An infinite dimensional umbral calculus

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Abstract

The aim of this paper is to develop umbral calculus on the space $\mathcal{D}'$ of distributions on $\mathbb{R}^d$, which leads to a general theory of Sheffer sequences on $\mathcal{D}'$. We define a sequence of monic polynomials on $\mathcal{D}'$, a polynomial sequence of binomial type, and a Sheffer sequence. We present equivalent conditions for a sequence of monic polynomials on $\mathcal{D}'$ to be of binomial type or a Sheffer sequence, respectively. Our theory has remarkable similarities to the classical setting of polynomials on $\mathbb{R}$. For example, the form of the generating function of a Sheffer sequence on $\mathcal{D}'$ is similar to the generating function of a Sheffer sequence on $\mathbb{R}$, albeit the constants appearing in the latter function are replaced in the former function by appropriate linear continuous operators. We construct a lifting of a sequence of monic polynomials on $\mathbb{R}$ of binomial type to a polynomial sequence of binomial type on $\mathcal{D}'$, and a lifting of a Sheffer sequence on $\mathbb{R}$ to a Sheffer sequence on $\mathcal{D}'$. Examples of lifted polynomial sequences include the falling and rising factorials on $\mathcal{D}'$, Abel, Hermite, Charlier, and Laguerre polynomials on $\mathcal{D}'$. Some of these polynomials have already appeared in different branches of infinite dimensional analysis and played there a fundamental role.

Keywords: Generating function; polynomial sequence of binomial type; Sheffer sequence; shift-invariance; umbral calculus.

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1 Introduction

In its modern form, umbral calculus is a study of shift-invariant linear operators acting on polynomials, their associated polynomial sequences of binomial type, and Sheffer sequences (including Appell sequences). We refer to the seminal papers [27, 34, 35], see also the monographs [22, 33]. Umbral calculus has applications in combinatorics, theory of special functions, approximation theory, probability and statistics, topology, and physics, see e.g. the survey paper [11] for a long list of references.

Many extensions of umbral calculus to the case of polynomials of several, or even infinitely many variables were discussed e.g. in [5,10,13,26,30–32,37,38], for a longer list of such papers see the introduction to [12]. Appell and Sheffer sequences of polynomials of several noncommutative variables arising in the context of free probability, Bollean probability, and conditionally free probability were discussed in [2–4], see also the references therein.

The paper [12] was a pioneering (and seemingly unique) work in which elements of basis-free umbral calculus were developed on an infinite dimensional space, more precisely, on a real separable Hilbert space $\mathcal{H}$. This paper discussed, in particular, shift-invariant linear operators acting on the space of polynomials on $\mathcal{H}$, Appell sequences, and examples of polynomial sequences of binomial type.

In fact, examples of Sheffer sequences, i.e., polynomial sequences with generating function of a certain exponential type, have appeared in infinite dimensional analysis on numerous occasions. Some of these polynomial sequences are orthogonal with respect to a given probability measure on an infinite dimensional space, while others are related to analytical structures on such spaces. Typically, these polynomials are either defined on a co-nuclear space Φ' (i.e, the dual of a nuclear space Φ), or on an appropriate subset of Φ'. Furthermore, in majority of examples, the nuclear space Φ consists of (smooth) functions on an underlying space X. For simplicity, we choose to work in this paper with the Gel'fand triple

$$\Phi = \mathcal{D} \subset L^2(\mathbb{R}^d, dx) \subset \mathcal{D}' = \Phi'.$$

Here $\mathcal{D}$ is the nuclear space of smooth compactly supported functions on $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathcal{D}'$ is the dual space of $\mathcal{D}$, where the dual pairing between $\mathcal{D}'$ and $\mathcal{D}$ is obtained by continuously extending the inner product in $L^2(\mathbb{R}^d, dx)$.

Let us mention several known examples of Sheffer sequences on $\mathcal{D}'$ or its subsets:

(i) In infinite dimensional Gaussian analysis, also called white noise analysis, Hermite polynomial sequences on $\mathcal{D}'$ (or rather on $S' \subset \mathcal{D}'$, the Schwartz space of tempered distributions) appear as polynomials orthogonal with respect to Gaussian white noise measure, see e.g. [6, 14, 15, 29].

(ii) Charlier polynomial sequences on the configuration space of counting Radon measures on $\mathbb{R}^d$, $\Gamma \subset \mathcal{D}'$, appear as polynomials orthogonal with respect to Poisson point process on $\mathbb{R}^d$, see [17, 20, 23].

(iii) Laguerre polynomial sequences on the cone of discrete Radon measures on $\mathbb{R}^d$, $\mathbb{K} \subset \mathcal{D}'$, appear as polynomials orthogonal with respect to the gamma random measure, see [19, 20].

(iv) Meixner polynomial sequences on $\mathcal{D}'$ appear as polynomials orthogonal with respect to the Meixner white noise measure, see [24, 25].

(v) Special polynomials on the configuration space $\Gamma \subset \mathcal{D}'$ are used to construct the K-transform, see e.g. [7, 16, 18]. Recall that the K-transform determines the duality between point processes on $\mathbb{R}^d$ and their correlation measures. These polynomials will be identified in this paper as the infinite dimensional analog of the falling factorials (a special case of the Newton polynomials).

(vi) Polynomial sequences on $\mathcal{D}'$ with generating function of a certain exponential type are used in biorthogonal analysis related to general measures on $\mathcal{D}'$, see [1, 21].

Note, however, that even the very notion of a general polynomial sequence on an infinite dimensional space has never been discussed!

The classical umbral calculus on the real line gives a general theory of Sheffer sequences. So our aim in this paper is to develop umbral calculus on the space $\mathcal{D}'$, which will eventually lead to a general theory of Sheffer sequences on $\mathcal{D}'$. In fact, our theory will have remarkable similarities to the classical setting of polynomials on $\mathbb{R}$. For example, the form of the generating function of a Sheffer sequence on $\mathcal{D}'$ will be similar to the generating function of a Sheffer sequence on $\mathbb{R}$, albeit the constants appearing in the latter function will be replaced in the former function by appropriate linear continuous operators.

There is a principal point in our approach that we would like to stress. The paper [12] deals with polynomials on a general Hilbert space $\mathcal{H}$, while the monograph [6] develops Gaussian analysis on a general co-nuclear space Φ', without the assumption that Φ' consists of generalized functions on $\mathbb{R}^d$ (or on a general underlying space). In fact, we will discuss in Remark 7.5 below that the case of the infinite dimensional Hermite polynomials is, in a sense, exceptional and does not require from the co-nuclear space Φ' any special structure. In all other cases, the choice $\Phi' = \mathcal{D}'$ is crucial. Having said this, let us note that our ansatz can still be applied to a rather general co-nuclear space of generalized functions over a topological space X, equipped with a reference measure.

The origins of the classical umbral calculus are in combinatorics. So, by analogy, one can think of umbral calculus on $\mathcal{D}'$ as a kind of spatial combinatorics. To give the reader a better feeling of this, let us consider the following example. Let $\gamma = \sum_{i=1}^{\infty} \delta_{x_i} \in \Gamma$ be a configuration. Here δ_{x_i} denotes the Dirac measure with mass at x_i. We will construct (the kernel of) the falling factorial, denoted by $(\gamma)_n$, as a function from Γ to $\mathcal{D}'^{\odot n}$. (Here and below $\odot$ denotes the symmetric tensor product.) This will allow us to define 'γ *choose* n' by $\binom{\gamma}{n} := \frac{1}{n!}(\gamma)_n$. And we will get the following explicit formula which supports this term:

$$\binom{\gamma}{n} = \sum_{\{i_1,\dots,i_n\} \subset \mathbb{N}} \delta_{x_{i_1}} \odot \delta_{x_{i_2}} \odot \cdots \odot \delta_{x_{i_n}}, \tag{1.1}$$

i.e., the sum is obtained by choosing all possible n-point subsets from the (locally finite) set $\{x_i\}_{i\in\mathbb{N}}$. The latter set can be obviously identified with the configuration γ.

The paper is organized as follows. In Section 2 we discuss preliminaries. In particular, we recall the construction of a general Gel'fand triple $\Phi \subset \mathcal{H}_0 \subset \Phi'$, where Φ is a nuclear space and Φ' is the dual of Φ (a co-nuclear space) with respect to the center Hilbert space $\mathcal{H}_0$. We discuss continuous linear operators acting on Φ and Φ'. We further define the space $\mathcal{P}(\Phi')$ of polynomials on Φ'. We equip $\mathcal{P}(\Phi')$ with a nuclear space topology and consider the dual space of $\mathcal{P}(\Phi')$, denoted by $\mathcal{F}(\Phi')$. The space $\mathcal{F}(\Phi')$ has a natural (commutative) algebraic structure with respect to the symmetric tensor product, $\odot$. We also realize $\mathcal{F}(\Phi')$ as a space of formal series in tensor powers of $\xi \in \Phi$. We define differentiation on Φ' and a family of shift operators, $(E(\zeta))_{\zeta\in\Phi'}$. The shift operators are related to the differentiation operators by Boole's formula. We finally define shift-invariant continuous linear operators on $\mathcal{P}(\Phi')$. We denote the space of such operators by $\mathbb{S}(\mathcal{P}(\Phi'))$.

In Section 3, we give the definitions of a polynomial sequence on Φ', a monic polynomial sequence on Φ', and a monic polynomial sequence on Φ' of binomial type.

Starting from Section 4, we choose the Gel'fand triple as $\mathcal{D} \subset L^2(\mathbb{R}^d, dx) \subset \mathcal{D}'$. We formulate and prove the first main result of the paper, Theorem 4.1. In this theorem, we present three equivalent conditions for a monic polynomial sequence to be of binomial type. The first equivalent condition is that the corresponding lowering operators are shift-invariant. The second condition gives a representation of each lowering operator through directional derivatives in directions of delta functions, δ_x ($x \in \mathbb{R}^d$). The third condition gives the form of the generating function of a polynomial sequence of binomial type.

To prove Theorem 4.1, we derive two essential results. The first one is an operator expansion theorem (Theorem 4.7), which gives a description of any operator $T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$ in terms of the lowering operators in directions δ_x. The second result is an isomorphism theorem (Theorem 4.9): we construct a bijection $J : \mathbb{S}(\mathcal{P}(\mathcal{D}')) \to \mathcal{F}(\mathcal{D}')$ such that, for any two operators $S, T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$, we have $J(ST) = J(S) \odot J(T)$. This implies, in particular, that any two operators from $\mathbb{S}(\mathcal{P}(\mathcal{D}'))$ commute.

Next, we define a family of delta operators on $\mathcal{D}'$ and prove that, for each such family, there exists a unique monic polynomial sequence of binomial type for which these delta operators are the lowering operators.

In Section 5, we identify a procedure of the lifting of a polynomial sequence of binomial type on $\mathbb{R}$ to a polynomial sequence of binomial type on $\mathcal{D}'$. Using this procedure, we identify, on $\mathcal{D}'$, the falling factorials, the rising factorials, the Abel polynomials, and the Laguerre polynomials of binomial type. We stress that the polynomial sequences lifted from $\mathbb{R}$ to $\mathcal{D}'$ form a subset of a (much larger) set of all polynomial sequences of binomial type on $\mathcal{D}'$.

In Section 6, we define a Sheffer sequence on $\mathcal{D}'$ as a monic polynomial sequence on $\mathcal{D}'$ whose lowering operators are delta operators. Thus, to every Sheffer sequence, there corresponds a (unique) polynomial sequence of binomial type. In particular, if the corresponding binomial sequence is just the set of monomials (i.e., their delta operators are differential operators), we call such a Sheffer sequence an Appell sequence. The second main result of the paper, Theorem 6.2, gives several equivalent conditions for a monic polynomial sequence to be a Sheffer sequence. In particular, we find the generating function of a Sheffer sequence on $\mathcal{D}'$.

In Section 7, we extend the procedure of the lifting described in Section 5 to Sheffer sequences. Thus, for each Sheffer sequence on $\mathbb{R}$, we define a Sheffer sequence on $\mathcal{D}'$. Using this procedure, we recover, on $\mathcal{D}'$, the Hermite polynomials, the Charlier polynomials, and the orthogonal Laguerre polynomials.

Finally, in Appendix, we discuss several properties of formal tensor power series. We also introduce and discuss there the space of 'Φ-valued' formal series in tensor powers of $\xi \in \mathcal{D}$.

2 Preliminaries

2.1 Nuclear and co-nuclear spaces

Let us first recall the definition of a nuclear space, for details see e.g. [8, Chapter 14, Section 2.2]. Consider a family of real separable Hilbert spaces $(\mathcal{H}_\tau)_{\tau\in T}$, where T is an arbitrary indexing set. Assume that the set $\Phi := \bigcap_{\tau\in T}\mathcal{H}_\tau$ is dense in each Hilbert space $\mathcal{H}_\tau$ and the family $(\mathcal{H}_\tau)_{\tau\in T}$ is directed by embedding, i.e., for any $\tau_1, \tau_2 \in T$ there exists a $\tau_3 \in T$ such that $\mathcal{H}_{\tau_3} \subset \mathcal{H}_{\tau_1}$ and $\mathcal{H}_{\tau_3} \subset \mathcal{H}_{\tau_2}$ and both embeddings are continuous. We introduce in Φ the projective limit topology of the $\mathcal{H}_\tau$ spaces:

$$\Phi = \operatorname*{proj\,lim}_{\tau\in T}\mathcal{H}_\tau.$$

By definition, the sets $\{\varphi \in \Phi \mid \|\varphi - \psi\|_{\mathcal{H}_\tau} < \varepsilon\}$ with $\psi \in \Phi$, $\tau \in T$, and $\varepsilon > 0$ form a system of base neighborhoods in this topology. Here $\|\cdot\|_{\mathcal{H}_\tau}$ denotes the norm in $\mathcal{H}_\tau$.

Assume that, for each $\tau_1 \in T$, there exists a $\tau_2 \in T$ such that $\mathcal{H}_{\tau_2} \subset \mathcal{H}_{\tau_1}$, and the operator of embedding of $\mathcal{H}_{\tau_2}$ into $\mathcal{H}_{\tau_1}$ is of the Hilbert–Schmidt class. Then the linear topological space Φ is called *nuclear*.

Next, let us assume that, for some $\tau_0 \in T$, each Hilbert space $\mathcal{H}_\tau$ with $\tau \in T$ is continuously embedded into $\mathcal{H}_0 := \mathcal{H}_{\tau_0}$. We will call $\mathcal{H}_0$ the *center space*.

Let Φ' denote the dual space of Φ with respect to the center space $\mathcal{H}_0$, i.e., the dual pairing between Φ' and Φ is obtained by continuously extending the inner product in $\mathcal{H}_0$, see e.g. [8, Chapter 14, Section 2.3]. The space Φ' is often called *co-nuclear*.

By the Schwartz theorem (e.g. [8, Chapter 14, Theorem 2.1]), $\Phi' = \bigcup_{\tau\in T}\mathcal{H}_{-\tau}$, where $\mathcal{H}_{-\tau}$ denotes the dual space of $\mathcal{H}_\tau$ with respect to the center space $\mathcal{H}_0$. We endow Φ' with the Mackey topology—the strongest topology in Φ' consistent with the duality between Φ and Φ' (i.e., the set of continuous linear functionals on Φ' coincides with Φ). The Mackey topology in Φ' coincides with the topology of the inductive limit of the $\mathcal{H}_{-\tau}$ spaces, see e.g. [36, Chapter IV, Proposition 4.4] or [6, Chapter 1, Section 1]. Thus, we obtain the *Gel'fand triple* (also called the *standard triple*)

$$\Phi = \operatorname{proj}\lim_{\tau \in T} \mathcal{H}_\tau \subset \mathcal{H}_0 \subset \operatorname{ind}\lim_{\tau \in T} \mathcal{H}_{-\tau} = \Phi'. \tag{2.1}$$

Let X and Y be linear topological spaces that are locally convex and Hausdorff. (Both Φ and Φ' are such spaces.) We denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators acting from X into Y. We will also denote $\mathcal{L}(X) := \mathcal{L}(X, X)$. We denote by X' and Y' the dual space of X and Y, respectively. We endow X' with the Mackey topology with respect to the duality between X and X'. We similarly endow Y' with the Mackey topology.

Each operator $A \in \mathcal{L}(X, Y)$ has the *adjoint operator* $A^* \in \mathcal{L}(Y', X')$ (also called the transpose of A or the dual of A), see e.g. [28, Theorem 8.11.3].

Remark 2.1. Note that, since we chose the Mackey topology on Φ', for an operator $A \in \mathcal{L}(\Phi')$, we have $A^* \in \mathcal{L}(\Phi)$.

Proposition 2.2. *Consider the Gel'fand triple* (2.1). *Let $A : \Phi \to \Phi$ and $B : \Phi' \to \Phi'$ be linear operators.*

(i) *We have $A \in \mathcal{L}(\Phi)$ if and only if, for each $\tau_1 \in T$, there exists a $\tau_2 \in T$ such that the operator A can be extended by continuity to an operator $\hat{A} \in \mathcal{L}(\mathcal{H}_{\tau_2}, \mathcal{H}_{\tau_1})$.*

(ii) *We have $B \in \mathcal{L}(\Phi')$ if and only if, for each $\tau_1 \in T$, there exists a $\tau_2 \in T$ such that the operator $\hat{B} := B \restriction \mathcal{H}_{-\tau_1}$ takes on values in $\mathcal{H}_{-\tau_2}$ and $\hat{B} \in \mathcal{L}(\mathcal{H}_{-\tau_1}, \mathcal{H}_{-\tau_2})$.*

Remark 2.3. Proposition 2.2 admits a sraightforward generalization to the case of two Gel'fand triples, $\Phi \subset \mathcal{H}_0 \subset \Phi'$ and $\Psi \subset \mathcal{G}_0 \subset \Psi'$, and linear operators $A : \Phi \to \Psi$ and $B : \Phi' \to \Psi'$.

Remark 2.4. Part (ii) of Proposition 2.2 is related to the universal property of an inductive limit, which states that any linear operator from an inductive limit of a family of locally convex spaces to another locally convex space is continuous if and only if the restriction of the operator to any member of the family is continuous, see e.g. [9, II.29].

Proof of Proposition 2.2. (i) By the definition of the topology in Φ, the linear operator $A : \Phi \to \Phi$ is continuous if and only if, for any $\tau_1 \in T$ and $\varepsilon_1 > 0$, there exist $\tau_2 \in T$ and $\varepsilon_2 > 0$ such that the pre-image of the set

$$\{\varphi \in \Phi \mid \|\varphi\|_{\mathcal{H}_{\tau_1}} < \varepsilon_1\}$$

contains the set

$$\{\varphi \in \Phi \mid \|\varphi\|_{\mathcal{H}_{\tau_2}} < \varepsilon_2\}.$$

But this implies the statement.

(ii) Assume $B \in \mathcal{L}(\Phi')$. Then, by Remark 2.1, we have $B^* \in \mathcal{L}(\Phi)$. Hence, for each $\tau_1 \in T$, there exists a $\tau_2 \in T$ such that the operator B^* can be extended by continuity to an operator $\hat{B}^* \in \mathcal{L}(\mathcal{H}_{\tau_2}, \mathcal{H}_{\tau_1})$. But the adjoint of the operator $\hat{B}^*$ is $\hat{B} := B \restriction \mathcal{H}_{-\tau_1}$. Hence $\hat{B} \in \mathcal{L}(\mathcal{H}_{-\tau_1}, \mathcal{H}_{-\tau_2})$.

Conversely, assume that, for each $\tau_1 \in T$, there exists a $\tau_2 \in T$ such that the operator $\hat{B} := B \restriction \mathcal{H}_{-\tau_1}$ takes on values in $\mathcal{H}_{-\tau_2}$ and $\hat{B} \in \mathcal{L}(\mathcal{H}_{-\tau_1}, \mathcal{H}_{-\tau_2})$. Therefore, $\hat{B}^* \in \mathcal{L}(\mathcal{H}_{\tau_2}, \mathcal{H}_{\tau_1})$. Denote $A := \hat{B}^* \restriction \Phi$. As easily seen, the definition of the operator A does not depend on the choice of $\tau_1, \tau_2 \in T$. Hence, $A : \Phi \to \Phi$, and by part (i) we conclude that $A \in \mathcal{L}(\Phi)$. But $B = A^*$ and hence $B \in \mathcal{L}(\Phi')$. □

In what follows, $\otimes$ will denote the tensor product. In particular, for a real separable Hilbert space $\mathcal{H}$, $\mathcal{H}^{\otimes n}$ denotes the nth tensor power of $\mathcal{H}$. We will denote by $\mathrm{Sym}_n \in \mathcal{L}(\mathcal{H}^{\otimes n})$ the symmetrization operator, i.e., the orthogonal projection satisfying

$$\mathrm{Sym}_n f_1 \otimes f_2 \otimes \cdots \otimes f_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}(n)} f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \cdots \otimes f_{\sigma(n)} \tag{2.2}$$

for $f_1, f_2, \dots, f_n \in \mathcal{H}$. Here $\mathfrak{S}(n)$ denotes the symmetric group acting on $\{1, \dots, n\}$. We will denote the symmetric tensor product by $\odot$. In particular,

$$f_1 \odot f_2 \odot \cdots \odot f_n := \mathrm{Sym}_n f_1 \otimes f_2 \otimes \cdots \otimes f_n, \quad f_1, f_2, \dots, f_n \in \mathcal{H},$$

and $\mathcal{H}^{\odot n} := \mathrm{Sym}_n \mathcal{H}^{\otimes n}$ is the nth symmetric tensor power of $\mathcal{H}$. Note that, for each $f \in \mathcal{H}$, we have $f^{\odot n} = f^{\otimes n}$.

