

Positive definite functions on Coxeter groups with applications to operator spaces and noncommutative probability

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Uffe Haagerup (1949–2015) in Memoriam

ABSTRACT. A new class of positive definite functions related to colour-length function on arbitrary Coxeter group is introduced. Extensions of positive definite functions, called the Riesz-Coxeter product, from the Riesz product on the Rademacher (Abelian Coxeter) group to arbitrary Coxeter group is obtained. Applications to harmonic analysis, operator spaces and noncommutative probability is presented. Characterization of radial and colour-radial functions on dihedral groups and infinite permutation group are shown.

Introduction

In 1979 Uffe Haagerup in his seminal paper [HAA79], essentially proved the positive definiteness, for $0 \leq q \leq 1$, of the function $P_q(x) = q^{|\cdot|} = \exp(-t|\cdot|)$, where $|\cdot|$ is the word length on a free Coxeter group $W = \mathbf{Z}/2 * \cdots * \mathbf{Z}/2$. From this he deduced also Khinchine type inequalities. He has shown that the regular C^* -algebra of W has bounded approximation property and later [DCH85] completely bounded approximation property (CBAP). These results of Uffe Haagerup have had significant impact on harmonic analysis on free groups and, more generally, on Coxeter groups; they also influenced the free probability theory and other noncommutative probability theories.

In the paper [BJS88] it was shown that for all Coxeter groups the function $P_q(x) = q^{|\cdot|}$ is also positive definite for $q \in [-1, 1]$, where the length $|\cdot|$ is the natural word length function on Coxeter group with respect to the set of its Coxeter generators. This fact implies that infinite Coxeter groups have Haagerup property and do not have Kazhdan's property (T).

Later, Januszkiewicz [JAN02] and Fendler [FEN02B] showed, in the spirit of Haagerup proof, that $w \mapsto z^{|w|}$ is a coefficient of a uniformly bounded Hilbert representation of W for all $z \in \mathbf{C}$ such that $|z| < 1$.

As shown in a very short paper of Valette [VAL93], this implies CBAP. See the book [BO08] for further extension of Uffe Haagerup results for a big class of groups.

In the paper [BS96] Bożejko and Speicher considered the free product (convolution) of classic normal distribution $N(0, 1)$ and the new length function on the permutation group \mathfrak{S}_n (i.e. the Coxeter group of type A) was rediscovered, which we shall call the *colour-length function* $\|\cdot\|$. It is defined as follows: for $w \in \mathfrak{S}_n$ in the minimal (reduced) representations $w = s_1 \dots s_k$, where each s_j belong to the set S of transpositions of the form $(i, i + 1)$, we put $\|w\| := \#\{s_1, s_2, \dots, s_k\}$.

For our study one of the most important results of this paper is that the function called Riesz-Coxeter product R_q defined on all Coxeter groups (W, S) as

$$R_q(s) = q_s, \text{ for } s \in S, \text{ and } R_q(xy) = R_q(x)R_q(y), \text{ if } \|xy\| = \|x\| + \|y\|$$

is positively defined for $0 \leq q_s \leq 1$.

This implies, in particular, that in arbitrary Coxeter group the set of its Coxeter generators is weak Sidon set and also it is completely bounded Λ_p^{cb} -set, see Theorems 6.1 and 6.2. Equivalently, the span of linear operators $\{\lambda(s) | s \in S\}$ in the noncommutative L^p -space $L^p(W)$ is completely boundedly isomorphic with row and column operator Hilbert space (see Theorem 6.2).

Another interesting connection between the two length functions $|\cdot|$ and $\|\cdot\|$ appeared in [BS96] in the formula for the moments of the free additive convolution power of the Bernoulli law $\mu_{-1} = 1/2(\delta_{-1} + \delta_1)$ (cf. Corollary 6 in cited paper):

$$m_{2n}(\mu_{-1}^{\boxplus q}) = q^n \sum_{\pi \in \mathcal{P}_2(2n)} (-1)^{|\pi|} q^{-\|\pi\|},$$

for $q = 1, 2, \dots$ (See also Section 7 of the present paper.)

Also, in [BBL11] the colour-length function on the permutation group \mathfrak{S}_n was studied. Some of its extensions to pairpartitions appeared in the presentation of the proof that classical normal law $N(0, 1)$ is free infinitely divisible under free additive convolution \boxplus .

Since we have recent extensions of the free probability (which is related to type A Coxeter groups) to the free probability of type B Coxeter groups (see [BEH15]), it seems to be interesting to determine the role of the colour-length functions for the Coxeter groups of type B and D.

The plan of the paper is as follows.

In Section 1 we recall definitions of Coxeter groups and of the length and the colour-length functions.

In Section 2 we show the following formula characterizing the radial normalised positive definite functions on these Coxeter groups which contain the infinite Rademacher group $\bigoplus_{i=1}^{\infty} \mathbf{Z}/2$ as a parabolic subgroup (these include the infinite permutation group \mathfrak{S}_{∞}): every radial positive definite function φ is of the form

$$\varphi(w) = \int_{-1}^1 q^{|w|} \mu(dq)$$

for a probability measure μ .

That characterisation is a variation on classical de Finetti theorem. A noncommutative version was shown by Kötler and Speicher [KS09].

We also show in Theorem 2.4, that the function $\exp(-t|w|^p)$ is positive definite for all $t \geq 0$ if and only if $p \in [0, 1]$.

In Section 3 we give a short proof of the equivalence of the two known results concerning positive definite functions on finite Coxeter groups.

In Section 4, we present main properties of the colour-dependent positive definite functions on Coxeter groups, in particular we show in Proposition 4.4. that on \mathfrak{S}_{∞} and some other Coxeter groups, the function $w \mapsto r^{\|w\|}$ is positive definite if and only if $r \in [0, 1]$.

The Section 5 gives characterization of all colour-length functions on finite and infinite dihedral groups \mathbf{D}_m , for $m = 1, 2, \dots, \infty$.

In Section 6 we prove that the set S of Coxeter generators is a weak Sidon set in arbitrary Coxeter groups (W, S) with constant 2 and that it is also a completely bounded $\Lambda(p)$ set with constants as $C\sqrt{p}$, for $p > 2$.

In Section 7 we prove for arbitrary finitely generated Coxeter group an identity involving both lengths $|\cdot|$ and $\|\cdot\|$ (see Proposition 7.1). We apply it to give a proof of Corollary 7 from [BS96], (see Equation (7.1)) where the proof, involving probabilistic considerations, was not presented in [BS96].

1. Coxeter groups

In this part we recall the basic facts regarding Coxeter groups and introduce notation which will be used throughout the rest of the paper. For more details we refer to [BOU68, HUM90].

A group W is called a *Coxeter group* if it admits the following presentation:

$$W := \left\langle S \mid \left\{ (s_1 s_2)^{m(s_1, s_2)} = 1 : s_1, s_2 \in S, m(s_1, s_2) \neq \infty \right\} \right\rangle,$$

where $S \subset W$ is a set and m is a function $m : S \times S \rightarrow \{\infty, 1, 2, 3, \dots\}$ such that $m(s_1, s_2) = m(s_2, s_1)$ for all $s_1, s_2 \in S$ and $m(s_1, s_2) = 1$ if and only if $s_1 = s_2$. The pair (W, S) is called a *Coxeter system*. In particular, every generator $s \in S$ has order two and every element $w \in W$ can be represented as

$$(1.1) \quad w = s_1 s_2 \dots s_m$$

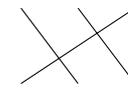
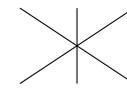
for some $s_1, s_2, \dots, s_m \in S$. If the sequence $(s_1, \dots, s_m) \in S^m$ is chosen in such a way that m is minimal then we write $|w| = m$ and call it the *length* of w . The right hand side of (1.1) is called a *reduced representation* or *reduced word* of w . This is not unique in general, but the set of involved generators is unique [BOU68, Ch. IV, §1, Prop. 7], i.e. if $w = s_1 s_2 \dots s_m = t_1 t_2 \dots t_m$ are two reduced representations of $w \in W$ then $\{s_1, s_2, \dots, s_m\} = \{t_1, t_2, \dots, t_m\}$. This set $\{s_1, s_2, \dots, s_m\} \subseteq S$ will be denoted S_w and called the *colour* of w .

