xi_{t_i^n}$  on  $[t_i^n, t_{i+1}^n)$ . The claim now follows from Proposition 4.9.  $\square$

**4.4. Proof of Theorem 2.3.** Let  $t_i^n = \frac{ti}{n}$  and

$$u^n(t) := S_H(\xi_{t_{n-1}^n, t_n^n}) \circ S_F\left(\frac{t}{n}\right) \circ \cdots \circ S_H(\xi_{t_0^n, t_1^n}) \circ S_F\left(\frac{t}{n}\right) u^0.$$

By Corollary 4.10, one has

$$u(t, \cdot) = \lim_{n \rightarrow \infty} u^n(t, \cdot).$$

Proposition 4.2 combined with Assumption 2.2 implies

$$D^2 u^n(t, \cdot) \leq \frac{Id}{L^n(t)},$$

where  $L^n$  is defined by the induction

$$L^n(0) = \ell_0, \quad L^n(t_i^n) = \left( \varphi^{V_F}\left(\frac{t}{n}\right)(L^n(t_{i-1}^n)) - \xi_{t_{i+1}^n, t_i^n} \right)_+.$$

Now If  $V_F$  admits a Lipschitz extension to  $[0, \infty)$ , then as  $n \rightarrow \infty$   $L^n$  converges to  $L$  by Proposition 4.5 and we are done.

Let now  $V$  be only locally Lipschitz continuous. First assume that  $L > \varepsilon > 0$  on  $[0, t]$  for some  $\varepsilon > 0$ . Let  $\tilde{V}$  be Lipschitz continuous on  $[0, \infty)$  with  $\tilde{V} = V$  on  $(\varepsilon, +\infty)$  and let  $\tilde{L}, \tilde{L}^n$  be the solutions to (2.4), (4.5) with  $V$  replaced by  $\tilde{V}$  respectively. Then  $L = \tilde{L}$  and  $\tilde{L} = \lim_n \tilde{L}^n$  by Proposition 4.5. Thus,  $\tilde{L}^n > \varepsilon$  for  $n$  large enough, which implies  $L^n = \tilde{L}^n$  and  $\lim_n L^n = L$ .

Now assume that  $L(s) = 0$  for some  $s \in [0, t]$  and  $L(t) > 0$  (otherwise there is nothing to prove). Hence, for all  $\varepsilon > 0$  (small enough), there exists an  $s_\varepsilon \in (0, t)$  with  $L_{s_\varepsilon} = \varepsilon$ , and  $L \geq \varepsilon$  on  $[s_\varepsilon, t]$ . Let now  $u^\varepsilon$  be the solution to (2.2) on  $(s_\varepsilon, t] \times \mathbb{R}^n$  with  $u^\varepsilon(s_\varepsilon, \cdot) = S_H(-\varepsilon)u(s_\varepsilon, \cdot)$ . By Proposition 4.2,  $D^2 u^\varepsilon(s_\varepsilon, \cdot) \leq \varepsilon Id$ , and since  $L > 0$  on  $[s_\varepsilon, t)$ , we may apply the Trotter-Kato formula as in the previous case to conclude that  $D^2 u^\varepsilon(t, \cdot) \leq \frac{Id}{L(t)}$ . Finally, note that  $u^\varepsilon(t)$  is the solution to (2.2) driven by  $\xi^\varepsilon = \xi + \varepsilon 1_{[s_\varepsilon, t]}$ . Since  $\xi^\varepsilon \rightarrow \xi$  uniformly as  $\varepsilon \rightarrow 0$ , we conclude the proof by Proposition 4.9.

## 5. OPTIMALITY

In this section we prove the optimality of the estimates given in Example 3.4 and thereby also the ones given in Theorem 2.3 by providing an example of an SPDE and suitable initial conditions for which these estimates are shown to be sharp.