Starting with Gel'fand triple (2.1), one constructs its nth symmetric tensor power as follows:

$$\Phi^{\odot n} := \operatorname{proj\,lim}_{\tau \in T} \mathcal{H}_\tau^{\odot n} \subset \mathcal{H}_0^{\odot n} \subset \operatorname{ind\,lim}_{\tau \in T} \mathcal{H}_{-\tau}^{\odot n} =: \Phi'^{\odot n},$$

see e.g. [6, Section 2.1] for details. In particular, $\Phi^{\odot n}$ is a nuclear space and $\Phi'^{\odot n}$ is its dual with respect to the center space $\mathcal{H}_0^{\odot n}$. We will also denote $\Phi^{\odot 0} = \mathcal{H}_0^{\odot 0} = \Phi'^{\odot 0} := \mathbb{R}$. The dual pairing between $F^{(n)} \in \Phi'^{\odot n}$ and $g^{(n)} \in \Phi^{\odot n}$ will be denoted by $\langle F^{(n)}, g^{(n)} \rangle$.

Remark 2.5. Consider the set $\{\xi^{\otimes n} \mid \xi \in \Phi\}$. By the polarization identity, the linear span of this set is dense in every space $\mathcal{H}_\tau^{\odot n}$, $\tau \in T$.

The following lemma will be very important for our considerations.

Lemma 2.6. (i) *Let $F^{(n)}, G^{(n)} \in \Phi'^{\odot n}$ be such that*

$$\langle F^{(n)}, \xi^{\otimes n} \rangle = \langle G^{(n)}, \xi^{\otimes n} \rangle \quad \text{for all } \xi \in \Phi,$$

then $F^{(n)} = G^{(n)}$.

(ii) *Let Φ and Ψ be nuclear spaces and let $A, B \in \mathcal{L}(\Phi^{\odot n}, \Psi)$. Assume that*

$$A\xi^{\otimes n} = B\xi^{\otimes n} \quad \text{for all } \xi \in \Phi.$$

Then $A = B$.

Proof. Statement (i) follows from Remark 2.5, statement (ii) follows from Proposition 2.2, (i) and Remarks 2.3 and 2.5. □

2.2 Polynomials on a co-nuclear space

Below we fix the Gel'fand triple (2.1).

Definition 2.7. A function $P : \Phi' \to \mathbb{R}$ is called a *polynomial on* Φ' if

$$P(\omega) = \sum_{k=0}^{n} \langle \omega^{\otimes k}, f^{(k)} \rangle, \quad \omega \in \Phi', \tag{2.3}$$

where $f^{(k)} \in \Phi^{\odot k}$, $k = 0, 1, \dots, n$, $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, and $\omega^{\otimes 0} := 1$. If $f^{(n)} \neq 0$, one says that the *polynomial* P *is of degree* n. We denote by $\mathcal{P}(\Phi')$ the set of all polynomials on Φ'.

Remark 2.8. For each $P \in \mathcal{P}(\Phi')$, its representation in form (2.3) is evidently unique.

For any $f^{(k)} \in \Phi^{\odot k}$ and $g^{(n)} \in \Phi^{\odot n}$, $k, n \in \mathbb{N}_0$, we have

$$\langle \omega^{\otimes k}, f^{(k)} \rangle \langle \omega^{\otimes n}, g^{(n)} \rangle = \langle \omega^{\otimes (k+n)}, f^{(k)} \odot g^{(n)} \rangle, \quad \omega \in \Phi'. \tag{2.4}$$

Hence $\mathcal{P}(\Phi')$ is an algebra under point-wise multiplication of polynomials on Φ'.

We will now define a topology on $\mathcal{P}(\Phi')$. Let $\mathcal{F}_{\text{fin}}(\Phi)$ denote the topological direct sum of the nuclear spaces $\Phi^{\odot n}$, $n \in \mathbb{N}_0$. Hence, $\mathcal{F}_{\text{fin}}(\Phi)$ is a nuclear space, see e.g. [6, Section 5.1]. This space consists of all finite sequences $f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$, where $f^{(k)} \in \Phi^{\odot k}$, $k = 0, 1, \dots, n$, $n \in \mathbb{N}_0$. The convergence in $\mathcal{F}_{\text{fin}}(\Phi)$ means the uniform finiteness of non-zero elements and the coordinate-wise convergence in each $\Phi^{\odot k}$.

Remark 2.9. Below we will often identify $f^{(n)} \in \Phi^{\odot n}$ with

$$(0, \dots, 0, f^{(n)}, 0, 0, \dots) \in \mathcal{F}_{\text{fin}}(\Phi).$$

We define a natural bijective mapping $I : \mathcal{F}_{\text{fin}}(\Phi) \to \mathcal{P}(\Phi')$ by

$$(If)(\omega) := \sum_{k=0}^{n} \langle \omega^{\otimes k}, f^{(k)} \rangle, \tag{2.5}$$

for $f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots) \in \mathcal{F}_{\text{fin}}(\Phi)$. We define a nuclear space topology on $\mathcal{P}(\Phi')$ as the image of the topology on $\mathcal{F}_{\text{fin}}(\Phi)$ under the mapping I.

The space $\mathcal{F}_{\text{fin}}(\Phi)$ may be endowed with the structure of an algebra with respect to the symmetric tensor product

$$f \odot g = \left(\sum_{i=0}^{k} f^{(i)} \odot g^{(k-i)} \right)_{k=0}^{\infty}, \tag{2.6}$$

where $f = (f^{(k)})_{k=0}^{\infty}$, $g = (g^{(k)})_{k=0}^{\infty} \in \mathcal{F}_{\text{fin}}(\Phi)$. The unit element of this (commutative) algebra is the *vacuum vector* $\Omega := (1, 0, 0 \dots)$.

By (2.4)–(2.6), the bijective mapping I provides an isomorphism between the algebras $\mathcal{F}_{\mathrm{fin}}(\Phi)$ and $\mathcal{P}(\Phi')$, namely, for any $f, g \in \mathcal{F}_{\mathrm{fin}}(\Phi)$,

$$\big(I(f \odot g)\big)(\omega) = (If)(\omega)(Ig)(\omega), \quad \omega \in \Phi'.$$

Let

$$\mathcal{F}(\Phi') := \prod_{k=0}^{\infty} \Phi'^{\odot k}$$

denote the topological product of the spaces $\Phi'^{\odot k}$. The space $\mathcal{F}(\Phi')$ consists of all sequences $F = (F^{(k)})_{k=0}^{\infty}$, where $F^{(k)} \in \Phi'^{\odot k}$, $k \in \mathbb{N}_0$. Note that the convergence in this space means the coordinate-wise convergence in each space $\Phi'^{\odot k}$.

Each element $F = (F^{(k)})_{k=0}^{\infty} \in \mathcal{F}(\Phi')$ determines a continuous linear functional on $\mathcal{F}_{\mathrm{fin}}(\Phi)$ by

$$\langle F, f\rangle := \sum_{k=0}^{\infty} \langle F^{(k)}, f^{(k)}\rangle, \quad f = (f^{(k)})_{k=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(\Phi) \tag{2.7}$$

(note that the sum in (2.7) is, in fact, finite). The dual of $\mathcal{F}_{\mathrm{fin}}(\Phi)$ is equal to $\mathcal{F}(\Phi')$, and the topology on $\mathcal{F}(\Phi')$ coincides with the Mackey topology on $\mathcal{F}(\Phi')$ that is consistent with the duality between $\mathcal{F}_{\mathrm{fin}}(\Phi)$ and $\mathcal{F}(\Phi')$, see e.g. [6]. In view of the definition of the topology on $\mathcal{P}(\Phi')$, we may also think of $\mathcal{F}(\Phi')$ as the dual space of $\mathcal{P}(\Phi')$, equipped with the Mackey topology on $\mathcal{F}(\Phi')$ that is consistent with the duality between $\mathcal{P}(\Phi')$ and $\mathcal{F}(\Phi')$.

Similarly to (2.6), one can introduce the symmetric tensor product on $\mathcal{F}(\Phi')$:

$$F \odot G = \left(\sum_{i=0}^{k} F^{(i)} \odot G^{(k-i)}\right)_{k=0}^{\infty}, \tag{2.8}$$

where $F = (F^{(k)})_{k=0}^{\infty}$, $G = (G^{(k)})_{k=0}^{\infty} \in \mathcal{F}(\Phi')$. The unit element of this algebra is again $\Omega = (1, 0, 0, \dots)$.

We will now discuss another realization of the space $\mathcal{F}(\Phi')$. We denote by $\mathcal{S}(\mathbb{R}, \mathbb{R})$ the vector space of formal series $R(t) = \sum_{n=0}^{\infty} r_n t^n$ in powers of $t \in \mathbb{R}$, where $r_n \in \mathbb{R}$ for $n \in \mathbb{N}_0$. The $\mathcal{S}(\mathbb{R}, \mathbb{R})$ is an algebra under the product of formal power series. Similarly to $\mathcal{S}(\mathbb{R}, \mathbb{R})$, we give the following

Definition 2.10. Each $(F^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\Phi')$ identifies a *'real-valued' formal series* $\sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n}\rangle$ *in tensor powers of* $\xi \in \Phi$. We denote by $\mathcal{S}(\Phi, \mathbb{R})$ the vector space of such formal series with natural operations. We define a product on $\mathcal{S}(\Phi, \mathbb{R})$ by

$$\left(\sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n}\rangle\right) \left(\sum_{n=0}^{\infty} \langle G^{(n)}, \xi^{\otimes n}\rangle\right) = \sum_{n=0}^{\infty} \left\langle \sum_{i=0}^{n} F^{(i)} \odot G^{(n-i)}, \xi^{\otimes n} \right\rangle, \tag{2.9}$$

where $(F^{(n)})_{n=0}^{\infty}, (G^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\Phi')$.

Remark 2.11. Assume that, for some $(F^{(n)})_{n=0}^{\infty}, (G^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\Phi')$ and $\xi \in \Phi$, both series $\sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n} \rangle$ and $\sum_{n=0}^{\infty} \langle G^{(n)}, \xi^{\otimes n} \rangle$ converge absolutely. Then also the series on the right hand side of (2.9) converges absolutely and (2.9) holds as an equality of two real numbers.

Remark 2.12. Let $t \in \mathbb{R}$ and $\xi \in \Phi$. Then, $t\xi \in \Phi$ and for $(F^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\Phi')$,

$$\sum_{n=0}^{\infty} \langle F^{(n)}, (t\xi)^{\otimes n} \rangle = \sum_{n=0}^{\infty} t^n \langle F^{(n)}, \xi^{\otimes n} \rangle, \tag{2.10}$$

the expression on the right hand side of equality (2.10) being the formal power series in t that has coefficient $\langle F^{(n)}, \xi^{\otimes n} \rangle$ by t^n.

According to the definition of $\mathcal{S}(\Phi, \mathbb{R})$, there exists a natural bijective mapping $\mathcal{I} : \mathcal{F}(\Phi') \to \mathcal{S}(\Phi, \mathbb{R})$ given by

$$(\mathcal{I}F)(\xi) := \sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n} \rangle, \quad F = (F^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\Phi'), \quad \xi \in \Phi. \tag{2.11}$$

The mapping $\mathcal{I}$ provides an isomorphism between the algebras $\mathcal{F}(\Phi')$ and $\mathcal{S}(\Phi, \mathbb{R})$, namely, for any $F, G \in \mathcal{F}(\Phi')$,

$$\big(\mathcal{I}(F \odot G)\big)(\xi) = (\mathcal{I}F)(\xi)(\mathcal{I}G)(\xi) \tag{2.12}$$

see (2.8) and (2.9).

Remark 2.13. In view of the isomorphism $\mathcal{I}$, we may think of $\mathcal{S}(\Phi, \mathbb{R})$ as the dual space of $\mathcal{P}(\Phi')$.

Analogously to Definition 2.10 and Remark 2.12, we can introduce a space of Φ-valued tensor power series.

Definition 2.14. Let $(A_n)_{n=1}^{\infty}$ be a sequence of operators $A_n \in \mathcal{L}(\Phi^{\odot n}, \Phi)$. Then the operators $(A_n)_{n=1}^{\infty}$ identify a *'Φ-valued' formal series* $\sum_{n=1}^{\infty} A_n \xi^{\otimes n}$ *in tensor powers of* $\xi \in \Phi$. We denote by $\mathcal{S}(\Phi, \Phi)$ the vector space of such formal series.

Remark 2.15. Let $t \in \mathbb{R}$ and $\xi \in \Phi$. Then, $t\xi \in \Phi$ and for a sequence $(A_n)_{n=1}^{\infty}$ as in Definition 2.14

$$\sum_{n=1}^{\infty} A_n (t\xi)^{\otimes n} = \sum_{n=1}^{\infty} t^n A_n \xi^{\otimes n}$$

is the formal power series in t that has coefficient $A_n \xi^{\otimes n} \in \Phi$ by t^n. Recall that, by Lemma 2.6, (ii), the values of the operator A_n on the vectors $\xi^{\otimes n} \in \Phi^{\odot n}$ uniquely identify the operator A_n.

In Appendix, we discuss several properties of formal tensor power series.

2.3 Shift-invariant operators

Definition 2.16. For each $\zeta \in \Phi'$, we define the *operator* $D(\zeta) : \mathcal{P}(\Phi') \to \mathcal{P}(\Phi')$ *of differentiation in direction* ζ by

$$(D(\zeta)P)(\omega) := \lim_{t\to 0} \frac{P(\omega + t\zeta) - P(\omega)}{t}, \quad P \in \mathcal{P}(\Phi'),\ \omega \in \Phi'.$$

Definition 2.17. For each $\zeta \in \Phi'$, we define the *annihilation operator* $\mathfrak{A}(\zeta) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\Phi))$ by

$$\mathfrak{A}(\zeta)\Omega := 0, \quad \mathfrak{A}(\zeta)\xi^{\otimes n} := n\langle \zeta, \xi\rangle \xi^{\otimes(n-1)} \quad \text{for } \xi \in \Phi,\ n \in \mathbb{N}.$$

Lemma 2.18. *For each $\zeta \in \Phi'$, we have $D(\zeta) \in \mathcal{L}(\mathcal{P}(\Phi'))$.*

Proof. Using the bijection $I : \mathcal{F}_{\mathrm{fin}}(\Phi) \to \mathcal{P}(\Phi')$ defined by (2.5), we easily obtain

$$D(\zeta) = I\mathfrak{A}(\zeta)I^{-1}, \quad \zeta \in \Phi',$$

which implies the statement. □

Definition 2.19. For each $\zeta \in \Phi'$, we define the *operator* $E(\zeta) : \mathcal{P}(\Phi') \to \mathcal{P}(\Phi')$ *of shift by* ζ by

$$(E(\zeta)P)(\omega) := P(\omega + \zeta), \quad P \in \mathcal{P}(\Phi'),\ \omega \in \Phi'.$$

Lemma 2.20. *(Boole's formula) For each $\zeta \in \Phi'$,*

$$E(\zeta) = \sum_{k=0}^{\infty} \frac{1}{k!} D(\zeta)^k.$$

Proof. Note that the infinite sum is, in fact, a finite sum when applied to a polynomial, and thus it is a well-defined operator on $\mathcal{P}(\Phi')$. For each $\xi \in \Phi$ and $n \in \mathbb{N}$,

$$\begin{aligned}\big(E(\zeta)\langle \cdot^{\otimes n}, \xi^{\otimes n}\rangle\big)(\omega) &= \langle (\omega+\zeta)^{\otimes n}, \xi^{\otimes n}\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle \omega, \xi\rangle^k \langle \zeta, \xi\rangle^{n-k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \big(D(\zeta)^k \langle \cdot^{\otimes n}, \xi^{\otimes n}\rangle\big)(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \big(D(\zeta)^k \langle \cdot^{\otimes n}, \xi^{\otimes n}\rangle\big)(\omega),\end{aligned}$$

which implies the statement. □

Note that Lemmas 2.18 and 2.20 imply that $E(\zeta) \in \mathcal{L}(\mathcal{P}(\Phi'))$ for each $\zeta \in \Phi'$.

Definition 2.21. We say that an operator $T \in \mathcal{L}(\mathcal{P}(\Phi'))$ is *shift-invariant* if

$$TE(\zeta) = E(\zeta)T \quad \text{for all } \zeta \in \Phi'.$$

We denote the linear space of all shift-invariant operators by $\mathbb{S}(\mathcal{P}(\Phi'))$. The space $\mathbb{S}(\mathcal{P}(\Phi'))$ is an algebra under the usual product (composition) of operators.

Obviously, for each $\zeta \in \Phi'$, the operators $D(\zeta)$ and $E(\zeta)$ belong to $\mathbb{S}(\mathcal{P}(\Phi'))$.

3 Monic polynomial sequences on a co-nuclear space

For each $n \in \mathbb{N}_0$, we denote by $\mathcal{P}^{(n)}(\Phi')$ the subspace of $\mathcal{P}(\Phi')$ that consists of all polynomials on Φ' of degree $\le n$. We will now define a polynomial sequence on Φ' as a sequence $(P^{(n)})_{n=0}^{\infty}$ of continuous linear operators $P^{(n)}: \Phi^{\odot n} \to \mathcal{P}^{(n)}(\Phi')$ satisfying a certain condition on the 'leading coefficient.'

Let $P^{(n)} \in \mathcal{L}\big(\Phi^{\odot n}, \mathcal{P}^{(n)}(\Phi')\big)$, $n \in \mathbb{N}_0$. Then, by the definition of $\mathcal{P}^{(n)}(\Phi')$, there exist operators $V_{k,n} \in \mathcal{L}(\Phi^{\odot n}, \Phi^{\odot k})$, $k = 0, 1, \dots, n$, such that, for any $f^{(n)} \in \Phi^{\odot n}$ and $\omega \in \Phi'$

$$\big(P^{(n)} f^{(n)}\big)(\omega) = \sum_{k=0}^{n} \langle \omega^{\otimes k}, V_{k,n} f^{(n)} \rangle \tag{3.1}$$

$$= \sum_{k=0}^{n} \langle U_{n,k}\, \omega^{\otimes k}, f^{(n)} \rangle, \tag{3.2}$$

where $U_{n,k} := V_{k,n}^{*} \in \mathcal{L}(\Phi'^{\odot k}, \Phi'^{\odot n})$. With an abuse of notation, we define a mapping $P^{(n)} : \Phi' \to \Phi'^{\odot n}$ by

$$P^{(n)}(\omega) := \sum_{k=0}^{n} U_{n,k} \omega^{\otimes k}, \quad \omega \in \Phi'. \tag{3.3}$$

Then formula (3.2) becomes

$$\big(P^{(n)} f^{(n)}\big)(\omega) = \langle P^{(n)}(\omega), f^{(n)} \rangle. \tag{3.4}$$

Conversely, every mapping $P^{(n)} : \Phi' \to \Phi'^{\odot n}$ of the form (3.3) with operators $U_{n,k} \in \mathcal{L}(\Phi'^{\odot k}, \Phi'^{\odot n})$ determines an operator $P^{(n)} \in \mathcal{L}(\Phi^{\odot n}, \mathcal{P}^{(n)}(\Phi'))$ by formula (3.1) with $V_{k,n} := U_{n,k}^{*} \in \mathcal{L}(\Phi^{\odot n}, \Phi^{\odot k})$.

Definition 3.1. A *polynomial sequence on* Φ' is defined as a family of polynomials on Φ' of the form

$$\big(\langle P^{(n)}(\omega), f^{(n)} \rangle\big)_{f^{(n)} \in \Phi^{\odot n},\, n \in \mathbb{N}_0}. \tag{3.5}$$

Here, for each $n \in \mathbb{N}_0$, $P^{(n)} : \Phi' \to \Phi'^{\odot n}$ is a mapping of the form (3.3) with $U_{n,k} \in \mathcal{L}(\Phi'^{\odot k}, \Phi'^{\odot n})$, $k = 0, 1, \dots, n$, and $U_{n,n} \in \mathcal{L}(\Phi'^{\odot n})$ is a homeomorphism. If additionally, for each $n \in \mathbb{N}_0$, $U_{n,n} = \mathbf{1}$, the identity operator on $\Phi'^{\odot n}$, then we call (3.5) a *monic polynomial sequence on* Φ'.

Since we have a one-to-one correspondence between the monic polynomial sequences on Φ' of the form (3.5) and the sequences $(P^{(n)})_{n=0}^{\infty}$ of mappings of the form (3.3) with $U_{n,n} = \mathbf{1}$, we will also call such a sequence $(P^{(n)})_{n=0}^{\infty}$ a *monic polynomial sequence on* Φ'.

Remark 3.2. Below, to simplify notations, we will only deal with monic polynomial sequences. The results of this paper can be extended to the case of a general polynomial sequence on $\Phi' = \mathcal{D}'$.

Remark 3.3. By the definition of a monic polynomial sequence we get

$$\langle P^{(n)}(\omega), f^{(n)}\rangle = \langle \omega^{\otimes n}, f^{(n)}\rangle + \sum_{k=0}^{n-1}\langle \omega^{\otimes k}, V_{k,n} f^{(n)}\rangle, \tag{3.6}$$

where $V_{k,n} := U_{n,k}^* \in \mathcal{L}(\Phi^{\odot n}, \Phi^{\odot k})$.