Given $T \subset S$ by W_T we denote the subgroup generated by T . To see that S_w is independent of the reduced representation of w notice that

$$(1.2) \quad s \in S_w \iff w \notin W_{S \setminus \{s\}}.$$

We define *colour-length* of w putting $\|w\| := \#S_w$ (the cardinality of S_w). Both lengths satisfy the triangle inequality and we have $\|w\| \leq |w|$.

In the case of the permutation group the colour-length has the following pictorial interpretation. If σ is a permutation in \mathfrak{S}_{n+1} then $|\sigma|$ equals to n minus the number of connected components of the diagram representing σ . Notice, that $|\sigma|$ equals to the number of crossings in the diagram (the number of pairs of chords that cross).

σ	e	(12)	(12)(23)	(12)(23)(12)
				
$ \sigma $	0	1	2	3
$\ \sigma\ $	0	1	2	2

It would be convenient to define

$$(1.3) \quad \|w\|_s := \begin{cases} 0 & \text{if } s \notin S_w, \\ 1 & \text{if } s \in S_w, \end{cases}$$

then, clearly, $\|w\| = \sum_{s \in S} \|w\|_s$.

Countable Abelian Coxeter groups are $\text{Rad}_n := \bigoplus_{i=1}^n \mathbf{Z}/2$, for $n = 1, 2, \dots, \infty$; we shall call them the *Rademacher groups*. Note, that on Rademacher groups the functions $|\cdot|$ and $\|\cdot\|$ coincide.

2. Remarks on radial positive definite functions on some infinitely generated Coxeter groups

A complex function φ on a group Γ is called *positive definite* if we have

$$\sum_{x, y \in \Gamma} \varphi(y^{-1}x) \alpha(x) \overline{\alpha(y)} \geq 0$$

for every finitely supported function $\alpha: \Gamma \rightarrow \mathbf{C}$.

A positive definite φ function is Hermitian and satisfies $|\varphi(x)| \leq \varphi(e)$ for all $x \in \Gamma$. Usually it is assumed, that φ is *normalised*, i.e. that $\varphi(e) = 1$.

In this and the following sections we discuss the *radial functions* on Coxeter groups. These are functions which depend on $|w|$ rather than on w .

First we start with a version of the classical de Finetti Theorem (see [FEL71, p. 223]) for the infinite Rademacher group.

Theorem 2.1. *Assume that φ is a radial function on the Rademacher group Rad_∞ , with $\varphi(e) = 1$. Then φ is positive definite if and only if there exists a probability measure μ on $[-1, 1]$ such that*

$$\varphi(x) = \int_{-1}^1 q^{|x|} \mu(dq).$$

This measure μ is unique.

Proof. Since the function $q^{|x|}$ is positive definite for $q \in [-1, 1]$, the “if” implication is obvious.

Assume that φ is positive definite. The group Rad_∞ is discrete and Abelian and its dual is the compact group $\widehat{\text{Rad}}_\infty = \prod_{i=1}^\infty \mathbf{Z}/2$. By Bochner theorem, there exists a probability measure η on $\widehat{\text{Rad}}_\infty$ such that

$$\varphi(x) = \int_{\widehat{\text{Rad}}_\infty} (-1)^{x \cdot y} d\eta(y),$$

where for $x = (x_1, x_2, \dots) \in \text{Rad}_\infty$, $y = (y_1, y_2, \dots) \in \widehat{\text{Rad}}_\infty$ we put $x \cdot y := x_1 y_1 + x_2 y_2 + \dots$. The radially of φ is equivalent to the fact that for every permutation $\sigma \in \mathfrak{S}_\infty$ we have $\varphi(x) = \varphi(\sigma(x))$, where $\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots)$. This, in turn, implies that η is σ -invariant for every $\sigma \in \mathfrak{S}_\infty$, i.e. we have $\eta(A) = \eta(\sigma(A))$ for every Borel subset $A \subset \widehat{\text{Rad}}_\infty$.

For a sequence $\epsilon = (\epsilon_i)_{i=1}^n \in \{0, 1\}^n$ we define $C_n(\epsilon) \subseteq \widehat{\text{Rad}}_\infty$ by

$$C_n(\epsilon) := \{y \in \widehat{\text{Rad}}_\infty \mid y_i = \epsilon_i : 1 \leq i \leq n\},$$

in particular $C_0(\emptyset) = \widehat{\text{Rad}}_\infty$. Then we have $\eta(C_n(\epsilon)) = \eta(C_n(\epsilon'))$ if $\epsilon'_i = \epsilon_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_n$ and every $1 \leq i \leq n$. For $\epsilon \in \mathbf{R}$ we put

$$\epsilon^n = \underbrace{\epsilon, \epsilon, \dots, \epsilon}_n$$

and $a_n := \eta(C_n(1^n))$. Moreover, for $n, k \geq 0$ we define the difference operators $\Delta^k a_n$ by induction: $\Delta^0 a_n := a_n$ and $\Delta^{k+1} a_n := \Delta^k a_{n+1} - \Delta^k a_n$. We claim that

$$(2.1) \quad (-1)^k \Delta^k a_n = \eta(C_{n+k}(1^n \circ^k)).$$

Denoting the right hand side of (2.1) by $c(n, k)$ we note that $c(n, 0) = a_n$ and

$$C_{n+k+1}(1^n \circ^k 0) \cup C_{n+k+1}(1^n \circ^k 1) = C_{n+k}(1^n \circ^k),$$

is a disjoint union. This implies

$$c(n, k+1) = c(n, k) - c(n+1, k).$$

This formula, by induction on k , leads to (2.1).

From (2.1) we see that the sequence (a_n) is *completely monotone*, i.e. that $(-1)^k \Delta^k a_n \geq 0$ for all $n, k \geq 0$. By the celebrated theorem of Hausdorff (see [HAU21, Satze II und III]), there exists a unique probability measure ρ on $[0, 1]$ such that

$$(2.2) \quad (-1)^k \Delta^k a_n = \int_0^1 u^n (1-u)^k d\rho(u).$$

(Note that Equation (2.2) for arbitrary $k \geq 0$ follows from the case $k = 0$.)

For $x = (1^n \circ^\infty) \in \text{Rad}_\infty$ so that $|x| = n$, we have

$$\begin{aligned} \varphi(x) &= \int_{\widehat{\text{Rad}}_\infty} (-1)^{x \cdot y} d\eta(y) = \int_{\widehat{\text{Rad}}_\infty} (-1)^{y_1 + \dots + y_n} d\eta(y) \\ &= \sum_{\epsilon \in \{0, 1\}^n} (-1)^{\epsilon_1 + \dots + \epsilon_n} \eta(C_n(\epsilon)) = \sum_{k=0}^n \binom{n}{k} (-1)^k \eta(C_n(1^k \circ^{n-k})) \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k \int_0^1 u^k (1-u)^{n-k} d\rho(u) = \int_0^1 (1-2u)^n d\rho(u) \\ &= \int_{-1}^1 q^n d\mu(q), \end{aligned}$$

where μ is defined by: $\mu(A) := \rho(\frac{1}{2} + \frac{1}{2}A)$ for a Borel set $A \subseteq [-1, 1]$. \square

We can extend previous theorem to a certain class of Coxeter groups.

Theorem 2.2. *Assume that (W, S) is a Coxeter system and that there is an infinite subset $S_o \subseteq S$ such that $st = ts$ for $s, t \in S_o$. Assume that φ is a radial function on W with $\varphi(e) = 1$. Then φ is positive definite if and only if there exists a probability measure μ on $[-1, 1]$ such that*

$$\varphi(\sigma) = \int_{-1}^1 q^{|\sigma|} \mu(dq).$$

This measure μ is unique.