We consider the class of functions

$$\mathcal{U} = \left\{ u \in BUC(\mathbb{R}) \text{ is 2-periodic with } u(x) = u(-x), u(1+x) = u(1-x), \forall x \in \mathbb{R} \right. \\ \left. \text{and s.t. } 0 \leq u_x \leq 1, u_{xxx} \leq 0 \text{ in the sense of distributions on } (0, 1) \right\}.$$

Note that if  $u \in \mathcal{U}$ , then

$$\begin{aligned} \|(u_{xx})_+\|_\infty &= u_{xx}(0) = \sup_{\delta \in (0,1)} \frac{u(\delta) - u(0)}{\delta^2} \\ \|(u_{xx})_-\|_\infty &= -u_{xx}(1) = - \sup_{\delta \in (0,1)} \frac{u(1) - u(1-\delta)}{\delta^2}, \end{aligned} \quad (5.1)$$

where both of them may take the value  $+\infty$ .

**Theorem 5.1.** *Let  $u^0 \in \mathcal{U}$ ,  $\xi \in C_0([0, T])$  and let  $u$  be the solution to*

$$du + \frac{1}{2}|u_x|^2 \circ d\xi(t) = \frac{1}{4}|u_x|^2 u_{xx} dt, \quad u(0, \cdot) = u^0. \quad (5.2)$$

*Then,  $u(t, \cdot) \in \mathcal{U}$  for all  $t \geq 0$ , and*

$$u_{xx}(t, 0) = \frac{1}{L^+(t)}, \quad u_{xx}(t, 1) = -\frac{1}{L^-(t)},$$

*where  $L^+$ ,  $L^-$  are the maximal continuous solutions to*

$$dL^+(t) = -\frac{1}{2L^+(t)} dt + d\xi(t) \text{ on } \{L^+ > 0\}, \quad L^+(t) \geq 0, \quad L^+(0) = \frac{1}{\|(u_{xx}^0)_+\|_\infty}, \quad (5.3)$$

$$dL^-(t) = -\frac{1}{2L^-(t)} dt - d\xi(t) \text{ on } \{L^- > 0\}, \quad L^-(t) \geq 0, \quad L^-(0) = \frac{1}{\|(u_{xx}^0)_+\|_\infty}. \quad (5.4)$$

An application of Proposition 4.8 yields

**Corollary 5.2.** *In Theorem 5.1 let  $\xi = \sigma B$  where  $B$  is a Brownian motion. Then*

- (1) *If  $\sigma \leq 1$ : a.s. there exists a  $T^*$  such that  $\|D^2 u(t, \cdot)\|_\infty = +\infty$  for all  $t > T^*$ .*
- (2) *If  $\sigma > 1$ : for each  $t > 0$ , a.s.  $\|D^2 u(t, \cdot)\|_\infty < +\infty$ .*

We next proceed to the proof of Theorem 5.1. We shall concentrate on proving  $u_{xx}(t, 0) = \frac{1}{L^+(t)}$ , the other equality can be obtained analogously. By Example 3.4 we already know that  $L^+(t) \leq \frac{1}{u_{xx}(t, 0)}$ . Since also  $L^+$  is the maximal solution to (5.3), it only remains to prove that  $t \mapsto \frac{1}{u_{xx}(t, 0)} \in \mathcal{S}(V, \frac{1}{u_{xx}(t, 0)}, \xi)$ , which is a consequence of Proposition 5.5 and Proposition 5.7 below.

**Lemma 5.3.** *Let  $u^0 \in C_b^6 \cap \mathcal{U}$  and  $\xi \in W^{1,1}([0, T]) \cap C^1(0, T)$ . Let  $L^+, L^-$  be the maximal solutions to (5.3), (5.4), let  $\tau^\pm = \inf\{t > 0, L^\pm(t) = 0\}$  and  $\tau = \tau^+ \wedge \tau^-$ . Then  $u \in C^{1,4}((0, \tau) \times \mathbb{R})$  with  $u(t, \cdot) \in \mathcal{U}$  for all  $t \in [0, \tau)$ .*

*Proof.* Without loss of generality, we assume that  $u$  is smooth and obtain  $L^\infty$  estimates from the PDE applied to the derivatives of  $u$ . This can be easily justified by considering solutions  $u^\varepsilon$  to the equations with an additional viscosity  $\varepsilon u_{xx}$  in the right-hand side, and noting that the bounds obtained from the arguments below are uniform in  $\varepsilon$ .