Lemma 3.4. *Let $(P^{(n)})_{n=0}^\infty$ be a monic polynomial sequence on Φ'. The following statements hold.*

(i) *There exist operators $R_{k,n} \in \mathcal{L}(\Phi^{\odot n}, \Phi^{\odot k})$, $k = 0, 1, \dots, n-1$, $n \in \mathbb{N}$, such that, for all $\omega \in \Phi'$ and $f^{(n)} \in \Phi^{\odot n}$,*

$$\langle \omega^{\otimes n}, f^{(n)}\rangle = \langle P^{(n)}(\omega), f^{(n)}\rangle + \sum_{k=0}^{n-1}\langle P^{(k)}(\omega), R_{k,n} f^{(n)}\rangle. \tag{3.7}$$

(ii) *We have*

$$\mathcal{P}(\Phi') = \left\{ \sum_{k=0}^{n}\langle P^{(k)}, f^{(k)}\rangle \mid f^{(k)} \in \Phi^{\odot k},\ k = 0, 1, \dots, n,\ n \in \mathbb{N}_0 \right\}. \tag{3.8}$$

Proof. (i) We prove by induction on n. For $n = 1$, the statement trivially holds. Assume that the statement holds for $1, 2, \dots, n$. Then, by using (3.6) and the induction assumption, we get

$$\langle \omega^{\otimes(n+1)}, f^{(n+1)}\rangle = \langle P^{(n+1)}(\omega), f^{(n+1)}\rangle - \sum_{k=0}^{n}\langle \omega^{\otimes k}, V_{k,n+1} f^{(n+1)}\rangle$$

$$= \langle P^{(n+1)}(\omega), f^{(n+1)}\rangle - \sum_{k=0}^{n}\left(\langle P^{(k)}(\omega), V_{k,n+1} f^{(n+1)}\rangle + \sum_{i=0}^{k-1}\langle P^{(i)}(\omega), R_{i,k} V_{k,n+1} f^{(n+1)}\rangle\right),$$

which implies the statement for $n+1$.

(ii) This follows immediately from (i). □

Definition 3.5. Let $(P^{(n)})_{n=0}^\infty$ be a monic polynomial sequence on Φ'. For each $\zeta \in \Phi'$, we define a *lowering operator* $Q(\zeta)$ as the linear operator on $\mathcal{P}(\Phi')$ (cf. (3.8)) satisfying

$$Q(\zeta)\langle P^{(n)}, f^{(n)}\rangle := \langle P^{(n-1)}, \mathfrak{A}(\zeta) f^{(n)}\rangle, \quad f^{(n)} \in \Phi^{\odot n},\ n \in \mathbb{N},$$
$$Q(\zeta)\langle P^{(0)}, f^{(0)}\rangle := 0, \quad f^{(0)} \in \mathbb{R},$$

where the operator $\mathfrak{A}(\zeta)$ is defined by Definition 2.17.

Lemma 3.6. *For every $\zeta \in \Phi'$, we have $Q(\zeta) \in \mathcal{L}(\mathcal{P}(\Phi'))$.*

Proof. We define an operator $R \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\Phi))$ by setting, for each $f^{(n)} \in \Phi^{\odot n}$,

$$\big(Rf^{(n)}\big)^{(k)} := \begin{cases} R_{k,n} f^{(n)}, & k < n \\ f^{(n)}, & k = n, \\ 0, & k > n, \end{cases}$$

where the operators $R_{k,n}$ are as in (3.7). Similarly, using the operators $V_{k,n}$ from formula (3.6), we define an operator $V \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(\Phi))$. As easily seen,

$$Q(\zeta) = IV\mathfrak{A}(\zeta)RI^{-1},$$

where I is the homeomorphism defined by (2.5). This implies the required result. □

The simplest example of a monic polynomial sequence on Φ' is $P^{(n)}(\omega) = \omega^{\otimes n}$, $n \in \mathbb{N}_0$. In this case, for $f^{(n)} \in \Phi^{\odot n}$, $\langle P^{(n)}(\omega), f^{(n)}\rangle = \langle \omega^{\otimes n}, f^{(n)}\rangle$ is just a *monomial on* Φ' *of degree* n. For each $\zeta \in \Phi'$, we obviously have $Q(\zeta) = D(\zeta)$, i.e., the corresponding lowering operators are just differentiation operators. Furthermore, we trivially see in this case that, for any $n \in \mathbb{N}$ and any $\omega, \zeta \in \Phi'$,

$$P^{(n)}(\omega + \zeta) = \sum_{k=0}^{n} \binom{n}{k} P^{(k)}(\omega) \odot P^{(n-k)}(\zeta). \tag{3.9}$$

Definition 3.7. Let $(P^{(n)})_{n=0}^{\infty}$ be a monic polynomial sequence on Φ'. We say that $(P^{(n)})_{n=0}^{\infty}$ is of *binomial type* if, for any $n \in \mathbb{N}$ and any $\omega, \zeta \in \Phi'$, formula (3.9) holds.

Remark 3.8. A monic polynomial sequence $(P^{(n)})_{n=0}^{\infty}$ is of binomial type if and only if, for any $n \in \mathbb{N}$, $\omega, \zeta \in \Phi'$, and $\xi \in \Phi$,

$$\langle P^{(n)}(\omega + \zeta), \xi^{\otimes n}\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle P^{(k)}(\omega), \xi^{\otimes k}\rangle \langle P^{(n-k)}(\zeta), \xi^{\otimes (n-k)}\rangle.$$

The following lemma will be important for our considerations.

Lemma 3.9. *Let* $(P^{(n)})_{n=0}^{\infty}$ *be a monic polynomial sequence on* Φ' *of binomial type. Then, for each* $n \in \mathbb{N}$, $P^{(n)}(0) = 0$.

Proof. We proceed by induction on n. For $n = 1$, it follows from (3.9) that

$$P^{(1)}(\omega + \zeta) = P^{(1)}(\omega) + P^{(1)}(\zeta), \quad \omega, \zeta \in \Phi'.$$

Setting $\zeta = 0$, one obtains $P^{(1)}(0) = 0$. Assume that the statement holds for $1, 2, \dots n$. Then, for all $\omega, \zeta \in \Phi'$,

$$P^{(n+1)}(\omega + \zeta) = P^{(n+1)}(\omega) + \sum_{k=1}^{n} \binom{n+1}{k} P^{(k)}(\omega) \odot P^{(n+1-k)}(\zeta) + P^{(n+1)}(\zeta).$$

Setting $\zeta = 0$, we conclude $P^{(n+1)}(0) = 0$. □

4 Equivalent characterizations of a polynomial sequence of binomial type

Our next aim is to derive equivalent characterizations of a polynomial sequence on Φ' of binomial type. As mentioned in Introduction, it will be important for our considerations that Φ' will be chosen as a space of generalized functions.

So we fix $d \in \mathbb{N}$ and choose Φ to be the nuclear space $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$ of all real-valued smooth functions on $\mathbb{R}^d$ with compact support. More precisely, let T denote the set of all pairs (l, φ) with $l \in \mathbb{N}_0$ and $\varphi \in C^\infty(\mathbb{R}^d)$, $\varphi(x) \geq 1$ for all $x \in \mathbb{R}^d$. For each $\tau = (l, \varphi) \in T$, we denote by $\mathcal{H}_\tau$ the Sobolev space $W^{l,2}(\mathbb{R}^d, \varphi(x)\,dx)$. Then

$$\mathcal{D} = \operatorname*{proj\,lim}_{\tau \in T} \mathcal{H}_\tau,$$

see [8, Chapter 14, Subsec. 4.3] for details. As the center space $\mathcal{H}_0 = \mathcal{H}_{\tau_0}$ we choose $L^2(\mathbb{R}^d, dx)$ (i.e., $\tau_0 = (0, 1)$). Thus, we obtain the Gel'fand triple

$$\mathcal{D} \subset L^2(\mathbb{R}^d, dx) \subset \mathcal{D}'.$$

Note that nuclear space $\mathcal{D}^{\odot n}$ consists of all functions from $C_0^\infty((\mathbb{R}^d)^n)$ that are symmetric in the variables $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$.

For each $x \in \mathbb{R}^d$, the delta function δ_x belongs to $\mathcal{D}'$, and we will use the notations

$$D(x) := D(\delta_x), \quad E(x) := E(\delta_x), \quad Q(x) := Q(\delta_x)$$

(the latter operator being defined for a given fixed monic polynomial sequence on $\mathcal{D}'$).

Below, for any $F^{(k)} \in \mathcal{D}'^{\odot k}$ and $f^{(k)} \in \mathcal{D}^{\odot k}$, $k \in \mathbb{N}_0$, we denote

$$\langle F^{(k)}(x_1, \dots, x_k), f^{(k)}(x_1, \dots, x_k) \rangle := \langle F^{(k)}, f^{(k)} \rangle.$$

We are now ready to state the first main result.

Theorem 4.1. *Let $(P^{(n)})_{n=0}^\infty$ be a monic polynomial sequence on $\mathcal{D}'$ such that $P^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be the corresponding lowering operators. Then the following conditions are equivalent:*

(BT1) *The sequence $(P^{(n)})_{n=0}^\infty$ is of binomial type.*

(BT2) *For each $\zeta \in \mathcal{D}'$, $Q(\zeta)$ is shift-invariant.*

(BT3) *There exists a sequence $(B_k)_{k=1}^\infty$ with $B_k \in \mathcal{L}(\mathcal{D}', \mathcal{D}'^{\odot k})$, $k \geq 2$ and $B_1 = \mathbf{1}$, the identity operator on $\mathcal{D}'$, such that for all $\zeta \in \mathcal{D}'$ and $P \in \mathcal{P}(\mathcal{D}')$,*

$$(Q(\zeta)P)(\omega) = \sum_{k=1}^\infty \frac{1}{k!} \big\langle (B_k\zeta)(x_1, \dots, x_k), (D(x_1) \cdots D(x_k)P)(\omega) \big\rangle, \quad \omega \in \mathcal{D}'. \tag{4.1}$$

(BT4) *The monic polynomial sequence* $(P^{(n)})_{n=0}^{\infty}$ *has the generating function*

$$\sum_{n=0}^{\infty}\frac{1}{n!}\langle P^{(n)}(\omega),\xi^{\otimes n}\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}\langle\omega^{\otimes n},A(\xi)^{\otimes n}\rangle$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\langle\omega,A(\xi)\rangle^{n}=\exp\big[\langle\omega,A(\xi)\rangle\big],\quad \omega\in\mathcal{D}'. \tag{4.2}$$

Here

$$A(\xi)=\sum_{k=1}^{\infty}A_k\xi^{\otimes k}\in\mathcal{S}(\mathcal{D},\mathcal{D}), \tag{4.3}$$

where $A_k\in\mathcal{L}(\mathcal{D}^{\odot k},\mathcal{D})$, $k\ge 2$, *and* $A_1=\mathbf{1}$, *the identity operator on* $\mathcal{D}$, *while* (4.2) *is an equality in* $\mathcal{S}(\mathcal{D},\mathbb{R})$.

Remark 4.2. Note that $\sum_{n=0}^{\infty}\frac{1}{n!}\langle\omega^{\otimes n},A(\xi)^{\otimes n}\rangle\in\mathcal{S}(\mathcal{D},\mathbb{R})$ in formula (4.2) is the composition of $\exp[\langle\omega,\xi\rangle]:=\sum_{n=0}^{\infty}\frac{1}{n!}\langle\omega^{\otimes n},\xi^{\otimes n}\rangle\in\mathcal{S}(\mathcal{D},\mathbb{R})$ and $A(\xi)\in\mathcal{S}(\mathcal{D},\mathcal{D})$.

Remark 4.3. Let $A(\xi)\in\mathcal{S}(\mathcal{D},\mathcal{D})$ be as in (4.2). Denote

$$B(\xi):=\sum_{k=1}^{\infty}\frac{1}{k!}B_k^*\xi^{\otimes k}\in\mathcal{S}(\mathcal{D},\mathcal{D}),$$

where the operators B_k are as in (BT3). It will follow from the proof of Theorem 4.1 that

$$B(A(\xi))=\xi.$$

See Definition A.4 for the meaning of the composition $B(A(\xi))$ and Proposition A.5 for explicit formulas which connect the operators $(A_k)_{k=1}^{\infty}$ and $(B_k^*)_{k=1}^{\infty}$.

Remark 4.4. It will also follow from the proof of Theorem 4.1 that, in (BT3), for each $k\ge 2$, we have $B_k=R_{1,k}^*$, the adjoint of the operator $R_{1,k}\in\mathcal{L}(\mathcal{D}^{\odot k},\mathcal{D})$ from formula (3.7).

Before proving this theorem, let us first note its immediate corollary.

Corollary 4.5. *Consider any sequence* $(A_k)_{k=1}^{\infty}$ *with* $A_k\in\mathcal{L}(\mathcal{D}^{\odot k},\mathcal{D})$, $k\ge 2$, *and* $A_1=\mathbf{1}$. *Then there exists a unique sequence* $(P^{(n)})_{n=0}^{\infty}$ *of monic polynomials on* $\mathcal{D}'$ *of binomial type that has the generating function* (4.2) *with* $A(\xi)$ *given by* (4.3).

Proof. Define $A(\xi)\in\mathcal{S}(\mathcal{D},\mathcal{D})$ by formula (4.3). For each $\omega\in\mathcal{D}'$, define $(\frac{1}{n!}P^{(n)}(\omega))_{n=0}^{\infty}\in\mathcal{F}(\mathcal{D}')$ by formula (4.2). It easily follows from Definitions 3.1 and A.6 that $(P^{(n)})_{n=0}^{\infty}$ is a monic polynomial sequence on $\mathcal{D}'$. Furthermore, for $n\in\mathbb{N}$, in the representation (3.3) of $P^{(n)}(\omega)$, we obtain $U_{n,0}=0$ so that $P^{(n)}(0)=0$. Now the statement follows from Theorem 4.1. □

We will now prove Theorem 4.1.

Proof of (BT1) $\Rightarrow$ (BT2)*.* First, we note that, for any $\eta, \zeta \in \mathcal{D}'$,

$$E(\zeta)Q(\eta)1 = Q(\eta)E(\zeta)1 = 0. \tag{4.4}$$

Next, using the the binomial identity (3.9), we get, for all $\xi \in \mathcal{D}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
Q(\eta)E(\zeta)\langle P^{(n)}, \xi^{\otimes n}\rangle &= \sum_{k=0}^{n} \binom{n}{k} \langle P^{(n-k)}(\zeta), \xi^{\otimes(n-k)}\rangle Q(\eta)\langle P^{(k)}, \xi^{\otimes k}\rangle \\
&= \langle \eta, \xi\rangle \sum_{k=1}^{n} \binom{n}{k} k \langle P^{(n-k)}(\zeta), \xi^{\otimes(n-k)}\rangle \langle P^{(k-1)}, \xi^{\otimes(k-1)}\rangle \\
&= n\langle \eta, \xi\rangle \sum_{k=1}^{n} \binom{n-1}{k-1} \langle P^{(n-k)}(\zeta), \xi^{\otimes(n-k)}\rangle \langle P^{(k-1)}, \xi^{\otimes(k-1)}\rangle \\
&= n\langle \eta, \xi\rangle \sum_{k=0}^{n-1} \binom{n-1}{k} \langle P^{(n-k-1)}(\zeta), \xi^{\otimes(n-k-1)}\rangle \langle P^{(k)}, \xi^{\otimes k}\rangle \\
&= n\langle \eta, \xi\rangle E(\zeta)\langle P^{(n-1)}, \xi^{\otimes(n-1)}\rangle \\
&= E(\zeta)Q(\eta)\langle P^{(n)}, \xi^{\otimes n}\rangle.
\end{aligned} \tag{4.5}$$

By (4.4), (4.5), and Lemma 3.4, we get $E(\zeta)Q(\eta)P = Q(\eta)E(\zeta)P$ for all $P \in \mathcal{P}(\mathcal{D}')$. $\square$

In order to prove the implication (BT2) $\Rightarrow$ (BT1), we first need the following

Theorem 4.6 (Polynomial expansion theorem)**.** *Let $(P^{(n)})_{n=0}^{\infty}$ be a monic polynomial sequence on $\mathcal{D}'$ such that $P^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, or, equivalently, for $P^{(n)}$ being of the form* (3.3)*, $U_{n,0} = 0$. Let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be the corresponding lowering operators. Then, for each $P \in \mathcal{P}(\mathcal{D}')$, we have*

$$P(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \big\langle P^{(k)}(\omega)(x_1, \dots, x_k), (Q(x_1) \cdots Q(x_k)P)(0)\big\rangle, \quad \omega \in \mathcal{D}'. \tag{4.6}$$

Here, for $k = 0$, we set $Q(x_1) \cdots Q(x_k)P := P$.

Proof. For $x_1, \dots, x_k \in \mathbb{R}^d$, $k \in \mathbb{N}$, $\xi \in \mathcal{D}$, and $n \in \mathbb{N}$, we have

$$Q(x_1) \cdots Q(x_k)\langle P^{(n)}, \xi^{\otimes n}\rangle = (n)_k \xi(x_1) \cdots \xi(x_k)\langle P^{(n-k)}, \xi^{\otimes(n-k)}\rangle, \tag{4.7}$$

where $(n)_k := n(n-1) \cdots (n-k+1)$. Note that $(n)_k = 0$ for $k > n$. Hence, for $k, n \in \mathbb{N}_0$, one finds

$$\big(Q(x_1) \cdots Q(x_k)\langle P^{(n)}, \xi^{\otimes n}\rangle\big)(0) = \delta_{k,n} n!\, \xi^{\otimes n}(x_1, \dots, x_n),$$

where $\delta_{k,n}$ denotes the Kronecker symbol. Thus,

$$\langle P^{(n)}(\omega), \xi^{\otimes n}\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \big\langle P^{(k)}(\omega)(x_1, \dots, x_k), \left(Q(x_1) \cdots Q(x_k) \langle P^{(n)}, \xi^{\otimes n}\rangle\right)(0)\big\rangle.$$

Hence, by Lemma 3.4, formula (4.6) holds for a generic $P \in \mathcal{P}(\mathcal{D}')$. $\square$

Proof of (BT2) $\Rightarrow$ (BT1). Let $\zeta \in \mathcal{D}'$, $\xi \in \mathcal{D}$, and $n \in \mathbb{N}$. An application of Theorem 4.6 to the polynomial $P = E(\zeta)\langle P^{(n)}, \xi^{\otimes n}\rangle$ yields

$$\langle P^{(n)}(\omega+\zeta), \xi^{\otimes n}\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \big\langle P^{(k)}(\omega)(x_1, \dots, x_k), (Q(x_1) \cdots Q(x_k) E(\zeta) \langle P^{(n)}, \xi^{\otimes n}\rangle)(0)\big\rangle, \tag{4.8}$$

and by (BT2) and (4.7), we have, for $k \in \mathbb{N}$,

$$\begin{aligned} \left(Q(x_1) \cdots Q(x_k) E(\zeta) \langle P^{(n)}, \xi^{\otimes n}\rangle\right)(0) &= \left(E(\zeta) Q(x_1) \cdots Q(x_k) \langle P^{(n)}, \xi^{\otimes n}\rangle\right)(0) \\ &= \left(Q(x_1) \cdots Q(x_k) \langle P^{(n)}, \xi^{\otimes n}\rangle\right)(\zeta) \\ &= (n)_k \zeta(x_1) \cdots \zeta(x_k) \langle P^{(n-k)}(\zeta), \xi^{\otimes (n-k)}\rangle. \end{aligned}$$

Hence,

$$\langle P^{(n)}(\omega+\zeta), \xi^{\otimes n}\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle P^{(k)}(\omega), \xi^{\otimes k}\rangle \langle P^{(n-k)}(\zeta), \xi^{\otimes (n-k)}\rangle. \quad \square$$

Thus, we have proved the equivalence of (BT1) and (BT2). To continue the proof of Theorem 4.1, we will need the following result.

Theorem 4.7 (Operator expansion theorem)**.** *Let $(P^{(n)})_{n=0}^{\infty}$ be a monic polynomial sequence on $\mathcal{D}'$ of binomial type, and let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be the corresponding lowering operators. A linear operator T acting on $\mathcal{P}(\mathcal{D}')$ is continuous and shift-invariant if and only if there is a $(G^{(k)})_{k=0}^{\infty} \in \mathcal{F}(\mathcal{D}')$ such that, for each $P \in \mathcal{P}(\mathcal{D}')$,*

$$(TP)(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \left\langle G^{(k)}(x_1, \dots, x_k), (Q(x_1) \cdots Q(x_k) P)(\omega)\right\rangle, \quad \omega \in \mathcal{D}'. \tag{4.9}$$

In the latter case, for each $k \in \mathbb{N}_0$ and $f^{(k)} \in \mathcal{D}^{\odot k}$,

$$\langle G^{(k)}, f^{(k)}\rangle = \left(T\langle P^{(k)}, f^{(k)}\rangle\right)(0). \tag{4.10}$$

Remark 4.8. Below we will sometimes write formula (4.9) in the form

$$T = \sum_{k=0}^{\infty} \frac{1}{k!} \left\langle G^{(k)}(x_1, \dots, x_k), Q(x_1) \cdots Q(x_k)\right\rangle.$$

Proof of Theorem 4.7. Let $n \in \mathbb{N}$, $\omega, \zeta \in \mathcal{D}'$, and $\xi \in \mathcal{D}$. By (4.8), we have

$$\big(E(\zeta)\langle P^{(n)}, \xi^{\otimes n}\rangle\big)(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!}\big\langle P^{(k)}(\omega)(x_1, \dots, x_k), (Q(x_1)\cdots Q(x_k)\langle P^{(n)}, \xi^{\otimes n}\rangle)(\zeta)\big\rangle. \tag{4.11}$$

Note that formula (4.11) remains true when $n = 0$. Hence, by Lemma 3.4, for each $P \in \mathcal{P}(\mathcal{D}')$,

$$\big(E(\zeta)P\big)(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!}\big\langle P^{(k)}(\omega)(x_1, \dots, x_k), (Q(x_1)\cdots Q(x_k)P)(\zeta)\big\rangle. \tag{4.12}$$

Assume $T \in \mathcal{L}(\mathcal{P}(\mathcal{D}'))$ is shift-invariant. Swapping ω and ζ in (4.12) and applying T to this equality, we get, for any $\omega, \zeta \in \mathcal{D}'$ and $P \in \mathcal{P}(\mathcal{D}')$,

$$(TE(\omega)P)(\zeta) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(T\big\langle P^{(k)}(\cdot)(x_1, \dots, x_k), (Q(x_1)\cdots Q(x_k)P)(\omega)\big\rangle\right)(\zeta). \tag{4.13}$$

By shift-invariance, the left hand side of (4.13) is equal to $(TP)(\omega + \zeta)$. In particular, this holds for $\zeta = 0$:

$$(TP)(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(T\big\langle P^{(k)}(\cdot)(x_1, \dots, x_k), (Q(x_1)\cdots Q(x_k)P)(\omega)\big\rangle\right)(0). \tag{4.14}$$

Let $G^{(k)} \in \mathcal{D}'^{\odot k}$, $k \in \mathbb{N}_0$, be defined by (4.10). Then (4.9) follows from (4.14).