Proof. It is sufficient to note that the group generated by S_o is a parabolic subgroup isomorphic with Rad_∞ . \square

Example. For $W = \mathfrak{S}_\infty$ we have $S = \{(n, n+1) : n \in \mathbf{N}\}$. Then we can take $S_o = \{(2n-1, 2n) : n \in \mathbf{N}\}$. Similar S_o can be found in infinitely generated groups of type B and D.

Problem 2.3. When $-1 \leq q \leq 1$, $q \neq 0$ is the positive definite function $q^{|\sigma|}$ on \mathfrak{S}_∞ an extreme point in the set of normalised positive definite functions?

Theorem 2.4. *The function $\varphi_p(\sigma) := e^{-t|\sigma|^p}$ is positive definite on \mathfrak{S}_∞ if and only if $0 \leq p \leq 1$.*

Proof. *A contrario.* Assume that for some $p > 1$ and $t_o > 0$ the function $\psi_p(\sigma) = e^{-t_o|\sigma|^p}$ is positive definite on \mathfrak{S}_∞ .

For $q_o = e^{-t_o}$, choosing σ such that $|\sigma| = 2n$ we have $q_o^{(2n)^p} = \int_{-1}^1 q^{2n} d\mu_o(q)$ for some probability measure μ_o on $[-1, 1]$. Since $\left(\int_{-1}^1 q^{2n} d\mu_o(q)\right)^{1/n}$ tends to $\max\{q^2 | q \in \text{supp } \mu_o\}$ while $\left(q_o^{(2n)^p}\right)^{1/n} \rightarrow 0$, we conclude that μ_o is the Dirac measure at 0, which is a contradiction.

The “if” part is standard. We need to show that $f(x) = e^{-tx^p}$ is the Laplace transform of some probability measure supported on $[0, \infty)$, so f is a convex combination of functions of the form e^{-sx} .

By characterisation of Laplace transforms (see [HAU21, Satz III]) this is equivalent to *complete monotonicity*, that is $(-1)^n f^{(n)} > 0$ for all $n = 0, 1, \dots$. And indeed, by induction, $(-1)^n f^{(n)}$ is a positive combination of positive functions of the form $x^{pj-n} f(x)$ for $1 \leq j \leq n$. \square

The measures with Laplace transforms e^{-tx^p} for $t \geq 0$ and $0 \leq p \leq 1$ are studied in detail in [YOS80, Ch. IX.11] (see Propositions 1 and 2 there).

Let us note that for such groups like \mathbf{Z}^k or \mathbf{R}^k with the Euclidean distance d the functions $\exp(-td^p)$ are positive definite for all $t \geq 0$ and $0 \leq p \leq 2$ (the case $p = 2$ corresponds to the Gaußian Law).

3. The longest element

If a Coxeter group W is finite, then it contains the unique element ω_\circ which is the longest with respect to $|\cdot|$.

From the definition it is clear, that a function φ on a group Γ with values in the field of complex numbers \mathbf{C} is *positive definite* if and only $\sum_{g \in \Gamma} \varphi(g)g$ is a nonnegative (bounded if the group is finite) operator on $\ell^2\Gamma$. (We will identify $g \in \Gamma$ with $\lambda(g) \in B(\ell^2\Gamma)$, where λ is the left regular representation, for short.)

Let W be a finite Coxeter group. The following two statements are well known.

- (A) The function $q^{|w|}$ is positive definite for any $0 \leq q \leq 1$.
- (B) The function $\Delta(w) = |\omega_\circ|/2 - |w|$ is positive definite.

The first one was proven in [BJS88] (even for infinite Coxeter groups and also for $-1 \leq q \leq 1$) while the second — in [BS03, Proposition 6]. Here we give a short direct prove of the following.

Proposition 3.1. *The above statements (A) and (B) are equivalent.*

Proof. Let $q = e^{-t}$ (with $t \geq 0$, as we assume $q \leq 1$). The case (A) is equivalent to $\Phi_t = \sum_{w \in W} e^{t\Delta(w)}w = e^{t|\omega_\circ|/2} \sum_{w \in W} q^{|w|}w$ being nonnegative.

Assume (A). Recall first, that $|\omega_\circ w| = |\omega_\circ| - |w| = |w\omega_\circ|$. Therefore $|\omega_\circ|/2 - |\omega_\circ w| = -(|\omega_\circ|/2 - |w|)$, ie. $\Delta(\omega_\circ w) = -\Delta(w)$ and similarly, $\Delta(w\omega_\circ) = -\Delta(w)$.

The equality $\Delta(\omega_\circ w) = \Delta(w\omega_\circ)$ implies that ω_\circ (and thus $Q = (1 - \omega_\circ)/2$) commutes with Δ (and thus Φ_t). Since $Q = Q^2$ is nonnegative we conclude that

$$t^{-1}\Phi_t Q = \sum_{w \in W} \frac{e^{t(|\omega_\circ|/2 - |w|)} - e^{t(|\omega_\circ|/2 - |w\omega_\circ|)}}{2t} w = \sum_{w \in W} \frac{\sinh(t\Delta(w))}{t} w.$$

is nonnegative. Therefore, taking the limit as $t \rightarrow 0$, we obtain that $\sum_{w \in W} \Delta(w)w$ is nonnegative. Thus (B).

Assuming (B) and using the Schur lemma, which says that the (point-wise) product of positive definite functions is positive definite, we

get that

$$\Phi_t = \sum_{w \in W} e^{t\Delta(w)} w = \sum_{n \geq 0} \frac{t^n}{n!} \left(\sum_{w \in W} \Delta(w)^n w \right)$$

is nonnegative. Thus (A). \square

4. Colour-dependent positive definite functions on Coxeter groups

We call a function φ on (W, S) *colour-dependent* if $\varphi(w)$ depends only on S_w . We call it *colour-radial* if it depends only on $\|w\|$.

The main example would be *Riesz–Coxeter function*. Given a sequence $\mathbf{q} = (q_s)_{s \in S}$ we define $R_{\mathbf{q}}(w) = \prod_{s \in S} q_s^{\|w\|_s} = \prod_{s \in S_w} q_s$. We will abuse the notation and denote by $R_{\mathbf{q}}$ also the associated operator $\sum_{w \in W} R_{\mathbf{q}}(w)w$. That is

$$R_{\mathbf{q}} = 1 + \sum_{s \in S} q_s s + \sum_{w: S_w = \{s_1, s_2\}} q_{s_2} q_{s_1} w + \sum_{w: S_w = \{s_1, s_2, s_3\}} q_{s_2} q_{s_1} q_{s_3} w + \dots$$

In the case all $q_s = q$ we get $R_{\mathbf{q}} = \sum q^{\|w\|} w$.

This generalises the classical case of Rademacher–Walsh functions in the Rademacher group Rad_n . If we denote the generator of the i -th factor $\mathbf{Z}/2$ by r_i then, by definition, $r_i^2 = 1$ and $r_i r_j = r_j r_i$ and

$$R_{\mathbf{q}} = \prod_{i=1}^n (1 + q_i r_i).$$

The question which colour-dependent or colour-radial functions are positive functions on Coxeter groups is wide open. In this section we provide some sufficient conditions. In the next section we will examine the dihedral groups in full details.

Lemma 4.1. *Let H be a subgroup of a group Γ of index d . Then the function φ_r defined by $\varphi_r(x) = 1$ if $x \in H$ and $\varphi_r(x) = r$ otherwise is positive definite on Γ if and only if $r \in [-1/(d-1), 1]$, with natural convention that if $d = \infty$ then $-1/(d-1) = 0$.*

Note, that if $H = \{1\}$ then $d = |\Gamma|$.

