

ON THE p -ADIC STARK CONJECTURE AT $s = 1$ AND APPLICATIONS

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ABSTRACT. Let E/F be a finite Galois extension of totally real number fields and let p be a prime. The ‘ p -adic Stark conjecture at $s = 1$ ’ relates the leading terms at $s = 1$ of p -adic Artin L -functions to those of the complex Artin L -functions attached to E/F . We prove this conjecture unconditionally when E/\mathbb{Q} is abelian. Moreover, we also show that for certain non-abelian extensions E/F the p -adic Stark conjecture at $s = 1$ is implied by Leopoldt’s conjecture for E at p . As an application, we provide strong new evidence for special cases of the ‘equivariant Tamagawa number conjecture’ for Tate motives and the closely related ‘leading term conjectures’ at $s = 0$ and $s = 1$.

1. INTRODUCTION

Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Let p be a prime. In the case that G is abelian, the fundamental work of Deligne and Ribet [DR80] and of Pierrette Cassou-Noguès [CN79] shows that one can attach to each irreducible character of G a p -adic Artin L -function that interpolates values of the corresponding complex Artin L -function at negative integers. This construction can be generalised to the case that G is non-abelian by using Brauer induction.

The ‘ p -adic Stark conjecture at $s = 1$ ’ originated with Serre [Ser78], was discussed by Tate [Tat84], and was clarified by Burns and Venjakob [BV06, BV11]. Roughly speaking, this conjecture relates the leading terms at $s = 1$ of the complex Artin L -function and the p -adic Artin L -function attached to a character of G . For the trivial character it is equivalent to the ‘ p -adic class number formula’ of Colmez [Col88] together with Leopoldt’s conjecture for F at p . More generally, the leading terms of the two L -functions attached to a character of G are related by a certain ‘comparison period’, which is non-zero if Leopoldt’s conjecture holds for E at p .

In the present article, we prove the p -adic Stark conjecture at $s = 1$ unconditionally when E/\mathbb{Q} is abelian by building on work of Ritter and Weiss [RW97] and using standard results on Dirichlet L -functions and their p -adic analogues. Moreover, we show that it is implied by Leopoldt’s conjecture for E at p when every character of G is a virtual permutation character (in particular, the symmetric groups S_n have this property). By combining the proofs of these two results, we also show that if $F = \mathbb{Q}$ and $G = \text{Aff}(q) := \mathbb{F}_q \rtimes \mathbb{F}_q^\times$ where \mathbb{F}_q is a finite field with q elements the semidirect product is defined by the natural action, then Leopoldt’s conjecture for E at p again implies the p -adic Stark conjecture at $s = 1$ for E/\mathbb{Q} (note that $\text{Aff}(4) \cong A_4$, the alternating group on 4 letters).

These results are motivated by their applications to the equivariant Tamagawa number conjecture (ETNC) for Tate motives and the closely related leading term conjectures, both at $s = 0$ and at $s = 1$. Building on work of Bloch and Kato [BK90], Fontaine and Perrin-Riou [FPR94], and Kato [Kat93], Burns and Flach [BF01] formulated the ETNC for any motive over \mathbb{Q} with the action of a semisimple \mathbb{Q} -algebra, describing the leading

term at $s = 0$ of an equivariant motivic L -function in terms of certain cohomological Euler characteristics. This is a powerful and unifying formulation which, in particular, recovers the Birch and Swinnerton-Dyer conjecture. We refer the reader to the survey article [Fla04] for a more detailed overview.

Let L/K be a finite Galois extension of number fields (not necessarily totally real) and let $G = \text{Gal}(L/K)$. In the case of Tate motives, the ETNC relates certain arithmetic complexes to the leading terms at integers of the equivariant complex Artin L -function attached to L/K . Burns [Bur01] formulated the leading term conjecture (LTC) at $s = 0$ and he showed that this conjecture for L/K is equivalent to the ETNC for the pair $(h^0(\text{Spec}(L)), \mathbb{Z}[G])$. The advantage of this new formulation is that it is more explicit. Moreover, the LTC at $s = 0$ recovers Stark's conjecture at $s = 0$ (as interpreted by Tate in [Tat84]), the 'strong Stark conjecture' of Chinburg [Chi83] and Chinburg's ' $\Omega(3)$ -conjecture' [Chi83, Chi85]. It is also equivalent to the 'lifted root number conjecture' of Gruenberg, Ritter and Weiss [GRW99] and implies numerous other conjectures involving leading terms of Artin L -functions at $s = 0$ (see [Bur11, Lecture 3] for a partial list of such conjectures). Breuning and Burns [BB07] formulated the LTC at $s = 1$, which simultaneously refines both Stark's conjecture at $s = 1$ (as formulated by Tate in [Tat84]) and Chinburg's ' $\Omega(1)$ -conjecture' [Chi85]. They also showed [BB10] that, under the assumption of Leopoldt's conjecture (at all primes), the LTC at $s = 1$ for L/K is equivalent to the ETNC for the pair $(h^0(\text{Spec}(L))(1), \mathbb{Z}[G])$. If the 'global epsilon constant conjecture' of Bley and Burns [BB03] holds then the LTCs at $s = 0$ and $s = 1$ are equivalent.

Again let E/F be a finite Galois extension of totally real number fields. Let $G = \text{Gal}(E/F)$ and let p be a prime. Burns [Bur15] has shown that under the assumption that certain μ -invariants attached to E and p vanish for all odd prime divisors p of $|G|$, the p -adic Stark conjecture for all characters of G and for all odd primes p implies the ETNC for the pair $(h^0(\text{Spec}(E))(1), \mathbb{Z}[\frac{1}{2}][G])$. The proof relies crucially on the descent machinery of Burns and Venjakob [BV11], and on the equivariant Iwasawa main conjecture, which has been proven by Ritter and Weiss [RW11] and by Kakde [Kak13] independently, under the assumption that the relevant μ -invariant vanishes.

We prove a refinement of the above result that allows us to work prime-by-prime. In other words, we show that for a fixed odd prime p , the vanishing of the relevant μ -invariant attached to E and p together with the p -adic Stark conjecture for all characters of G imply the ETNC for the pair $(h^0(\text{Spec}(E))(1), \mathbb{Z}_{(p)}[G])$, where $\mathbb{Z}_{(p)}$ is the localisation of \mathbb{Z} at p . A key ingredient is a direct proof that for any fixed prime p , the p -adic Stark conjecture at $s = 1$ for all characters of G implies Stark's conjecture at $s = 1$ for all such characters. By combining this prime-by-prime descent result with our new results on the p -adic Stark conjecture at $s = 1$, we obtain new evidence for the relevant case of the ETNC. Thus by tweaking the aforementioned result of Breuning and Burns [BB10] to work prime-by-prime, we also obtain results for the LTC at $s = 1$. Then by using known cases (and by proving a new case) of the global epsilon constant conjecture, we obtain results on the LTC at $s = 0$ and thus the ETNC for the pair $(h^0(\text{Spec}(E)), \mathbb{Z}[G])$.

We now give three concrete examples of the new results obtained. For the first example, let p be an odd prime and let m be a positive integer. If E/\mathbb{Q} is any totally real Galois extension with $\text{Gal}(E/\mathbb{Q}) \cong \text{Aff}(p^m)$ (see definition above) then under the assumption that Leopoldt's conjecture for E at p holds, the LTC for E/\mathbb{Q} at $s = 1$ holds. We note that Leopoldt's conjecture for a given number field and prime can be verified computationally (see Remark 10.9). For the second example, let C_n denote the cyclic group of order n and let $G = (C_3)^m \rtimes C_2$ where m is a positive integer and C_2 acts on $(C_3)^m$ by inversion

(in the case $m = 1$ we have $G \cong S_3$). If E/\mathbb{Q} is any totally real Galois extension with $\text{Gal}(E/\mathbb{Q}) \cong G$ then under the assumption that Leopoldt's conjecture for E at 3 holds, the LTCs for E/\mathbb{Q} at both $s = 0$ and $s = 1$ hold. For the final example, let G be any finite group. Then there exist infinitely many Galois extensions of totally real number fields E/F with $\text{Gal}(E/F) \cong G$ such that, if for all odd prime divisors p of $|G|$ both Leopoldt's conjecture holds for E at p and a certain μ -invariant attached to E and p vanishes, then the LTCs for E/F at $s = 0$ and $s = 1$ both hold outside their 2-primary parts.

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Notation and conventions. All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. We fix the following notation:

S_n	the symmetric group of degree n
A_n	the alternating group of degree n
C_n	the cyclic group of order n
D_{2n}	the dihedral group of order $2n$
V_4	the subgroup of A_4 generated by double transpositions
\mathbb{F}_q	the finite field with q elements, where q is a prime power
$\text{Aff}(q)$	the affine group isomorphic to $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$ defined in §10.2
ζ_n	a primitive n -th root of unity
$\text{Tr}_{L/K}$	the trace map for the field extension L/K
$\text{Quot}(R)$	the field of fractions of an integral domain R
$M_{n \times n}(R)$	the ring of $n \times n$ matrices over a ring R
$\Sigma_\infty(K)$	the set of infinite places of a number field K
$\Sigma_p(K)$	the set of places of a number field K above a rational prime p

A finite Galois extension of totally real number fields will usually be denoted by E/F . By contrast, L/K will usually denote a finite Galois extension of number fields, neither of which is necessarily totally real.

2. ALGEBRAIC PRELIMINARIES

2.1. Representations and characters of finite groups. Let G be a finite group and let K be a field of characteristic 0. We write $R_K^+(G)$ for the set of characters attached to finite-dimensional K -valued representations of G , and $R_K(G)$ for the ring of virtual characters generated by $R_K^+(G)$. Moreover, we let $\text{Irr}_K(G)$ denote the subset of irreducible characters in $R_K^+(G)$ and let $\text{Char}_K(G)$ denote the ring of K -valued virtual characters of G . Thus we have containments

$$\text{Irr}_K(G) \subset R_K^+(G) \subset R_K(G) \subset \text{Char}_K(G).$$

We let 1_G denote the trivial character of G and for $\chi, \psi \in R_K(G)$ we write $\langle \psi, \chi \rangle_G$ for the usual inner product of virtual characters. For a subgroup H of G and $\psi \in R_K^+(H)$

we write $\text{ind}_H^G \psi \in R_K^+(G)$ for the induced character; for a normal subgroup N of G and $\chi \in R_K^+(G/N)$ we write $\text{infl}_{G/N}^G \chi \in R_K^+(G)$ for the inflated character. For $\sigma \in \text{Aut}(K)$ and $\chi \in \text{Char}_K(G)$ we set $\chi^\sigma := \sigma \circ \chi$ and note that this defines a group action from the left even though we write exponents on the right of χ .

We write $\text{Perm}(G)$ for the ring of characters of virtual permutation representations of G , that is, \mathbb{Z} -linear combinations of characters of the form $\text{ind}_H^G 1_H$ where H ranges over subgroups of G . It is important to note that each of the inclusions

$$\text{Perm}(G) \subset R_{\mathbb{Q}}(G) \subset \text{Char}_{\mathbb{Q}}(G)$$

may be strict.

2.2. Endomorphisms of modules over group algebras. Let G be a finite group and let K be a field of characteristic 0. For any $\chi \in R_K^+(G)$ we fix a $K[G]$ -module V_χ with character χ . For any $K[G]$ -module M and any $\alpha \in \text{End}_{K[G]}(M)$ we write M^α for the K -vector space

$$\text{Hom}_{K[G]}(V_\chi, M) \cong \text{Hom}_K(V_\chi, M)^\alpha$$

and α^χ for the induced map $(f \mapsto \alpha \circ f) \in \text{End}_K(M^\alpha)$. We note that $\det_K(\alpha^\chi)$ is independent of the choice of V_χ . The following is similar to [Tat84, Chapitre I, 6.4].

Lemma 2.1. *Let M be an $K[G]$ -module and let $\alpha \in \text{End}_{K[G]}(M)$. Let H be a subgroup of G and let $M|_H$ denote M considered as an $K[H]$ -module. Let N be a normal subgroup of G and let M^N denote the $K[G/N]$ -module of N -invariants of M .*

- (i) *If $\chi_1, \chi_2 \in R_K^+(G)$ then $\det_K(\alpha^{\chi_1 + \chi_2}) = \det_K(\alpha^{\chi_1}) \det_K(\alpha^{\chi_2})$.*
- (ii) *If $\chi \in R_K^+(H)$ then $M^{\text{ind}_H^G \chi} \cong (M|_H)^\alpha$ and $\det_K(\alpha^{\text{ind}_H^G \chi}) = \det_K(\alpha^\chi)$.*
- (iii) *If $\chi \in R_K^+(G/N)$ then $M^{\text{infl}_{G/N}^G \chi} \cong (M^N)^\alpha$ and $\det_K(\alpha^{\text{infl}_{G/N}^G \chi}) = \det_K((\alpha|_{M^N})^\alpha)$.*

Proof. In the appropriate bases, the matrix for $\alpha^{\chi_1 + \chi_2}$ is a block matrix whose blocks are the matrices for α^{χ_1} and α^{χ_2} , and this gives claim (i). Claim (ii) follows from Frobenius reciprocity, i.e., the natural isomorphism $\text{Hom}_{K[G]}(\text{ind}_H^G V_\chi, M) \cong \text{Hom}_{K[H]}(V_\chi, M|_H)$. Similarly, the natural isomorphism $\text{Hom}_{K[G]}(\text{infl}_{G/N}^G V_\chi, M) \cong \text{Hom}_{K[G/N]}(V_\chi, M^N)$ gives claim (iii). \square

3. STARK'S CONJECTURE AT $s = 1$

3.1. Artin L -functions. Let L/K be a finite Galois extension of number fields and let $G = \text{Gal}(L/K)$. Let Σ be a finite set of places of K containing the set of infinite places $\Sigma_\infty(K)$. For each character $\chi \in R_{\mathbb{C}}(G)$ we write $L_\Sigma(s, \chi)$ for the Σ -truncated (complex) Artin L -function attached to χ (see [Neu99, Chapter VII, §10]). We recall that $L_\Sigma(s, \chi_1 + \chi_2) = L_\Sigma(s, \chi_1) L_\Sigma(s, \chi_2)$ and that $L_\Sigma(s, \chi)$ is invariant under induction and inflation of characters. Moreover, $L_\Sigma(s, 1_G) = \zeta_{K, \Sigma}(s)$, the Σ -truncated Dedekind zeta-function attached to K , which has a simple pole at $s = 1$. In fact, writing $L_\Sigma^*(1, \chi)$ for the leading term at $s = 1$ of $L_\Sigma(s, \chi)$, it is well-known that

$$(3.1) \quad L_\Sigma^*(1, \chi) = \lim_{s \rightarrow 1} (s-1)^{\langle 1_G, \chi \rangle_G} \cdot L_\Sigma(s, \chi)$$

(see the discussion of [Hei67, p. 225], for instance).