Conversely, let $(G^{(k)})_{k=0}^{\infty} \in \mathcal{F}(\mathcal{D}')$ be fixed, and let T be given by (4.9). As easily seen, $T \in \mathcal{L}(\mathcal{P}(\mathcal{D}'))$. For each $P \in \mathcal{P}(\mathcal{D}')$ and $\zeta \in \mathcal{D}'$, we get from (4.9) and (BT2):

$$\begin{aligned}(TE(\zeta)P)(\omega) &= \sum_{k=0}^{\infty} \frac{1}{k!}\left\langle G^{(k)}(x_1, \dots, x_k), (E(\zeta)Q(x_1)\cdots Q(x_k)P)(\omega)\right\rangle \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left\langle G^{(k)}(x_1, \dots, x_k), (Q(x_1)\cdots Q(x_k)P)(\omega + \zeta)\right\rangle \\ &= (TP)(\omega + \zeta) = (E(\zeta)TP)(\omega).\end{aligned}$$

Therefore, the operator T is shift-invariant. Moreover, it easily follows from (4.9) and (4.7) that (4.10) holds. □

Note that the statement (BT3) ⇒ (BT2) follows immediately from Theorem 4.7.

Proof of (BT2) ⇒ (BT3). Let $\zeta \in \mathcal{D}'$. We apply Theorem 4.7 to the sequence of monomials and its family of lowering operators, $(D(\eta))_{\eta \in \mathcal{D}'}$, and the shift-invariant operator $Q(\zeta)$. By using also formula (3.7), we obtain

$$Q(\zeta) = \sum_{k=1}^{\infty} \frac{1}{k!}\big\langle G^{(k)}(\zeta, x_1, \dots, x_k), D(x_1)\cdots D(x_k)\big\rangle, \tag{4.15}$$

where

$$\langle G^{(k)}(\zeta, x_1, \dots, x_k), f^{(k)}(x_1, \dots, x_k)\rangle = \left(Q(\zeta)\langle \cdot^{\otimes k}, f^{(k)}\rangle\right)(0) = \langle \zeta, R_{1,k} f^{(k)}\rangle \qquad (4.16)$$

for all $k \in \mathbb{N}$ and $f^{(k)} \in \mathcal{D}^{\odot k}$. Here, we set $R_{1,1} := \mathbf{1}$, the identity operator on $\mathcal{D}$. For $k \in \mathbb{N}$, we denote $B_k := R_{1,k}^* \in \mathcal{L}(\mathcal{D}', \mathcal{D}'^{\odot k})$. Note that $B_1 = \mathbf{1}$, the identity operator on $\mathcal{D}'$. By (4.16),

$$G^{(k)}(\zeta, \cdot) = B_k \zeta, \quad k \geq 1. \qquad (4.17)$$

Formulas (4.15), (4.17) imply (BT3). □

Thus, we have proved the equivalence of (BT1), (BT2), and (BT3).

According to Theorem 4.7, under the conditions assumed therein, there is a one-to-one correspondence between shift-invariant operators T and sequences $(\frac{1}{k!}\, G^{(k)})_{k=0}^{\infty} \in \mathcal{F}(\mathcal{D}')$. We noted above that the space $\mathbb{S}(\mathcal{P}(\mathcal{D}'))$ of shift-invariant operators is an algebra under the product of operators, while $\mathcal{F}(\mathcal{D}')$ is a commutative algebra under the symmetric tensor product.

Theorem 4.9 (The isomorphism theorem)**.** *Let $(P^{(n)})_{n=0}^{\infty}$ be a sequence of monic polynomials on $\mathcal{D}'$ of binomial type, and let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be the corresponding lowering operators. Then, the correspondence given by Theorem 4.7,*

$$\mathbb{S}(\mathcal{P}(\mathcal{D}')) \ni T \mapsto JT := \left(\frac{1}{k!}\, G^{(k)}\right)_{k=0}^{\infty} \in \mathcal{F}(\mathcal{D}'),$$

is an algebra isomorphism.

Proof. In view of Theorem 4.7, we only have to prove that, for any $S, T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$,

$$J(ST) = JS \odot JT. \qquad (4.18)$$

Let

$$JS = \left(\frac{1}{k!}\, F^{(k)}\right)_{k=0}^{\infty}, \quad JT = \left(\frac{1}{k!}\, G^{(k)}\right)_{k=0}^{\infty}.$$

By Theorem 4.7, for all $\xi \in \mathcal{D}$ and $\omega \in \mathcal{D}'$,

$$\left(T\langle P^{(n)}, \xi^{\otimes n}\rangle\right)(\omega) = \sum_{k=0}^{n} \binom{n}{k} \langle G^{(k)}, \xi^{\otimes k}\rangle \langle P^{(n-k)}(\omega), \xi^{\otimes (n-k)}\rangle, \qquad (4.19)$$

and a similar expression holds for S. Therefore,

$$\left(ST\langle P^{(n)}, \xi^{\otimes n}\rangle\right)(\omega)$$
$$= \sum_{k=0}^{n} \binom{n}{k} \langle G^{(k)}, \xi^{\otimes k}\rangle \left(S\langle P^{(n-k)}, \xi^{\otimes (n-k)}\rangle\right)(\omega)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \langle G^{(k)}, \xi^{\otimes k} \rangle \sum_{i=0}^{n-k} \binom{n-k}{i} \langle F^{(i)}, \xi^{\otimes i} \rangle \langle P^{(n-k-i)}(\omega), \xi^{\otimes (n-k-i)} \rangle$$

$$= \sum_{k=0}^{n} \sum_{i=0}^{n-k} \frac{n!}{k!\, i!\, (n-k-i)!} \langle G^{(k)} \odot F^{(i)}, \xi^{\otimes (k+i)} \rangle \langle P^{(n-k-i)}(\omega), \xi^{\otimes (n-k-i)} \rangle$$

$$= \sum_{j=0}^{n} \binom{n}{j} \left\langle \sum_{k=0}^{j} \binom{j}{k} G^{(k)} \odot F^{(j-k)}, \xi^{\otimes j} \right\rangle \langle P^{(n-j)}(\omega), \xi^{\otimes (n-j)} \rangle.$$

From here and (4.19), formula (4.18) follows. □

As an immediate consequence of Theorem 4.9, we conclude

Corollary 4.10. *Any two shift-invariant operators commute.*

Corollary 4.11. *Let the conditions of Theorem 4.9 be satisfied and let the operator $J : \mathbb{S}(\mathcal{P}(\mathcal{D}')) \to \mathcal{F}(\mathcal{D}')$ be defined as in that theorem. Define $\mathcal{J} : \mathbb{S}(\mathcal{P}(\mathcal{D}')) \to \mathcal{S}(\mathcal{D}, \mathbb{R})$ by $\mathcal{J} := \mathcal{I}J$. Here $\mathcal{I} : \mathcal{F}(\mathcal{D}') \to \mathcal{S}(\mathcal{D}, \mathbb{R})$ is defined by (2.11). Then, for each $T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$, we have*

$$(\mathcal{J}T)(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(T \langle P^{(n)}, \xi^{\otimes n} \rangle \right)(0). \tag{4.20}$$

Furthermore, $\mathcal{J}$ is an algebra isomorphism, i.e., for any $S, T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$, we have

$$\big(\mathcal{J}(ST)\big)(\xi) = (\mathcal{J}S)(\xi)(\mathcal{J}T)(\xi). \tag{4.21}$$

Proof. Formula (4.20) follows Theorem 4.7 and the definition of $\mathcal{J}$. Formula (4.21) is a consequence of (2.12) and Theorem 4.9. □

Corollary 4.12. *Let $T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$. The operator T is invertible if and only if $T1 \neq 0$. Furthermore, if $T1 \neq 0$, then $T^{-1} \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$.*

Proof. If $T1 = 0$, then the kernel of T is not equal to $\{0\}$. Hence, T is not invertible.

Assume $T1 \neq 0$. Let the isomorphism J from Theorem 4.9 be constructed through the monomials and the corresponding lowering operators $D(\zeta)$, $\zeta \in \mathcal{D}'$. So formula (4.20) becomes

$$(\mathcal{J}T)(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(T \langle \cdot^{\otimes n}, \xi^{\otimes n} \rangle \right)(0). \tag{4.22}$$

Since $T1 \neq 0$, formula (4.22), Corollary 4.11, and Proposition A.1 imply the existence of an operator $S \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$ such that $ST = TS = \mathbf{1}$. Hence, the operator T is invertible and $T^{-1} = S \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$. □

Proof of (BT3) $\Rightarrow$ (BT4). Let the isomorphism J from Theorem 4.9 be constructed through the monomials and the corresponding lowering operators $D(\zeta)$, $\zeta \in \mathcal{D}'$. By (4.22),

$$(\mathcal{J}D(\zeta))(\xi) = \langle \zeta, \xi \rangle.$$

Thus, by Lemma 2.20 and the isomorphism theorem,

$$(\mathcal{J}E(\zeta))(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathcal{J}D(\zeta)^n)(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \zeta^{\otimes n}, \xi^{\otimes n} \rangle = \exp[\langle \zeta, \xi \rangle]. \tag{4.23}$$

Let $G^{(k)} \in \mathcal{D}'^{\odot k}$, $k \in \mathbb{N}$. Then formula (4.22) with

$$T = \big\langle G^{(k)}(x_1, \dots, x_k), D(x_1) \cdots D(x_k) \big\rangle$$

yields

$$(\mathcal{J}\big\langle G^{(k)}(x_1, \dots, x_k), D(x_1) \cdots D(x_k) \big\rangle)(\xi) = \langle G^{(k)}, \xi^{\otimes k} \rangle.$$

Therefore, condition (BT3) gives, for each $\zeta \in \mathcal{D}'$,

$$(\mathcal{J}Q(\zeta))(\xi) = \sum_{k=1}^{\infty} \frac{1}{k!} \langle B_k \zeta, \xi^{\otimes k} \rangle = \sum_{k=1}^{\infty} \frac{1}{k!} \langle \zeta, R_{1,k} \xi^{\otimes k} \rangle. \tag{4.24}$$

In the latter equality we used the fact that $R_{1,k}^* = B_k$, see the proof of (BT2) $\Rightarrow$ (BT3). Choosing in (4.24) $\zeta = \delta_x$, $x \in \mathbb{R}^d$, we obtain

$$(\mathcal{J}Q(x))(\xi) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(R_{1,k} \xi^{\otimes k} \right)(x),$$

and, more generally, by Theorem 4.9, for any $x_1, \dots, x_k \in \mathbb{R}^d$, $k \in \mathbb{N}$,

$$(\mathcal{J}Q(x_1) \cdots Q(x_k))(\xi) = \prod_{i=1}^{k} \left(\sum_{n=1}^{\infty} \frac{1}{n!} \left(R_{1,n} \xi^{\otimes n} \right)(x_i) \right). \tag{4.25}$$

By (4.22) and (4.25), we get, for each $G^{(k)} \in \mathcal{D}'^{\odot k}$, $k \in \mathbb{N}$,

$$\begin{aligned}
&\big(\mathcal{J}\langle G^{(k)}(x_1, \dots, x_k), Q(x_1) \cdots Q(x_k) \rangle\big)(\xi) \\
&\quad = \big\langle G^{(k)}(x_1, \dots, x_k), \mathcal{J}(Q(x_1) \cdots Q(x_k))(\xi) \big\rangle \\
&\quad = \left\langle G^{(k)}(x_1, \dots, x_k), \prod_{i=1}^{k} \left(\sum_{n=1}^{\infty} \frac{1}{n!} \left(R_{1,n} \xi^{\otimes n} \right)(x_i) \right) \right\rangle \\
&\quad = \left\langle G^{(k)}, \left(\sum_{n=1}^{\infty} \frac{1}{n!} R_{1,n} \xi^{\otimes n} \right)^{\otimes k} \right\rangle.
\end{aligned} \tag{4.26}$$

By (4.12) and (4.26),

$$(\mathcal{J}E(\zeta))(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \left\langle P^{(k)}(\zeta), \left(\sum_{n=1}^{\infty} \frac{1}{n!} R_{1,n} \xi^{\otimes n} \right)^{\otimes k} \right\rangle. \tag{4.27}$$

Formulas (4.23) and (4.27) imply

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left\langle P^{(k)}(\zeta), \left(\sum_{n=1}^{\infty} \frac{1}{n!} R_{1,n} \xi^{\otimes n} \right)^{\otimes k} \right\rangle = \exp[\langle \zeta, \xi \rangle]. \tag{4.28}$$

According to Proposition A.5, (i), we find operators $A_k \in \mathcal{L}(\mathcal{D}^{\odot k}, \mathcal{D})$, $k \in \mathbb{N}$, with $A_1 = \mathbf{1}$, that satisfy

$$\sum_{n=1}^{\infty} \frac{1}{n!} R_{1,n} \left(\sum_{k=1}^{\infty} A_k \xi^{\otimes k} \right)^{\otimes n} = \xi. \tag{4.29}$$

For $k \geq 2$, these operators are given by the recurrence formula

$$A_k = -\sum_{n=2}^{k} \frac{1}{n!} R_{1,n} \sum_{\substack{(l_1, \dots, l_n) \in \mathbb{N}^n \\ l_1 + \dots + l_n = k}} A_{l_1} \odot \dots \odot A_{l_n}. \tag{4.30}$$

Formula (4.2) now follows from Corollary A.10 and formulas (4.28), (4.29). □

Proof of (BT4) ⇒ (BT1). By (BT4), we have, for any $\omega, \zeta \in \mathcal{D}'$,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\omega + \zeta), \xi^{\otimes n} \rangle &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\omega), \xi^{\otimes n} \rangle \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} \langle P^{(k)}(\zeta), \xi^{\otimes k} \rangle \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{k=0}^{n} \binom{n}{k} P^{(k)}(\omega) \odot P^{(n-k)}(\zeta), \xi^{\otimes n} \right\rangle, \end{aligned}$$

which implies (BT1). This concludes the proof of Theorem 4.1. □

Definition 4.13. Let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be a family of operators from $\mathcal{L}(\mathcal{P}(\mathcal{D}'))$. We say that $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ is a *family of delta operators* if the following conditions are satisfied:

(i) Each $Q(\zeta)$ is shift-invariant;

(ii) For each $\zeta \in \mathcal{D}'$ and each $\xi \in \mathcal{D}$,

$$Q(\zeta) \langle \cdot, \xi \rangle = \langle \zeta, \xi \rangle; \tag{4.31}$$

(iii) $Q(\zeta)$ linearly depends on $\zeta \in \mathcal{D}'$. Furthermore, for each $k \geq 2$, the mapping $\mathcal{D}' \ni \zeta \mapsto B_k\zeta \in \mathcal{D}'^{\odot k}$ defined by

$$\langle B_k\zeta, f^{(k)}\rangle := \left(Q(\zeta)\langle \cdot^{\otimes k}, f^{(k)}\rangle\right)(0), \quad f^{(k)} \in \mathcal{D}^{\odot k}, \tag{4.32}$$

belongs to $\mathcal{L}(\mathcal{D}', \mathcal{D}'^{\odot k})$.

It is a straightforward consequence of Theorem 4.1 that, for any monic polynomial sequence $(P^{(n)})_{n=0}^{\infty}$ of binomial type, the corresponding family $(Q(\zeta))_{\zeta\in\mathcal{D}'}$ of lowering operators is a family of delta operators.

Proposition 4.14. *Let $(Q(\zeta))_{\zeta\in\mathcal{D}'}$ be a family of delta operators. Then, there exists a unique monic polynomial sequence $(P^{(n)})_{n=0}^{\infty}$ of binomial type for which $(Q(\zeta))_{\zeta\in\mathcal{D}'}$ is the family of lowering operators.*

Proof. Let $\zeta, \omega \in \mathcal{D}'$ and $\xi \in \mathcal{D}$. By shift-invariance of $Q(\zeta)$ and (4.31), we get

$$\begin{aligned}\langle \zeta, \xi\rangle &= E(\omega)\langle \zeta, \xi\rangle = E(\omega)Q(\zeta)\langle \cdot, \xi\rangle = Q(\zeta)E(\omega)\langle \cdot, \xi\rangle \\ &= Q(\zeta)\langle \cdot + \omega, \xi\rangle = Q(\zeta)\langle \cdot, \xi\rangle + \langle \omega, \xi\rangle Q(\zeta)1 = \langle \zeta, \xi\rangle + \langle \omega, \xi\rangle Q(\zeta)1,\end{aligned}$$

which implies that $Q(\zeta)1 = 0$. Hence, by Theorem 4.7, (4.31), and (4.32), we have

$$Q(\zeta) = \sum_{k=1}^{\infty} \frac{1}{k!}\big\langle (B_k\zeta)(x_1, \dots, x_k), D(x_1)\cdots D(x_k)\big\rangle,$$

with $B_1 := \mathbf{1}$, the identity operator on $\mathcal{D}'$.

Let $A_1 := \mathbf{1}$ be the identity operator on $\mathcal{D}$, and for $k \geq 2$, let the operators $A_k \in \mathcal{L}(\mathcal{D}^{\odot k}, \mathcal{D})$ be defined by the recurrence formula (4.30) with $R_{1,k} := B_k^*$. Thus, (4.29) holds.

Let $(P^{(n)})_{n=0}^{\infty}$ be the monic polynomial sequence on $\mathcal{D}'$ of binomial type that has the generating function (4.2) with $A(\xi)$ given by (4.3), see Corollary 4.5. By Remark 4.3, $(P^{(n)})_{n=0}^{\infty}$ is the unique required polynomial sequence. □

Definition 4.15. Let $(Q(\zeta))_{\zeta\in\mathcal{D}'}$ be a family of delta operators. The corresponding monic polynomial sequence $(P^{(n)})_{n=0}^{\infty}$ of binomial type given by Proposition 4.14 is called the *basic sequence for* $(Q(\zeta))_{\zeta\in\mathcal{D}'}$.

Proposition 4.16. *$(Q(\zeta))_{\zeta\in\mathcal{D}'}$ is a family of delta operators if and only if there exists a sequence $(B_k)_{k=1}^{\infty}$, with $B_k \in \mathcal{L}(\mathcal{D}', \mathcal{D}'^{\odot k})$, such that $B_1 = \mathbf{1}$ and (4.1) holds.*

Proof. The statement follows immediately from Theorem 4.1 and Proposition 4.14. □

5 Lifting of polynomials on $\mathbb{R}$ of binomial type

Let $(p_n)_{n=0}^{\infty}$ be a monic polynomial sequence on $\mathbb{R}$ of binomial type, and let Q be its delta operator, that is, $Qp_n = np_{n-1}$ for each $n \in \mathbb{N}_0$. According to the one-dimensional (classical) version of Theorem 4.1 (see e.g. [22]), Q has a formal expansion

$$Q = \sum_{k=1}^{\infty} \frac{b_k}{k!} D^k = q(D), \tag{5.1}$$

where $(b_k)_{k\in\mathbb{N}}$ is a sequence of real numbers such that $b_1 = 1$, D is the differentiation operator and

$$q(t) := \sum_{k=1}^{\infty} \frac{b_k}{k!} t^k \tag{5.2}$$

is a formal power series in $t \in \mathbb{R}$. Furthermore,

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} p_n(t) = \exp[ta(u)], \tag{5.3}$$

where the formal power series in $u \in \mathbb{R}$,

$$a(u) = \sum_{k=1}^{\infty} a_k u^k \tag{5.4}$$

is the compositional inverse of q. In particular, $a_1 = 1$. We will now lift the sequence of polynomials $(p_n)_{n=0}^{\infty}$ to a monic polynomial sequence on $\mathcal{D}'$ of binomial type.

For each $k \in \mathbb{N}$, we define an operator $\mathbb{D}_k \in \mathcal{L}(\mathcal{D}^{\odot k}, \mathcal{D})$ by

$$(\mathbb{D}_k f^{(k)})(x) := f^{(k)}(x, \dots, x), \quad f^{(k)} \in \mathcal{D}^{\odot k},\ x \in \mathbb{R}^d \tag{5.5}$$

($\mathbb{D}_1$ being the identity operator on $\mathcal{D}$). The adjoint operator $\mathbb{D}_k^* \in \mathcal{L}(\mathcal{D}', \mathcal{D}'^{\odot k})$ satisfies

$$\langle \mathbb{D}_k^* \zeta, f^{(k)} \rangle = \langle \zeta(x), f^{(k)}(x, \dots, x) \rangle, \quad \zeta \in \mathcal{D}',\ f^{(k)} \in \mathcal{D}^{\odot k}.$$

In particular,

$$\langle \mathbb{D}_k^* \zeta, \xi^{\otimes k} \rangle = \langle \zeta, \xi^k \rangle, \quad \zeta \in \mathcal{D}',\ \xi \in \mathcal{D}. \tag{5.6}$$

We now define an operator $B_k : \mathcal{D}' \to \mathcal{D}'^{\odot k}$ by

$$B_k := b_k \mathbb{D}_k^*, \quad k \in \mathbb{N}, \tag{5.7}$$

where the numbers b_k are as in (5.1). Let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be the family of delta operators given by (4.1), see Proposition 4.16. By (5.6) and (5.7), we then have

$$Q(x) = \sum_{k=1}^{\infty} \frac{b_k}{k!} D^k(x) = q(D(x)), \quad x \in \mathbb{R}^d, \tag{5.8}$$

and moreover,

$$Q(\zeta) = \left\langle \zeta(x), \sum_{k=1}^{\infty} \frac{b_k}{k!}\, D^k(x) \right\rangle = \big\langle \zeta(x), q(D(x)) \big\rangle. \tag{5.9}$$

Let $(P^{(n)})_{n=0}^{\infty}$ be the basic sequence for $(Q(\zeta))_{\zeta\in\mathcal{D}'}$. Thus, in view of (5.1) and (5.8), we may think of $(P^{(n)})_{n=0}^{\infty}$ as the *lifting of the monic polynomial sequence* $(p_n)_{n=0}^{\infty}$ *of binomial type*.