Now we first note that the fact that  $0 \leq u_x \leq 1$ ,  $u_{xx} \geq 0$  is clear by (5.7), (5.8) and the maximum principle, and so is the fact that  $u(t, \cdot)$ ,  $u(t, 1 + \cdot)$  are even for all  $t \geq 0$ . In addition, we already know from Example 3.4 that  $u_{xx}(t, \cdot)$  is bounded for  $t \in [0, \tau)$ . We set  $u_i := (\partial_x)^i u$  and observe that

$$\begin{cases} \partial_t u_3 = \frac{3}{2} u_3^2 u_1 + 3u_3 u_2^2 + 2u_4 u_2 u_1 + \frac{1}{4} u_5 u_1^2 - \dot{\xi}(t) (3u_3 u_2 + u_1 u_4) \\ u_3(0, x) = (u^0)_3(x), \quad u_3(t, 0) = 0, \quad u_3 \text{ bounded.} \end{cases} \quad (5.5)$$

One first checks that  $\sup_{x \in \mathbb{R}} u_3(0, x) \leq 0$  implies  $\sup_{x \in \mathbb{R}} u_3(t, x) \leq 0$ , by a maximum principle argument. Since the only nonlinear term in the right hand side of (5.5) is  $3u_3^2 u_1 \geq 0$ , the maximum principle implies that on  $[0, \tau) \times \mathbb{R}_+$ ,

$$0 \geq u_3 \geq -\|u_0\| \exp\left(6\|u_2\|_\infty^2 \tau + \|u_2\|_\infty \int_0^\tau |\dot{\xi}(s)| ds\right).$$

Then one writes in a similar way the equation for  $u_4$  (and then  $u_5, u_6$ ), noting that this time they are linear with coefficients depending on  $u_1, u_2, u_3$ , (resp.  $u_1$  to  $u_4$ , and  $u_1$  to  $u_5$ ) so that  $u_4, u_5$  and  $u_6$  also stay bounded for  $t < \tau$ .

Finally, from (5.2), (5.7), (5.8), (5.5) one gets that boundedness of  $u_1, \dots, u_6$  implies continuity of  $\partial_t u, \dots, \partial_t u_4$ , i.e.  $u \in C^{1,4}([0, \tau) \times \mathbb{R})$ .  $\square$

**Lemma 5.4.** *Let  $u_0 \in \mathcal{U}$ ,  $\xi \in C([0, \infty])$  and  $u$  be the solution to (5.2). Then,  $u(t, \cdot) \in \mathcal{U}$  for all  $t \geq 0$ .*

*Proof.* Let  $u^{0,\varepsilon} \in \mathcal{U}$  be smooth approximations of  $u^0$ ,  $\xi^\varepsilon$  be smooth approximations of  $\xi$  and  $u^\varepsilon$  be the unique smooth solution (cf. [24]) to

$$\partial_t u^\varepsilon = \left(\varepsilon + \frac{1}{4} |u_x^\varepsilon|^2\right) u_{xx}^\varepsilon - \frac{1}{2} |u_x^\varepsilon|^2 \dot{\xi}^\varepsilon(t), \quad u(0, \cdot) = u^{0,\varepsilon}(\cdot). \quad (5.6)$$

Since  $u^\varepsilon$  is smooth, as in the proof of the previous lemma we may differentiate (5.6) and use the maximum principle to obtain that for each  $\varepsilon > 0$ ,  $u^\varepsilon$  is 2-periodic, symmetric in  $x$  around 0 and 1, and  $0 \leq u_x^\varepsilon \leq 1$ ,  $u_{xx}^\varepsilon \leq 0$  on  $[0, +\infty) \times (0, 1)$ . Since  $u^\varepsilon \rightarrow u$  uniformly and  $\mathcal{U}$  is stable under uniform convergence, we can conclude.  $\square$

**Proposition 5.5.** *Assume that  $u_{xx}^0(0) < \infty$ , then  $u_{xx}(t, 0) = \frac{1}{L^+(t)}$  for  $t \leq \tau^+ := \inf\{s > 0, L^+(s) = 0\}$ .*