3.2. Stark's conjecture at $s = 1$. For any place w of L we write L_w for the completion of L at w . Let $L_\infty := \prod_{w \in \Sigma_\infty(L)} L_w$. Define $\text{Tr}_\infty : L_\infty \rightarrow \mathbb{R}$ by $(x_w)_{w \in \Sigma_\infty(L)} \mapsto \sum_{w \in \Sigma_\infty(L)} \text{Tr}_{L_w/\mathbb{R}}(x_w)$ and denote the kernels of the trace maps by $L_\infty^0 := \ker(\text{Tr}_\infty)$ and $L^0 := \ker(\text{Tr}_{L/\mathbb{Q}})$. Then we have a commutative diagram of $\mathbb{R}[G]$ -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R} \otimes_{\mathbb{Q}} L^0 & \xrightarrow{\subset} & \mathbb{R} \otimes_{\mathbb{Q}} L & \xrightarrow{\mathbb{R} \otimes_{\mathbb{Q}} \text{Tr}_{L/\mathbb{Q}}} & \mathbb{R} \longrightarrow 0 \\ & & \downarrow \mu_L & & \downarrow \mu'_L & & \parallel \\ 0 & \longrightarrow & L_\infty^0 & \xrightarrow{\subset} & L_\infty & \xrightarrow{\text{Tr}_\infty} & \mathbb{R} \longrightarrow 0 \end{array}$$

where μ_L is the restriction of the $\mathbb{R}[G]$ -module isomorphism $\mu'_L : \mathbb{R} \otimes_{\mathbb{Q}} L \xrightarrow{\sim} L_\infty$ given by $x \otimes y \mapsto (x\sigma_w(y))_{w \in \Sigma_\infty(L)}$ and $\sigma_w : L \rightarrow L_w$ is an embedding for each infinite place w . Let $\text{exp}_\infty : L_\infty \rightarrow L_\infty^\times$ denote the product of exponential maps and let $\Delta_\infty : L^\times \rightarrow L_\infty^\times$ denote the diagonal embedding. Set $\log_\infty(\mathcal{O}_L^\times) := \{x \in L_\infty : \text{exp}_\infty(x) \in \Delta_\infty(\mathcal{O}_L^\times)\}$. Then one can use the proof of the Dirichlet unit theorem to show that $\log_\infty(\mathcal{O}_L^\times)$ is a full lattice in $L_\infty^0 \cong \mathbb{R} \otimes_{\mathbb{Q}} L^0$ (see [Deb11, Lemma 6.3]), and so there is an isomorphism of $\mathbb{C}[G]$ -modules

$$(3.2) \quad \mu_\infty : \mathbb{C} \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_L^\times) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} L^0.$$

(Note that μ'_L , μ_L and μ_∞ are canonical when L is totally real; otherwise they depend of the choice of embeddings $\sigma_w : L \rightarrow L_w$ for complex places w .) Hence there exists a (non-canonical) $\mathbb{Q}[G]$ -isomorphism $g : L^0 \xrightarrow{\sim} \mathbb{Q} \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_L^\times)$ (use [NSW08, Lemma 8.7.1], for instance). For any such g and any $\chi \in R_{\mathbb{C}}^+(G)$ we define

$$R_1(\chi, g) := \det_{\mathbb{C}}(\mu_\infty \circ (\mathbb{C} \otimes_{\mathbb{Q}} g))^{\chi} \in \mathbb{C}^\times.$$

Conjecture 3.1 (Stark's conjecture at $s = 1$). *Let L/K be a finite Galois extension of number fields and let $G = \text{Gal}(L/K)$. Let Σ be a finite set of places of K containing $\Sigma_\infty(K)$. Let $\chi \in R_{\mathbb{C}}^+(G)$. Then for every $\mathbb{Q}[G]$ -isomorphism $g : L^0 \xrightarrow{\sim} \mathbb{Q} \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_L^\times)$ and every $\sigma \in \text{Aut}(\mathbb{C})$ we have*

$$\frac{R_1(\chi^\sigma, g)}{L_\Sigma^*(1, \chi^\sigma)} = \sigma \left(\frac{R_1(\chi, g)}{L_\Sigma^*(1, \chi)} \right).$$

Remark 3.2. Conjecture 3.1 is in fact a reformulation of Stark's conjecture at $s = 1$ as stated in [Tat84, Chapitre I, Conjecture 8.2]; the two formulations are indeed equivalent as explained in the second paragraph of the proof of [BB07, Proposition 3.6]. Moreover, by [Tat84, Chapitre I, Théorème 8.4] Stark's conjecture at $s = 1$ is equivalent to Stark's conjecture at $s = 0$ ([Tat84, Chapitre I, Conjecture 5.4]).

Remark 3.3. Stark's conjecture at $s = 1$ is independent of certain choices as follows. (i) By [Tat84, Chapitre I, Proposition 8.3], if it is true for some choice of g then it is true for every choice of g . (ii) By considering Euler factors one can show that if it is true for some choice of Σ then it is true for every choice of Σ . (iii) A straightforward substitution shows that if it is true for χ then it is true for χ^τ for every choice of $\tau \in \text{Aut}(\mathbb{C})$.

Remark 3.4. The following facts are proven in [Tat84, Chapitre II]. Using the analytic class number formula, one can show that Stark's conjectures at $s = 0$ and $s = 1$ hold for the trivial character. Moreover, the truth of these conjectures is invariant under induction and respects addition of characters, and from this it is straightforward to deduce that they hold for all $\chi \in \text{Perm}(G)$. With more effort, one can show that in fact they hold for all $\chi \in \text{Char}_{\mathbb{Q}}(G)$ (recall that the inclusion $\text{Perm}(G) \subset \text{Char}_{\mathbb{Q}}(G)$ can be strict).

4. THE p -ADIC STARK CONJECTURE AT $s = 1$

4.1. Leopoldt's conjecture. For a comprehensive discussion of Leopoldt's conjecture, we refer the reader to [NSW08, Chapter X, §3]. Let K be a number field. If w is a finite place of K , let U_{K_w} denote the group of units of K_w and let $U_{K_w}^1$ denote the subgroup of principal units. Let p be a prime and let $\Sigma_p(K)$ denote the set of places of K above p . After taking p -adic completions of abelian groups, the diagonal embedding $\mathcal{O}_K^\times \longrightarrow \prod_{w \in \Sigma_p(K)} U_{K_w}$ gives rise to a canonical homomorphism

$$\lambda_p : \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K^\times \longrightarrow \prod_{w \in \Sigma_p(K)} U_{K_w}^1.$$

We say that $\text{Leo}(K, p)$ holds when λ_p is injective and we take our formulation of Leopoldt's conjecture for K at p to be that of [NSW08, (10.3.5)]. If F is a totally real number field we write $R_{F,p}$ for the p -adic regulator of F . We recall the following results that we shall use throughout this article, often without further reference.

Theorem 4.1. *Let p be prime and let K be a number field.*

- (i) *Leopoldt's conjecture for K at p holds if and only if $\text{Leo}(K, p)$ holds.*
- (ii) *The homomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lambda_p$ is injective if and only if $\text{Leo}(K, p)$ holds.*
- (iii) *If K/\mathbb{Q} is a finite abelian extension then $\text{Leo}(K, p)$ holds.*
- (iv) *If M is a subfield of K then $\text{Leo}(K, p)$ implies $\text{Leo}(M, p)$.*
- (v) *If F is a totally real number field, then $R_{F,p} \neq 0$ if and only if $\text{Leo}(F, p)$ holds.*

Proof. The map λ_p is always injective on the p -torsion part of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K^\times$ and thus the injectivity of λ_p is equivalent to the injectivity of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lambda_p$, establishing assertion (ii). Hence assertion (i) follows from [NSW08, Theorem 10.3.6 (iii)] after observing that (using the notation of op. cit.) $\hat{U}_{\mathfrak{p}} \cong U_{\mathfrak{p}}^1$ for $\mathfrak{p} \in \Sigma_p(K)$ and $\hat{U}_{\mathfrak{p}}$ is finite for $\mathfrak{p} \notin \Sigma_p(K)$. Assertion (iii) was proved by Brumer [Bru67] (also see [NSW08, Theorem 10.3.16]). For assertions (iv) and (v), see [NSW08, Theorem 10.3.11] and [NSW08, p. 627], respectively. \square

4.2. The p -adic analytic class number formula. We follow the exposition of [NSW08, Chapter XI, §6.2], to which we refer the reader for further details and references. Let F be a totally real number field and let $\zeta_F(s)$ denote the Dedekind zeta function attached to F . Then by the Siegel-Klingen theorem $\zeta_F(1-n) \in \mathbb{Q}$ for integers $n \geq 1$ and these values are non-zero when n is even. Moreover, for each prime p there exists a unique continuous function $\zeta_{F,p} : \mathbb{Z}_p \setminus \{1\} \rightarrow \mathbb{Q}_p$ satisfying

$$\zeta_{F,p}(1-n) = \zeta_F(1-n) \prod_{v \in \Sigma_p(F)} (1 - \text{Norm}_{F/\mathbb{Q}}(v)^{n-1})$$

for all $n > 1$ with $d \mid n$ where $d = [F(\zeta_{2p}) : F]$. The function $\zeta_{F,p}(s)$ is called the p -adic zeta function attached to F , and is p -adic analytic, having at most a simple pole at $s = 1$. Colmez proved the following result analogous to the usual analytic class number formula.

Theorem 4.2 ([Col88, §5]). *Let p be a prime and let F be a totally real number field. Then*

$$\lim_{s \rightarrow 1} (s-1) \zeta_{F,p}(s) = \frac{2^{[F:\mathbb{Q}]} h_F R_{F,p}}{w_F \sqrt{|d_F|}} \prod_{v \in \Sigma_p(F)} (1 - \text{Norm}_{F/\mathbb{Q}}(v)^{-1})$$

where h_F , $R_{F,p}$ and d_F are the class number, p -adic regulator and discriminant of F , respectively, and $w_F (= 2)$ is the number of roots of unity contained in F .

Remark 4.3. The quantities $R_{F,p}$ and $\sqrt{|d_F|}$ are only defined up to sign. However, their quotient can be well-defined as follows (we recall the explanation of [AF72, §2.3]). Let $n = [F : \mathbb{Q}]$ and let $\sigma_1, \dots, \sigma_n$ be the embeddings of F into \mathbb{R} . Let $\varepsilon_1, \dots, \varepsilon_{n-1}$ be a system of fundamental units in \mathcal{O}_F^\times and let $\omega_1, \dots, \omega_n$ be a \mathbb{Z} -basis of \mathcal{O}_F . Both the regulator $R_{F,\infty} := \det(\log|\sigma_i(\varepsilon_j)|)_{1 \leq i,j \leq n-1}$ and $\sqrt{|d_F|} := \det(\sigma_i(\omega_k))_{1 \leq i,k \leq n}$ are well-defined up to sign. Moreover, one can make choices of the tuples $(\sigma_i), (\varepsilon_j)$ and (ω_k) such that the quotient $R_{F,\infty}/\sqrt{|d_F|}$ is positive. Note that it is this quotient that appears in the usual analytic class number formula at $s = 1$. With the same choices, for any field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ the quotient

$$\frac{R_{F,p}}{\sqrt{|d_F|}} := \frac{\det(\log_p j \circ \sigma_i(\varepsilon_j))_{1 \leq i,j \leq n-1}}{\det(j \circ \sigma_i(\omega_k))_{1 \leq i,k \leq n}} = \frac{\det(\log_p j \circ \sigma_i(\varepsilon_j))_{1 \leq i,j \leq n-1}}{j(\sqrt{|d_F|})}$$

is well-defined and does not depend on the choice of j .

Corollary 4.4. $\zeta_{F,p}(s)$ has a (simple) pole at $s = 1$ if and only if $\text{Leo}(F, p)$ holds.

4.3. A certain ‘comparison period’. Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Recall from (3.2) that

$$\mu_\infty : \mathbb{C} \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_E^\times) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} E^0$$

is a canonical isomorphism of $\mathbb{C}[G]$ -modules. Let $\exp'_\infty : \log_\infty(\mathcal{O}_E^\times) \rightarrow \mathcal{O}_E^\times$ denote the map induced by $\exp_\infty : E_\infty \rightarrow E_\infty^\times$ and the inverse of the restriction of the diagonal embedding $\Delta_\infty : E^\times \rightarrow E_\infty^\times$ to \mathcal{O}_E^\times . Note that \exp'_∞ is injective since E is totally real.