As easily seen, the generating function of $(P^{(n)})_{n=0}^{\infty}$ is given by (4.2) with the operators $A_k \in \mathcal{L}(\mathcal{D}^{\odot k}, \mathcal{D})$ given by

$$A_k = a_k \mathbb{D}_k, \quad k \in \mathbb{N},$$

where the numbers a_k are as in (5.4). Therefore, by (4.3),

$$A(\xi) = \sum_{k=1}^{\infty} a_k \xi^k = a(\xi). \tag{5.10}$$

Hence,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\omega), \xi^{\otimes n} \rangle = \exp\left[\left\langle \omega(x), \sum_{k=1}^{\infty} a_k \xi^k(x) \right\rangle \right] = \exp\big[\langle \omega, a(\xi)\rangle\big]. \tag{5.11}$$

Thus, this generating function can be though of as the lifting of the generating function (5.3).

Recall that a *set partition* π *of a set* $\mathcal{X} \neq \varnothing$ is an (unordered) collection of disjoint nonempty subsets of $\mathcal{X}$ whose union equals $\mathcal{X}$. We denote by $\mathfrak{P}(n)$ the collection of all set partitions of $\mathcal{X} = \{1, 2, \dots, n\}$. For a set $B \subset \{1, \dots, n\}$, we denote by $|B|$ the cardinality of B.

Proposition 5.1. *Let* $(P^{(n)})_{n=0}^{\infty}$ *be the monic polynomial sequence on* $\mathcal{D}'$ *of binomial type that has the generating function* (5.11)*. For* $k \in \mathbb{N}$*, denote* $\alpha_k := a_k k!$*, so that*

$$a(u) = \sum_{k=1}^{\infty} \frac{\alpha_k}{k!}\, u^k. \tag{5.12}$$

Then, for any $n \in \mathbb{N}$*,* $\omega \in \mathcal{D}'$*, and* $\xi \in \mathcal{D}$*,*

$$\langle P^{(n)}(\omega), \xi^{\otimes n} \rangle = \sum_{\pi\in\mathfrak{P}(n)} \prod_{B\in\pi} \alpha_{|B|} \langle \omega, \xi^{|B|} \rangle, \tag{5.13}$$

or equivalently

$$P^{(n)}(\omega) = \sum_{k=1}^{n} \sum_{\{B_1, B_2, \dots, B_k\}\in\mathfrak{P}(n)} \alpha_{|B_1|}\alpha_{|B_1|}\cdots\alpha_{|B_k|} \times \big(\mathbb{D}^*_{|B_1|}\omega^{\otimes|B_1|}\big) \odot \big(\mathbb{D}^*_{|B_2|}\omega^{\otimes|B_2|}\big) \odot \cdots \odot \big(\mathbb{D}^*_{|B_k|}\omega^{\otimes|B_k|}\big). \tag{5.14}$$

Proof. It follows immediately from the form of the generating function (5.11) that

$$\begin{aligned}\langle P^{(n)}(\omega), \xi^{\otimes n}\rangle &= \sum_{k=1}^{n} \frac{n!}{k!} \sum_{\substack{(i_1,\dots,i_k)\in\mathbb{N}^k\\ i_1+\dots+i_k=n}} a_{i_1}\cdots a_{i_k}\langle\omega,\xi^{i_1}\rangle\cdots\langle\omega,\xi^{i_k}\rangle\\ &= \sum_{\substack{(j_1,j_2\dots,j_n)\in\mathbb{N}_0^n\\ j_1+2j_2+\dots+nj_n=n}} \frac{n!}{j_1!j_2!\cdots j_n!\,(1!)^{j_1}(2!)^{j_2}\cdots(n!)^{j_n}}\,\alpha_1^{j_1}\alpha_2^{j_2}\cdots\alpha_n^{j_n}\\ &\quad\times\langle\omega,\xi\rangle^{j_1}\langle\omega,\xi^2\rangle^{j_2}\cdots\langle\omega,\xi^n\rangle^{j_n}.\end{aligned}$$

which implies (5.13), hence also (5.14). □

We denote by $\mathbb{M}(\mathbb{R}^d)$ the space of all signed Radon measures on $\mathbb{R}^d$, i.e., the set of all signed measures η on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $|\eta|(\Lambda) < \infty$ for all $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$. Here $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra on $\mathbb{R}^d$, $\mathcal{B}_0(\mathbb{R}^d)$ denotes the collection of all bounded sets $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, and $|\eta|$ denotes the variation of $\eta \in \mathbb{M}(\mathbb{R}^d)$.

Further, let $\mathcal{B}_{\mathrm{sym}}((\mathbb{R}^d)^n)$ denote the sub-σ-algebra of $\mathcal{B}((\mathbb{R}^d)^n)$ that consists of all symmetric sets $\Delta \in \mathcal{B}((\mathbb{R}^d)^n)$, i.e., for each permutation $\sigma \in \mathfrak{S}(n)$, Δ is an invariant set for the mapping

$$(\mathbb{R}^d)^n \ni (x_1,\dots,x_n) \mapsto (x_{\sigma(1)},\dots,x_{\sigma(n)}) \in (\mathbb{R}^d)^n.$$

We denote by $\mathbb{M}_{\mathrm{sym}}((\mathbb{R}^d)^n)$ the space of all signed Radon measures on $((\mathbb{R}^d)^n, \mathcal{B}_{\mathrm{sym}}((\mathbb{R}^d)^n))$.

Corollary 5.2. *Let $(P^{(n)})_{n=0}^\infty$ be the monic polynomial sequence on $\mathcal{D}'$ of binomial type that has the generating function* (5.11)*. Then, for each $\eta \in \mathbb{M}(\mathbb{R}^d)$ and $n \in \mathbb{N}$, we have $P^{(n)}(\eta) \in \mathbb{M}_{\mathrm{sym}}((\mathbb{R}^d)^n)$. Furthermore, for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$,*

$$\big(P^{(n)}(\eta)\big)(\Lambda^n) = p_n(\eta(\Lambda)), \quad n \in \mathbb{N}, \tag{5.15}$$

where $(p_n)_{n=0}^\infty$ is the polynomial sequence on $\mathbb{R}$ with generating function (5.3)*.*

Proof. Let $\eta \in \mathbb{M}(\mathbb{R}^d)$. For each $j \in \mathbb{N}$, $\mathbb{D}_j^*\eta \in \mathbb{M}_{\mathrm{sym}}((\mathbb{R}^d)^j)$, since for each $f^{(j)} \in \mathcal{D}^{\odot j}$

$$\langle \mathbb{D}_j^*\eta, f^{(j)}\rangle = \int_{\mathbb{R}^d} f^{(j)}(x,\dots,x)\,d\eta(x).$$

Note that the measure $\mathbb{D}_j^*\eta$ is concentrated on the set

$$\{(x_1,x_2,\dots,x_j) \in (\mathbb{R}^d)^j \mid x_1 = x_2 = \dots = x_j\}.$$

By formula (5.14), we therefore get $P^{(n)}(\eta) \in \mathbb{M}_{\mathrm{sym}}((\mathbb{R}^d)^n)$.

Fix any $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$. Set $\xi := u\chi_\Lambda$, where $u \in \mathbb{R}$ and χ_Λ denotes the indicator function of Λ. It easily follows from (5.11) by an approximation argument that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{u^n}{n!} \big(P^{(n)}(\eta)\big)(\Lambda^n) &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\eta), \xi^{\otimes n}\rangle \\ &= \exp\left[\int_{\mathbb{R}^d} a(\xi(x))\, d\eta(x)\right] \\ &= \exp\left[\eta(\Lambda)a(u)\right]. \end{aligned} \tag{5.16}$$

In formula (5.16), $\langle P^{(n)}(\eta), \xi^{\otimes n}\rangle$ denotes the integral of the function $\xi^{\otimes n}$ with respect to the measure $P^{(n)}(\eta)$. Formula (5.15) now follows from (5.3) and (5.16). □

The following proposition shows that the lifted polynomials $(P^{(n)})_{n=0}^{\infty}$ have an additional property of binomial type.

Proposition 5.3. *Let $(P^{(n)})_{n=0}^{\infty}$ be the monic polynomial sequence on $\mathcal{D}'$ of binomial type that has the generating function* (5.11)*. Let $\xi, \phi \in \mathcal{D}$ be such that*

$$\{x \in \mathbb{R}^d \mid \xi(x) \neq 0\} \cap \{x \in \mathbb{R}^d \mid \phi(x) \neq 0\} = \varnothing. \tag{5.17}$$

Then, for any $\omega \in \mathcal{D}'$ and $k, n \in \mathbb{N}$,

$$\langle P^{(k+n)}(\omega), \xi^{\otimes k} \odot \phi^{\otimes n}\rangle = \langle P^{(k)}(\omega), \xi^{\otimes k}\rangle\langle P^{(n)}(\omega), \phi^{\otimes n}\rangle. \tag{5.18}$$

Therefore, for each $n \in \mathbb{N}$,

$$\langle P^{(n)}(\omega), (\xi+\phi)^{\otimes n}\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle P^{(k)}(\omega), \xi^{\otimes k}\rangle\langle P^{(n-k)}(\omega), \phi^{\otimes (n-k)}\rangle. \tag{5.19}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!}\langle P^{(n)}(\omega), (\xi+\phi)^{\otimes n}\rangle &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\, \langle P^{(n)}(\omega), \xi^{\otimes k} \odot \phi^{\otimes (n-k)}\rangle \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{k!\, n!}\langle P^{(n+k)}(\omega), \phi^{\otimes n} \odot \xi^{\otimes k}\rangle, \end{aligned} \tag{5.20}$$

and by (5.11) and (5.17)

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!}\langle P^{(n)}(\omega), (\xi+\phi)^{\otimes n}\rangle &= \exp\left[\langle \omega, a(\xi+\phi)\rangle\right] \\ &= \exp\left[\langle \omega, a(\xi)\rangle\right] \exp\left[\langle \omega, a(\phi)\rangle\right] \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{k!\, n!}\langle P^{(k)}(\omega), \xi^{\otimes k}\rangle\langle P^{(n)}(\omega), \phi^{\otimes n}\rangle. \end{aligned} \tag{5.21}$$

Formulas (5.20), (5.21) imply (5.18), hence also (5.19). □

We will now consider examples of sequences of lifted polynomials of binomial type.

5.1 Falling factorials on $\mathcal{D}'$

The classical falling factorials is the sequence $(p_n)_{n=0}^\infty$ of monic polynomials on $\mathbb{R}$ of binomial type that are explicitly given by

$$p_n(t) = (t)_n := t(t-1)(t-2)\cdots(t-n+1).$$

The corresponding delta operator is $Q = e^D - 1$, so that Q is the difference operator $(Qp)(t) = p(t+1) - p(t)$. Here p belongs to $\mathcal{P}(\mathbb{R})$, the space of polynomials on $\mathbb{R}$. The generating function of the falling factorials is

$$\sum_{n=0}^\infty \frac{u^n}{n!}(t)_n = \exp[t\log(1+u)] = (1+u)^t.$$

One also defines an extension of the binomial coefficient,

$$\binom{t}{n} := \frac{1}{n!}\,(t)_n = \frac{t(t-1)(t-2)\cdots(t-n+1)}{n!}, \tag{5.22}$$

which becomes the classical binomial coefficient for $t \in \mathbb{N}$, $t \geq n$.

Let us now consider the corresponding lifted sequence of polynomials, $(P^{(n)})_{n=0}^\infty$. We will call these polynomials the *falling factorials on* $\mathcal{D}'$. By analogy with the one-dimensional case, we will write $(\omega)_n := P^{(n)}(\omega)$ for $\omega \in \mathcal{D}'$.

By (5.8), $Q(x) = e^{D(x)} - 1$. Hence, by Boole's formula,

$$(Q(x)P)(\omega) = P(\omega + \delta_x) - P(\omega), \quad x \in \mathbb{R}^d,\ P \in \mathcal{P}(\mathcal{D}'), \tag{5.23}$$

and by (5.9),

$$(Q(\zeta)P)(\omega) = \big\langle \zeta(x), P(\omega+\delta_x) - P(\omega)\big\rangle, \quad \zeta \in \mathcal{D}',\ P \in \mathcal{P}(\mathcal{D}').$$

Further, by (5.11), the generating function is given by

$$\sum_{n=0}^\infty \frac{1}{n!}\langle (\omega)_n, \xi^{\otimes n}\rangle = \exp\big[\langle \omega, \log(1+\xi)\rangle\big]. \tag{5.24}$$

Proposition 5.4. *The falling factorials on $\mathcal{D}'$ have the following explicit form:*

$$(\omega)_0 = 1;$$
$$(\omega)_1 = \omega;$$
$$(\omega)_n(x_1,\dots,x_n) = \omega(x_1)(\omega(x_2) - \delta_{x_1}(x_2))$$
$$\times (\omega(x_3) - \delta_{x_1}(x_3) - \delta_{x_2}(x_3))\cdots(\omega(x_n) - \delta_{x_1}(x_n) - \cdots - \delta_{x_{n-1}}(x_n)) \tag{5.25}$$

for $n \geq 2$.

Proof. Let $((\omega)_n)_{n=0}^\infty$ denote the monic polynomial sequence on $\mathcal{D}'$ defined by formula (5.25). Note that, for $n \in \mathbb{N}$, $(\omega)_n = 0$ for $\omega = 0$. It can be easily shown by induction that the polynomials $(\omega)_n$ satisfy the following recurrence relation:

$$\begin{aligned}
&(\omega)_0 = 1;\\
&\langle(\omega)_{n+1}, \xi^{\otimes(n+1)}\rangle = \langle(\omega)_n \odot \omega, \xi^{\otimes(n+1)}\rangle - n\langle(\omega)_n, \xi^2 \odot \xi^{\otimes(n-1)}\rangle,\\
&\qquad \xi \in \mathcal{D},\ n \in \mathbb{N}_0.
\end{aligned} \tag{5.26}$$

Furthermore, this recurrence relation uniquely determines the polynomials $((\omega)_n)_{n=0}^\infty$. It suffices to prove that, for each $\omega \in \mathcal{D}'$, $n \in \mathbb{N}$, $\xi \in \mathcal{D}$, and $x \in \mathbb{R}^d$,

$$\big(Q(x)\langle(\cdot)_n, \xi^{\otimes n}\rangle\big)(\omega) = n\xi(x)\langle(\omega)_{n-1}, \xi^{\otimes(n-1)}\rangle,$$

or equivalently, by (5.23),

$$\langle(\omega + \delta_x)_n, \xi^{\otimes n}\rangle = \langle(\omega)_n, \xi^{\otimes n}\rangle + n\xi(x)\langle(\omega)_{n-1}, \xi^{\otimes(n-1)}\rangle. \tag{5.27}$$

We prove this formula by induction. It trivially holds for $n = 1$. Assume that (5.27) holds for $1, 2, \dots, n$ and let us prove it for $n+1$. Using our assumption, the recurrence relation (5.26) and the polarization identity, we get

$$\begin{aligned}
&\langle(\omega + \delta_x)_{n+1}, \xi^{\otimes(n+1)}\rangle = \langle(\omega + \delta_x)_n, \xi^{\otimes n}\rangle\langle\omega + \delta_x, \xi\rangle - n\langle(\omega + \delta_x)_n, \xi^2 \odot \xi^{\otimes(n-1)}\rangle\\
&\quad = \big(\langle(\omega)_n, \xi^{\otimes n}\rangle + n\xi(x)\langle(\omega)_{n-1}, \xi^{\otimes(n-1)}\rangle\big)\big(\langle\omega, \xi\rangle + \xi(x)\big)\\
&\qquad - n\big(\langle(\omega)_n, \xi^2 \odot \xi^{\otimes(n-1)}\rangle + \xi^2(x)\langle(\omega)_{n-1}, \xi^{\otimes(n-1)}\rangle\\
&\qquad\qquad + (n-1)\xi(x)\langle(\omega)_{n-1}, \xi^2 \odot \xi^{\otimes(n-2)}\rangle\big)\\
&\quad = \big(\langle(\omega)_n \odot \omega, \xi^{\otimes(n+1)}\rangle - n\langle(\omega)_n, \xi^2 \odot \xi^{\otimes(n-1)}\rangle\big) + \xi(x)\langle(\omega)_n, \xi^{\otimes n}\rangle\\
&\qquad + n\xi(x)\big(\langle(\omega)_{n-1} \odot \omega, \xi^{\otimes n}\rangle - (n-1)\langle(\omega)_{n-1}, \xi^2 \odot \xi^{\otimes(n-2)}\rangle\big)\\
&\quad = \langle(\omega)_{n+1}, \xi^{\otimes(n+1)}\rangle + n\xi(x)\langle(\omega)_n, \xi^{\otimes n}\rangle.
\end{aligned}$$

□

Remark 5.5. Note that the recurrence relation (5.26) satisfied by the polynomials on $\mathcal{D}'$ with generating function (5.24) was already discussed in [7].

Since we have interpreted $(\omega)_n$ as a falling factorial on $\mathcal{D}'$, we naturally define *'ω choose n'* by $\binom{\omega}{n} := \frac{1}{n!}(\omega)_n$, compare with (5.22).

We denote by Γ the configuration space over $\mathbb{R}^d$, i.e., the space of all Radon measures $\gamma \in \mathbb{M}(\mathbb{R}^d)$ that are of the form $\gamma = \sum_{i=1}^\infty \delta_{x_i}$, where $x_i \neq x_j$ if $i \neq j$. (We can obviously identify the configuration $\gamma = \sum_{i=1}^\infty \delta_{x_i}$ with the (locally finite) set $\{x_i\}_{i\in\mathbb{N}}$.) The following result is immediate.

Corollary 5.6. *For each $\gamma = \sum_{i=1}^\infty \delta_{x_i}$, formula (1.1) holds.*

Remark 5.7. Polynomials $\binom{\gamma}{n}$ play a crucial role in the theory of point process (i.e., Γ-valued random variables), see e.g. [16]. More precisely, given a probability space $(\Xi, \mathcal{F}, \mathbb{P})$ and a point process $\gamma : \Xi \to \Gamma$, the nth correlation measure of γ is defined as the (unique) measure $\sigma^{(n)}$ on $\big((\mathbb{R}^d)^n, \mathcal{B}_{\mathrm{sym}}((\mathbb{R}^d)^n)\big)$ that satisfies

$$\mathbb{E}\left\langle \binom{\gamma}{n}, f^{(n)} \right\rangle = \int_{(\mathbb{R}^d)^n} f^{(n)}(x_1, \dots, x_n)\, d\sigma^{(n)}(x_1, \dots, x_n) \quad \text{for all } f^{(n)} \in \mathcal{D}^{\odot n},\ f^{(n)} \geq 0.$$

Here $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$. Under very mild conditions on the point process γ, the correlation measures $(\sigma^{(n)})_{n=1}^{\infty}$ uniquely identify the distribution of γ on Γ. In the case where each measure $\sigma^{(n)}$ is absolutely continuous with respect to the Lebesgue measure, one defines the nth correlation function of the point process γ, denote by $k^{(n)}(x_1, \dots, x_n)$, as follows:

$$d\sigma^{(n)}(x_1, \dots, x_n) = \frac{1}{n!}\, k^{(n)}(x_1, \dots, x_n)\, dx_1 \cdots dx_n,$$

or equivalently

$$\mathbb{E}\big\langle (\gamma)_n, f^{(n)} \big\rangle = \int_{(\mathbb{R}^d)^n} f^{(n)}(x_1, \dots, x_n) k^{(n)}(x_1, \dots, x_n)\ dx_1 \cdots dx_n.$$

Corollary 5.8. *For each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, $\gamma \in \Gamma$ and $n \in \mathbb{N}$, we have,*

$$\binom{\gamma}{n}(\Lambda^n) = \binom{\gamma(\Lambda)}{n}.$$

Proof. The result is obvious by Corollary 5.6, or alternatively, by Corollary 5.2. □

Remark 5.9. In view of Corollaries 5.6 and 5.8, for the falling factorials on $\mathcal{D}'$, the set Γ plays a role similar to that played by the set $\mathbb{N}$ for the falling factorials on $\mathbb{R}$.

5.2 Rising factorials on $\mathcal{D}'$

The classical rising factorials is the sequence $(p_n)_{n=0}^{\infty}$ of monic polynomials on $\mathbb{R}$ of binomial type that are explicitly given by

$$p_n(t) = (t)^n := t(t+1)(t+2)\cdots(t+n-1).$$

The corresponding delta operator is $Q = 1 - e^{-D}$, so that Q is the difference operator $(Qp)(t) = p(t) - p(t-1)$ for $p \in \mathcal{P}(\mathbb{R})$. The generating function of the rising factorials is given by

$$\sum_{n=0}^{\infty} \frac{u^n}{n!}(t)^n = \exp[-t\log(1-u)] = (1-u)^{-t}.$$

One also has the following connection between the rising factorials and the falling factorials: $(t)^n = (-1)^n (-t)_n$.

Let us now consider the corresponding lifted sequence of polynomials, $(P^{(n)})_{n=0}^\infty$. We will call these polynomials the *rising factorials on* $\mathcal{D}'$, and we will write $(\omega)^n := P^{(n)}(\omega)$ for $\omega \in \mathcal{D}'$.