*Proof.* In the case of  $\xi \in C^1$  and  $u \in C^{1,4}$  with  $u(t, \cdot) \in \mathcal{U}$  for all  $t \geq 0$ , the result follows from differentiating (5.2) twice

$$\partial_t u_x = \frac{1}{4} u_{xxx} u_x^2 + \frac{1}{2} u_{xx}^2 u_x - \dot{\xi}(t) u_{xx} u_x, \quad (5.7)$$

$$\partial_t u_{xx} = \frac{1}{4} u_{xxxx} u_x^2 + \frac{3}{2} u_{xxx} u_{xx} u_x + \frac{1}{2} u_{xx}^3 - \dot{\xi}(t) (u_{xx}^2 + u_{xxx} u_x), \quad (5.8)$$

and noting that  $u_x(t, 0) = u_{xxx}(t, 0) = 0$  for all  $t \geq 0$ .

Let  $\xi^\eta \in W^{1,1}([0, T]) \cap C^1(0, T)$  with  $\xi^\eta \uparrow \xi$ ,  $\xi^\eta(0) = \xi(0)$ . Further let  $u^{0,\eta} \in C_b^6 \wedge \mathcal{U}$  with  $u^{0,\eta} \rightarrow u^0$  uniformly,  $u^{0,\eta}(0) = u^0(0)$ ,  $u^{0,\eta} \leq u^0$  and such that  $u_{xx}^{0,\eta}(0) \uparrow u_{xx}^0(0)$ . Also assume that  $u_{xx}^{0,\eta}(1)$  is chosen small enough that if  $L^{+,\eta}$ ,  $L^{-,\eta}$  are the solution to (2.4) driven by  $\xi^\eta$  and starting from  $\frac{1}{u_{xx}^{0,\eta}(0)}$ ,  $-\frac{1}{u_{xx}^{1,\eta}(0)}$ , the hitting times of 0 satisfy  $\tau^{-,\eta} > \tau^{+,\eta}$ . Let  $u^\eta$  be the solution to (5.2) driven by  $\xi^\eta$  and starting from  $u^{0,\eta}$ . By Lemma 5.3, for  $t \in [0, T]$ ,

$$u_{xx}^\eta(t, 0) = \frac{1}{L^{+,\eta}(t)}.$$

We note that  $L^{+,\eta}(t) \uparrow_{\eta \rightarrow 0} L^+(t)$  uniformly in  $[0, \tau^+]$  and, by Lemma 5.4,  $u_{xx}^\eta(t, 0) = \sup_{\delta \in (0,1)} \frac{u^\eta(t, \delta) - u^\eta(t, 0)}{\delta^2}$ . Finally, from (A.4) it follows that  $u^\eta \uparrow u$  with  $u^\eta(t, 0) = u(t, 0) (= u^0(0))$ , and we get

$$u_{xx}(t, 0) = \sup_{\delta \in (0,1)} \sup_{\eta > 0} \frac{u^\eta(t, \delta) - u^\eta(t, 0)}{\delta^2} = \sup_{\eta > 0} u_{xx}^\eta(t, 0) = \frac{1}{L^+(t)}.$$

□

**Lemma 5.6.** *Let  $\xi \in C([0, T])$ ,  $u_0 \in (BUC \cap W^{1,1})([0, 2])$  periodic and  $u$  be the corresponding viscosity solution to (5.2). Then  $v = \partial_x u$  is the pathwise entropy solution<sup>2</sup> to*

$$\begin{aligned} dv + \frac{1}{2} \partial_x v^2 \circ d\xi(t) &= \frac{1}{12} \partial_{xx} v^{[3]} dt \\ v(0) &= \partial_x u_0. \end{aligned} \quad (5.9)$$

Let  $u_0^1, u_0^2 \in (BUC \cap W^{1,1})([0, 2]) \cap \mathcal{U}$  and  $u^1, u^2$  be the corresponding viscosity solutions to (5.2) such that  $\partial_x u_0^1 \geq \partial_x u_0^2$  a.e. on  $(0, 1)$ . Then for all  $t \geq 0$ ,

$$\partial_x u^1(t, \cdot) \geq \partial_x u^2(t, \cdot) \quad \text{a.e. on } (0, 1).$$