Now let p be a prime and let $\Sigma_p = \Sigma_p(E)$ be the set of places of E above p . For each $w \in \Sigma_p(E)$ let $\log_w : U_{E_w}^1 \rightarrow E_w$ be the restriction of the p -adic logarithm \log_p and recall that $\ker(\log_w)$ consists of the p -power roots of unity in E_w (see [Was97, Proposition 5.6]). The composite $\mathbb{Z}_p[G]$ -module homomorphism

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_E^\times) \xrightarrow{\exp'_\infty} \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times \xrightarrow{\lambda_p} \prod_{w \in \Sigma_p(E)} U_{E_w}^1 \xrightarrow{\prod_w \log_w} \prod_{w \in \Sigma_p(E)} E_w \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} E$$

factors through the inclusion $\mathbb{Q}_p \otimes_{\mathbb{Q}} E^0 \subset \mathbb{Q}_p \otimes_{\mathbb{Q}} E$ and hence induces a homomorphism of $\mathbb{C}_p[G]$ -modules

$$\mu_p : \mathbb{C}_p \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_E^\times) \longrightarrow \mathbb{C}_p \otimes_{\mathbb{Q}} E^0.$$

Observe that

$$(4.1) \quad \text{Leo}(E, p) \text{ holds} \iff \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lambda_p \text{ is injective} \iff \mu_p \text{ is an isomorphism,}$$

where the first equivalence is Theorem 4.1 (ii) and the second equivalence follows from the definition of μ_p , the injectivity of \exp'_∞ , the fact that $\ker(\log_w)$ is torsion for each w , and that both the domain and codomain of μ_p are of \mathbb{C}_p -dimension $[E : \mathbb{Q}] - 1$.

Definition 4.5. Let $j : \mathbb{C} \cong \mathbb{C}_p$ be a field isomorphism and let $\rho \in R_{\mathbb{C}_p}^+(G)$. We define the comparison period attached to ρ and j to be

$$\Omega_j(\rho) := \det_{\mathbb{C}_p}(\mu_p \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_\infty)^{-1})^\rho \in \mathbb{C}_p.$$

We record some basic properties of $\Omega_j(-)$.

Lemma 4.6. Let H, N be subgroups of G with N normal in G .

- (i) Let $\rho_1, \rho_2 \in R_{\mathbb{C}_p}^+(G)$. Then $\Omega_j(\rho_1 + \rho_2) = \Omega_j(\rho_1)\Omega_j(\rho_2)$.
- (ii) Let $\rho \in R_{\mathbb{C}_p}^+(H)$. Then $\Omega_j(\text{ind}_H^G \rho) = \Omega_j(\rho)$.

(iii) Let $\rho \in R_{\mathbb{C}_p}^+(G/N)$. Then $\Omega_j(\text{infl}_{G/N}^G \rho) = \Omega_j(\rho)$.

Proof. Each part follows from the corresponding part of Lemma 2.1. \square

Remark 4.7. Since μ_∞ is an isomorphism, for any two choices of field isomorphism $j_1, j_2 : \mathbb{C} \cong \mathbb{C}_p$ we have that $\Omega_{j_1}(\rho) = 0$ if and only if $\Omega_{j_2}(\rho) = 0$.

Remark 4.8. For any fixed choice of field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ we have

$$\begin{aligned} \text{Leo}(E, p) \text{ holds} &\iff \mu_p \text{ is an isomorphism} \\ &\iff \Omega_j(\rho) \neq 0 \quad \forall \rho \in \text{Irr}_{\mathbb{C}_p}(G) \\ &\iff \Omega_j(\rho) \neq 0 \quad \forall \rho \in R_{\mathbb{C}_p}^+(G), \end{aligned}$$

where the first equivalence is (4.1) and the last follows from Lemma 4.6 (i). Thus the non-vanishing of $\Omega_j(\rho)$ can be thought of as the ‘ ρ -part’ of $\text{Leo}(E, p)$. Moreover, if we assume $\text{Leo}(E, p)$ then we may set $\Omega_j(-\rho) := \Omega_j(\rho)^{-1}$ and Lemma 4.6 (i) shows that the definition of $\Omega_j(\rho)$ naturally extends to any virtual character $\rho \in R_{\mathbb{C}_p}(G)$.

4.4. p -adic Artin L -functions. Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Let p be a prime and let Σ be a finite set of places of F containing $\Sigma_p(F) \cup \Sigma_\infty(F)$. For each character $\rho \in R_{\mathbb{C}_p}(G)$ the Σ -truncated p -adic Artin L -function attached to ρ is the unique p -adic meromorphic function $L_{p,\Sigma}(s, \rho) : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ with the property that for each strictly negative integer n and each field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ we have

$$L_{p,\Sigma}(n, \rho) = j \left(L_\Sigma(n, (\rho \otimes \omega^{n-1})^{j^{-1}}) \right),$$

where $\omega : \text{Gal}(\overline{F}/F) \rightarrow \mathbb{Z}_p^\times$ is the Teichmüller character. (By a result of Siegel [Sie70] the right-hand side does not depend on the choice of j .) These functions satisfy the same properties with respect to induction, inflation and addition of characters as complex Artin L -functions. In the case that ρ is linear, $L_{p,\Sigma}(s, \rho)$ was constructed independently by Deligne and Ribet [DR80], Barsky [Bar78] and Cassou-Nogués [CN79]. Greenberg [Gre83] then extended the construction to the general case using Brauer induction. Note that when $\Sigma = \Sigma_p(F) \cup \Sigma_\infty(F)$ we have $L_{p,\Sigma}(s, 1_G) = \zeta_{F,p}(s)$ (see §4.2). If we assume $\text{Leo}(E, p)$ and write $L_{p,\Sigma}^*(1, \rho)$ for the leading term at $s = 1$ of $L_{p,\Sigma}(s, \rho)$, then in analogy with (3.1) one can show that

$$(4.2) \quad L_{p,\Sigma}^*(1, \rho) = \lim_{s \rightarrow 1} (s-1)^{\langle 1_G, \rho \rangle_G} \cdot L_{p,\Sigma}(s, \rho).$$

4.5. Statement of the p -adic Stark conjecture at $s = 1$. The following conjecture originates with Serre in [Ser78], is discussed by Tate in [Tat84, Chapitre VI, §5], and is clarified by Burns and Venjakob in [BV06, §5.2] and [BV11, §7.2]; the statement given here differs slightly from those given previously.

Conjecture 4.9 (The p -adic Stark conjecture at $s = 1$). *Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Let p be a prime and let Σ be a finite set of places of F containing $\Sigma_p(F) \cup \Sigma_\infty(F)$. Let $\rho \in R_{\mathbb{C}_p}^+(G)$. Then for every choice of field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ we have*

$$(4.3) \quad L_{p,\Sigma}^*(1, \rho) = \Omega_j(\rho) \cdot j \left(L_\Sigma^*(1, \rho^{j^{-1}}) \right).$$

Remark 4.10. If we assume $\text{Leo}(E, p)$ then Remark 4.8 shows that the statement of Conjecture 4.9 naturally extends to any virtual character $\rho \in R_{\mathbb{C}_p}(G)$. Moreover, (4.3) for ρ and j implies that $\Omega_j(\rho) \neq 0$ and hence Remark 4.8 also shows that if (4.3) holds for a fixed j and for all $\rho \in \text{Irr}_{\mathbb{C}_p}(G)$, then $\text{Leo}(E, p)$ holds.

Remark 4.11. It is clear that $\Omega_j(\rho)$ does not depend on Σ . Thus, by considering Euler factors, it is straightforward to show that the truth of Conjecture 4.9 is independent of the choice of Σ .

Remark 4.12. Since both complex and p -adic Artin L -functions satisfy properties analogous to those of $\Omega_j(-)$ given in Lemma 4.6, the truth of Conjecture 4.9 is invariant under induction and inflation. Moreover, if Conjecture 4.9 holds for $\rho_1, \rho_2 \in R_{\mathbb{C}_p}(G)$ then it also holds for $\rho_1 + \rho_2$, and hence if it holds for all $\rho \in \text{Irr}_{\mathbb{C}_p}(G)$ then (noting Remark 4.10) it holds for all $\rho \in R_{\mathbb{C}_p}(G)$.

Remark 4.13. If we assume $\text{Leo}(E, p)$ then Remark 4.12 together with Brauer's theorem on induced characters (see [CR81, §15B], for example) shows that the proof of Conjecture 4.9 for E/F and a fixed choice of p reduces to certain cyclic sub-extensions of E/F .

4.6. The relation to Stark's conjecture at $s = 1$.

Theorem 4.14. *Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Let p be a prime and let Σ be a finite set of places of F containing $\Sigma_p(F) \cup \Sigma_\infty(F)$. Let $\rho \in R_{\mathbb{C}_p}^+(G)$. If $\Omega_j(\rho) \neq 0$ for some (and hence every) choice of $j : \mathbb{C} \cong \mathbb{C}_p$ (in particular, this is the case if $\text{Leo}(E, p)$ holds) then the following statements are equivalent.*

- (i) $\Omega_j(\rho) \cdot j(L_\Sigma^*(1, \rho^{j^{-1}}))$ is independent of the choice of $j : \mathbb{C} \cong \mathbb{C}_p$.
- (ii) Stark's conjecture at $s = 1$ holds for $\rho^{j^{-1}} \in R_{\mathbb{C}}^+(G)$ and some (and hence every) choice of $j : \mathbb{C} \cong \mathbb{C}_p$.

Remark 4.15. That (ii) implies (i) in Theorem 4.14 was already shown by Serre (see [Tat84, Chapitre VI, Théorème 5.2]); it is clear that this does not require the hypothesis that $\Omega_j(\rho) \neq 0$ for some (and hence every) choice of j .

Remark 4.16. The first and second occurrence of 'and hence every' in Theorem 4.14 follow from Remark 3.3 and Remark 4.7 (iii), respectively.

Proof of Theorem 4.14. Let $j, j' : \mathbb{C} \cong \mathbb{C}_p$ be field isomorphisms and let $\chi := \rho^{j^{-1}}$. Then $j = j' \circ \sigma$ for some $\sigma \in \text{Aut}(\mathbb{C})$ and so $\rho^{j'^{-1}} = \chi^\sigma$. For every $\mathbb{Q}[G]$ -isomorphism $g : E^0 \xrightarrow{\sim} \mathbb{Q} \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_E^\times)$ we have

$$j(R_1(\chi, g)) = j(\det_{\mathbb{C}}(\mu_\infty \circ (\mathbb{C} \otimes_{\mathbb{Q}} g))^{\chi}) = \det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_\infty) \circ (\mathbb{C}_p \otimes_{\mathbb{Q}} g))^\rho,$$

and thus

$$\Omega_j(\rho) \cdot j(R_1(\chi, g)) = \det_{\mathbb{C}_p}(\mu_p \circ (\mathbb{C}_p \otimes_{\mathbb{Q}} g))^\rho,$$

which does not depend on j . In particular, $\Omega_j(\rho) \cdot j(R_1(\chi, g)) = \Omega_{j'}(\rho) \cdot j'(R_1(\chi^\sigma, g))$. Hence

$$\frac{\Omega_j(\rho) \cdot j(L_\Sigma^*(1, \rho^{j^{-1}}))}{\Omega_{j'}(\rho) \cdot j'(L_\Sigma^*(1, \rho^{(j')^{-1}}))} = \frac{j'(R_1(\chi^\sigma, g)) \cdot j(L_\Sigma^*(1, \chi))}{j(R_1(\chi, g)) \cdot j'(L_\Sigma^*(1, \chi^\sigma))} = j' \left(\frac{\sigma(L_\Sigma^*(1, \chi)) \cdot R_1(\chi^\sigma, g)}{\sigma(R_1(\chi, g)) \cdot L_\Sigma^*(1, \chi^\sigma)} \right),$$

which is equal to 1 if and only if Stark's conjecture at $s = 1$ (Conjecture 3.1) holds for the character χ . \square

Corollary 4.17. *Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Fix a prime p . If the p -adic Stark conjecture at $s = 1$ holds for all $\rho \in R_{\mathbb{C}_p}^+(G)$ then Stark's conjecture at $s = 1$ holds for all $\chi \in R_{\mathbb{C}}^+(G)$.*

Proof. This follows from Theorem 4.14, Remark 4.10 and the statement of the p -adic Stark conjecture at $s = 1$ (Conjecture 4.9). \square

4.7. An alternative description of the comparison period. Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Let p be a prime. For any $j : \mathbb{C} \cong \mathbb{C}_p$ and any $\rho \in R_{\mathbb{C}_p}^+(G)$, we shall define a period $\Psi_j(\rho)$ and show that $\Psi_j(\rho) = \Omega_j(\rho)$. This alternative description of $\Omega_j(\rho)$ will be used in §6 to prove the p -adic Stark conjecture at $s = 1$ for absolutely abelian characters. However, we emphasise that the results of this section are valid for all characters.

Viewing $\mathbb{C} \otimes_{\mathbb{Q}} E^0 \subset \mathbb{C} \otimes_{\mathbb{Q}} E \cong \prod_{\text{Hom}(E, \mathbb{C})} \mathbb{C}$, the usual Dirichlet map

$$\begin{aligned} \varphi_{\infty} : \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{O}_E^{\times} &\xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} E^0 \\ z \otimes \epsilon &\mapsto (z \log |\sigma(\epsilon)|)_{\sigma \in \text{Hom}(E, \mathbb{C})}, \end{aligned}$$

is a canonical isomorphism of $\mathbb{C}[G]$ -modules. We likewise define a p -adic Dirichlet map

$$\begin{aligned} \varphi_p : \mathbb{C}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^{\times} &\longrightarrow \mathbb{C}_p \otimes_{\mathbb{Q}} E^0 \\ z \otimes \epsilon &\mapsto (z \log_p(\tau(\epsilon)))_{\tau \in \text{Hom}(E, \mathbb{C}_p)}, \end{aligned}$$

which is a homomorphism of $\mathbb{C}_p[G]$ -modules. By Theorem 4.1 (v) and the definition of φ_p we have

$$\text{Leo}(E, p) \text{ holds} \iff R_p(E) \neq 0 \iff \varphi_p \text{ is an isomorphism.}$$

Definition 4.18. Let $j : \mathbb{C} \cong \mathbb{C}_p$ be a field isomorphism and let $\rho \in R_{\mathbb{C}_p}^+(G)$. We define

$$\Psi_j(\rho) := \det_{\mathbb{C}_p}(\varphi_p \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \varphi_{\infty})^{-1})^{\rho} \in \mathbb{C}_p.$$

Lemma 4.19. *Let $j : \mathbb{C} \cong \mathbb{C}_p$ be a field isomorphism and let $\rho \in R_{\mathbb{C}_p}^+(G)$. Then*

$$\varphi_p \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \varphi_{\infty})^{-1} = \mu_p \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_{\infty})^{-1},$$

and so in particular $\Psi_j(\rho) = \Omega_j(\rho)$.