By (5.8), $Q(x) = 1 - e^{-D(x)}$. Hence, by Boole's formula,

$$(Q(x)P)(\omega) = P(\omega) - P(\omega - \delta_x), \quad x \in \mathbb{R}^d,\ P \in \mathcal{P}(\mathcal{D}'),$$

and by (5.9),

$$(Q(\zeta)P)(\omega) = \big\langle \zeta(x), P(\omega) - P(\omega - \delta_x)\big\rangle, \quad \zeta \in \mathcal{D}',\ P \in \mathcal{P}(\mathcal{D}').$$

By (5.11), the generating function is equal to

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle (\omega)^n, \xi^{\otimes n}\rangle = \exp\big[\langle \omega, -\log(1-\xi)\rangle\big]. \tag{5.28}$$

Proposition 5.10. *We have* $(\omega)^n = (-1)^n(-\omega)_n$ *for all* $n \in \mathbb{N}$*, and the following explicit formulas hold:*

$$(\omega)^0 = 1;$$
$$(\omega)^1 = \omega;$$
$$(\omega)^n(x_1, \dots, x_n) = \omega(x_1)(\omega(x_2) + \delta_{x_1}(x_2))$$
$$\times (\omega(x_3) + \delta_{x_1}(x_3) + \delta_{x_2}(x_3)) \cdots (\omega(x_n) + \delta_{x_1}(x_n) + \dots + \delta_{x_{n-1}}(x_n))$$

for $n \geq 2$.

Proof. By (5.24),

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle (-1)^n (-\omega)_n, \xi^{\otimes n}\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle (-\omega)_n, (-\xi)^{\otimes n}\rangle = \exp\big[\langle \omega, -\log(1-\xi)\rangle\big].$$

Hence, by (5.28), $(\omega)^n = (-1)^n(-\omega)_n$ for all $\omega \in \mathcal{D}'$ and $n \in \mathbb{N}$. From this and Proposition 5.4, the statement follows. $\square$

5.3 Abel polynomials on $\mathcal{D}'$

Let us fix a parameter $\alpha \in \mathbb{R} \setminus \{0\}$. The classical Abel polynomials on $\mathbb{R}$ corresponding to the parameter α is the monic polynomial sequence $(p_n)_{n=0}^\infty$ of binomial type that has the delta operator $Q = De^{\alpha D}$, i.e., $(Qp)(t) = p'(t+\alpha)$ for $p \in \mathcal{P}(\mathbb{R})$. Thus, $Q = q(D)$, where $q(u) = ue^{\alpha u}$. The generating function of the Abel polynomials is given by

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} p_n(t) = \exp\big[t\alpha^{-1} W(\alpha u)\big],$$

where W is the inverse function of $u \mapsto ue^u$ (around 0), the so-called Lambert W-function.

Consider the corresponding lifted sequence of polynomials $(P^{(n)})_{n=0}^\infty$, the *Abel polynomials on $\mathcal{D}'$*. By Lemma 2.20 and (5.8),

$$Q(x) = D(x)e^{\alpha D(x)} = D(x)E(\alpha\delta_x), \quad x \in \mathbb{R}^d,$$

and by (5.9),

$$Q(\zeta)P = \big\langle \zeta(x), D(x)P(\cdot + \alpha\delta_x)\big\rangle, \quad \zeta \in \mathcal{D}',\ P \in \mathcal{P}(\mathcal{D}').$$

By (5.11), the generating function is equal to

$$\sum_{n=0}^\infty \frac{1}{n!}\langle P^{(n)}(\omega), \xi^{\otimes n}\rangle = \exp\left[\big\langle \omega, \alpha^{-1}W(\alpha\xi)\big\rangle\right].$$

We have

$$\alpha^{-1}W(\alpha u) = \sum_{k=1}^\infty \frac{(-\alpha k)^{k-1}}{k!}u^k.$$

Hence, by Proposition 5.1,

$$\langle P^{(n)}(\omega), \xi^{\otimes n}\rangle = \sum_{\pi \in \mathfrak{P}(n)} \prod_{B \in \pi} (-\alpha|B|)^{|B|-1}\langle \omega, \xi^{|B|}\rangle.$$

5.4 Laguerre polynomials on $\mathcal{D}'$ of binomial type

Let us recall that the (monic) Laguerre polynomials on $\mathbb{R}$ corresponding to a parameter $k \geq -1$, $(p_n^{[k]})_{n=0}^\infty$, have the generating function

$$\sum_{n=0}^\infty \frac{u^n}{n!}\, p_n^{[k]}(t) = \exp\left[\frac{tu}{1+u}\right](1+u)^{-(k+1)}. \tag{5.29}$$

In particular, for the parameter $k=-1$, the Laguerre polynomial sequence $(p_n)_{n=0}^\infty := (p_n^{[-1]})_{n=0}^\infty$ has the generating function

$$\sum_{n=0}^\infty \frac{u^n}{n!}\, p_n(t) = \exp\left[\frac{tu}{1+u}\right]. \tag{5.30}$$

Hence, the polynomial sequence $(p_n)_{n=0}^\infty$ is of binomial type and its delta operator is $Q = q(D)$ with

$$q(u) = \frac{u}{1-u} = \sum_{k=1}^\infty u^k.$$

The corresponding lifted sequence $(P^{(n)})_{n=0}^{\infty}$ will be called the *Laguerre polynomial sequence on $\mathcal{D}'$ of binomial type*. Thus, for each $x \in \mathbb{R}^d$, the delta operator $Q(x)$ of $(P^{(n)})_{n=0}^{\infty}$ is given by

$$Q(x) = q(D(x)) = \frac{D(x)}{1 - D(x)} = \sum_{k=1}^{\infty} D(x)^k,$$

and the corresponding generating function is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\omega), \xi^{\otimes n} \rangle = \exp\left[\left\langle \omega, \frac{\xi}{1+\xi} \right\rangle\right]. \tag{5.31}$$

By Proposition 5.1, the polynomial $\langle P^{(n)}(\omega), \xi^{\otimes n} \rangle$ has representation (5.13) with $\alpha_k = (-1)^{k+1} k!$. In view of the factor $k!$ in α_k, we can also give the following combinatorial formula for $\langle P^{(n)}(\omega), \xi^{\otimes n} \rangle$.

Let $\mathfrak{W}(n)$ denote the collection of all sets $\beta = \{b_1, b_2, \dots, b_k\}$ such that each $b_i = (j_1, \dots, j_{l_i})$ is an element of $\{1, 2, \dots, n\}^{l_i}$ with $j_u \neq j_v$ if $u \neq v$, and each $j \in \{1, 2, \dots, n\}$ is a coordinate of exactly one $b_i \in \beta$. For $\beta \in \mathfrak{W}(n)$ and $b_i = (j_1, \dots, j_{l_i}) \in \beta$, we denote $|b_i| := l_i$.

By (5.13), we now get, for the Laguerre polynomials,

$$\langle P^{(n)}(\omega), \xi^{\otimes n} \rangle = \sum_{\beta \in \mathfrak{W}(n)} \prod_{b \in \beta} \langle -\omega, (-\xi)^{|b|} \rangle. \tag{5.32}$$

6 Sheffer sequences

Definition 6.1. Let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be a family of delta operators. We say that a monic polynomial sequence on $\mathcal{D}'$, $(S^{(n)})_{n=0}^{\infty}$, is a *Sheffer sequence for the family of delta operators* $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ if for each $\zeta \in \mathcal{D}'$ and $f^{(n)} \in \mathcal{D}^{\odot n}$, $n \in \mathbb{N}$,

$$Q(\zeta) \langle S^{(n)}, f^{(n)} \rangle = \langle S^{(n-1)}, \mathfrak{A}(\zeta) f^{(n)} \rangle, \tag{6.1}$$

where $\mathfrak{A}(\zeta)$ is the annihilation operator, see Definition 2.17.

Of course, any basic sequence for a family of delta operators is a Sheffer sequence for that family of delta operators. The following theorem is the second main result of the paper.

Theorem 6.2. *Let $(Q(\zeta))_{\zeta \in \mathcal{D}'}$ be a family of delta operators and let $(P^{(n)})_{n=0}^{\infty}$ be its basic sequence, which has the generating function (4.2). Let $(S^{(n)})_{n=0}^{\infty}$ be a monic polynomial sequence on $\mathcal{D}'$. Then the following conditions are equivalent:*

(SS1) *$(S^{(n)})_{n=0}^{\infty}$ is a Sheffer sequence for the family $(Q(\zeta))_{\zeta \in \mathcal{D}'}$.*

(SS2) *There is a unique operator* $T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$ *such that, for each* $f^{(n)} \in \mathcal{D}^{\odot n}$, $n \in \mathbb{N}_0$,

$$T\langle S^{(n)}, f^{(n)}\rangle = \langle P^{(n)}, f^{(n)}\rangle. \tag{6.2}$$

(SS3) *The sequence* $(S^{(n)})_{n=0}^{\infty}$ *has the generating function*

$$\sum_{n=0}^{\infty} \frac{1}{n!}\langle S^{(n)}(\omega), \xi^{\otimes n}\rangle = \frac{\exp[\langle \omega, A(\xi)\rangle]}{\tau(A(\xi))}, \quad \omega \in \mathcal{D}',\ \xi \in \mathcal{D}, \tag{6.3}$$

where $A(\xi) \in \mathcal{S}(\mathcal{D},\mathcal{D})$ *is given by* (4.3) *and* $\tau \in \mathcal{S}(\mathcal{D},\mathbb{R})$ *is such that* $\tau(0) = 1$.

(SS4) *For each* $n \in \mathbb{N}$ *and* $\omega, \zeta \in \mathcal{D}'$,

$$S^{(n)}(\omega+\zeta) = \sum_{k=0}^{n} \binom{n}{k} S^{(k)}(\omega) \odot P^{(n-k)}(\zeta). \tag{6.4}$$

(SS5) *There is a* $(\rho^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\mathcal{D}')$ *with* $\rho^{(0)} = 1$ *such that, for each* $n \in \mathbb{N}$,

$$S^{(n)}(\omega) = \sum_{k=0}^{n} \binom{n}{k} \rho^{(k)} \odot P^{(n-k)}(\omega), \quad \omega \in \mathcal{D}'.$$

Remark 6.3. For the meaning of the right hand side of formula (6.3), see Proposition A.1 and Remark A.2.

Proof of Theorem 6.2. (SS1)⇒(SS2). We define a linear operator $T : \mathcal{P}(\mathcal{D}') \to \mathcal{P}(\mathcal{D}')$ by formula (6.2). Since $(P^{(n)})_{n=0}^{\infty}$ and $(S^{(n)})_{n=0}^{\infty}$ are monic polynomial sequences on $\mathcal{D}'$, we have $T \in \mathcal{L}(\mathcal{P}(\mathcal{D}'))$, see formulas (3.6) and (3.7). Thus, we only have to prove that T is shift-invariant. For this purpose, fix any $G^{(k)} \in \mathcal{D}'^{\odot k}$, $\xi \in \mathcal{D}$, and $k \in \mathbb{N}_0$. By (6.1) and (6.2), we obtain

$$\begin{aligned}
&T\big\langle G^{(k)}(x_1,\dots,x_k), Q(x_1)\cdots Q(x_k)\langle S^{(n)}, \xi^{\otimes n}\rangle\big\rangle = (n)_k\langle G^{(k)}, \xi^{\otimes k}\rangle T\langle S^{(n-k)}, \xi^{\otimes(n-k)}\rangle\\
&\quad = (n)_k\langle G^{(k)}, \xi^{\otimes k}\rangle\langle P^{(n-k)}, \xi^{\otimes(n-k)}\rangle\\
&\quad = \big\langle G^{(k)}(x_1,\dots,x_k), Q(x_1)\cdots Q(x_k)\langle P^{(n)}, \xi^{\otimes n}\rangle\big\rangle\\
&\quad = \big\langle G^{(k)}(x_1,\dots,x_k), Q(x_1)\cdots Q(x_k)T\langle S^{(n)}, \xi^{\otimes n}\rangle\big\rangle.
\end{aligned}$$

Therefore,

$$T\langle G^{(k)}(x_1,\dots,x_k), Q(x_1)\cdots Q(x_k)\rangle = \langle G^{(k)}(x_1,\dots,x_k), Q(x_1)\cdots Q(x_k)\rangle T.$$

Hence, by Theorem 4.7, T is shift-invariant.

(SS2)⇒(SS1). By (6.2) with $n = 0$, we have $T1 = 1$. Hence, by Corollary 4.12, the operator T is invertible and $T^{-1} \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$. By Corollary 4.10, T^{-1} commutes with each $Q(\zeta)$, $\zeta \in \mathcal{D}'$. Therefore, for each $n \in \mathbb{N}$ and $\xi \in \mathcal{D}$, we get

$$\begin{aligned} Q(\zeta)\langle S^{(n)}, \xi^{\otimes n}\rangle &= Q(\zeta)T^{-1}\langle P^{(n)}, \xi^{\otimes n}\rangle = T^{-1}Q(\zeta)\langle P^{(n)}, \xi^{\otimes n}\rangle \\ &= T^{-1}(n\langle \zeta, \xi\rangle\langle P^{(n-1)}, \xi^{\otimes(n-1)}\rangle) = n\langle \zeta, \xi\rangle\langle S^{(n-1)}, \xi^{\otimes(n-1)}\rangle. \end{aligned}$$

(SS2)⇒(SS3). For a fixed $\zeta \in \mathcal{D}'$, we apply Theorem 4.7 to the shift-invariant operator $E(\zeta)T^{-1}$. This gives

$$(E(\zeta)T^{-1}P)(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!}\big\langle G^{(k)}(x_1, \dots, x_k), (Q(x_1)\cdots Q(x_k)P)(\omega)\big\rangle, \quad P \in \mathcal{P}(\mathcal{D}'),$$

where

$$\langle G^{(k)}, f^{(k)}\rangle = \left(E(\zeta)T^{-1}\langle P^{(k)}, f^{(k)}\rangle\right)(0) = \left(E(\zeta)\langle S^{(k)}, f^{(k)}\rangle\right)(0) = \langle S^{(k)}(\zeta), f^{(k)}\rangle.$$

Thus,

$$(E(\zeta)T^{-1}P)(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!}\big\langle S^{(k)}(\zeta)(x_1, \dots, x_k), (Q(x_1)\cdots Q(x_k)P)(\omega)\big\rangle, \quad P \in \mathcal{P}(\mathcal{D}'). \tag{6.5}$$

For the family of delta operators $(D(\zeta))_{\zeta\in\mathcal{D}'}$ and its basic sequence (monomials), consider the isomorphism $\mathcal{J}$ defined in Corollary 4.11. Hence, by (4.26) and (6.5),

$$(\mathcal{J}(E(\zeta)T^{-1}))(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}\left\langle S^{(k)}(\zeta), \left(\sum_{n=1}^{\infty} \frac{1}{n!}\left(R_{1,n}\xi^{\otimes n}\right)\right)^{\otimes k}\right\rangle. \tag{6.6}$$

Furthermore, by (4.22),

$$(\mathcal{J}T)(\xi) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\langle \tau^{(k)}, \xi^{\otimes k}\rangle =: \tau(\xi), \tag{6.7}$$

where

$$\langle \tau^{(k)}, \xi^{\otimes k}\rangle := \left(T\langle \cdot^{\otimes k}, \xi^{\otimes k}\rangle\right)(0), \quad \xi \in \mathcal{D},\ k \in \mathbb{N}.$$

(Note that the first term in (6.7) is indeed equal to 1, because T maps 1 into 1.) By (4.23) and the isomorphism theorem,

$$\exp[\langle \zeta, \xi\rangle] = (\mathcal{J}E(\zeta))(\xi) = \left(\mathcal{J}(E(\zeta)T^{-1})\right)(\xi)(\mathcal{J}T)(\xi). \tag{6.8}$$

By Proposition A.1 (see also Remark A.2) and (6.6)–(6.8), we get

$$\sum_{k=0}^{\infty} \frac{1}{k!}\left\langle S^{(k)}(\zeta), \left(\sum_{n=1}^{\infty} \frac{1}{n!}\left(R_{1,n}\xi^{\otimes n}\right)\right)^{\otimes k}\right\rangle = \frac{\exp[\langle \zeta, \xi\rangle]}{\tau(\xi)}, \tag{6.9}$$

compare with (4.28). We define operators $A_k \in \mathcal{L}(\mathcal{D}^{\odot k}, \mathcal{D})$ by formula (4.30) for $k \geq 2$ and $A_1 := \mathbf{1}$. Formulas (4.29) and (6.9) imply (SS3).

(SS3)⇒(SS2). For the family of delta operators $(D(\zeta))_{\zeta \in \mathcal{D}'}$ and its basic sequence (monomials), we construct the isomorphism $\mathcal{J}$. We define an operator $T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$ by $T := \mathcal{J}^{-1}\tau$. Since $\tau^{(0)} = 1$, by Proposition A.1 and Corollary 4.11, there exists an operator $S \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$ satisfying $ST = TS = \mathbf{1}$. Hence, the operator T is invertible and $T^{-1} = S$.

Since $T^{-1}1 = 1/\tau^{(0)} = 1$, for each $f^{(n)} \in \mathcal{D}^{\odot n}$, we obtain

$$\big(T^{-1}\langle \cdot^{\otimes n}, f^{(n)}\rangle\big)(\omega) = \langle \omega^{\otimes n}, f^{(n)}\rangle + \sum_{i=0}^{n-1} \langle \omega^{\otimes i}, g^{(i)}\rangle \tag{6.10}$$

for some $g^{(i)} \in \mathcal{D}^{\odot i}$, $i = 0, 1, \dots, n-1$.

Consider the linear operator

$$\mathcal{D}^{\odot n} \ni f^{(n)} \mapsto \tilde{S}^{(n)} f^{(n)} := T^{-1}\langle P^{(n)}, f^{(n)}\rangle \in \mathcal{P}(\mathcal{D}'). \tag{6.11}$$

Since $T^{-1} \in \mathcal{L}(\mathcal{P}(\mathcal{D}'))$ and the linear operator

$$\mathcal{D}^{\odot n} \ni f^{(n)} \mapsto \langle P^{(n)}, f^{(n)}\rangle \in \mathcal{P}(\mathcal{D}')$$

is continuous (see Section 3), we conclude that $\tilde{S}^{(n)} \in \mathcal{L}(\mathcal{D}^{\odot n}, \mathcal{P}(\mathcal{D}'))$.

Using formula (6.10), just as in Section 3, we conclude the existence of a monic polynomial sequence $(\tilde{S}^{(n)})_{n=0}^{\infty}$ that satisfies

$$(\tilde{S}^{(n)} f^{(n)})(\omega) = \langle \tilde{S}^{(n)}(\omega), f^{(n)}\rangle, \tag{6.12}$$

cf. (3.4). By (6.11) and (6.12), we obtain

$$\langle P^{(n)}, f^{(n)}\rangle = T\langle \tilde{S}^{(n)}, f^{(n)}\rangle.$$

It follows from the proof of the implication (SS2)⇒(SS3) that the monic polynomial sequence $(\tilde{S}^{(n)})_{n=0}^{\infty}$ has the same generating function as $(S^{(n)})_{n=0}^{\infty}$, so they coincide. But this implies (6.2).

Thus, we have proved that the conditions (SS1), (SS2), and (SS3) are equivalent.

(SS2)⇒(SS4). For $\omega, \zeta \in \mathcal{D}'$, $\xi \in \mathcal{D}$, and $n \in \mathbb{N}$ we have

$$\begin{aligned}
\langle S^{(n)}(\omega + \zeta), \zeta^{\otimes n}\rangle &= \big(E(\zeta)\langle S^{(n)}, \xi^{\otimes n}\rangle\big)(\omega) \\
&= \big(E(\zeta)T^{-1}\langle P^{(n)}, \xi^{\otimes n}\rangle\big)(\omega) \\
&= \big(T^{-1}E(\zeta)\langle P^{(n)}, \xi^{\otimes n}\rangle\big)(\omega) \\
&= \big(T^{-1}\langle P^{(n)}(\cdot + \zeta), \xi^{\otimes n}\rangle\big)(\omega)
\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \big(T^{-1}\langle P^{(k)}, \xi^{\otimes k}\rangle\big)(\omega)\langle P^{(n-k)}(\zeta), \xi^{\otimes(n-k)}\rangle$$
$$= \sum_{k=0}^{n} \binom{n}{k} \langle S^{(k)}(\omega), \xi^{\otimes k}\rangle\langle P^{(n-k)}(\zeta), \xi^{\otimes(n-k)}\rangle.$$

(SS4)⇒(SS5). In formula (6.4), swap ω and ζ, set $\zeta = 0$, and denote $\rho^{(n)} := S^{(n)}(0)$. Note that $\rho^{(0)} = S^{(0)}(0) = 1$.

(SS5)⇒(SS1). For each $\zeta \in \mathcal{D}'$, $\xi \in \mathcal{D}$, and $n \in \mathbb{N}$, we get from (SS5)

$$\begin{aligned}
Q(\zeta)\langle S^{(n)}, \xi^{\otimes n}\rangle &= \sum_{k=0}^{n} \binom{n}{k} \langle \rho^{(k)}, \xi^{\otimes k}\rangle\, Q(\zeta)\langle P^{(n-k)}, \xi^{\otimes(n-k)}\rangle \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \langle \rho^{(k)}, \xi^{\otimes k}\rangle (n-k)\langle \zeta, \xi\rangle \langle P^{(n-k-1)}, \xi^{\otimes(n-k-1)}\rangle \\
&= n\langle \zeta, \xi\rangle \sum_{k=0}^{n-1} \binom{n-1}{k} \langle \rho^{(k)}, \xi^{\otimes k}\rangle \langle P^{(n-k-1)}, \xi^{\otimes(n-k-1)}\rangle \\
&= n\langle \zeta, \xi\rangle \langle S^{(n-1)}, \xi^{\otimes(n-1)}\rangle. \qquad \square
\end{aligned}$$

Corollary 6.4. *Let the conditions of Theorem 6.2 be satisfied. Then $(S^{(n)})_{n=0}^{\infty}$ is a Sheffer sequence for the family $(Q(\zeta))_{\zeta\in\mathcal{D}'}$ if and only if the sequence $(S^{(n)})_{n=0}^{\infty}$ has the generating function*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle S^{(n)}(\omega), \xi^{\otimes n}\rangle = \exp\big[\langle \omega, A(\xi)\rangle - C(A(\xi))\big], \quad \omega \in \mathcal{D}',\ \xi \in \mathcal{D}, \tag{6.13}$$

where $A(\xi) := \sum_{k=1}^{\infty} A_k \xi^{\otimes k} \in \mathcal{S}(\mathcal{D}, \mathcal{D})$ is as in (4.2) and $C(\xi) := \sum_{k=1}^{\infty} \langle C^{(k)}, \xi^{\otimes k}\rangle \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ is such that $C(0) = 0$.