Proof. First assume that d is finite and let us enumerate the left cosets:

$$\{gH : g \in \Gamma\} = \{H_1, H_2, \dots, H_d\}.$$

Note, that for $x \in H_i, y \in H_j$ we have $y^{-1}x \in H$ if and only if $i = j$. Therefore, for $r_0 = -1/(d-1)$ and for a finitely supported complex

function f on Γ we have

$$\sum_{x,y \in \Gamma} \varphi_{r_0}(y^{-1}x) f(x) \overline{f(y)} = \frac{1}{d-1} \sum_{1 \leq i < j \leq d} \left| \sum_{x \in H_i} f(x) - \sum_{y \in H_j} f(y) \right|^2,$$

which proves that φ_{r_0} is positive definite. For $r \in [-1/(d-1), 1]$ the function φ_r is positive definite as a convex combination of φ_{r_0} and the constant function φ_1 .

On the other hand, if we choose $x_i \in H_i$ for each $i \leq d$ and define f as the characteristic function of the set $\{x_1, \dots, x_d\}$ then

$$(4.1) \quad \sum_{x,y \in \Gamma} \varphi_r(y^{-1}x) f(x) \overline{f(y)} = d + (d^2 - d)r,$$

which proves that $r \geq -1/(d-1)$ is a necessary condition for positive definiteness of φ_r .

If $d = \infty$ then $r_0 = 0$ and the function φ_0 is positive definite as the characteristic function of the subgroup H . For “only if” part we chose an arbitrarily long sequence $x_1, \dots, x_{d'}$ of elements from different left cosets and use (4.1) with d' instead of d . \square

Proposition 4.2. *Assume that for every $s \in S$ we are given a number q_s ,*

$$\frac{-1}{d_s - 1} \leq q_s \leq 1,$$

where d_s denotes the index of the parabolic subgroup generated by $S \setminus \{s\}$ in W : $d_s := [W : W_{S \setminus \{s\}}]$. Then the Riesz–Coxeter $R_{\mathbf{q}}$ is positive definite on W .

Proof. From Lemma 4.1 the function $w \mapsto q_s^{\|w\|_s}$ is positive definite for $s \in S$ and $-1/(d_s - 1) \leq q_s \leq 1$. Since the pointwise product of positive definite functions is positive definite, the statement holds. \square

Example. Take $W = \mathfrak{S}_n$, the permutation group on the set $\{1, 2, \dots, n\}$. It is generated by the transpositions $S = \{s_i = (i, i+1), 1 \leq i \leq n-1\}$. For $1 \leq i \leq n-1$ the parabolic subgroup generated by $S \setminus \{s_i\}$ is isomorphic with $\mathfrak{S}_{i-1} \times \mathfrak{S}_{n-i-1}$, so its index is $n \binom{n-1}{i-1}$.

It would be interesting to determine for which r the function $w \mapsto r^{\|w\|}$ is positive definite. By Proposition 4.2 this holds for $r \in [-1/(d-1), 1]$, where d is the maximal index of the parabolic subgroups of the form $\langle S \setminus \{s\} \rangle$. We note a necessary condition.

Proposition 4.3. *Assume that we have distinct generators $s_0, s_1, \dots, s_n \in S$ such that $s_0 s_k \neq s_k s_0$ (i.e. $m(s_0, s_k) > 2$) for $1 \leq k \leq n$. If the function $w \mapsto r^{\|w\|}$ is positive definite on W , then $-1/n-1 \leq r^3 \leq 1$.*

If there is an element $s_o \in S$ for which there are infinitely many $s \in S$ such that $s_o s \neq s s_o$ then $r^{\|w\|}$ is positive definite on W if and only if $0 \leq r \leq 1$.

Proof. Consider elements $w_k = s_o s_k s_o$. Note, that for $k \neq l$ we have $\|w_l^{-1} w_k\| = 3$. If φ_r is positive definite on W then we have

$$0 \leq \sum_{k,l=1}^n \varphi_r(x_l^{-1} x_k) = n + (n^2 - n)r^3,$$

which implies $r^3 \geq -1/(n-1)$. \square

Corollary 4.4. The function $w \mapsto q^{\|w\|}$ on \mathfrak{S}_∞ is positive definite if and only if $0 \leq q \leq 1$.

Problem 4.5. Thus, it is valid to ask the following. Is it true that every normalised positive definite colour-length-radial function $\phi: \mathfrak{S}_\infty \rightarrow \mathbf{R}$ is of the form $\phi(\sigma) = \int_0^1 q^{\|\sigma\|} d\mu(q)$ for some probability measure μ on $[0, 1]$?

5. Dihedral groups

In this part we are going to examine the class of colour-dependent positive definite functions on the case the simplest nontrivial Coxeter groups. Assume that $W = \mathbf{D}_{2n} = \langle s, t | (st)^n \rangle$ (i.e. the group of symmetries of a regular n -gon), and define a colour-dependent function on W :

$$(5.1) \quad \phi(w) := \begin{cases} 1 & \text{if } w = e, \\ p & \text{if } w = s, \\ q & \text{if } w = t, \\ r & \text{otherwise.} \end{cases}$$

If $p = q$ then ϕ is colour radial. We are going to determine for which parameters p, q, r the function ϕ is positive definite on W . It is easy to observe necessary conditions: $p, q, r \in [-1, 1]$. Moreover, since $\langle st \rangle$ is a cyclic subgroup of order n , Lemma 4.1, implies a necessary condition: $-1/(n-1) \leq r \leq 1$.

Finite dihedral groups. Assume that W is a finite dihedral group, $W = \mathbf{D}_{2n}$, so that $(st)^n = 1$. We will use the following version of the Bochner theorem: A function f on a compact group G is positive definite if and only if its Fourier transform:

$$\widehat{f}(\pi) = \int_G f(x) \pi(x^{-1}) dx$$

is a positive operator for every $\pi \in \widehat{G}$, where \widehat{G} denotes the dual object of G , i.e. the family of all equivalency classes of unitary irreducible representations of G , see [SIM96]. Then we have

$$f(x) = \sum_{\pi \in \widehat{G}} d_{\pi} \operatorname{tr} [\widehat{f}(\pi) \pi(x)].$$

Therefore, for every irreducible representation π of \mathbf{D}_{2n} we are going to find

$$\widehat{\phi}(\pi) := \frac{1}{2n} \sum_{g \in G} \phi(g) \pi(g^{-1}).$$

We will identify s with $(0, -1)$ and t with $(1, -1)$. If n is odd then \mathbf{D}_{2n} possesses two characters: $\chi_{+,+}$ such that $\chi_{+,+}(w) = 1$ for every $w \in \mathbf{D}_{2n}$ and $\chi_{-,-}$ such that $\chi_{-,-}(s) = \chi_{-,-}(t) = -1$. If n is even then we have two additional characters $\chi_{+,-}$ and $\chi_{-,+}$ such that $\chi_{+,-}(s) = \chi_{-,+}(t) = 1$ and $\chi_{+,-}(t) = \chi_{-,+}(s) = -1$. It is easy to check that

$$2n\widehat{\phi}(\chi_{+,+}) = 1 + p + q + (2n - 3)r,$$

$$2n\widehat{\phi}(\chi_{-,-}) = 1 - p - q + r,$$

which gives

$$-1 - (2n - 3)r \leq p + q \leq 1 + r$$

and, for n even,

$$2n\widehat{\phi}(\chi_{+,-}) = 1 + p - q - r,$$

$$2n\widehat{\phi}(\chi_{-,+}) = 1 - p + q - r,$$

which implies

$$|p - q| \leq 1 - r.$$

We have also the family of two dimensional representations U_a :

$$U_a(k, 1) = \begin{pmatrix} e^{2\pi i k a/n} & 0 \\ 0 & e^{-2\pi i k a/n} \end{pmatrix},$$

$$U_a(k, -1) = \begin{pmatrix} 0 & e^{2\pi i k a/n} \\ e^{-2\pi i k a/n} & 0 \end{pmatrix},$$

where $a = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$. Then for the function given by (5.1) we have

$$\begin{aligned} 2n\widehat{\phi}(U_a) &= (1 - r)\operatorname{Id} + (p - r)U_a(0, -1) + (q - r)U_a(1, -1) \\ &= \begin{pmatrix} 1 - r & p - r + (q - r)e^{2\pi i a/n} \\ p - r + (q - r)e^{-2\pi i a/n} & 1 - r \end{pmatrix}. \end{aligned}$$

This matrix is positive definite if and only if $r \leq 1$ and

$$|p - r + (q - r)e^{2\pi i a/n}| \leq 1 - r.$$

Therefore we have

Proposition 5.1. *The function ϕ given by (5.1) is positive definite on \mathbf{D}_{2n} if and only if*

$$1 + p + q + (2n - 3)r \geq 0, \quad 1 - p - q + r \geq 0$$

(plus

$$1 + p - q - r \geq 0, \quad 1 - p + q - r \geq 0$$

whenever n is even) and

$$|p - r + (q - r)e^{2\pi ia/n}| \leq 1 - r.$$

for $a = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$.