Proof. Let $\epsilon \in \mathcal{O}_F^{\times}$. It suffices to show that

$$\varphi_p(1 \otimes \epsilon) = \mu_p \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_{\infty})^{-1} \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \varphi_{\infty})(1 \otimes \epsilon).$$

In fact, since the subgroup of \mathcal{O}_F^{\times} of totally positive units is of finite index, we can and do assume without loss of generality that ϵ is totally positive. Thus we have

$$(\mathbb{C}_p \otimes_{\mathbb{C}, j} \varphi_{\infty})(1 \otimes \epsilon) = (j(\log |\sigma(\epsilon)|))_{\sigma \in \text{Hom}(E, \mathbb{C})} = (j(\log \sigma(\epsilon)))_{\sigma \in \text{Hom}(E, \mathbb{C})} \in \mathbb{C}_p \otimes_{\mathbb{Q}} E_0,$$

where we have used the identification

$$\mathbb{C}_p \otimes_{\mathbb{Q}} E_0 \subset \mathbb{C}_p \otimes_{\mathbb{Q}} E \cong \mathbb{C}_p \otimes_{\mathbb{C}, j} \prod_{\text{Hom}(E, \mathbb{C})} \mathbb{C} \cong \prod_{\text{Hom}(E, \mathbb{C}_p)} \mathbb{C}_p.$$

However, we also have

$$\mathbb{C}_p \otimes_{\mathbb{Z}} \log_{\infty}(\mathcal{O}_E^{\times}) \subset \mathbb{C}_p \otimes_{\mathbb{Q}} E \cong \prod_{\text{Hom}(E, \mathbb{C}_p)} \mathbb{C}_p$$

and the canonical isomorphism $(\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_{\infty})^{-1} : \mathbb{C}_p \otimes_{\mathbb{Q}} E_0 \cong \mathbb{C}_p \otimes_{\mathbb{Z}} \log_{\infty}(\mathcal{O}_E^{\times})$ can be considered as an equality inside $\mathbb{C}_p \otimes_{\mathbb{Q}} E \cong \prod_{\text{Hom}(E, \mathbb{C}_p)} \mathbb{C}_p$. Therefore

$$\begin{aligned} (4.4) \quad (\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_{\infty})^{-1} \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \varphi_{\infty})(1 \otimes \epsilon) &= (j(\log \sigma(\epsilon)))_{\sigma \in \text{Hom}(E, \mathbb{C})} \\ &\in \mathbb{C}_p \otimes_{\mathbb{Z}} \log_{\infty}(\mathcal{O}_E^{\times}) \subset \prod_{\text{Hom}(E, \mathbb{C}_p)} \mathbb{C}_p. \end{aligned}$$

Since ϵ is totally positive we have

$$\begin{aligned} \prod_{\sigma \in \text{Hom}(E, \mathbb{R})} \mathbb{R} \cong E_{\infty} &\xrightarrow{\text{exp}_{\infty}} E_{\infty}^{\times} \subset E_{\infty} \cong \prod_{\sigma \in \text{Hom}(E, \mathbb{R})} \mathbb{R} \\ (\log \sigma(\epsilon))_{\sigma} &\mapsto (\sigma(\epsilon))_{\sigma}. \end{aligned}$$

Moreover, we also have

$$\begin{aligned} \mathcal{O}_E^\times &\hookrightarrow E_\infty^\times \subset E_\infty \cong \prod_{\sigma \in \text{Hom}(E, \mathbb{R})} \mathbb{R} \\ \epsilon &\mapsto (\sigma(\epsilon))_\sigma. \end{aligned}$$

Hence

$$(4.5) \quad (\text{id}_{\mathbb{C}_p} \otimes_{\mathbb{Z}} \exp'_\infty)((j(\log \sigma(\epsilon)))_{\sigma \in \text{Hom}(E, \mathbb{C})}) = 1 \otimes \epsilon \in \mathbb{C}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times$$

where $\exp'_\infty : \log_\infty(\mathcal{O}_E^\times) \rightarrow \mathcal{O}_E^\times$ was defined in §4.3. Therefore combining (4.4) and (4.5) gives

$$(\text{id}_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} \exp'_\infty) \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \mu_\infty)^{-1} \circ (\mathbb{C}_p \otimes_{\mathbb{C}, j} \varphi_\infty) = \text{id}_{\mathbb{C}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times}.$$

Now

$$\lambda_p(1 \otimes \epsilon) = (\sigma_w(\epsilon))_{w \in \Sigma_p(E)} \in \prod_{w \in \Sigma_p(E)} U_{E_w}^1$$

where $\sigma_w : E \rightarrow E_w$ is the embedding of E into its completion E_w . Moreover, we have

$$\begin{aligned} \prod_{w \in \Sigma_p(E)} U_{E_w}^1 &\xrightarrow{\prod_w \log_w} \prod_{w \in \Sigma_p(E)} E_w \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} E \subset \mathbb{C}_p \otimes_{\mathbb{Q}} E \cong \prod_{w \in \Sigma_p(E)} \prod_{\tau_w \in \text{Hom}(E_w, \mathbb{C}_p)} \mathbb{C}_p \\ (\sigma_w(\epsilon))_w &\longmapsto (\log_w \sigma_w(\epsilon))_w \longmapsto (\tau_w \log_w \sigma_w(\epsilon))_{w, \tau_w}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \prod_{w \in \Sigma_p(E)} \prod_{\tau_w \in \text{Hom}(E_w, \mathbb{C}_p)} \mathbb{C}_p &\cong \prod_{\tau \in \text{Hom}(E, \mathbb{C}_p)} \mathbb{C}_p \quad \text{via} \quad (w, \tau_w) \mapsto \tau = \tau_w \circ \sigma_w \\ (\tau_w \log_w \sigma_w(\epsilon))_{w, \tau_w} &\mapsto (\log_p(\tau(\epsilon)))_\tau = \varphi_p(1 \otimes \epsilon), \end{aligned}$$

which gives the desired result. \square

5. RATIONAL-VALUED CHARACTERS

5.1. The trivial character. Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Let p be a prime.

Proposition 5.1 ([Tat84, Remark p. 138]). *Leo(F, p) holds if and only if the p -adic Stark conjecture at $s = 1$ holds for the trivial character 1_G .*

Proof. Using Remark 4.3 and Corollary 4.4, this follows by comparing the p -adic analytic class number formula at $s = 1$ (Theorem 4.2) with the usual analytic class number formula at $s = 1$. \square

Corollary 5.2 ([BV06, Remark 5.4]). *Let H be a subgroup of G and let $\rho = \text{ind}_H^G 1_H$. Then Leo(E^H, p) holds if and only if the p -adic Stark conjecture at $s = 1$ holds for ρ .*

Proof. This is just the combination of Proposition 5.1 and the fact that the truth of the p -adic Stark conjecture at $s = 1$ is invariant under induction (see Remark 4.12). \square

5.2. Permutation characters and rational-valued characters. Let G be a finite group and let $\rho \in \text{Char}_{\mathbb{Q}}(G)$. Then by Artin's induction theorem (see [CR81, (15.4)], for example) there exists a natural number n_ρ dividing $|G|$ such that

$$(5.1) \quad n_\rho \cdot \rho = \sum n_H \cdot \text{ind}_H^G 1_H$$

where the sum runs over all subgroups H of G and each n_H is an integer. By definition $\rho \in \text{Perm}(G)$ if and only if one can take $n_\rho = 1$.

The following result is analogous to but different from [BV06, Corollary 5.7].

Theorem 5.3. *Let E/F be a finite Galois extension of totally real number fields with Galois group G . Let p be a prime and let Σ be a finite set of places of F containing $\Sigma_p(F) \cup \Sigma_\infty(F)$. Suppose that $\text{Leo}(E, p)$ holds. Let $\rho \in \text{Char}_\mathbb{Q}(G) \subset R_{\mathbb{C}_p}(G)$ and suppose that the expression (5.1) holds for ρ . Then for every field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ we have*

$$(L_{p,\Sigma}^*(1, \rho))^{n_\rho} = \left(\Omega_j(\rho) \cdot j \left(L_\Sigma^*(1, \rho^{j^{-1}}) \right) \right)^{n_\rho}.$$

In particular, if $\rho \in \text{Perm}(G)$, then the p -adic Stark conjecture at $s = 1$ holds for ρ .

Proof. By Theorem 4.1 (iv), $\text{Leo}(E^H, p)$ holds for every subgroup $H \leq G$. Thus by Corollary 5.2, the p -adic Stark conjecture at $s = 1$ holds for every $\text{ind}_H^G 1_H$. In other words, for every $j : \mathbb{C} \cong \mathbb{C}_p$ and every subgroup $H \leq G$ we have

$$L_{p,\Sigma}^*(1, \text{ind}_H^G 1_H) = \Omega_j(\text{ind}_H^G 1_H) \cdot j \left(L_\Sigma^*(1, \text{ind}_H^G 1_H) \right).$$

The desired result now follows from the expression (5.1) for ρ and the properties of $\Omega_j(-)$ and both complex and p -adic Artin L -functions with respect to addition of characters. \square

Corollary 5.4. *Let E/F be a finite Galois extension of totally real number fields with Galois group G such that $\text{Perm}(G) = R_\mathbb{C}(G)$. Let p be a prime and suppose that $\text{Leo}(E, p)$ holds. Then the p -adic Stark conjecture at $s = 1$ holds for all $\rho \in R_{\mathbb{C}_p}(G)$.*

Remark 5.5. The condition $\text{Perm}(G) = R_\mathbb{C}(G)$ is discussed in Remark 8.10.

6. ABSOLUTELY ABELIAN CHARACTERS

We now prove the p -adic Stark conjecture at $s = 1$ for absolutely abelian characters by building on work of Ritter and Weiss [RW97] and using standard results on Dirichlet L -functions and their p -adic analogues (see [Was97], for example).

Theorem 6.1. *Let E/F be a finite Galois extension of totally real number fields and let $G = \text{Gal}(E/F)$. Let p be a prime. Suppose that $\rho \in R_{\mathbb{C}_p}(G)$ is an absolutely abelian character, i.e., there exists a normal subgroup N of G such that ρ factors through $G/N \cong \text{Gal}(E^N/F)$ and E^N/\mathbb{Q} is abelian. Then the p -adic Stark conjecture at $s = 1$ holds for ρ .*

Proof. We first observe that we can make a number of simplifying assumptions. Recall from Remark 4.12 that the truth of Conjecture 4.9 is invariant under induction and inflation. By invariance under induction we may assume that $F = \mathbb{Q}$. By the Kronecker-Weber theorem and invariance under inflation we may further assume that ρ is a Dirichlet character and that $E = \mathbb{Q}(\zeta_n)^+$, the maximal totally real subfield of $\mathbb{Q}(\zeta_n)$, where n is the conductor of ρ and ζ_n denotes a fixed primitive n -th root of unity. Note that $\text{Leo}(E, p)$ holds by Theorem 4.1 (iii). Moreover, again by Remark 4.12, we may also assume that ρ is irreducible. Finally, by Proposition 5.1 and the fact that $\text{Leo}(\mathbb{Q}, p)$ holds, we may also assume that ρ is non-trivial.

Note that $n \not\equiv 2 \pmod{4}$ and let $n = \prod_{i=1}^s p_i^{e_i}$ be its prime factorisation. Following [Was97, Theorem 8.3] and [RW97, §10], let I run through all proper subsets of $\{1, \dots, s\}$ and set $n_I := \prod_{i \in I} p_i^{e_i}$. Let

$$\xi_n := \prod_I (1 - \zeta_n^{n_I})(1 - \zeta_n^{-n_I}) \in E^\times.$$

Observe that $\xi_n^{\sigma^{-1}} := \sigma(\xi_n)/\xi_n$ is a totally positive unit for every $\sigma \in G$.

Let S_∞ denote the set of archimedean places of E . The map $\zeta_n \mapsto \exp(2\pi i/n)$ embeds $\mathbb{Q}(\zeta_n)$ into \mathbb{C} , and its restriction to E gives a distinguished archimedean place $\infty \in S_\infty$.