Proof. We note that $\langle \omega, A(\xi)\rangle \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ as the composition of $\langle \omega, \xi\rangle \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ and $A(\xi) \in \mathcal{S}(\mathcal{D}, \mathcal{D})$. As easily seen, $\exp[\langle \omega, A(\xi)\rangle] \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ in formula (6.3) can be understood as the composition of $\exp t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ and $\langle \omega, A(\xi)\rangle \in \mathcal{S}(\mathcal{D}, \mathbb{R})$, see Definition A.3.

Consider $\log(1 + t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n} \in \mathcal{S}(\mathbb{R}, \mathbb{R})$. Define $C(\xi) \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ as the composition of $\log(1+t) \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ and $\tau(A(\xi))-1 \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ (note that $\tau(A(0))-1 = 0$). Then $C(0) = 0$. By Remark A.9, we obtain $\tau(A(\xi)) = \exp(C(\xi))$, the equality in $\mathcal{S}(\mathcal{D}, \mathbb{R})$. Hence, using again Remark A.9, we conclude that formula (6.3) can be written as (6.13). $\square$

Remark 6.5. According to Theorem 6.2 and Corollary 6.4, any Sheffer sequence $(S^{(n)})_{n=0}^{\infty}$ is completely identified by a family of delta operators, $(Q(\zeta))_{\zeta\in\mathcal{D}'}$, and a formal series $C \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ with $C(0) = 0$.

Corollary 6.6. *Under the conditions of Theorem 6.2, assume that* $(S^{(n)})_{n=0}^{\infty}$ *is a Sheffer sequence. Let* $(\varkappa^{(n)})_{n=0}^{\infty} \in \mathcal{F}(\mathcal{D}')$ *with* $\varkappa^{(0)} = 1$ *satisfy*

$$\tau(A(\xi)) = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \varkappa^{(k)}, \xi^{\otimes k} \rangle. \tag{6.14}$$

Then, for each $n \in \mathbb{N}$,

$$P^{(n)}(\omega) = \sum_{k=0}^{n} \binom{n}{k} \varkappa^{(k)} \odot S^{(n-k)}(\omega), \quad \omega \in \mathcal{D}'. \tag{6.15}$$

Proof. By (4.2), (SS3) and (6.14),

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P^{(n)}(\omega), \xi^{\otimes n} \rangle = \left(\sum_{k=0}^{\infty} \frac{1}{k!} \langle \varkappa^{(k)}, \xi^{\otimes k} \rangle \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} \langle S^{(n)}(\omega), \xi^{\otimes n} \rangle \right),$$

which implies (6.15). □

Corollary 6.7. *Under the conditions of Theorem 6.2, assume that* $(S^{(n)})_{n=0}^{\infty}$ *is a Sheffer sequence. Then,* $(S^{(n)})_{n=0}^{\infty} = (P^{(n)})_{n=0}^{\infty}$ *if and only if* $S^{(n)}(0) = 0$ *for each* $n \in \mathbb{N}$.

Proof. By (SS4), for each $n \in \mathbb{N}$ and $\omega \in \mathcal{D}'$, we have

$$S^{(n)}(\omega) = P^{(n)}(\omega) + \sum_{k=1}^{n} \binom{n}{k} S^{(k)}(0) \odot P^{(n-k)}(\omega).$$

From here the statement follows. □

Let $\mathcal{C}(\mathcal{D}')$ denote the cylinder σ-algebra on $\mathcal{D}'$, i.e., the minimal σ-algebra on $\mathcal{D}'$ with respect to which each monomial $\langle \cdot, \xi \rangle$ ($\xi \in \mathcal{D}$) is measurable.

Definition 6.8. Let $(S^{(n)})_{n=0}^{\infty}$ be a Sheffer sequence on $\mathcal{D}'$ and let μ be a probability measure on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$. The polynomials $(S^{(n)})_{n=0}^{\infty}$ are said to be *orthogonal with respect to* μ if for any $m, n \in \mathbb{N}_0$, $m \neq n$, $f^{(m)} \in \mathcal{D}^{\odot m}$, and $g^{(n)} \in \mathcal{D}^{\odot n}$,

$$\int_{\mathcal{D}'} \langle S^{(m)}(\omega), f^{(m)} \rangle \langle S^{(n)}(\omega), g^{(n)} \rangle \, d\mu(\omega) = 0.$$

Recall the operator $T \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$ given by (6.2). Then the operator $T^{-1} \in \mathbb{S}(\mathcal{P}(\mathcal{D}'))$ satisfies

$$T^{-1} \langle P^{(n)}, f^{(n)} \rangle = \langle S^{(n)}, f^{(n)} \rangle, \quad f^{(n)} \in \mathcal{D}^{\odot n}, \ n \in \mathbb{N}_0.$$

Corollary 6.9. *Let $(S^{(n)})_{n=0}^\infty$ be a Sheffer sequence on $\mathcal{D}'$. Let μ be a probability measure on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$ such that the polynomials $(S^{(n)})_{n=0}^\infty$ are orthogonal with respect to μ. Then the corresponding operator T^{-1} has the following representation:*

$$(T^{-1}P)(\omega) = \int_{\mathcal{D}'} P(\omega + \zeta)\, d\mu(\zeta), \quad P \in \mathcal{P}(\mathcal{D}'). \tag{6.16}$$

Proof. Note that formula (6.16) holds for $P = 1$. Now, for each $n \in \mathbb{N}$ and $\xi \in \mathcal{D}$, we obtain by (SS4):

$$\begin{aligned}\int_{\mathcal{D}'} \langle S^{(n)}(\omega+\zeta), \xi^{\otimes n}\rangle\, d\mu(\zeta) &= \sum_{k=0}^{n} \binom{n}{k} \langle P^{(n-k)}(\omega), \xi^{\otimes(n-k)}\rangle \int_{\mathcal{D}'} \langle S^{(k)}(\zeta), \xi^{\otimes k}\rangle\, d\mu(\zeta) \\ &= \langle P^{(n)}(\omega), \xi^{\otimes n}\rangle = \big(T^{-1}\langle S^{(n)}, \xi^{\otimes n}\rangle\big)(\omega). \qquad \square\end{aligned}$$

Remark 6.10. Note that, in the proof of Corollary 6.9, we only use the fact that $\int_{\mathcal{D}'} \langle S^{(n)}(\omega), f^{(n)}\rangle\, d\mu(\omega) = 0$ for all $n \in \mathbb{N}$.

Recall that a Sheffer sequence on $\mathbb{R}$ whose delta operator is the operator of differentiation is called an Appell sequence on $\mathbb{R}$.

Definition 6.11. Let $(S^{(n)})_{n=0}^\infty$ be a Sheffer sequence for the family of delta operators $(Q(\zeta))_{\zeta\in\mathcal{D}'} = (D(\zeta))_{\zeta\in\mathcal{D}'}$. Then we call $(S^{(n)})_{n=0}^\infty$ an *Appell sequence on $\mathcal{D}'$*.

By (4.2), $A(\xi) = \xi$ in the case of an Appell sequence. Hence, an Appell sequence $(S^{(n)})_{n=0}^\infty$ has the generating function

$$\sum_{n=0}^\infty \frac{1}{n!} \langle S^{(n)}(\omega), \xi^{\otimes n}\rangle = \exp\big[\langle \omega, \xi\rangle - C(\xi)\big], \quad \omega \in \mathcal{D}',\ \xi \in \mathcal{D}.$$

7 Lifting of Sheffer sequences on $\mathbb{R}$

We can extend the procedure described in Section 5 to a lifting of Sheffer sequences on $\mathbb{R}$. Let $(s_n)_{n=0}^\infty$ be a Sheffer sequence of monic polynomials on $\mathbb{R}$ for the delta operator Q, i.e., $Qs_n = ns_{n-1}$ for each $n \in \mathbb{N}_0$. Thus, Q has representation (5.1) and the polynomial sequence $(s_n)_{n=0}^\infty$ has the generating function

$$\sum_{n=0}^\infty \frac{u^n}{n!} s_n(t) = \exp[ta(u) - c(a(u))],$$

where $a(u) \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ is given by (5.3) (being the compositional inverse of the q given by (5.2)) and

$$c(u) = \sum_{k=1}^\infty c_k u^k \in \mathcal{S}(\mathbb{R}, \mathbb{R}).$$

We now consider the family of delta operators, $(Q(\zeta))_{\zeta\in\mathcal{D}'}$, given by (5.9). Then $A(\xi)$ is given by (5.10). Furthermore, for $k\in\mathbb{N}$, we define $C^{(k)}\in\mathcal{D}'^{\odot k}$ by

$$\langle C^{(k)}, f^{(k)}\rangle := c_k\langle \mathbb{D}_k f^{(k)}\rangle, \quad f^{(k)}\in\mathcal{D}^{\odot k}.$$

Here, the operator $\mathbb{D}_k$ is defined by (5.5) and for $\xi\in\mathcal{D}$, we denote $\langle\xi\rangle := \int_{\mathbb{R}^d}\xi(x)\,dx$. Thus, for $\xi\in\mathcal{D}$, we define

$$C(\xi) := \sum_{k=1}^{\infty}\langle C^{(k)},\xi^{\otimes k}\rangle = \sum_{k=1}^{\infty} c_k\langle \xi^k\rangle =: \langle c(\xi)\rangle.$$

We now consider the Sheffer sequence $(S^{(n)})_{n=0}^{\infty}$ for the family of delta operators $(Q(\zeta))_{\zeta\in\mathcal{D}'}$ that has the generating function

$$\sum_{n=0}^{\infty}\frac{1}{n!}\langle S^{(n)}(\omega),\xi^{\otimes n}\rangle = \exp\big[\langle\omega, a(\xi)\rangle - \langle c(a(\xi))\rangle\big], \quad \omega\in\mathcal{D}',\ \xi\in\mathcal{D}, \tag{7.1}$$

see (5.4) and Corollary 6.4. Thus, we may think of the Sheffer sequence $(S^{(n)})_{n=0}^{\infty}$ on $\mathcal{D}'$ as the lifting of the Sheffer sequence $(s_n)_{n=0}^{\infty}$ on $\mathbb{R}$.

Proposition 7.1. *Let $(S^{(n)})_{n=0}^{\infty}$ be a Sheffer sequence with generating function* (7.1). *Then, for each $n\in\mathbb{N}$, $\rho^{(n)} := S^{(n)}(0)\in\mathcal{D}'^{\odot n}$ satisfies*

$$\langle\rho^{(n)},\xi^{\otimes n}\rangle = \sum_{\pi\in\mathfrak{P}(n)}\prod_{B\in\pi}\lambda_{|B|}\langle\xi^{|B|}\rangle, \quad \xi\in\mathcal{D}, \tag{7.2}$$

where $\lambda_k\in\mathbb{R}$ are defined by

$$\lambda(u) := -c(a(u)) = \sum_{k=1}^{\infty}\frac{\lambda_k}{k!}\,u^k, \quad u\in\mathbb{R}. \tag{7.3}$$

Furthermore,

$$\langle S^{(n)}(\omega),\xi^{\otimes n}\rangle = \sum_{k=0}^{n}\binom{n}{k}\langle\rho^{(k)},\xi^{\otimes k}\rangle\langle P^{(n-k)}(\omega),\xi^{\otimes(n-k)}\rangle, \quad \omega\in\mathcal{D}',\xi\in\mathcal{D}, \tag{7.4}$$

where $\langle P^n(\omega),\xi^{\otimes n}\rangle$ is given by formula (5.13).

Proof. By (7.1) with $\omega=0$ and (7.3), we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}\,\langle\rho^{(n)},\xi^{\otimes n}\rangle = \exp\left[\sum_{k=1}^{\infty}\frac{\lambda_k}{k!}\langle\xi^k\rangle\right]. \tag{7.5}$$

Using Faà di Bruno's formula for the nth derivative of composition of functions, we deduce (7.2) from (7.5). Formula (7.4) immediately follows from (SS4) with $\omega=0$ (or (SS5)), (5.11), and Proposition 5.1. □

We can also write down the result of Proposition 7.1 in the following form. By a *marked partition of the set* $\{1, 2, \dots, n\}$ we will mean a pair $(\pi, \mathfrak{m}_\pi)$ in which $\pi = \{B_1, B_2, \dots, B_k\} \in \mathfrak{P}(n)$ and $\mathfrak{m}_\pi : \pi \to \{-, +\}$. (The value $\mathfrak{m}_\pi(B_i) \in \{-, +\}$ may be interpreted as the mark of the element B_i of the partition π). We will denote by $\mathfrak{MP}(n)$ the collection of all marked partitions of $\{1, 2, \dots, n\}$.

Corollary 7.2. *Let* $(S^{(n)})_{n=0}^{\infty}$ *be a Sheffer sequence with generating function* (7.1). *Then, for each* $n \in \mathbb{N}$, $\omega, \in \mathcal{D}'$, *and* $\xi \in \mathcal{D}$,

$$\langle S^{(n)}(\omega), \xi^{\otimes n} \rangle = \sum_{(\pi, \mathfrak{m}_\pi) \in \mathfrak{MP}(n)} \left(\prod_{B \in \pi:\, \mathfrak{m}_\pi(B) = +} \alpha_{|B|} \langle \omega, \xi^{|B|} \rangle \right) \left(\prod_{B \in \pi:\, \mathfrak{m}_\pi(B) = -} \lambda_{|B|} \langle \xi^{|B|} \rangle \right),$$

see formula (5.12) *for the definition of* α_k.

Proof. Immediate from Proposition 7.1. □

Using Proposition 7.1, we can now immediately extend Corollary 5.2 to the case of a lifted Sheffer sequence.

Corollary 7.3. *Let* $(S^{(n)})_{n=0}^{\infty}$ *be a Sheffer sequence with generating function* (7.1). *Then, for each* $\eta \in \mathbb{M}(\mathbb{R}^d)$ *and* $n \in \mathbb{N}$, *we have* $S^{(n)}(\eta) \in \mathbb{M}_{\mathrm{sym}}((\mathbb{R}^d)^n)$. *Furthermore, for each* $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, *and* $n \in \mathbb{N}$, *we have*

$$\big(S^{(n)}(\eta)\big)(\Lambda^n) = \tilde{s}_n(\eta(\Lambda)), \tag{7.6}$$

where $(\tilde{s}_n)_{n=0}^{\infty}$ *is the Sheffer sequence on* $\mathbb{R}$ *with generating function*

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \tilde{s}_n(t) = \exp\big[ta(u) - \mathrm{vol}(\Lambda) c(a(u))\big],$$

where $\mathrm{vol}(\Lambda) := \int_\Lambda dx$. *In particular,* $(\tilde{s}_n)_{n=0}^{\infty} = (s_n)_{n=0}^{\infty}$ *if* $\mathrm{vol}(\Lambda) = 1$.

Proposition 7.4. *The statement of Proposition 5.3 remains true for a Sheffer sequence* $(S^{(n)})_{n=0}^{\infty}$ *with generating function* (7.1).

Proof. Analogously to the proof of Proposition 5.3, we note that, for any $\xi, \phi \in \mathcal{D}$ satisfying (5.17),

$$\begin{aligned} &\exp\big[\langle \omega, a(\xi + \phi) \rangle - \langle c(a(\xi + \phi)) \rangle\big] \\ &\quad = \exp\big[\langle \omega, a(\xi) \rangle - \langle c(a(\xi)) \rangle\big] \exp\big[\langle \omega, a(\phi) \rangle - \langle c(a(\phi)) \rangle\big]. \end{aligned}$$

The rest of the proof is similar to that of Proposition 5.3. □

We will now consider examples of lifted Sheffer sequences.

7.1 Hermite polynomials on $\mathcal{D}'$

The sequence of the Hermite polynomials on $\mathbb{R}$, $(s_n)_{n=0}^\infty$, is the Appell sequence on $\mathbb{R}$ with $c(u)=\frac{u^2}{2}$. The Hermite polynomials are orthogonal with respect to the standard Gaussian (normal) distribution on $\mathbb{R}$. The lifting of $(s_n)_{n=0}^\infty$ is the sequence of *Hermite polynomials on* $\mathcal{D}'$, $(S^{(n)})_{n=0}^\infty$, that has the generating function

$$\sum_{n=0}^\infty \frac{1}{n!}\langle S^{(n)}(\omega),\xi^{\otimes n}\rangle = \exp\left[\langle\omega,\xi\rangle - \frac{\langle\xi^2\rangle}{2}\right] = \exp\left[\langle\omega,\xi\rangle - \frac{1}{2}\|\xi\|^2_{L^2(\mathbb{R}^d,dx)}\right]. \tag{7.7}$$

Remark 7.5. For each $\xi\in\mathcal{D}$ with $\|\xi\|_{L^2(\mathbb{R}^d,dx)}=1$, we get from (7.7) that

$$\langle S^{(n)}(\omega),\xi^{\otimes n}\rangle = s_n(\langle\omega,\xi\rangle). \tag{7.8}$$

We see that the Hermite polynomials, $S^{(n)}$, do not actually make use of the spatial structure of the underlying space, $\mathcal{D}'$, but essentially use only the Hilbert space structure of $L^2(\mathbb{R}^d,dx)$. Formula (7.8) is an exceptional property of the infinite-dimensional Hermite polynomials, compare with the general case discussed in Corollary 7.3.

Using either Proposition 7.1 or Corollary 7.2, we easily get an explicit formula

$$\langle S^{(n)}(\omega),\xi^{\otimes n}\rangle = \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\frac{(2k)!}{k!\,2^k}(-\langle\xi^2\rangle)^k\langle\omega,\xi\rangle^{n-2k},$$

where $[\frac{n}{2}]$ denotes the largest integer $\le\frac{n}{2}$. (Note that $\frac{(2k)!}{k!\,2^k}$ is the number of all partitions $\pi\in\mathfrak{P}(2k)$ such that each set from the partition π has precisely two elements.)

Let μ be the probability measure on $\mathcal{D}'$ that has Fourier transform

$$\int_{\mathcal{D}'}\exp[i\langle\omega,\xi\rangle]\,d\mu(\omega) = \exp\left[-\frac{1}{2}\|\xi\|^2_{L^2(\mathbb{R}^d,dx)}\right],\quad \xi\in\mathcal{D}.$$

The measure μ is called the *Gaussian white noise measure*. The Hermite polynomials $(S^{(n)})_{n=0}^\infty$ are orthogonal with respect to μ, and furthermore, for any $m,n\in\mathbb{N}$, $f^{(m)}\in\mathcal{D}^{\odot m}$, and $g^{(n)}\in\mathcal{D}^{\odot n}$,

$$\int_{\mathcal{D}'}\langle S^{(m)}(\omega),f^{(m)}\rangle\langle S^{(n)}(\omega),g^{(n)}\rangle\,d\mu(\omega) = \delta_{m,n}n!(f^{(m)},g^{(n)})_{L^2(\mathbb{R}^d,dx)^{\odot n}}, \tag{7.9}$$

see e.g. [6, 14].

As pointed out in the Introduction, the infinite dimensional Hermite polynomials are well-known and play a fundamental role in Gaussian white noise analysis, see e.g. [6, 14, 15, 29] and the references therein. In white noise analysis, one usually writes $:\omega^{\otimes n}:$ for $S^{(n)}(\omega)$ and call it the nth Wick power of ω. In that context, the transformation T^{-1} given by formula (6.16) is known as the C-transform.

7.2 Charlier polynomials on $\mathcal{D}'$

The sequence of the Charlier polynomials on $\mathbb{R}$, $(s_n)_{n=0}^\infty$, is the Sheffer sequence with $a(u) = \log(1+u)$ and $c(u) = e^u - 1$, so that $\lambda(u) = -u$. The Charlier polynomials are orthogonal with respect to the Poisson distribution corresponding to the intensity parameter 1. The lifting of $(s_n)_{n=0}^\infty$ is the sequence of the *Charlier polynomials on* $\mathcal{D}'$, $(S^{(n)})_{n=0}^\infty$, that has the generating function

$$\sum_{n=0}^\infty \frac{1}{n!}\langle S^{(n)}(\omega), \xi^{\otimes n}\rangle = \exp\left[\langle \omega, \log(1+\xi)\rangle - \langle \xi\rangle\right].$$

Note that the corresponding binomial sequence is $((\omega)_n)_{n=0}^\infty$, the falling factorials on $\mathcal{D}'$.

By Proposition 7.1,

$$\langle S^{(n)}(\omega), \xi^{\otimes n}\rangle = \sum_{k=0}^n \binom{n}{k} \langle -\xi\rangle^k \langle (\omega)_{n-k}, \xi^{\otimes(n-k)}\rangle. \tag{7.10}$$

Furthermore, by Corollary 6.6, we obtain

$$\langle (\omega)_n, \xi^{\otimes n}\rangle = \sum_{k=0}^n \binom{n}{k} \langle \xi\rangle^k \langle S^{(n-k)}(\omega), \xi^{\otimes(n-k)}\rangle. \tag{7.11}$$

Compare formulas (7.10) and (7.11) with Corollaries 2.9 and 2.10 in [18], respectively. Note that the latter results were obtained only for ω from the configuration space Γ.