Let us confine ourselves to colour-radial functions.

Corollary 5.2. *Assuming that $p = q$, the function ϕ defined by (5.1) is positive definite on $W = \mathbf{D}_{2n}$ if and only if*

$$\max \left\{ \frac{-2p-1}{2n-3}, 2p-1 \right\} \leq r \leq \frac{1+2p \cos(\pi/n)}{1+2 \cos(\pi/n)},$$

i.e. if and only if the point (p, r) belongs to the triangle which has vertices at

$$\left(\frac{1-n-\cos(\pi/n)}{1+(2n-1)\cos(\pi/n)}, \frac{1-\cos(\pi/n)}{1+(2n-1)\cos(\pi/n)} \right), \quad \left(\frac{n-2}{2n-2}, \frac{-1}{n-1} \right), \quad (1, 1).$$

Proof. For $p = q$ the conditions from Proposition 5.1 reduce to

$$2p-1 \leq r, \quad -1-2p \leq (2n-3)r, \quad \text{and} \quad 2 \cos(\pi/n)|p-r| \leq 1-r.$$

It is sufficient to note that $2p-1 \leq r$ implies $2 \cos(\pi/n)(p-r) \leq 1-r$ for $p \leq 1$. \square

Example. For \mathbf{D}_4 we have the positive definiteness of ϕ is equivalent to

$$-1 + |p + q| \leq r \leq 1 - |p - q|,$$

which means that the set of all possible (p, q, r) forms a tetrahedron with vertices $(-1, 1, -1)$, $(1, -1, -1)$, $(-1, -1, 1)$, $(1, 1, 1)$. For $p = q$ the condition reduces to $2|p| - 1 \leq r \leq 1$.

In the case of \mathbf{D}_6 Proposition 5.1 leads to the following conditions:

$$1 - p - q + r \geq 0, \quad 1 + p + q + 3r \geq 0,$$

$$1 - r \geq \sqrt{p^2 + q^2 + r^2 - pq - pr - qr},$$

which can be expressed as

$$\max \left\{ \frac{-1-p-q}{3}, p+q-1 \right\} \leq r \leq \frac{1-p^2-q^2+pq}{2-p-q}.$$

The infinite dihedral group. Here we are going to study $W = \mathbf{D}_\infty$.

Proposition 5.3. *The function ϕ given by (5.1) is positive definite on $W = \mathbf{D}_\infty$ if and only if $0 \leq r$ and $|p - r| + |q - r| \leq 1 - r$, i.e.*

$$(5.2) \quad \max\{0, p + q - 1\} \leq r \leq \min\left\{1 - |p - q|, \frac{1 + p + q}{3}\right\}.$$

Proof. First we note that the set of $(p, q, r) \in \mathbf{R}^3$ satisfying (5.2) constitutes a pyramid which is the convex hull of the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(1, 1, 1)$ (apex). For these particular parameters it is easy to see that ϕ is positive definite: $(1, 1, 1)$ corresponds to the constant function 1, $(1, 0, 0)$ to the characteristic function of the subgroup $\langle s \rangle = \{1, s\}$, and $(-1, 0, 0)$ to the character $\chi_{-, -}$ times the characteristic function of $\langle s \rangle$. Similarly for $(0, \pm 1, 0)$. This, by convexity, proves that (5.2) is a sufficient condition.

On the other hand, we know already that $r \geq 0$ is a necessary condition. Let us fix n and define $W^+(n) := \{x \in W : |sx| < |x| \leq 2n\}$, $BW^-(n) := \{x \in W : |tx| < |x| \leq 2n\}$ and

$$f(x) := \begin{cases} \pm 1 & \text{if } x \in W^\pm(n), \\ 0 & \text{otherwise.} \end{cases}$$

For $x, y \in W^+(n)$ we have $S_{y^{-1}x} = \emptyset$ in $2n$ cases (namely, if $x = y$) $S_{y^{-1}x} = \{s\}$ in $2n - 2$ cases (namely if $|x| = 2k$, $|y| = 2k + 1$ or vice-versa, $k = 1, \dots, n - 1$) $S_{y^{-1}x} = \{t\}$ in $2n$ cases (namely if $|x| = 2k$, $|y| = 2k - 1$ or vice-versa, $k = 1, \dots, n$) and $S_{y^{-1}x} = \{s, t\}$ in all the other $(2n - 1)(2n - 2)$ cases. Similarly, for $x, y \in W^-(n)$ we have $S_{y^{-1}x} = \emptyset$ in $2n$ cases, $S_{y^{-1}x} = \{s\}$ in $2n$ cases, $S_{y^{-1}x} = \{t\}$ in $2n - 2$ cases and $S_{y^{-1}x} = \{s, t\}$ in $(2n - 1)(2n - 2)$ cases. If $x \in W^+(n)$, $y \in W^-(n)$ or vice-versa then $S_{y^{-1}x} = \{s, t\}$. Summing up, we get

$$\begin{aligned} & \sum_{x, y \in W} \phi(y^{-1}x) f_n(x) f_n(y) \\ &= 4n + (4n - 2)p + (4n - 2)q + (4n - 2)(2n - 2)r - 8n^2r \\ &= 4n + (4n - 2)p + (4n - 2)q - (12n - 4)r. \end{aligned}$$

Therefore for every $n \in \mathbf{N}$ we have a necessary condition

$$1 + \left(1 - \frac{1}{2n}\right)p + \left(1 - \frac{1}{2n}\right)q - \left(3 - \frac{1}{n}\right)r \geq 0.$$

Letting $n \rightarrow \infty$ we get $1 + p + q \geq 3r$.

Put $x_k := stst\dots$, $|x_k| = k$. Fix n and define

$$g(x) := \begin{cases} \chi_{-, +}(x) & \text{if } x = x_k \text{ for } 1 \leq k \leq 4n, \\ 0 & \text{otherwise,} \end{cases}$$

where, as before, $\chi_{-,+}$ is the character on W for which $\chi_{-,+}(s) = -1$, $\chi_{-,+}(t) = 1$. Then

$$\sum_{x,y \in W} \phi(y^{-1}x)g(x)g(y) = \sum_{k,l=1}^{4n} \phi(x_l^{-1}x_k)g(x_k)g(x_l).$$

Denote $c_{k,l} := \phi(x_l^{-1}x_k)g(x_k)g(x_l)$. Then we have $c_{k,k} = 1$, $1 \leq k \leq 4n$, $c_{k,k-1} = q$ if k is even, $c_{k,k-1} = -p$ if k is odd, $2 \leq k \leq 4n$ and $c_{k,l} = c_{l,k}$ for all $1 \leq k, l \leq 4n$. If $1 \leq k, l \leq 4n$ and $|k-l| \geq 2$ then $c_{k,l} = (-1)^j r$, where j is the total number of s appearing in x_k and x_l . Now it is not difficult to check that

$$\sum_{l=1}^{4n} c_{k,l} = \begin{cases} 1 + q - 2r & \text{if } k = 1 \text{ or } k = 4n, \\ 1 - p + q - r & \text{if } 1 < k < 4n, \end{cases}$$

which implies

$$\sum_{x,y \in W} \phi(y^{-1}x)g(x)g(y) = 4n - (4n-2)p + 4nq - (4n+2)r$$

and leads to necessary condition $r \leq 1 - p + q$. In a similar manner we get $r \leq 1 + p - q$.