Then $\mathbb{Z}S_\infty$ is a free $\mathbb{Z}[G]$ -module of rank 1 with basis ∞ . Let ΔS_∞ denote the kernel of the augmentation map $\mathbb{Z}S_\infty \rightarrow \mathbb{Z}$ which sends each place in S_∞ to 1. Following [RW97, §10], the $\mathbb{Z}[G]$ -linear map $\mathbb{Z}S_\infty \rightarrow E^\times$ which sends ∞ to ξ_n induces an embedding

$$\phi : \Delta S_\infty \hookrightarrow \mathcal{O}_E^\times.$$

Let $e_\rho := |G|^{-1} \sum_{\sigma \in G} \rho(\sigma^{-1}) \sigma \in \mathbb{C}_p[G]$ be the primitive idempotent corresponding to ρ . As ρ is a non-trivial linear character, we have

$$(\mathbb{C}_p \otimes_{\mathbb{Z}} \Delta S_\infty)^\rho \cong e_\rho \cdot (\mathbb{C}_p \otimes_{\mathbb{Z}} \Delta S_\infty) = e_\rho \cdot \mathbb{C}_p S_\infty = e_\rho \mathbb{C}_p[G] \cdot \infty.$$

Treating the first isomorphism as an identification we may view $e_\rho \infty$ as a \mathbb{C}_p -basis of $(\mathbb{C}_p \otimes_{\mathbb{Z}} \Delta S_\infty)^\rho$. Let $j : \mathbb{C} \cong \mathbb{C}_p$ be a field isomorphism and recall the definitions of φ_∞ and φ_p from §4.7. We denote the composite isomorphism

$$\mathbb{C}_p \otimes_{\mathbb{Z}} \Delta S_\infty \xrightarrow{1 \otimes \phi} \mathbb{C}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times \xrightarrow{1 \otimes_{\mathbb{C}, j} \varphi_\infty} \mathbb{C}_p \otimes_{\mathbb{Q}} E_0 \cong \mathbb{C}_p \otimes_{\mathbb{Z}} \Delta S_\infty$$

again by $\varphi_\infty \phi$ and compute the image of $e_\rho \infty$ under this map:

$$\begin{aligned} \varphi_\infty \phi(e_\rho \infty) &= \frac{1}{[E : \mathbb{Q}]} \sum_{\sigma \in G} \rho(\sigma^{-1}) (1 \otimes_{\mathbb{C}, j} \varphi_\infty)(\xi_n^{\sigma^{-1}}) \\ &= \frac{1}{[E : \mathbb{Q}]} \sum_{\sigma \in G} \rho(\sigma^{-1}) \sum_{\sigma' \in G} j(\log((\sigma')^{-1}(\xi_n^{\sigma^{-1}}))) \cdot \sigma' \infty \\ &= \frac{1}{[E : \mathbb{Q}]} \sum_{\sigma \in G} \sum_{\sigma' \in G} \rho((\sigma \sigma')^{-1}) j(\log((\sigma')^{-1}(\xi_n^{\sigma \sigma'^{-1}}))) \cdot \sigma' \infty \\ (6.1) \quad &= \frac{1}{[E : \mathbb{Q}]} \sum_{\sigma \in G} \sum_{\sigma' \in G} \rho(\sigma^{-1}) \rho((\sigma')^{-1}) j(\log \xi_n^\sigma) \cdot \sigma' \infty \\ &\quad - \frac{1}{[E : \mathbb{Q}]} \sum_{\sigma \in G} \sum_{\sigma' \in G} \rho(\sigma^{-1}) \rho((\sigma')^{-1}) j(\log \xi_n^{(\sigma')^{-1}}) \cdot \sigma' \infty \\ &= \left(\sum_{\sigma \in G} \rho(\sigma^{-1}) j(\log \xi_n^\sigma) \right) e_\rho \infty. \end{aligned}$$

Here, the first equality holds because $\sum_{\sigma \in G} \rho(\sigma^{-1})$ vanishes and we thus have an equality $e_\rho = [E : \mathbb{Q}]^{-1} \sum_{\sigma \in G} \rho(\sigma^{-1})(\sigma - 1)$; the second equality holds by the definition of φ_∞ and the fact that $\xi_n^{\sigma^{-1}}$ is totally positive; the third and fourth equalities are clear; and the last equality again follows from the vanishing of $\sum_{\sigma \in G} \rho(\sigma^{-1})$. A similar computation shows that

$$(6.2) \quad \varphi_p \phi(e_\rho \infty) = \left(\sum_{\sigma \in G} \rho(\sigma^{-1}) \log_p(j(\xi_n^\sigma)) \right) e_\rho \infty.$$

Using equations (6.1) and (6.2) and Lemma 4.19 we obtain

$$(6.3) \quad \Omega_j(\rho) = \Psi_j(\rho) = \frac{\sum_{\sigma \in G} \rho(\sigma^{-1}) \log_p(j(\xi_n^\sigma))}{\sum_{\sigma \in G} \rho(\sigma^{-1}) j(\log \xi_n^\sigma)}.$$

We identify $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^\times$ as usual and view ρ as an even \mathbb{C}_p -valued Dirichlet character $(\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \mathbb{C}_p^\times$. Let $\check{\rho}$ be the contragredient of ρ and let $L_p(s, \rho)$ be the (non-truncated) p -adic L -function attached to ρ . Let $\chi := \rho^{j^{-1}} : (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be the classical even Dirichlet character corresponding to ρ via j , let $\tau(\chi)$ be its Gauss sum

and let $L(s, \chi)$ be the (non-truncated) Dirichlet L -function attached to χ . Finally, let Σ be the set containing p and the unique infinite place of \mathbb{Q} .

We now compute the denominator of the right-hand side in (6.3):

$$\begin{aligned}
\sum_{\sigma \in G} \rho(\sigma^{-1}) j(\log \xi_n^\sigma) &= \sum_{\sigma \in G} \rho(\sigma^{-1}) j \left(\sum_I \log |(1 - \zeta_n^{n_I})^{2\sigma}| \right) \\
&= \frac{1}{2} \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \check{\rho}(a) j \left(\sum_I \log |(1 - \zeta_n^{an_I})^2| \right) \\
&= \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \check{\rho}(a) j \left(\sum_I \log |1 - \zeta_n^{an_I}| \right) \\
(6.4) \quad &= \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \check{\rho}(a) j(\log |1 - \zeta_n^a|) \\
&= -j \left(\frac{n}{\tau(\chi)} L(1, \chi) \right) \\
&= -j \left(\frac{n}{\tau(\chi)} L_\Sigma(1, \chi) \right) \left(1 - \frac{\rho(p)}{p} \right)^{-1}.
\end{aligned}$$

Here, the first equality holds as $\xi_n = \prod_I (-\zeta_n^{n_I})(1 - \zeta_n^{n_I})^2$, the fourth is [Was97, Lemma 8.4] and the fifth is [Was97, Theorem 4.9]. A similar computation using [Was97, Theorem 5.18] shows that we have

$$(6.5) \quad \sum_{\sigma \in G} \rho(\sigma^{-1}) \log_p(j(\xi_n^\sigma)) = -j \left(\frac{n}{\tau(\chi)} \right) L_p(1, \rho) \left(1 - \frac{\rho(p)}{p} \right)^{-1}.$$

Moreover, $L(1, \chi) \neq 0$ by [Was97, Corollary 4.4] and $L_p(1, \rho) \neq 0$ by [Was97, Corollary 5.30]. Hence $L_\Sigma(1, \chi) = L_\Sigma^*(1, \chi)$ and $L_p(1, \rho) = L_p^*(1, \rho)$. The equality (4.3) for the choice of Σ above now follows by inserting (6.4) and (6.5) into (6.3). Finally, by Remark 4.11 we obtain the desired result for any choice of Σ . \square

7. THE ETNC AND THE EQUIVARIANT IWASAWA MAIN CONJECTURE

7.1. Algebraic K -theory. For a left noetherian ring Λ we write $K_0(\Lambda)$ for the Grothendieck group of the category of finitely generated projective Λ -modules (see [CR87, §38]) and $K_1(\Lambda)$ for the Whitehead group (see [CR87, §40]). Moreover, we denote the relative algebraic K -group associated to a ring homomorphism $\Lambda \rightarrow \Lambda'$ by $K_0(\Lambda, \Lambda')$. We recall that $K_0(\Lambda, \Lambda')$ is an abelian group with generators $[X, g, Y]$ where X and Y are finitely generated projective Λ -modules and $g : \Lambda' \otimes_\Lambda X \rightarrow \Lambda' \otimes_\Lambda Y$ is an isomorphism of Λ' -modules; for a full description in terms of generators and relations, see [Swa68, p. 215]. Furthermore, there is a long exact sequence of relative K -theory

$$(7.1) \quad K_1(\Lambda) \longrightarrow K_1(\Lambda') \xrightarrow{\partial} K_0(\Lambda, \Lambda') \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Lambda')$$

(see [Swa68, Chapter 15]).

Let R be a noetherian integral domain of characteristic 0, let E be any extension of the field of fractions of R , and let G be a finite group. We then write $K_0(R[G], E)$ for the relative algebraic K -group associated to the ring homomorphism $R[G] \hookrightarrow E[G]$ and

write $K_0(R[G], E)_{\text{tors}}$ for its torsion subgroup. If H is a subgroup of G then the inclusion map $R[H] \hookrightarrow R[G]$ induces canonical restriction and induction maps

$$\text{res}_H^G : K_0(R[G], E) \longrightarrow K_0(R[H], E), \quad \text{ind}_H^G : K_0(R[H], E) \longrightarrow K_0(R[G], E).$$

Moreover, if N is a normal subgroup of G then the quotient map $R[G] \twoheadrightarrow R[G/N]$ induces a canonical quotient map

$$\text{q}_{G/N}^G : K_0(R[G], E) \longrightarrow K_0(R[G/N], E).$$

Finally, the maps $K_0(\mathbb{Z}[G], \mathbb{Q}) \rightarrow K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ induce a canonical isomorphism

$$(7.2) \quad K_0(\mathbb{Z}[G], \mathbb{Q}) \cong \bigoplus_p K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$$

where p ranges over all primes (see the discussion following [CR87, (49.12)]).

7.2. The equivariant Tamagawa number conjecture (ETNC). We give a very brief description of the statement and properties of the equivariant Tamagawa number conjecture (ETNC) for Tate motives formulated by Burns and Flach [BF01]; we omit all details except those necessary for proofs in later sections.

Let L/K be a finite Galois extension of number fields with Galois group G . For each integer r we set $\mathbb{Q}(r)_L := h^0(\text{Spec}(L))(r)$, which we regard as a motive defined over K and with coefficients in the semisimple algebra $\mathbb{Q}[G]$. The conjecture ‘ETNC($\mathbb{Q}(r)_L, \mathbb{Z}[G]$)’ formulated in [BF01, Conjecture 4(iv)] for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ asserts that a certain canonical element $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{R})$ vanishes. (As observed in [BF03, §1], the element $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ is indeed well-defined.) We define the following notation.

- ETNC($L/K, r$) means $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G]) = 0$.
- ETNC^{tors}($L/K, r$) means $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G]) \in K_0(\mathbb{Z}[G], \mathbb{Q})_{\text{tors}}$.
- ETNC^{rat}($L/K, r$) means $T\Omega(\mathbb{Q}(r)_K, \mathbb{Z}[G]) \in K_0(\mathbb{Z}[G], \mathbb{Q})$.

Thus if ETNC^{rat}($L/K, r$) holds then by (7.2) we have elements $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])_p$ in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ for each prime p . In this situation, we define the following notation.

- ETNC _{p} ($L/K, r$) means $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])_p = 0$.
- ETNC _{p} ^{tors}($L/K, r$) means $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])_p \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$.

We now observe that several conjectures encountered thus far are in fact equivalent.

Proposition 7.1. *Let L/K be a finite Galois extension of number fields with Galois group G . Then the following are equivalent:*

- (i) ETNC^{rat}($L/K, 0$);
- (ii) ETNC^{rat}($L/K, 1$);
- (iii) Stark’s conjecture at $s = 0$ for every $\chi \in R_{\mathbb{C}}^+(G)$;
- (iv) Stark’s conjecture at $s = 1$ for every $\chi \in R_{\mathbb{C}}^+(G)$.

Proof. As already observed in Remark 3.2, (iii) and (iv) are equivalent by [Tat84, Chapitre I, Théorème 8.4]. The equivalence of (i) and (ii) follows from [BF01, Theorem 5.2]. Finally, [BF03, Corollary 1] gives the equivalence of (i) and (iii). \square

7.3. Reduction steps for the ETNC. A Brauer induction argument shows that to prove either ETNC^{rat}($L/K, r$) or ETNC^{tors}($L/K, r$) it suffices to consider certain cyclic sub-extensions of L/K . Now fix a prime p . Burns [Bur04, Theorem 4.1] showed that

$$(7.3) \quad K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}} = \bigcap \ker(\text{q}_Q^H \circ \text{res}_H^G : K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \longrightarrow K_0(\mathbb{Z}_p[Q], \mathbb{Q}_p)),$$

where the intersection runs over all cyclic groups H of G and over all quotients Q of H that are of order prime to p . Thus by the functorial properties of $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ with respect

to restriction and quotient maps (see [BF01, Theorem 4.1]), to prove $\text{ETNC}_p^{\text{tors}}(L/K, r)$ it suffices to consider cyclic sub-extensions of L/K of degree prime to p . Moreover, if we assume $\text{ETNC}_p^{\text{tors}}(L/K, r)$ then [GRW99, Proposition 9] and functoriality show that $\text{ETNC}_p(L/K, r)$ reduces to certain p -elementary Galois sub-extensions of L/K (a finite group is p -elementary if it is isomorphic to $C_m \times P$ for some p -group P and some $m \in \mathbb{N}$ with $p \nmid m$). Therefore $\text{ETNC}_p(L/K, r)$ reduces to the case of abelian (resp. cyclic) sub-extensions if G has an abelian (resp. cyclic) Sylow p -subgroup. However, $\text{ETNC}_p(L/K, r)$ does not reduce to abelian sub-extensions via functoriality in general. For example, using the algorithm of Bley and Wilson [BW09] (implemented by Bley using Magma [BCP97]), one can show that if G is the Heisenberg group of order 27 then $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$ has exponent 18 whereas if Q is any proper subquotient of G then $K_0(\mathbb{Z}_p[Q], \mathbb{Q}_p)_{\text{tors}}$ has exponent dividing 6. Thus if Q runs over all proper subquotients of G then the map $\prod q_Q^H \circ \text{res}_H^G$ cannot be injective on $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$.