*Proof.* We consider  $u_0^n$  smooth, periodic such that  $u_0^n \rightarrow u_0$  uniformly and in  $W^{1,1}([0, 2])$ . Further let  $\xi^n$  smooth with  $\xi^n \rightarrow \xi$  uniformly. For  $\varepsilon > 0$  let  $u^{\varepsilon,n}$  be the unique classical solution to

$$\begin{aligned} du^{\varepsilon,n} &= \left( \varepsilon u_{xx}^{\varepsilon,n} + \frac{1}{4} |u_x^{\varepsilon,n}|^2 u_{xx}^{\varepsilon,n} \right) dt - \frac{1}{2} (u_x^{\varepsilon,n})^2 \dot{\xi}^n(t) \\ u^{\varepsilon,n}(0) &= u_0^n. \end{aligned} \quad (5.10)$$

<sup>2</sup>For a theory of pathwise entropy solutions to (5.9) we refer to [18].

Then  $v^{\varepsilon,n} := \partial_x u^{\varepsilon,n}$  is the unique solution to

$$\begin{aligned} dv^{\varepsilon,n} &= \left( \varepsilon v_{xx}^{\varepsilon,n} + \frac{1}{12} \partial_x (v^{\varepsilon,n})^3 \right) dt - \frac{1}{2} \partial_x (v^{\varepsilon,n})^2 \dot{\xi}^n(t) \\ v^{\varepsilon,n}(0) &= \partial_x u_0^n. \end{aligned} \quad (5.11)$$

By stability of viscosity solutions we have  $u^{\varepsilon,n} \rightarrow u^n$  uniformly and  $v^{\varepsilon,n} \rightarrow v^n$  in  $C([0;T];L^1)$  by [30], where  $u^n$  is the viscosity solution to (5.10) and  $v^n$  is the kinetic solution to (5.11) with  $\varepsilon = 0$  respectively. By Theorem A.1 we have  $u^n \rightarrow u$  uniformly and by [18, Theorem 2.3, Proposition 2.5] we have  $v^n \rightarrow v$  in  $C([0;T];L^1)$ , where  $u$  is the viscosity solution to (5.2) and  $v$  is the kinetic solution to (5.9).

Let now  $u_0^1, u_0^2 \in (BUC \cap W^{1,1})([0,2]) \cap \mathcal{U}$  with  $\partial_x u_0^1 \geq \partial_x u_0^2$  a.e. on  $(0,1)$ . As above, consider the respective approximations  $u^{1,\varepsilon,n}, u^{2,\varepsilon,n}$ , with  $u_0^{1,n}, u_0^{2,n}$  smooth elements of  $\mathcal{U}$  with  $\partial_x u_0^{1,n} \geq \partial_x u_0^{2,n}$  in  $[0,1]$ . Then, as in Lemma 5.4,  $u^{1,\varepsilon,n}(t, \cdot), u^{2,\varepsilon,n}(t, \cdot) \in \mathcal{U}$  for all  $t \geq 0$ . Note that for  $u \in C^1 \cap \mathcal{U}$ ,  $\partial_x u(0) = \partial_x u(1) = 0$ . Hence,  $\partial_x u^{1,\varepsilon,n}(t, \cdot) \geq \partial_x u^{2,\varepsilon,n}(t, \cdot)$  on  $[0,1]$  by the comparison principle for (5.11) with Dirichlet boundary conditions on  $(0,1)$ . Taking limits implies the claim.  $\square$

**Proposition 5.7.** *The map  $t \mapsto u_{xx}(t,0) \in (0, \infty]$  is continuous.*

*Proof.* First note that  $t \mapsto u_{xx}(t,0)$  is lower semicontinuous as supremum of continuous functions by (5.1), and taking also into account Proposition 5.5, we only need to prove that

$$t_n \nearrow t, \quad u_{xx}(t_n,0) \rightarrow +\infty \quad \Rightarrow \quad u_{xx}(t,0) = +\infty. \quad (5.12)$$