Finally, define a function h similarly like g , but now we use the character $\chi_{-,-}$:

$$h(x) := \begin{cases} \chi_{-,-}(x) = (-1)^k & \text{if } x = x_k \text{ for } 1 \leq k \leq 4n, \\ 0 & \text{otherwise.} \end{cases}$$

Putting $d_{k,l} := \phi(x_l^{-1}x_k)h(x_k)h(x_l)$ we have $d_{k,k} = 1$, $d_{k,k-1} = -p$ if $2 \leq k \leq 4n$ is even and $d_{k,k-1} = -q$ if k is odd. Moreover, if $|k-l| \geq 2$, $1 \leq k, l \leq 4n$ then $d_{k,l} = (-1)^{k+l} r$. Now one can check that

$$\sum_{l=1}^{4n} d_{k,l} = \begin{cases} 1 - q & \text{if } k = 1 \text{ or } k = 4n, \\ 1 - p - q + r & \text{if } 1 < k < 4n, \end{cases}$$

which yields $1 - p - q + r \geq 0$ and completes the proof that the conditions (5.2) are necessary. \square

6. Weak Sidon sets and operator Khinchin inequality

The aim of this section is to show that the set of Coxeter generators S in an arbitrary Coxeter group W is a *weak Sidon set*, ie. an interpolation set for the Fourier–Stieltjes algebra $B(W)$.

Given a group Γ , the Fourier–Stieltjes algebra consists of linear combinations of positive definite functions on Γ , ie. every element of

$B(\Gamma)$ is of the form $f = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ for some positive definite functions φ_i ($1 \leq i \leq 4$) on Γ . The norm on $B(\Gamma)$ is defined as

$$\|f\|_{B(\Gamma)} = \inf \left\{ \sum \varphi_i(e) \mid \text{where } f \text{ decomposes as above} \right\}$$

Theorem 6.1. *The set of Coxeter generators S in an arbitrary Coxeter group W is a weak Sidon set, ie. for every bounded function $f: S \rightarrow [-1, 1]$ there exists positive definite functions φ_{\pm} , such that $f(s) = \varphi_+(s) - \varphi_-(s)$ for any $s \in S$. One can take $\varphi_{\pm} = R_{\mathbf{q}^{\pm}}$ for a suitable choice of \mathbf{q}^{\pm} . Moreover*

$$\|\varphi_+ - \varphi_-\|_{B(W)} \leq 2$$

Proof. Put $S_{\pm}(f) = \{s \in S \mid \pm f(s) > 0\}$. Set

$$q_s^{\pm} = \begin{cases} \pm f(s) & \text{for } s \in S_{\pm}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(s) = R_{\mathbf{q}^+}(s) - R_{\mathbf{q}^-}(s)$ as claimed. The rest of the statement hold as the Riesz-Coxeter function at the identity element equals to one. \square

Given a matrix $A \in M_n(\mathbf{C})$ and $p \geq 1$ the Schatten p -class norm $\|A\|_{\mathcal{S}_p}$ is defined as $\|A\|_{\mathcal{S}_p}^p = (\text{tr } |A|)^{1/p}$, where $|A| = (A^*A)^{1/2}$.

Let λ denote the left regular representation of a group Γ . Given a finite sum $f = \sum c_g \lambda(g) \in \mathbf{C}[\Gamma]$ we define noncommutative L^p -norm

$$\|f\|_{L^p(\Gamma)}^p = \left(\tau \left((f^* * f)^{1/2} \right) \right)^{1/p}$$

where $\tau(f) = c_e$ is the von Neumann trace and $L^p(\Gamma)$ is a completion of $\mathbf{C}[\Gamma]$ with respect to the above norm.

We recall, that a scalar-valued map φ on a group Γ is called a *completely bounded Fourier multiplier* on $L^p(\Gamma)$ if the associated operator

$$M_{\varphi}(\lambda(g)) = \varphi(g)\lambda(g), \quad g \in \Gamma$$

extends to a completely bounded operator on $L^p(\Gamma)$.

We let $M_{\text{cb}}(L^p(\Gamma))$ to be an algebra of completely bounded Fourier multipliers equipped with the norm

$$\|\varphi\|_{M_{\text{cb}}(L^p(\Gamma))} = \|M_{\varphi} \otimes \text{id}_{\mathcal{S}^p}\|.$$

Following Pisier [PISo3], for $a_s \in M_n(\mathbf{C})$, where $s \in S$, we define

$$\|(a_s)_{s \in S}\|_{\mathbf{R} \cap \mathbf{C}} = \max \left\{ \left\| \sum_{s \in S} (a_s a_s^*)^{1/2} \right\|_{\mathcal{S}^p}, \left\| \sum_{s \in S} (a_s^* a_s)^{1/2} \right\|_{\mathcal{S}^p} \right\}.$$

For a set $E \in \Gamma$ we define the *completely bounded* constant $\Lambda_p^{\text{cb}}(E)$ as infimum of C such that

$$\left\| \sum_{s \in S} a_s \otimes \lambda(s) \right\|_{L^p(W)} \leq C \|(a_s)_{s \in S}\|_{\mathbb{R} \cap \mathbb{C}}$$

for all matrices $a_s \in M_n(\mathbb{C})$ and $n \in \mathbb{N}$.

Theorem 6.2. *If $a_s \in M_n(\mathbb{C})$, then for all $p \geq 2$ and any Coxeter system (W, S) we have*

$$\|(a_s)_{s \in S}\|_{\mathbb{R} \cap \mathbb{C}} \leq \left\| \sum_{s \in S} a_s \otimes \lambda(s) \right\|_{L^p(W)} \leq 2A' \sqrt{p} \|(a_s)_{s \in S}\|_{\mathbb{R} \cap \mathbb{C}}.$$

Proof. It was shown by Harcharras [LP86, Prop. 1.8] that $\Lambda_p^{\text{cb}}(E)$ is finite if and only if E is an interpolation set for $M_{\text{cb}}(L^p(\Gamma))$, i.e. every bounded function on E can be extended to a multiplier, and

$$\Lambda_p^{\text{cb}}(E) \leq \Lambda_p^{\text{cb}}(\mathbb{R}) \mu_p^{\text{cb}}(E),$$

where \mathbb{R} is the generating set in the Rademacher group Rad_∞ and $\mu_p^{\text{cb}}(E)$ is the interpolation constant.

As shown by Buchholz [BUC05, Thm. 5] for $p = 2n$, and S the standard generating set in Rad_∞ , $\Lambda_{2n}^{\text{cb}}(\mathbb{R}) = ((2n-1)!!)^{1/2n} \leq A\sqrt{p}$ for some absolute A . This was extended by Pisier [PIS03, Thm. 9.8.2] for any $p \geq 2$, i.e

$$\Lambda_p^{\text{cb}}(\mathbb{R}) \leq A' \sqrt{p},$$

for an absolute constant A' .

We have shown in Theorem 6.1 that in an arbitrary Coxeter group W its Coxeter generating set S is a weak Sidon set, i.e. it is interpolation set for the Fourier–Stieltjes algebra $B(W)$. Since for $p \geq 1$, $B(\Gamma)$ is a subalgebra of $M_{\text{cb}}(L^p(\Gamma))$ and

$$\|\varphi\|_{M_{\text{cb}}(L^p(\Gamma))} \leq \|\varphi\|_{B(\Gamma)},$$

we see that $\mu_p^{\text{cb}}(S) \leq 2$. Thus $\Lambda_p^{\text{cb}}(S) \leq 2A' \sqrt{p}$. This finishes the proof of the right inequality.