7.4. The equivariant Iwasawa main conjecture. Let p be an odd prime and let E/F be a finite Galois extension of totally real fields with Galois group G . Let E^{cyc} be the cyclotomic \mathbb{Z}_p -extension of E and let Σ be a finite set of places of F containing $\Sigma_p(F) \cup \Sigma_\infty(F)$. Let M_Σ be the maximal abelian pro- p -extension of E^{cyc} unramified outside Σ . Let $X_\Sigma = \text{Gal}(M_\Sigma/E^{\text{cyc}})$. Then $\mathcal{G} := \text{Gal}(E^{\text{cyc}}/F)$ acts on X_Σ by $g \cdot x = \tilde{g}x\tilde{g}^{-1}$ where $g \in \mathcal{G}$ and \tilde{g} is any lift of g to $\text{Gal}(M_\Sigma/F)$. Thus X_Σ is a $\Lambda(\mathcal{G})$ -module, where $\Lambda(\mathcal{G}) := \mathbb{Z}_p[[\mathcal{G}]]$ denotes the Iwasawa algebra of \mathcal{G} over \mathbb{Z}_p . We denote the total ring of fractions of $\Lambda(\mathcal{G})$ by $\mathcal{Q}(\mathcal{G})$.

Since E is totally real, a result of Iwasawa [Iwa73] shows that X_Σ is finitely generated and torsion as a $\mathbb{Z}_p[[\Gamma]]$ -module, where $\Gamma = \Gamma_E := \text{Gal}(E^{\text{cyc}}/E) \cong \mathbb{Z}_p$. We let $\mu_p(E)$ denote the Iwasawa μ -invariant of X_Σ and note that this does not depend on the choice of Σ (see [NSW08, Corollary 11.3.6]). Thus $\mu_p(E) = 0$ if and only if X_Σ is finitely generated as a \mathbb{Z}_p -module. It is conjectured that we always have $\mu_p(E) = 0$ and as explained in [JN, Remark 4.3], this is closely related to the classical Iwasawa ‘ $\mu = 0$ ’ conjecture for $E(\zeta_p)$ at p . Thus a result of Ferrero and Washington [FW79] on this latter conjecture implies that $\mu_p(E) = 0$ whenever E/\mathbb{Q} is abelian.

We now consider the canonical complex

$$C_\Sigma^\bullet(E^{\text{cyc}}/F) := R\text{Hom}(R\Gamma_{\text{ét}}(\text{Spec}(\mathcal{O}_{E^{\text{cyc}}, \Sigma}), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p).$$

Here, $\mathcal{O}_{E^{\text{cyc}}, \Sigma}$ denotes the ring of integers $\mathcal{O}_{E^{\text{cyc}}}$ in E^{cyc} localised at all primes above those in Σ and $\mathbb{Q}_p/\mathbb{Z}_p$ denotes the constant sheaf of the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$ on the étale site of $\text{Spec}(\mathcal{O}_{E^{\text{cyc}}, \Sigma})$. The only non-trivial cohomology groups occur in degree -1 and 0 and we have

$$H^{-1}(C_\Sigma^\bullet(E^{\text{cyc}}/F)) \cong X_\Sigma, \quad H^0(C_\Sigma^\bullet(E^{\text{cyc}}/F)) \cong \mathbb{Z}_p.$$

It follows from [FK06, Proposition 1.6.5] that $C_\Sigma^\bullet(E^{\text{cyc}}/F)$ is a perfect complex of $\Lambda(\mathcal{G})$ -modules. In particular, $C_\Sigma^\bullet(E^{\text{cyc}}/F)$ defines a class $[C_\Sigma^\bullet(E^{\text{cyc}}/F)]$ in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$ (see [Suj13, §2], for example). Note that $C_\Sigma^\bullet(E^{\text{cyc}}/F)$ and the complex used by Ritter and Weiss (as constructed in [RW04]) become isomorphic in the derived category of $\Lambda(\mathcal{G})$ -modules by [Nic13, Theorem 2.4] (see also [Ven13] for more on this topic). Hence it makes no essential difference which of these complexes we use in the following.

Let χ_{cyc} be the p -adic cyclotomic character

$$\chi_{\text{cyc}} : \text{Gal}(E^{\text{cyc}}(\zeta_p)/F) \longrightarrow \mathbb{Z}_p^\times,$$

defined by $\sigma(\zeta) = \zeta^{\chi_{\text{cyc}}(\sigma)}$ for any $\sigma \in \text{Gal}(E^{\text{cyc}}(\zeta_p)/F)$ and any p -power root of unity ζ . Let ω and κ denote the composition of χ_{cyc} with the projections onto the first and second

factors of the canonical decomposition $\mathbb{Z}_p^\times = \langle \zeta_{p-1} \rangle \times (1 + p\mathbb{Z}_p)$, respectively; thus ω is the Teichmüller character. We note that κ factors through both \mathcal{G} and $\Gamma_F := \text{Gal}(F^{\text{cyc}}/F)$; by abuse of notation we also use κ to denote the maps with either of these domains.

Now let $\pi_\rho : \mathcal{G} \rightarrow \text{GL}_n(\mathcal{O})$ be an Artin representation (i.e. π_ρ is continuous and has finite image) with character ρ , where \mathcal{O} denotes the ring of integers in some finite extension L of \mathbb{Q}_p . Let $\overline{\mathbb{Q}_p}$ denote an algebraic closure of \mathbb{Q}_p . Choose a topological generator γ_F of Γ_F and put $u := \kappa(\gamma_F)$. Each such choice permits the definition of a power series $G_{\rho, \Sigma}(T) \in \overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \text{Quot}(\mathbb{Z}_p[[T]])$ such that for every $s \in \mathbb{Z}_p$ we have

$$(7.4) \quad L_{p, \Sigma}(1 - s, \rho) = \frac{G_{\rho, \Sigma}(u^s - 1)}{H_\rho(u^s - 1)},$$

where, for irreducible ρ , we have

$$H_\rho(T) = \begin{cases} \rho(\gamma_F)(1 + T) - 1 & \text{if } \text{Gal}(E^{\text{cyc}}/F^{\text{cyc}}) \subset \ker \rho \\ 1 & \text{otherwise.} \end{cases}$$

We recall the following construction from [CFK⁺05, §3]. For $g \in \mathcal{G}$ we write \bar{g} for its image under the canonical projection $\mathcal{G} \rightarrow \Gamma_F$ and let $\mathcal{Q}^L(\Gamma_F) := L \otimes_{\mathbb{Q}_p} \mathcal{Q}(\Gamma_F)$. Each continuous representation $\pi : \mathcal{G} \rightarrow \text{GL}_n(\mathcal{O})$ gives rise to a continuous group homomorphism $\mathcal{G} \rightarrow \text{GL}_n(\mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_F))$ defined by $g \mapsto \pi(g)\bar{g}$. This extends to a ring homomorphism $\mathcal{Q}(\mathcal{G}) \rightarrow M_{n \times n}(\mathcal{Q}^L(\Gamma_F))$ which in turn induces a homomorphism of abelian groups

$$\Phi_\pi : K_1(\mathcal{Q}(\mathcal{G})) \longrightarrow K_1(M_{n \times n}(\mathcal{Q}^L(\Gamma_F))) \cong K_1(\mathcal{Q}^L(\Gamma_F)) \cong \mathcal{Q}^L(\Gamma_F)^\times \cong \text{Quot}(\mathcal{O}[[T]])^\times.$$

Here the first isomorphism is induced by Morita equivalence, the second by taking determinants and the third by using [RW04, Lemma 4] and mapping γ_F to $1 + T$. We define an evaluation map

$$\begin{aligned} \phi : \text{Quot}(\mathcal{O}[[T]]) &\rightarrow L \cup \{\infty\} \\ f(T) &\mapsto f(0). \end{aligned}$$

If ζ is an element of $K_1(\mathcal{Q}(\mathcal{G}))$ we define $\zeta(\pi) := \phi(\Phi_\pi(\zeta))$.

The following is a formulation of the equivariant Iwasawa main conjecture without its uniqueness statement (we assume the hypotheses and notation above).

Conjecture 7.2 (equivariant Iwasawa main conjecture). *There exists a $\zeta_\Sigma \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\partial(\zeta_\Sigma) = -[C_\Sigma^\bullet(E^{\text{cyc}}/F)]$ and for every irreducible Artin representation π_ρ of \mathcal{G} with character ρ and for each integer $r \geq 1$ divisible by $p - 1$ we have*

$$(7.5) \quad \zeta_\Sigma(\pi_\rho \kappa^r) = L_{p, \Sigma}(1 - r, \rho) = j \left(L_\Sigma(1 - r, \rho^{j^{-1}}) \right)$$

for any $j : \mathbb{C} \cong \mathbb{C}_p$.

Remark 7.3. It can be shown that the validity of Conjecture 7.2 is independent of the choice of Σ .

The following theorem has been shown by Ritter and Weiss [RW11] and by Kakde [Kak13] independently.

Theorem 7.4. *If $\mu_p(E) = 0$ then Conjecture 7.2 holds for E^{cyc}/F .*

If $f(T)$ belongs to $\text{Quot}(\mathcal{O}[[T]])$ and a is in \mathbb{Z}_p^\times , we put $(t_a f)(T) := f(a(1 + T) - 1)$. It is then clear from the definitions that for any $\zeta \in K_1(\mathcal{Q}(\mathcal{G}))$, any irreducible Artin representation π_ρ of \mathcal{G} and $r \in \mathbb{Z}$ we have

$$(7.6) \quad \Phi_{\pi_\rho \kappa^r}(\zeta) = t_{u^r}(\Phi_{\pi_\rho}(\zeta)).$$

Let $f_{\rho,\Sigma}(T) := G_{\rho,\Sigma}(T)/H_\rho(T)$. Then for each integer $r \geq 1$ divisible by $p-1$ we have

$$(7.7) \quad \phi((t_{ur} f_{\rho,\Sigma})(T)) = f_{\rho,\Sigma}(u^r - 1) = L_{p,\Sigma}(1-r, \rho).$$

Note that this property uniquely determines $f_{\rho,\Sigma}(T)$ by the p -adic Weierstrass preparation theorem (see [Was97, Theorem 7.3]). Thus Conjecture 7.2 implies that there exists a $\zeta_\Sigma \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\partial(\zeta_\Sigma) = -[C_\Sigma^\bullet(E^{\text{cyc}}/F)]$ and that for every irreducible Artin representation π_ρ of \mathcal{G} with character ρ we have

$$(7.8) \quad \Phi_{\pi_\rho}(\zeta_\Sigma) = f_{\rho,\Sigma}(T)$$

(to see this, use equations (7.5), (7.6) and (7.7)).

7.5. The interpolation property at $s = 1$. We keep the notation of the last subsection. In particular π_ρ is an irreducible Artin representation of \mathcal{G} with character ρ . We denote the order of vanishing of $L_{p,\Sigma}(s, \rho)$ at $s = 1$ by $r(\rho)$. Thus by (4.2), if $\text{Leo}(E, p)$ holds, we have $r(\rho) = -1$ when ρ is the trivial character and $r(\rho) = 0$ when ρ is non-trivial. For any $\xi \in K_1(\mathcal{Q}(\mathcal{G}))$ we let $\xi^*(\rho)$ be its leading term at ρ as defined in [BV11, §2], that is, $\xi^*(\rho)$ is the leading term at $T = 0$ of $\Phi_{\pi_\rho}(\xi)$. The following result is a variant of [Bur15, Theorem 9.1].

Proposition 7.5. *Suppose that Conjecture 7.2 holds for E^{cyc}/F . Then there exists a $\zeta_\Sigma \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\partial(\zeta_\Sigma) = -[C_\Sigma^\bullet(E^{\text{cyc}}/F)]$ and for every irreducible Artin character ρ of \mathcal{G} we have*

$$\zeta_\Sigma^*(\rho) = \log_p(u)^{-r(\rho)} L_{p,\Sigma}^*(1, \rho).$$

Proof. As Conjecture 7.2 holds, the leading term $\zeta_\Sigma^*(\rho)$ coincides with the leading term of $f_{\rho,\Sigma}(T)$ at $T = 0$ by (7.8). Now write $f_{\rho,\Sigma}(T) = T^{r(\rho)} \cdot h_{\rho,\Sigma}(T)$ so that $h_{\rho,\Sigma}(0)$ equals the leading term of $f_{\rho,\Sigma}(T)$ at $T = 0$. Thus $\zeta_\Sigma^*(\rho) = h_{\rho,\Sigma}(0)$. Moreover,

$$(7.9) \quad L_{p,\Sigma}(1-s, \rho) = f_{\rho,\Sigma}(u^s - 1) = (u^s - 1)^{r(\rho)} h_{\rho,\Sigma}(u^s - 1),$$

where the first equality follows from (7.4) and the definition of $f_{\rho,\Sigma}(T)$.

Since $u = \kappa(\gamma_F) \in 1 + p\mathbb{Z}_p$ and p is odd, by [Was97, Proposition 5.7] we have

$$(7.10) \quad u^s - 1 = \exp_p(\log_p(u^s)) - 1 = \sum_{n \geq 1} \frac{(\log_p(u))^n}{n!} s^n$$

for $s \in \mathbb{Z}_p$. Now we have

$$\begin{aligned} h_{\rho,\Sigma}(u^s - 1) &= (u^s - 1)^{-r(\rho)} L_{p,\Sigma}(1-s, \rho) \quad \text{by (7.9)} \\ &= \left(\sum_{n \geq 1} \frac{(\log_p(u))^n}{n!} s^n \right)^{-r(\rho)} L_{p,\Sigma}(1-s, \rho) \quad \text{by (7.10)} \\ &= \left(\sum_{n \geq 1} \frac{(\log_p(u))^n}{n!} s^{n-1} \right)^{-r(\rho)} s^{-r(\rho)} L_{p,\Sigma}(1-s, \rho). \end{aligned}$$

Therefore taking limits as $s \rightarrow 0$ gives

$$\zeta_\Sigma^*(\rho) = h_{\rho,\Sigma}(0) = \lim_{s \rightarrow 0} h_{\rho,\Sigma}(u^s - 1) = \log_p(u)^{-r(\rho)} L_{p,\Sigma}^*(1, \rho). \quad \square$$

8. A PRIME-BY-PRIME DESCENT THEOREM FOR THE ETNC AT $s = 1$

Theorem 8.1. *Let E/F be a finite Galois extension of totally real number fields with Galois group G . Fix a prime p and suppose that the p -adic Stark conjecture at $s = 1$ holds for all $\rho \in \text{Irr}_{\mathbb{C}_p}(G)$.*

- (i) $\text{ETNC}^{\text{rat}}(E/F, 1)$ holds.
- (ii) If p is odd then $\text{ETNC}_p^{\text{tors}}(E/F, 1)$ holds.