We fix  $M > 0$  and let  $u^n$  be solutions to (5.2) but starting from data  $u^{t_n,n}$  at time  $t_n$ , where  $u^{t_n,n} \in \mathcal{U}$  is such that  $u_{xx}^{t_n,n}(0) = M$  and  $u_x^{t_n,n} \leq u_x(t_n, \cdot)$  (this is possible at least for  $n$  large enough). By Proposition 5.5,  $u_{xx}^n(s,0) = \frac{1}{L^{+,n}(s)}$  for  $s \in [t_n, \tau^{+,n})$ , where

$$dL^{+,n}(s) = -\frac{1}{2L^{+,n}(s)} ds + d\xi(s), \quad L^{+,n}(t_n) = M^{-1}$$

and  $\tau^{+,n} = \inf \{s > t_n, L^{+,n}(s) = 0\}$ . By Lemma 5.3 one has  $\tau^{+,n} > t$  for  $n$  large enough, and, clearly,  $\lim_{n \rightarrow \infty} L^{+,n}(t) = M^{-1}$ . Since  $u_{xx}(t,0) \geq u_{xx}^n(t,0)$  by Lemma 5.6, it follows that  $u_{xx}(t,0) \geq M$ . Since  $M$  was arbitrary, this proves (5.12).  $\square$

## APPENDIX A. STOCHASTIC VISCOSITY SOLUTIONS

In this section we briefly recall the definition and main properties of stochastic viscosity solutions to fully nonlinear SPDE of the type

$$\begin{aligned} du + \frac{1}{2} |Du|^2 \circ d\xi(t) &= F(t, x, u, Du, D^2u) dt \quad \text{in } \mathbb{R}^N \times (0, T] \\ u(0, \cdot) &= u_0 \quad \text{on } \mathbb{R}^N \times \{0\}, \end{aligned} \quad (A.1)$$

where  $u_0 \in BUC(\mathbb{R}^N)$ ,  $F \in C([0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N)$  and  $\xi$  is a continuous path.

We recall from [16, Theorem 1.2, Theorem 1.3]

**Theorem A.1.** *Let  $u_0, v_0 \in BUC(\mathbb{R}^N)$ ,  $T > 0$ ,  $\xi, \zeta \in C_0^1([0, T]; \mathbb{R})$  and assume that Assumption 2.1 holds. If  $u \in BUSC([0, T] \times \mathbb{R}^N)$ ,  $v \in BLSC([0, T] \times \mathbb{R}^N)$  are viscosity sub- and super-solutions to (A.1) driven by  $\xi, \zeta$  respectively, then,*

$$\sup_{[0, T] \times \mathbb{R}^N} (u - v) \leq \sup_{\mathbb{R}^N} (u_0 - v_0)_+ + \Phi(\|\xi - \zeta\|_{C([0, T])}), \quad (\text{A.2})$$

where  $\Phi$  depends only on  $T$ , the sup-norms and moduli of continuity of  $u_0, v_0$  and the quantities appearing in Assumption 2.1, is non-decreasing and such that  $\Phi(0^+) = 0$ . In particular, the solution operator

$$S : BUC(\mathbb{R}^N) \times C_0^1([0, T]; \mathbb{R}^N) \rightarrow BUC([0, T] \times \mathbb{R}^N)$$

admits a unique continuous extension to

$$S : BUC(\mathbb{R}^N) \times C_0^0([0, T]; \mathbb{R}^N) \rightarrow BUC([0, T] \times \mathbb{R}^N).$$

We then call  $u = S(u_0, \xi)$  the unique viscosity solution to (A.1). One then has

$$\|S^\xi(u_0) - S^\zeta(v_0)\|_{C([0, T] \times \mathbb{R}^N)} \leq \|u_0 - v_0\|_{C(\mathbb{R}^N)} + \Phi(\|\xi - \zeta\|_{C([0, T])}). \quad (\text{A.3})$$

In the case where  $F = F(p, X)$  only depends on its last two arguments, the estimate simplifies to

$$\sup_{[0, T] \times \mathbb{R}^N} (u - v) \leq \sup_{x, y \in \mathbb{R}^N} \left( u_0(x) - v_0(y) - \frac{|x - y|^2}{\sup_{s \in [0, T]} (\xi(s) - \zeta(s))} \right) \quad (\text{A.4})$$

(with convention  $0/0 = 0$ ,  $1/0 = +\infty$ ).

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