The left inequality holds for any group Γ and any $S \subset \Gamma$ (see [LP86]). \square

Remark 6.3. Fendler [FEN02A] has shown that if for all $s, t \in S$, $s \neq t$, we have $m_{s,t} \geq 3$, then

$$\Lambda_p^{\text{cb}}(S) \leq 2\sqrt{2}.$$

See also [BOŽ75] and [BUC99] for related results in the case of free Coxeter groups. Also Haagerup and Pisier have shown that $\Lambda_\infty^{\text{cb}}(S) = 2$, where $\Lambda_\infty^{\text{cb}}(E) = \sup_{p \geq 2} \Lambda_p^{\text{cb}}(E)$ [HP93]. See the paper of Haagerup

[HAA81] where the best constant was calculated for the set of Coxeter generators of the Rademacher group in case when a_s are scalars.

7. Chromatic length function for Coxeter groups and pairpartitions

Let $[2n] = \{1, \dots, 2n\}$. Let $2^{[2n]}$ denote the set of subsets of $[2n]$. By a partition of $[2n]$ we mean $\pi \subset 2^{[2n]}$ such that $\bigcup \pi = [2n]$ and if $\pi', \pi'' \in \pi$ then $\pi' = \pi''$ or $\pi' \cap \pi'' = \emptyset$. We say, that partition ρ is a *coarsening* of a partition π if for any $\pi' \in \pi$ there exists $\rho' \in \rho$ such that $\pi' \subset \rho'$.

A partition is called *crossing* if there exist $1 \leq a < b < c < d \leq 2n$ and $\pi_1, \pi_2 \in \pi$ with $a, c \in \pi_1 \neq \pi_2 \ni b, d$; otherwise it is called *noncrossing*. For any partition π there exists the smallest noncrossing coarsening $\Phi(\pi)$ of π (ie. if ρ is a noncrossing coarsening of π then it is a coarsening of $\Phi(\pi)$). We define $\|\pi\| = n - \#\Phi(\pi)$. The notion for the map Φ was introduced in [BY06].

We say that π is a *pairpartition* if every member of π has cardinality two. The set of pairpartitions of $[2n]$ is denoted by $\mathcal{P}_2(2n)$. Given $\pi \in \mathcal{P}_2(2n)$ we write $|\pi|$ to denote the number of ordered quadruples $1 \leq a < b < c < d \leq 2n$ such that both $\{a, c\}$ and $\{b, d\}$ belong to π . Note, that $|\pi| = 0$ precisely when π is noncrossing. The set of noncrossing pairpartitions is denoted $\mathcal{NC}_2(2n)$.

Given a noncrossing pairpartition ω we call $\{b, c\} \in \omega$ an *inner block* if there exists $\{a, d\} \in \omega$ with $a < b < c < d$. The number of inner blocks of ω we denote as $\text{inn}(\omega)$.

In [BS96, Cor. 7] Bożejko and Speicher observed the following identity.

$$(7.1) \quad \sum_{\pi \in \mathcal{P}_2(2n)} (-1)^{|\pi|} q^{|\pi|} = \sum_{\omega \in \mathcal{NC}_2(2n)} (1 - q)^{\text{inn}(\omega)}.$$

Let $f_n(q) = \sum_{\omega \in \mathcal{NC}_2(2n)} (1 - q)^{\text{inn}(\omega)}$. It is elementary to derive

$$f_n(q) = C_n {}_2F_1 \left(\begin{matrix} n, 1 - n \\ n + 2 \end{matrix} \middle| q \right),$$

where $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \#\mathcal{NC}_2(2n)$ denote the n -th Catalan number and ${}_2F_1$ is the classical hypergeometric function. If we write $f(q) = \sum_{j=1}^{n-1} t_j^n q^j$, then the triangle $(t_j^n)_{0 \leq j < n}$ appears in [SLO01] as “sequence” A062991). Since we are not going to use this formula, we leave it as an exercise to the reader. For the expansion of $f_n(1 - t)$ and the Delanoy triangle appearing there the reader may consult [BW01, Prop 6.1].

In what follows, we prove a result about an arbitrary finitely generated Coxeter group, which for permutation groups implies the above one. Given a permutation $\sigma \in S_n$, we construct a pairpartition $\bar{\sigma} = \{\{i, 2n+1-\sigma(i)\} | 1 \leq i \leq n\}$. Note, that $|\bar{\sigma}|$ is equal to the length of σ with respect to the Coxeter generators $(1, 2), \dots, (n-1, n)$ of S_n . Therefore, denoting by $|w|$ the Coxeter length of an element w of some Coxeter group W will not lead into any confusion.

It is also clear that, with respect to the identification of permutations with a subset of pairpartitions, the two definitions of $\|\cdot\|$ agree (see Equations (1.3) and (1.2)).

By $W(t)$ we denote a growth series of a finitely generated Coxeter group W (length function). That is, a power series $W(t) = \sum_{w \in W} t^{|w|}$. (Note, that the coefficient at t equals to $\#S$. This explains why here and in the rest of this section we consider only finitely generated Coxeter groups. We will not repeat this assumption for short.) Moreover, for $X \subset W$ we write $X(t) := \sum_{w \in X} t^{|w|}$.

Let us define a multivariable formal power series (*chromatic length function*). For any $X \subset W$ define

$$X(t, \mathbf{q}) := \sum_{w \in X} t^{|w|} \prod_{s \in S_w} q_s.$$

In particular $X(t) = X(t, \mathbf{1})$, where $\mathbf{1} = (1)_{s \in S}$.

Proposition 7.1. *The polynomial (or formal power series, if W is infinite) $W(t, \mathbf{q})$ satisfies*

$$W(t, \mathbf{q}) = \sum_{T \subset S} W_T(t) \prod_{r \in T} r_s \prod_{s \in S \setminus T} (1 - q_s).$$

Proof. Let W_R° denote the set of all elements of W_R not contained in any proper parabolic subgroup of W_R , ie. $W_R^\circ := W_R - \bigcup_{T \subsetneq R} W_T$. Then, by inclusion-exclusion principle, $W_R^\circ(t) = \sum_{T \subsetneq R} (-1)^{\#(R-T)} W_T(t)$. Therefore,

$$\begin{aligned} W(t, \mathbf{q}) &= \sum_{w \in W} t^{|w|} \prod_{s \in S_w} q_s = \sum_{R \subset S} W_R^\circ(t) \prod_{r \in R} q_r \\ &= \sum_{R \subset S} \sum_{T \subsetneq R} W_T(t) (-1)^{\#(R-T)} \prod_{r \in R} q_r \\ &= \sum_{T \subset S} W_T(t) \prod_{r \in T} q_r \sum_{T \subsetneq R \subset S} \prod_{s \in R \setminus T} (-q_s) \\ &= \sum_{T \subset S} W_T(t) \prod_{r \in T} q_r \prod_{s \in S \setminus T} (1 - q_s). \end{aligned}$$

□

Corollary 7.2. *If W is a finite Coxeter group then*

$$(7.2) \quad W(-1, \mathbf{q}) = \prod_{s \in S} (1 - q_s).$$

Proof. Choose $s \in T$ and put $W_T^{\{s\}} = \{w \in W : |w| < |ws|\}$. Clearly, $W_T = W_T^{\{s\}} W_T$ therefore $W_T(t) = W_T^{\{s\}}(t) W_{\{s\}}(t)$. Since W is a finite group, $W_T^{\{s\}}$ is a polynomial. Thus $W_T(-1) = 0$ if T is nonempty (and $W_\emptyset(-1) = 1$). \square

In order to prove Equation (7.1) we define the *Wick map* $\mathcal{P}_2(2n) \ni \pi \mapsto : \pi :$ (related to the normal order in quantum field theory). Given a pairpartition π we define $: \pi :$ by repetitive resolving crossings. That is, we replace repetitively every crossing pair $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$ by $\{a, d\}$ and $\{b, c\}$. In order to see that the result is independent of the order of resolution we describe $: \pi :$ in an equivalent way.