- (iii) Suppose p is odd and if p divides $|G|$ then further suppose that $\mu_p(E) = 0$. Then $\text{ETNC}_p(E/F, 1)$ holds.

Remark 8.2. Theorem 8.1 refines Burns' descent result [Bur15, Corollary 2.8]. A key point is that Burns' result assumes the p -adic Stark conjecture at $s = 1$ for *all* odd primes and then uses a result on certain maps between relative algebraic K -groups [BB07, Lemma 2.1] to deduce $\text{ETNC}^{\text{rat}}(E/F, 1)$. By contrast, we assume the p -adic Stark conjecture at $s = 1$ for a single prime p and deduce $\text{ETNC}^{\text{rat}}(E/F, 1)$ using Corollary 4.17. There are several other differences between the two approaches, but we emphasise that both crucially rely on the descent theory of Burns and Venjakob [BV11].

Proof of Theorem 8.1. Claim (i) follows from Corollary 4.17 and Proposition 7.1. Now assume that p is odd. By the discussion around (7.3), to prove $\text{ETNC}_p^{\text{tors}}(E/F, 1)$ we may assume without loss of generality that E/F is cyclic of degree prime to p . Thus to prove (ii) it suffices to prove (iii), which we now do. Let E^{cyc} denote the cyclotomic \mathbb{Z}_p -extension of E and let $\mathcal{G} = \text{Gal}(E^{\text{cyc}}/F)$. Then the equivariant Iwasawa main conjecture for the extension E^{cyc}/F holds by [JN, Theorem 4.12] if $p \nmid |G|$ and by Theorem 7.4 otherwise. As the p -adic Stark conjecture at $s = 1$ holds for all $\rho \in \text{Irr}_{\mathbb{C}_p}(G)$ by assumption, it follows from Proposition 7.5 that there exists a $\zeta_{\Sigma} \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\partial(\zeta_{\Sigma}) = -[C_{\Sigma}^{\bullet}(E^{\text{cyc}}/F)]$ and we have

$$\zeta_{\Sigma}^*(\rho) = \log_p(u)^{\langle \rho, 1_G \rangle} \Omega_j(\rho) \cdot j \left(L_{\Sigma}^*(1, \rho^{j-1}) \right)$$

for all Artin representations ρ of \mathcal{G} that factor through G and for every $j : \mathbb{C} \cong \mathbb{C}_p$. Thus [BV11, Theorem 2.2] implies that [BV11, equation (8.8)] holds. (To see this, observe that the complex $C_{E^{\text{cyc}}}$ of [BV11] identifies with $C_{\Sigma}^{\bullet}(E^{\text{cyc}}/F)[-3]$ by Artin-Verdier duality; the shift implies that $[C_{E^{\text{cyc}}}] = -[C_{\Sigma}^{\bullet}(E^{\text{cyc}}/F)]$. Moreover, in the notation of loc. cit., the exponent $-\langle \rho, 1 \rangle$ that occurs in [BV11, equation (8.8)] should in fact be $+\langle \rho, 1 \rangle$ because the factor $c_{\gamma}^{\langle \rho, 1 \rangle}$ that appears on the left of [BV06, (29)] should actually be on the right of that formula.) Since $T\Omega(\mathbb{Q}(1)_E, \mathbb{Z}[G])$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$ by part (i), we can deduce as in the proof of [BV11, Theorem 8.1] that its image in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ vanishes. \square

Remark 8.3. It seems plausible that the result of Theorem 8.1 (ii) can be deduced using only the Iwasawa main conjecture for totally real fields as proven by Wiles [Wil90], as opposed to the more general equivariant Iwasawa main conjecture.

Remark 8.4. Let L/K be a finite Galois extension of number fields and fix a prime p . In §7.3, we saw that to prove $\text{ETNC}^{\text{rat}}(L/K, 1)$, $\text{ETNC}^{\text{tors}}(L/K, 1)$ or $\text{ETNC}_p^{\text{tors}}(L/K, 1)$, it suffices to consider certain abelian sub-extensions of L/K by functoriality, but that the proof $\text{ETNC}_p(L/K, 1)$ cannot always be reduced to abelian sub-extensions in this way. In the context of Theorem 8.1, it is interesting to contrast this with the fact that the proof of the p -adic Stark conjecture at $s = 1$ for E/F a Galois extension of totally real number fields always reduces to certain abelian sub-extensions of E/F (see Remark 4.13).

Remark 8.5. Let p be an odd prime and let L/K be a finite Galois p -extension of number fields. Then $\mu_p(L) = 0$ if and only if $\mu_p(K) = 0$ by [NSW08, Theorem 11.3.8]. Hence the hypothesis that $\mu_p(E) = 0$ in Theorem 8.1 (iii) can be weakened to “there exists a subfield E' of E such that E/E' is a Galois extension of p -power degree and $\mu_p(E') = 0$ ”. Moreover, since $\mu_p(E) = 0$ vanishes whenever E/\mathbb{Q} is abelian (see §7.4) we deduce that $\mu_p(E) = 0$ whenever E is a Galois p -extension of an abelian extension E'/\mathbb{Q} . The same remarks apply to Theorem 10.1 and Corollaries 10.3 and 10.4 below.

Remark 8.6. There are two reasons for the $\mu_p(E) = 0$ hypothesis in Theorem 8.1. First, it ensures the validity of the EIMC by Theorem 7.4. Using the theory of ‘hybrid Iwasawa algebras’, the present authors [JN] have proven the EIMC unconditionally in certain cases when it is not known that $\mu_p(E) = 0$. Unfortunately, it is not possible to use this in the present context because the second reason for the $\mu_p(E) = 0$ hypothesis in Theorem 8.1 is that it is required for the descent theory of Burns and Venjakob [BV11] when \mathcal{G} has an element of order p (i.e. when p divides $[E^{\text{cyc}} : F^{\text{cyc}}]$). However, it is still possible to weaken the $\mu_p(E) = 0$ hypothesis in certain situations by using the theory of p -adic hybrid group rings [JN16] at the finite level as illustrated by Example 8.11 below.

Corollary 8.7. *Let E/F be a finite Galois extension of totally real number fields and suppose that E/\mathbb{Q} is abelian. Then $\text{ETNC}^{\text{rat}}(E/F, 1)$ holds and $\text{ETNC}_p(E/F, 1)$ holds for every odd prime p .*

Proof. Let p be a prime. The p -adic Stark conjecture at $s = 1$ holds by Theorem 6.1 and a theorem of Ferrero and Washington [FW79] implies that $\mu_p(E) = 0$ (see §7.4). Therefore $\text{ETNC}^{\text{rat}}(E/F, 1)$ and $\text{ETNC}_p(E/F, 1)$ for p odd follow from Theorem 8.1. \square

Remark 8.8. The result of Corollary 8.7 (including the case $p = 2$) is certainly well-known, but our approach provides a new proof. The method of Burns and Flach [BF06] uses the validity of $\text{ETNC}(E/F, 0)$ (as proven outside the 2-part by Burns and Greither [BG03] and at $p = 2$ by Flach [Fla11]) and shows compatibility with the functional equation. One advantage of our approach is that we completely avoid the technically difficult issue of ‘trivial zeroes’ of p -adic L -functions. In some respects, our approach is closer to that of Huber and Kings [HK03], in which they prove the Bloch-Kato conjecture for Dirichlet characters; this implies (among other results) $\text{ETNC}_p^{\text{tors}}(E/F, 1)$ for odd primes p (of course, this is somewhat weaker than $\text{ETNC}_p(E/F, 1)$). They formulate a variant of the main conjecture and then descend at $s = 1$; however, they do not prove or refer to the p -adic Stark conjecture at $s = 1$.

Corollary 8.9. *Let E/F be a finite Galois extension of totally real number fields with Galois group G such that $\text{Perm}(G) = R_{\mathbb{C}}(G)$. Fix a prime p and suppose that $\text{Leo}(E, p)$ holds. Then conclusions (i), (ii) and (iii) of Theorem 8.1 hold.*

Proof. This is the combination of Corollary 5.4 and Theorem 8.1. \square

Remark 8.10. The collection of finite groups G such that $\text{Perm}(G) = R_{\mathbb{C}}(G)$ is closed under direct products and includes the symmetric groups, the hyperoctahedral groups (which include the dihedral group of order 8), and many others besides. See [Sol74] and [Kle84] for more on this topic.

Example 8.11. Let E be a totally real Galois extension of \mathbb{Q} with $\text{Gal}(E/\mathbb{Q}) \cong S_4$ and assume that $\text{Leo}(E, 3)$ holds. Then $\text{ETNC}^{\text{rat}}(E/\mathbb{Q}, 1)$ and $\text{ETNC}_3^{\text{tors}}(E/\mathbb{Q}, 1)$ hold by Corollary 8.9 (i) & (ii), respectively. Let F be the subfield of E fixed by V_4 , the subgroup of S_4 generated by the double transpositions. Then $\text{Gal}(F/\mathbb{Q}) \cong S_4/V_4 \cong S_3$. Since F is a cubic Galois extension of a quadratic extension of \mathbb{Q} , we have $\mu_3(F) = 0$ by Remark 8.5. Thus $\text{ETNC}_3(F/\mathbb{Q}, 1)$ holds by Corollary 8.9 (iii). Moreover, the group ring $\mathbb{Z}_3[S_4]$ is ‘ V_4 -hybrid’ by [JN16, Example 2.18] and so the map

$$q_{S_4/V_4}^{S_4} : K_0(\mathbb{Z}_3[S_4], \mathbb{Q}_3)_{\text{tors}} \longrightarrow K_0(\mathbb{Z}_3[S_4/V_4], \mathbb{Q}_3)_{\text{tors}}$$

is injective by [JN16, Proposition 3.8]. Therefore $\text{ETNC}_3(E/\mathbb{Q}, 1)$ holds by the functoriality properties of the ETNC with respect to quotient maps. Note that this result

cannot be deduced directly from Corollary 8.9 (iii) without additionally assuming that $\mu_3(E) = 0$, which illustrates the point made at the end of Remark 8.6.

9. THE LEADING TERM CONJECTURES AT $s = 0$ AND $s = 1$

9.1. Overview of the leading term conjectures. We give a brief overview of the leading term conjectures (LTCs) at $s = 0$ and $s = 1$ formulated by Breuning and Burns; we refer the reader to their article [BB07] and the references therein for further details.

Let L/K be a finite Galois extension of number fields with Galois group G . For $r \in \{0, 1\}$ one can define elements $T\Omega(L/K, r) \in K_0(\mathbb{Z}[G], \mathbb{R})$ that relate leading terms of equivariant Artin L -functions at $s = r$ to certain arithmetic complexes. The leading term conjecture at $s = r$ is the assertion that $T\Omega(L/K, r)$ vanishes. In analogy with §7.2, we define the following notation.

- $\text{LTC}(L/K, r)$ means $T\Omega(L/K, r) = 0$.
- $\text{LTC}^{\text{tors}}(L/K, r)$ means $T\Omega(L/K, r) \in K_0(\mathbb{Z}[G], \mathbb{Q})_{\text{tors}}$.
- $\text{LTC}^{\text{rat}}(L/K, r)$ means $T\Omega(L/K, r) \in K_0(\mathbb{Z}[G], \mathbb{Q})$.

Thus if $\text{LTC}^{\text{rat}}(L/K, r)$ holds then by (7.2) we have elements $T\Omega(L/K, r)_p$ in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ for each prime p . In this situation, we define the following notation.

- $\text{LTC}_p(L/K, r)$ means $T\Omega(L/K, r)_p = 0$.
- $\text{LTC}_p^{\text{tors}}(L/K, r)$ means $T\Omega(L/K, r)_p \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$.
- $\text{LTC}_{\text{odd}}(L/K, r)$ means $\text{LTC}_p(L/K, r)$ holds for all odd primes p .

The LTCs have the same functoriality properties as the ETNC and so the reduction steps for the ETNC described in §7.3 also apply to the LTCs. A brief discussion of the relation of the LTCs to other conjectures is given in §1; also see [BB07, Propositions 3.6 and 4.4]. In particular, $\text{LTC}(L/K, 0)$ implies Chinburg's ‘ $\Omega(3)$ -conjecture’ [Chi83, Chi85] and $\text{LTC}(L/K, 1)$ implies Chinburg's ‘ $\Omega(1)$ -conjecture’ [Chi85].

We note two important known cases of the LTCs.