Let μ be the probability measure on $\mathcal{D}'$ that has Fourier transform

$$\int_{\mathcal{D}'} \exp[i\langle \omega, \xi\rangle]\, d\mu(\omega) = \exp\left[\int_{\mathbb{R}^d} (e^{i\xi(x)} - 1)\, dx\right], \quad \xi \in \mathcal{D}.$$

The measure μ is concentrated on the configuration space Γ and is called the *Poisson point process*, or the *Poisson white noise measure*. The Charlier polynomials $(S^{(n)})_{n=0}^\infty$ are orthogonal with respect to the Poisson point process μ and formula (7.9) holds true in this case.

The Charlier polynomials $(S^{(n)})_{n=0}^\infty$ play a fundamental role in Poisson analysis, see e.g. [17,20,23]. In this analysis, the transformation T^{-1} given by formula (6.16) is also known as the C-transform.

7.3 Orthogonal Laguerre polynomials on $\mathcal{D}'$

It follows from (5.29) that, for each parameter $k > -1$, the sequence of the Laguerre polynomials $(p_n^{[k]})_{n=0}^\infty$ on $\mathbb{R}$ corresponding to the parameter k is a Sheffer sequence, whose corresponding binomial sequence is $(p_n)_{n=0}^\infty = (p_n^{[-1]})_{n=0}^\infty$, see (5.30). For each

$k > -1$, the Laguerre polynomials $(p_n^{[k]})_{n=0}^\infty$ are orthogonal with respect to the gamma distribution

$$\frac{1}{\Gamma(k+1)}\,\chi_{(0,\infty)}(t)t^k e^{-t}\,dt.$$

In particular, for $k=0$, the Laguerre polynomials $(s_n)_{n=0}^\infty := (p_n^{[0]})_{n=0}^\infty$ are orthogonal with respect to the exponential distribution $\chi_{(0,\infty)}(t)e^{-t}\,dt$ on $\mathbb{R}$. By (5.29), $(s_n)_{n=0}^\infty$ is the Sheffer sequence with $a(u) = \frac{u}{1+u}$ and $c(u) = -\log(1-u)$, so that $\lambda(u) = -\log(1+u)$.

The lifting of $(s_n)_{n=0}^\infty$ is the sequence $(S^{(n)})_{n=0}^\infty$ of the *Laguerre polynomials on* $\mathcal{D}'$ that has the generating function

$$\sum_{n=0}^\infty \frac{1}{n!}\langle S^{(n)}(\omega), \xi^{\otimes n}\rangle = \exp\left[\left\langle \omega, \frac{\xi}{1+\xi}\right\rangle - \langle \log(1+\xi)\rangle\right]. \tag{7.12}$$

Note that the corresponding polynomial sequence of binomial type is the Laguerre sequence $(P^{(n)})_{n=0}^\infty$ with generating function (5.31), see Subsection 5.4. Analogously to formula (5.32), we will now present a combinatorial formula for $\langle S^{(n)}(\omega), \xi^{\otimes n}\rangle$.

We can identify each permutation $\pi \in \mathfrak{S}(n)$ with $c(\pi) := \{\nu_1, \dots, \nu_k\}$, the set of the cycles in π. For each cycle $\nu_i \in c(\pi)$, we denote by $|\nu_i|$ the length of the cycle ν_i. We define

$$\mathfrak{M}\mathfrak{S}(n) := \big\{(\pi, \mathfrak{m}_\pi) \mid \pi \in \mathfrak{S}(n),\ \mathfrak{m}_\pi : c(\pi) \to \{+,-\}\big\},$$

compare with the definition of $\mathfrak{M}\mathfrak{P}(n)$ above.

Note that, for a given subset of $\mathbb{N}$ that has m elements, there are $(m-1)!$ cycles of length m that contain the points from this set. Hence, by Corollary 7.2 and (7.12), we get:

$$\langle S^{(n)}(\omega), \xi^{\otimes n}\rangle = \sum_{(\pi,\mathfrak{m}_\pi)\in\mathfrak{M}\mathfrak{S}(n)} \left(\prod_{\nu\in c(\pi):\, \mathfrak{m}_\pi(\nu)=+} |\nu|\big\langle -\omega, (-\xi)^{|\nu|}\big\rangle\right)\left(\prod_{\nu\in c(\pi):\, \mathfrak{m}_\pi(\nu)=-} \big\langle (-\xi)^{|\nu|}\big\rangle\right).$$

By Corollary 7.3, formula (7.6) holds with $(\tilde{s}_n)_{n=0}^\infty = (p_n^{[k]})_{n=0}^\infty$, the Laguerre polynomials on $\mathbb{R}$ corresponding to the parameter $k = \mathrm{vol}(\Lambda) - 1 > -1$.

Let μ be the probability measure on $\mathcal{D}'$ that has the Laplace transform

$$\int_{\mathcal{D}'} e^{-\langle\omega,\xi\rangle}\,d\mu(\omega) = \exp\left[-\int_{\mathbb{R}^d} \log(1+\xi(x))\,dx\right], \quad \xi \in \mathcal{D},\ \xi > -1.$$

The μ is called the *gamma measure*, or the *gamma completely random measure*. It is concentrated on the set of all (positive) discrete Radon measures $\eta = \sum_i s_i\delta_{x_i} \in \mathbb{M}(\mathbb{R}^d)$ with $s_i > 0$ for all i. Note that, with μ-probability one, the set of atoms of η, $\{x_i\}$, is dense in $\mathbb{R}^d$.

As follows from [19, 20], the Laguerre polynomials $(S^{(n)})_{n=0}^{\infty}$ are orthogonal with respect to the gamma measure μ, and furthermore, for any $\xi, \psi \in \mathcal{D}$ and $m, n \in \mathbb{N}$,

$$\int_{\mathcal{D}'} \langle S^{(m)}(\omega), \xi^{\otimes m}\rangle \langle S^{(n)}(\omega), \psi^{\otimes n}\rangle \, d\mu(\omega) = \delta_{m,n} n! \sum_{\pi \in \mathfrak{S}(n)} \prod_{\nu \in c(\pi)} \big\langle (\xi\psi)^{|\nu|} \big\rangle.$$

The Laguerre polynomials $(S^{(n)})_{n=0}^{\infty}$ play a fundamental role in gamma analysis, see e.g. [19, 20, 24, 25].

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Appendix: Formal tensor power series

We fix a general Gel'fand triple (2.1). The following proposition is a direct consequence of formula (2.9).

Proposition A.1. *Let $F(\xi) = \sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n}\rangle \in \mathcal{S}(\Phi, \mathbb{R})$ be such that $F(0) = F^{(0)} \neq 0$. Then there exists a unique $\sum_{n=0}^{\infty} \langle G^{(n)}, \xi^{\otimes n}\rangle \in \mathcal{S}(\Phi, \mathbb{R})$ such that*

$$\left(\sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n}\rangle \right) \left(\sum_{n=0}^{\infty} \langle G^{(n)}, \xi^{\otimes n}\rangle \right) = 1.$$

Explicitly, $G^{(0)} = 1/F^{(0)}$ and for $n \geq 1$, $G^{(n)}$ is recursively given by

$$G^{(n)} = -\frac{1}{F^{(0)}} \sum_{i=0}^{n-1} F^{(n-i)} \odot G^{(i)}.$$

We will denote

$$\left(\sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n}\rangle \right)^{-1} := \sum_{n=0}^{\infty} \langle G^{(n)}, \xi^{\otimes n}\rangle.$$

Remark A.2. It follows from Proposition A.1 that, for any $F(\xi), G(\xi) \in \mathcal{S}(\Phi, \mathbb{R})$ with $F(0) \neq 0$, we obtain

$$\frac{G(\xi)}{F(\xi)} \in \mathcal{S}(\Phi, \mathbb{R}).$$

Definition A.3. Let $R(t)=\sum_{n=0}^\infty r_nt^n\in\mathcal{S}(\mathbb{R},\mathbb{R})$. For each $F(\xi)=\sum_{n=1}^\infty\langle F^{(n)},\xi^{\otimes n}\rangle\in\mathcal{S}(\Phi,\mathbb{R})$ with $F(0)=0$, we define a *composition of R and F*, denoted by

$$R(F(\xi))=\sum_{n=0}^\infty r_n\left(\sum_{k=1}^\infty\langle F^{(k)},\xi^{\otimes k}\rangle\right)^n,$$

as the formal series $G(\xi)=\sum_{n=0}^\infty\langle G^{(n)},\xi^{\otimes n}\rangle\in\mathcal{S}(\Phi,\mathbb{R})$ with $G^{(0)}:=r_0$ and

$$G^{(n)}:=\sum_{m=1}^n\sum_{\substack{(k_1,\dots,k_m)\in\mathbb{N}^m\\k_1+\dots+k_m=n}}r_m\big(F^{(k_1)}\odot F^{(k_2)}\odot\dots\odot F^{(k_m)}\big),\quad n\in\mathbb{N}.$$

Definition A.4. Let $A(\xi)=\sum_{n=1}^\infty A_n\xi^{\otimes n},\ B(\xi)=\sum_{n=1}^\infty B_n\xi^{\otimes n}\in\mathcal{S}(\Phi,\Phi)$. We define a *composition of B and A*, denoted by

$$B(A(\xi))=\sum_{n=1}^\infty B_n\left(\sum_{k=1}^\infty A_k\xi^{\otimes k}\right)^{\otimes n},$$

as the formal series $C(\xi)=\sum_{n=1}^\infty C_n\xi^{\otimes n}\in\mathcal{S}(\Phi,\Phi)$ with

$$C_n:=\sum_{m=1}^n\sum_{\substack{(k_1,\dots,k_m)\in\mathbb{N}^m\\k_1+\dots+k_m=n}}B_m\big(A_{k_1}\odot A_{k_2}\odot\dots\odot A_{k_m}\big),\quad n\in\mathbb{N}.\tag{A.13}$$

Here, for $(k_1,\dots,k_m)\in\mathbb{N}^m$ with $k_1+\dots+k_m=n$, we denote

$$A_{k_1}\odot A_{k_2}\odot\dots\odot A_{k_m}:=\mathrm{Sym}_n(A_{k_1}\otimes A_{k_2}\otimes\dots\otimes A_{k_m}),$$

where $\mathrm{Sym}_n\in\mathcal{L}(\Phi^{\otimes n},\Phi^{\odot n})$ is the operator of symmetrization, see (2.2).

Proposition A.5. (i) *Let $B(\xi)=\sum_{n=1}^\infty B_n\xi^{\otimes n}\in\mathcal{S}(\Phi,\Phi)$ with $B_1=\mathbf{1}$. Then there exists a unique $A(\xi)=\sum_{n=1}^\infty A_n\xi^{\otimes n}\in\mathcal{S}(\Phi,\Phi)$ such that $B(A(\xi))=\xi$. The operators $(A_n)_{n=1}^\infty$ are given recursively by $A_1=\mathbf{1}$ and for $n\geq 2$,*

$$A_n=-\sum_{m=2}^n\sum_{\substack{(k_1,\dots,k_m)\in\mathbb{N}^m\\k_1+\dots+k_m=n}}B_m\big(A_{k_1}\odot A_{k_2}\odot\dots\odot A_{k_m}\big).\tag{A.14}$$

(ii) *Let $A(\xi)=\sum_{n=1}^\infty A_n\xi^{\otimes n}\in\mathcal{S}(\Phi,\Phi)$ with $A_1=\mathbf{1}$. Then there exists a unique $B(\xi):=\sum_{n=1}^\infty B_n\xi^{\otimes n}\in\mathcal{S}(\Phi,\Phi)$ such that $B(A(\xi))=\xi$. The operators $(B_n)_{n=1}^\infty$ are given recursively by $B_1=\mathbf{1}$ and for $n\geq 2$,*

$$B_n=-\sum_{m=1}^{n-1}\sum_{\substack{(k_1,\dots,k_m)\in\mathbb{N}^m\\k_1+\dots+k_m=n}}B_m\big(A_{k_1}\odot A_{k_2}\odot\dots\odot A_{k_m}\big).\tag{A.15}$$

Proof. First, it follows from formula (A.13) that $C_1 = B_1 A_1$. Hence, for $C_1 = \mathbf{1}$, we must have $A_1 = \mathbf{1}$ if $B_1 = \mathbf{1}$, and $B_1 = \mathbf{1}$ if $A_1 = \mathbf{1}$. Now, if $A_1 = B_1 = \mathbf{1}$, the operator C_n from formula (A.13) has the following form for $n \geq 2$:

$$C_n = A_n + \sum_{m=2}^{n-1} \sum_{\substack{(k_1,\dots,k_m)\in\mathbb{N}^m \\ k_1+\dots+k_m=n}} B_m\big(A_{k_1} \odot A_{k_2} \odot \dots \odot A_{k_m}\big) + B_n.$$

Hence, in both cases (i) and (ii), we get C_n=0 for $n \geq 2$ if and only if formulas (A.14) and (A.15) respectively hold. □

Similarly to Defintion A.4, we give the following

Definition A.6. Let $F(\xi) = \sum_{n=0}^{\infty} \langle F^{(n)}, \xi^{\otimes n}\rangle \in \mathcal{S}(\Phi, \mathbb{R})$ and $A(\xi) = \sum_{n=1}^{\infty} A_n \xi^{\otimes n} \in \mathcal{S}(\Phi, \Phi)$. We define a composition of F and A, denoted by

$$F(A(\xi)) = \sum_{n=0}^{\infty} \left\langle F^{(n)}, \left(\sum_{k=1}^{\infty} A_k \xi^{\otimes k}\right)^{\otimes n} \right\rangle$$

as the formal series $\sum_{n=0}^{\infty} \langle G^{(n)}, \xi^{\otimes n}\rangle \in \mathcal{S}(\Phi, \mathbb{R})$ with $G^{(0)} := F^{(0)}$ and

$$G^{(n)} := \sum_{m=1}^{n} \sum_{\substack{(k_1,\dots,k_m)\in\mathbb{N}^m \\ k_1+\dots+k_m=n}} (A^*_{k_1} \odot A^*_{k_2} \odot \dots \odot A^*_{k_m}) F^{(m)}, \quad n \in \mathbb{N}.$$

Here $A^*_k \in \mathcal{L}(\Phi', \Phi'^{\odot k})$ is the adjoint of A_k.

Remark A.7. It follows from Definition A.6 that

$$F(A(\xi)) = F^{(0)} + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \sum_{\substack{(k_1,\dots,k_m)\in\mathbb{N}^m \\ k_1+\dots+k_m=n}} \big\langle F^{(m)}, (A_{k_1}\xi^{\otimes k_1}) \odot (A_{k_2}\xi^{\otimes k_2}) \odot \dots \odot (A_{k_m}\xi^{\otimes k_m}) \big\rangle \right).$$

Proposition A.8. *Let $F(\xi) \in \mathcal{S}(\Phi, \mathbb{R})$ and let $A(\xi), B(\xi) \in \mathcal{S}(\Phi, \Phi)$. Denote $G(\xi) := F(B(\xi)) \in \mathcal{S}(\Phi, \mathbb{R})$ and $C(\xi) := B(A(\xi)) \in \mathcal{S}(\Phi, \Phi)$. Then $G(A(\xi)) = F(C(\xi))$, the equality in $\mathcal{S}(\Phi, \mathbb{R})$.*

Proof. The proposition follows from Definitions A.4 and A.6, see also Remark A.7. We leave the details to the interested reader. □

Remark A.9. As easily seen, a statement similar to Proposition A.8 also holds for the composition $S(R(F(\xi))) \in \mathcal{S}(\Phi, \mathbb{R})$, where $S, R \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ and $F \in \mathcal{S}(\Phi, \mathbb{R})$.

Proposition A.8 immediately implies the following result.

Corollary A.10. *Let $F(\xi) \in \mathcal{S}(\Phi, \mathbb{R})$ and let $A(\xi), B(\xi) \in \mathcal{S}(\Phi, \Phi)$ be such that $B(A(\xi)) = \xi$. Denote $G(\xi) := F(B(\xi)) \in \mathcal{S}(\Phi, \mathbb{R})$. Then $G(A(\xi)) = F(\xi)$.*

References

[1] Albeverio, S., Daletsky, Y.L., Kondratiev, Y.G., Streit, L.: Non-Gaussian infinite-dimensional analysis. J. Funct. Anal. 138 (1996), 311–350.

[2] Anshelevich, M.: Appell polynomials and their relatives. Int. Math. Res. Not. 2004, no. 65, 3469–3531.

[3] Anshelevich, M.: Appell polynomials and their relatives. II. Boolean theory. Indiana Univ. Math. J. 58 (2009), 929–968.

[4] Anshelevich, M.: Appell polynomials and their relatives. III. Conditionally free theory. Illinois J. Math. 53 (2009), 39–66.

[5] Barnabei, M., Brini, A., Nicoletti, G.: A general umbral calculus in infinitely many variables. Adv. Math. 50 (1983), 49–93.

[6] Berezansky, Y.M., Kondratiev, Y.G.: Spectral methods in infinite-dimensional analysis. Vol. 1, 2. Kluwer Academic Publishers, Dordrecht, 1995 (translated from the 1988 Russian original).

[7] Berezansky, Y.M., Kondratiev, Y.G., Kuna, T., Lytvynov, E.: On a spectral representation for correlation measures in configuration space analysis. Methods Funct. Anal. Topology 5 (1999), no. 4, 87–100.

[8] Berezansky, Y.M., Sheftel, Z.G., Us, G.F.: Functional analysis. Vol. II. Birkhäuser, Basel, 1996.

[9] Bourbaki, N.: Topological vector spaces. Chapters 1–5. Springer-Verlag, Berlin, 1987.

[10] Brown, J.: On multivariable Sheffer sequences. J. Math. Anal. Appl. 69 (1979), 398–410.

[11] Di Bucchianico, A., Loeb, D.: A selected survey of umbral calculus. Electron. J. Combin. 2 (1995), Dynamic Survey 3, 28 pp. (electronic).

[12] Di Bucchianico, A., Loeb, D., Rota, G.-C.: Umbral calculus in Hilbert space. In: Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996), Progr. Math., 161, Birkhäuser, Boston, 1998, pp. 213–238.

[13] Garsia, A., Joni, S.: Higher dimensional polynomials of binomial type and formal power series inversion. Comm. Algebra 6 (1978), 1187–1211.

[14] Hida, T., Kuo, H.-H., Potthoff, J., Streit, L.: White noise. An infinite dimensional calculus. Kluwer Academic Publishers, Dordrecht, 1993.

[15] Holden, H., Øksendal, B., Ubøe, J., Zhang, T.: Stochastic partial differential equations. A modeling, white noise functional approach. Birkhäuser, Boston, 1996.

[16] Kondratiev, Y.G., Kuna, T.: Harmonic analysis on configuration space. I. General theory. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5 (2002), 201–233.

[17] Kondratiev, Y.G., Kuna, T., Oliveira, M.J.: Analytic aspects of Poissonian white noise analysis. Methods Funct. Anal. Topology 8 (2002), no. 4, 15–48.

[18] Kondratiev, Y.G., Kuna, T., Oliveira, M.J.: On the relations between Poissonian white noise analysis and harmonic analysis on configuration spaces. J. Funct. Anal. 213 (2004), 1–30.

[19] Kondratiev, Y.G. , Lytvynov, E.W.: Operators of gamma white noise calculus. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (2000), 303–335.

[20] Kondratiev, Y.G., da Silva, J.L., Streit, L., Us, G.F.: Analysis on Poisson and gamma spaces. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 1 (1998), 91–117.

[21] Kondratiev, Y.G., Streit, L., Westerkamp, W., Yan, J.: Generalized functions in infinite-dimensional analysis. Hiroshima Math. J. 28 (1998), 213–260.

[22] Kung, J.P.S., Rota, G.-C., Yan, C.H.: Combinatorics: the Rota way. Cambridge University Press, Cambridge, 2009.

[23] Ito, Y., Kubo, I.: Calculus on Gaussian and Poisson white noises. Nagoya Math. J. 111 (1988), 41–84.

[24] Lytvynov, E.: Polynomials of Meixner's type in infinite dimensions—Jacobi fields and orthogonality measures. J. Funct. Anal. 200 (2003), 118–149.

[25] Lytvynov, E.: Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (2003), 73–102.

[26] Michor, P.: Contributions to finite operator calculus in several variables. J. Combin. Inform. System Sci. 4 (1979), 39–65.

[27] Mullin, R., Rota, G.-C.: On the foundations of combinatorial theory. III. Theory of binomial enumeration. In: Graph theory and its applications (Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969), Academic Press, New York, pp. 167–213.

[28] Narici, L., Beckenstein, E.: Topological vector spaces. Second edition. CRC Press, Boca Raton, FL, 2011.

[29] Obata, N.: White noise calculus and Fock space. Springer-Verlag, Berlin, 1994.

[30] Parrish, C.: Multivariate umbral calculus. J. Linear and Multilinear Algebra 6 (1978), 93–109.

[31] Reiner, D.: Multivariate sequences of binomial type. Stud. Appl. Math. 57 (1977), 119–133.

[32] Roman, S.: The algebra of formal series. III. Several variables. J. Approx. Theory 26 (1979), 340–381.

[33] Roman, S.: The umbral calculus. Academic Press, New York, 1984.

[34] Roman, S.M., Rota, G.-C.: The umbral calculus. Adv. Math. 27 (1978), 95–188.

[35] Rota, G.-C., Kahaner, D., Odlyzko, A.: On the foundations of combinatorial theory. VIII. Finite operator calculus. J. Math. Anal. Appl. 42 (1973), 684–760.

[36] Schaefer, H.H., Wolff, M.P.: Topological vector spaces. Second edition. Springer-Verlag, New York, 1999.

[37] Singh, S., Asthana, S.: Multivariate shift invariant operators. J. Math. Anal. Appl. 118 (1986), 422–442.

[38] Ueno, K.: General power umbral calculus in several variables. J. Pure Appl. Algebra 59 (1989), 299–308.