Let $\Phi(\pi)$ be the smallest noncrossing coarsening of π . For each block β of $\Phi(\pi)$ define $\beta^+ = \{y | (\exists x) x \in \beta, y > x, \{x, y\} \in \pi\}$ and $\beta_1^- = \{x | (\exists y) y \in \beta, y > x, \{x, y\} \in \pi\}$. Order $\beta^+ = \{y_1, \dots, y_k\}$ in increasing way and $\beta^- = \{x_1, \dots, x_k\}$ in decreasing way. Then all pairs $\{x_i, y_i\}$ will be parts of $: \pi :$.

Equation (7.1) will follow from a more refined statement.

Proposition 7.3. *For every $\omega \in \mathcal{NC}_2(2n)$*

$$(7.3) \quad \sum_{\substack{\pi \in \mathcal{P}_2(2n) \\ : \pi : = \omega}} (-1)^{|\pi|} q^{||\pi||} = (1 - q)^{\text{inn}(\omega)}.$$

Proof. Let us first consider the case of $\omega = \bar{1} = \{(i, 2n+1-i) | 1 \leq i \leq n\}$. Clearly, $\{\pi | : \pi : = \bar{1}\} = \{\bar{\sigma} | \sigma \in \mathfrak{S}_n\}$. And Equation (7.3) is equivalent to Equation (7.2) (as all q_s are set to q) for $W = \mathfrak{S}_n$.

In a general case observe, that $\Phi(\pi)$ is a coarsening of $\omega = : \pi :$. Yet, not every coarsening may appear. The obvious condition is that for each block β of $: \pi :$ the pair $\{\min \beta, \max \beta\}$ belong to ω . For the purpose of this proof we will call such a coarsening admissible. Its clear, that admissible coarsenings ρ are in one to one correspondence with subsets of ω containing all outer (not inner) parts of ω of the form $\{\{\min \rho', \max \rho'\} | \rho' \in \rho\}$.

Let us refine Equation (7.3) further. For every $\omega \in \mathcal{NC}_2(2n)$ and any admissible coarsening η of ω we have

$$(7.4) \quad \sum_{\substack{\pi \in \mathcal{P}_2(2n) \\ : \pi : = \omega, \Phi(\pi) = \eta}} (-1)^{|\pi|} = (-1)^{\#\eta}.$$

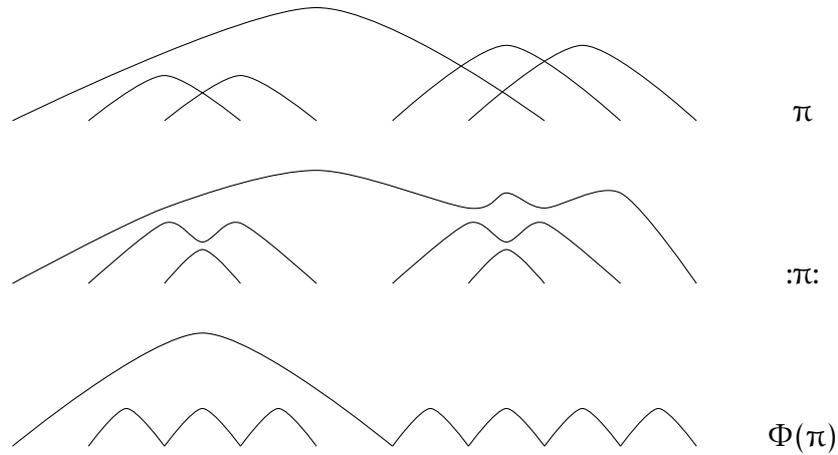


FIGURE. Examples of π , $:\pi:$, and $\Phi(\pi)$.

Equation (7.3) follows from (7.4) by multiplying by $q^{n-\#\eta}$ and summing over all admissible coarsenings η of ω .

Equation (7.4) is again equivalent to Equation (7.2) as both sides factor as a product over blocks of ρ . \square

Question 7.4. We have proven Equation 7.1 with the help of an embedding $\mathfrak{S}_n \ni \sigma \mapsto \bar{\sigma} \in \mathcal{NC}_2(2n)$ (or several such embeddings, one for each outer block of ω). Corollary 7.2 holds for any Coxeter group. Is there a corresponding formula concerning some generalization of pairpartitions?

In the proof of Proposition 7.1 we have not assumed that W was finite. Let us finish this section with a discussion of infinite Coxeter groups. Recall, that -1 does not lie in the radius of convergence on $W(t)$ if W is not finite. Nevertheless, $W(t)$ represents a rational function as follows from the following result.

Proposition 7.5. ([STE68],[SER71, Prop. 26]) *Let (W, S) be an infinite Coxeter system. Then*

$$(7.5) \quad \frac{1}{W(t)} = \sum_{T \in \mathcal{F}} \frac{(-1)^{\#T}}{W_T(1/t)}.$$

Where \mathcal{F} denote the family of subsets $T \subset S$, such that the group W_T generated by T is finite. In particular, $W(t)$ is a series of a rational function (i.e. a quotient of polynomials).

One may ask a question what is the class of (infinite) Coxeter groups such that $W_T(-1) = 0$ for any nonempty subset T of generators. A naïve argument that

$$W(t) = W_{\{s\}}(t)W^{[s]}(t) = (1+t)W^{[s]}(t)$$

shows, that the question if $W(-1) \neq 0$ is equivalent to whether $W^{\{s\}}(t)$ can have a pole at $t = -1$. On the other hand note, that if W is of type \tilde{A}_2 , ie. W is given by a presentation $\langle s_i : 1 \leq i \leq 3 | s_i^2, (s_i s_j)^3 : 1 \leq i < j \leq 3 \rangle$ then, by Equation (7.5), $W(t) = \frac{1+t+t^2}{(1-t)^2}$ and $W(-1) = 1/4$.

More generally, it is known ([BOU68]) that in each coset of W_T there exists the unique shortest element. Let W^T denote the set of those shortest representatives. Moreover if $w = w^T w_T$ with $w^T \in W^T$ and $w_T \in W_T$ then $|w| = |w^T| + |w_T|$. Therefore $W^T(t)W_T(t) = W(t)$. In particular, $W^T(t)$ represents a rational function, and it is legitimate to ask about the value of $W^T(-1)$.

In the case of finite Coxeter group W , Eng [ENG01] observed that

$$W^T(-1) = \#\{w \in W^T | w w_0 w \in W_T\},$$

where w_0 is the longest element in W . (Eng's proof was case-by-case. Later, a general classification-free proof of Eng's theorem was given in [RSW04]).

Subsequently, Reiner [REI02] has shown that if W is crystallographic (ie. the Weyl group in a compact Lie group G), then both sides of the above equality compute the signature of the corresponding flag variety G/Q_T , where Q_T is a parabolic subgroup associated to T .

What is the meaning of $W(-1)$ or $W^T(-1)$ for infinite W ?

We do not know if it possible for $W(-1)$ to be negative. If one takes $W = \langle s_i : 1 \leq i \leq 4 | s_i^2, (s_i s_j)^3 : 1 \leq i < j \leq 4 \rangle$. Then, by Equation (7.5), $W(t) = \frac{(1+t)(1+t+t^2)}{3t^3-2t^2-2t+1}$ and $W^{s_i}(-1) = -1/2$.

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Acknowledgements

Marek Bożejko was supported by NCN MAESTRO grant DEC-2011/02/A/ST1/00119. Marek Bożejko and Wojciech Młotkowski were supported by NCN grant 2016/21/B/ST1/00628.

The authors would like to thank Ryszard Szwarc and Janusz Wysoczański for many discussions and help with the preparation of this paper.

The first author would like to thank professor F. Götze for his kind invitation to SFB701 and his hospitality in Bielefeld in 2016.

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