Theorem 9.1. *Let L/K be a finite Galois extension of number fields with Galois group G and let $r \in \{0, 1\}$.*

- (i) *If L/\mathbb{Q} is abelian then $\text{LTC}(L/K, r)$ holds.*
- (ii) *If $\text{Char}_{\mathbb{Q}}(G) = R_{\mathbb{C}}(G)$ then $\text{LTC}^{\text{tors}}(L/K, r)$ holds.*

Proof. The first claim is [BB10, Corollary 1.3], which crucially depends on the results of [BG03], [BF06], and [Fla11]. The second claim is well-known to experts; we give a proof here for the convenience of the reader. The strong Stark conjecture (as formulated by Chinburg [Chi83, Conjecture 2.2]) for L/K and all $\chi \in R_{\mathbb{C}}^+(G)$ is known to be equivalent to $\text{LTC}^{\text{tors}}(L/K, 0)$ by [BB07, Proposition 4.4 (ii)]. However, the strong Stark conjecture holds for rational valued characters by [Tat84, Chapitre II, Théorème 6.8]. Thus $\text{LTC}^{\text{tors}}(L/K, 0)$ holds and so does $\text{LTC}^{\text{tors}}(L/K, 1)$ by Corollary 9.6 below. \square

9.2. The epsilon constant conjectures. The global epsilon constant conjecture formulated by Bley and Burns [BB03] is a natural conjecture for global epsilon constants arising from the compatibility of the leading term conjectures at $s = 0$ and $s = 1$ with respect to the functional equation of the equivariant Artin L -function. Here we consider an equivalent formulation of Breuning and Burns [BB07, §5]. An element $T\Omega^{\text{loc}}(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{R})$ is defined and the assertion of the conjecture is that this element vanishes. Moreover, it is known [BB07, Theorem 5.2] that

$$(9.1) \quad \psi_G^*(T\Omega(L/K, 0)) - T\Omega(L/K, 1) = T\Omega^{\text{loc}}(L/K, 1),$$

where ψ_G^* is a certain involution of $K_0(\mathbb{Z}[G], \mathbb{R})$ (see [BB07, §2.1.4]). Thus if $T\Omega^{\text{loc}}(L/K, 1)$ vanishes then $\text{LTC}(L/K, 0)$ holds if and only if $\text{LTC}(L/K, 1)$ holds. Furthermore, as shown in [BB03, Remark 4.2(iv)], the vanishing of $T\Omega^{\text{loc}}(L/K, 1)$ also implies Chinburg's ‘ $\Omega(2)$ -conjecture’ as formulated in [Chi85, Question 3.1].

It is known that we always have $T\Omega^{\text{loc}}(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ and thus (7.2) defines p -parts $T\Omega^{\text{loc}}(L/K, 1)_p \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ for every prime p . Breuning [Bre04a] has refined this further by formulating an independent conjecture for each finite Galois extension of p -adic fields N/M . He defined an element $R_{N/M} \in K_0(\mathbb{Z}_p[\text{Gal}(N/M)], \mathbb{Q}_p)$ incorporating local epsilon constants and conjectured that $R_{N/M}$ always vanishes. Moreover, this local conjecture is related to the global epsilon constant conjecture by the equation

$$(9.2) \quad T\Omega^{\text{loc}}(L/K, 1)_p = \sum_{v \in \Sigma_p(K)} \text{ind}_{G_w}^G(R_{L_w/K_v}),$$

where w is a fixed place of L above v , G_w denotes the decomposition group and $\text{ind}_{G_w}^G$ is the induction map defined between relative algebraic K -groups (see [Bre04a, Theorem 4.1]). In particular, for a fixed prime p , the validity of the local conjecture for all non-archimedean completions L_w/K_v with $v \in \Sigma_p(K)$ implies that $T\Omega^{\text{loc}}(L/K, 1)_p = 0$.

Theorem 9.2. *Let L/K be a finite Galois extension of number fields with Galois group G . Let p be a prime. Then $T\Omega^{\text{loc}}(L/K, 1)_p \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$. Moreover, we have that $T\Omega^{\text{loc}}(L/K, 1)_p = 0$ if for every $v \in \Sigma_p(K)$ there is some $w \in \Sigma_p(L)$ such that $w \mid v$ and at least one of the following holds:*

- (i) w is at most tamely ramified in L/K ;
- (ii) p is odd and L_w/\mathbb{Q}_p is abelian;
- (iii) p is odd and $[L_w : \mathbb{Q}_p] \leq 15$;
- (iv) p is odd, L_w/K_v is abelian and weakly ramified with cyclic ramification group, K_v/\mathbb{Q}_p is unramified and $[K_v : \mathbb{Q}_p]$ is coprime to the inertia degree of L_w/K_v .

Proof. By (9.2), each claim follows from the analogous claim for the local conjecture. The first claim follows from [Bre04a, Corollary 3.8]. Cases (i) and (ii) follow from [Bre04a, Theorem 3.6, Proposition 4.4]. Case (iii) follows from [BD13, Theorem 1(a)] and case (iv) follows from [BC16, Theorem 1]. \square

Theorem 9.3. *Let m be a positive integer and let $G = (C_3)^m \rtimes C_2$ where C_2 acts on $(C_3)^m$ by inversion (in the case $m = 1$ we have $G \cong S_3$). Let L/K be a Galois extension of number fields with $\text{Gal}(L/K) \cong G$ such that 3 splits completely in K/\mathbb{Q} . Then the global epsilon constant conjecture holds for L/K , that is, $T\Omega^{\text{loc}}(L/K, 1) = 0$.*

Remark 9.4. Breuning [Bre04b] was the first to show that the global epsilon constant conjecture holds for all S_3 -extensions of \mathbb{Q} .

Proof of Theorem 9.3. Note that $T\Omega^{\text{loc}}(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})_{\text{tors}}$ by Theorem 9.2. Moreover, $K_0(\mathbb{Z}_2[G], \mathbb{Q}_2)_{\text{tors}}$ is trivial by [JN16, Lemma 3.10]. Hence it suffices to show that $T\Omega^{\text{loc}}(L/K, 1)_3 = 0$ in $K_0(\mathbb{Z}_3[G], \mathbb{Q}_3)_{\text{tors}}$. Let w be a fixed place of L above 3. Then by (9.2) and the hypothesis that 3 splits completely in K/\mathbb{Q} , we are reduced to showing that $R_{L_w/\mathbb{Q}_3} = 0$ in $K_0(\mathbb{Z}_3[H], \mathbb{Q}_3)_{\text{tors}}$ where $H = \text{Gal}(L_w/\mathbb{Q}_3)$. If H is abelian or $H \cong S_3$ then we are done by Theorem 9.2 (ii) or (iii), respectively. Otherwise, $H \cong (C_3)^n \rtimes C_2$ for some $n \geq 2$. If $n \geq 3$ then H has $(3^n - 1)/2 \geq 13$ quotients isomorphic to S_3 (one for each of the quotients of $(C_3)^n$ of order 3). However, by [Bre04b, Proposition 4.3] there are only 6 Galois extensions M/\mathbb{Q}_3 with $\text{Gal}(M/\mathbb{Q}_3) \cong S_3$ (also see [JR06]). Therefore we are now reduced to the case $H = (C_3)^2 \rtimes C_2$. From the database of local

fields of Jones and Roberts [JR04, JR06], we see that there is precisely one Galois extension M/\mathbb{Q}_3 with $\text{Gal}(M/\mathbb{Q}_3) \cong (C_3)^2 \rtimes C_2$, namely the Galois closure of the extension with generating polynomial $x^9 + 3x^3 + 9x^2 + 9x + 3$. From the number fields database of Klüners and Malle [KM01], we see that the number field with generating polynomial $x^{18} + 45x^{12} + 27x^6 + 27$ is a global representative of M of minimal degree. Applying the algorithm of Bley and Debeerst [BD13] shows that $R_{M/\mathbb{Q}_3} = 0$. (The algorithm is implemented in Magma [BCP97] with source code bundled in Debeerst's PhD thesis [Deb11].) \square

9.3. A common notion of rationality. We note that all six notions of ‘rationality’ encountered in this article are in fact equivalent.

Proposition / Definition 9.5. *Let L/K be a finite Galois extension of number fields with Galois group G . Then the following are equivalent:*

- (i) $\text{LTC}^{\text{rat}}(L/K, 0)$; (ii) $\text{LTC}^{\text{rat}}(L/K, 1)$;
- (iii) $\text{ETNC}^{\text{rat}}(L/K, 0)$; (iv) $\text{ETNC}^{\text{rat}}(L/K, 1)$;
- (v) *Stark's conjecture at $s = 0$ for every $\chi \in R_{\mathbb{C}}^+(G)$;*
- (vi) *Stark's conjecture at $s = 1$ for every $\chi \in R_{\mathbb{C}}^+(G)$.*

We denote these equivalent conditions by $\text{Rat}(L/K)$.

Proof. As $T\Omega^{\text{loc}}(L/K, 1)$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$, the equivalence of (i) and (ii) follows from (9.1). Items (iii)-(vi) are equivalent by Proposition 7.1. Finally, [BB07, Proposition 3.6] says that (ii) and (vi) are equivalent. \square

Corollary 9.6. *Let L/K be a finite Galois extension of number fields with Galois group G . Suppose that $\text{Rat}(L/K)$ holds and let p be a prime.*

- (i) $\text{LTC}_p^{\text{tors}}(L/K, 0)$ holds if and only if $\text{LTC}_p^{\text{tors}}(L/K, 1)$ holds.
- (ii) *If $p \nmid |G|$ then the following are equivalent:*
 $\text{LTC}_p^{\text{tors}}(L/K, 0)$, $\text{LTC}_p^{\text{tors}}(L/K, 1)$, $\text{LTC}_p(L/K, 0)$ and $\text{LTC}_p(L/K, 1)$.

Proof. By Proposition / Definition 9.5, $T\Omega(L/K, r) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ for $r \in \{0, 1\}$. Recall that we always have $T\Omega^{\text{loc}}(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$. Thus we obtain claim (i) by projecting (9.1) from $K_0(\mathbb{Z}[G], \mathbb{Q})$ to $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ and using that $T\Omega^{\text{loc}}(L/K, 1)_p \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$. Now assume $p \nmid |G|$. Then $\mathbb{Z}_p[G]$ is a maximal \mathbb{Z}_p -order and so $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)_{\text{tors}}$ is trivial (see [BW09, Theorem 2.4 (ii)], for instance). Thus if $r \in \{0, 1\}$ then $\text{LTC}_p^{\text{tors}}(L/K, r)$ is equivalent to $\text{LTC}_p(L/K, r)$, giving claim (ii). \square

9.4. The relation between the leading term conjectures and the ETNC.

Proposition 9.7. *Let L/K be a finite Galois extension of number fields and let p be a prime. Suppose that $\text{Rat}(L/K)$ holds. Then*

- (i) $\text{LTC}_p^{\text{tors}}(L/K, 0)$ holds if and only if $\text{ETNC}_p^{\text{tors}}(L/K, 0)$ holds;
- (ii) $\text{LTC}_p(L/K, 0)$ holds if and only if $\text{ETNC}_p(L/K, 0)$ holds.

Suppose further that () there is a totally complex Galois extension M of \mathbb{Q} such that $L \subset M$ and $\text{Leo}(M, p)$ holds. Then*

- (iii) $\text{LTC}_p^{\text{tors}}(L/K, 1)$ holds if and only if $\text{ETNC}_p^{\text{tors}}(L/K, 1)$ holds;
- (iv) $\text{LTC}_p(L/K, 1)$ holds if and only if $\text{ETNC}_p(L/K, 1)$ holds.

Proof. By [BF03, (29)] the elements $T\Omega(\mathbb{Q}(0)_L, \mathbb{Z}[\text{Gal}(L/K)])$ and $T\Omega(L/K, 0)$ are equal up to an involution of $K_0(\mathbb{Z}[\text{Gal}(L/K)], \mathbb{R})$. Hence we obtain claims (i) and (ii).

Let $G = \text{Gal}(M/\mathbb{Q})$, let $H = \text{Gal}(M/K)$, and let $Q = \text{Gal}(L/K)$. Fix an embedding of fields $j : \mathbb{R} \rightarrow \mathbb{C}_p$. There is a commutative diagram

$$\begin{array}{ccc} K_0(\mathbb{Z}[G], \mathbb{R}) & \xrightarrow{j_*^G} & K_0(\mathbb{Z}_p[G], \mathbb{C}_p) \\ \text{q}_Q^H \circ \text{res}_H^G \downarrow & & \downarrow \text{q}_Q^H \circ \text{res}_H^G \\ K_0(\mathbb{Z}[Q], \mathbb{R}) & \xrightarrow{j_*^Q} & K_0(\mathbb{Z}_p[Q], \mathbb{C}_p) \end{array}$$

where the horizontal maps are induced by j . Moreover, under the assumption $(*)$, the proof of [BB10, Proposition 5.1] shows that $j_*^G(T\Omega(\mathbb{Q}(1)_M, \mathbb{Z}[G])) = j_*^G(T\Omega(M/\mathbb{Q}, 1))$. Therefore by the functoriality properties of the conjectures with respect to restriction and quotient maps, we have

$$\begin{aligned} j_*^Q(T\Omega(L/K, 1)) &= j_*^Q(\text{q}_Q^H \circ \text{res}_H^G(T\Omega(M/\mathbb{Q}, 1))) \\ &= \text{q}_Q^H \circ \text{res}_H^G(j_*^G(T\Omega(M/\mathbb{Q}, 1))) \\ &= \text{q}_Q^H \circ \text{res}_H^G(j_*^G(T\Omega(\mathbb{Q}(1)_M, \mathbb{Z}[G]))) \\ &= j_*^Q(\text{q}_Q^H \circ \text{res}_H^G(T\Omega(\mathbb{Q}(1)_M, \mathbb{Z}[G]))) \\ &= j_*^Q(T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[Q])). \end{aligned}$$

The assumption that $\text{Rat}(L/K)$ holds means that both $T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[Q])$ and $T\Omega(L/K, 1)$ belong to $K_0(\mathbb{Z}[Q], \mathbb{Q})$. Furthermore, the map j_*^Q restricts to the canonical projection $K_0(\mathbb{Z}[Q], \mathbb{Q}) \rightarrow K_0(\mathbb{Z}_p[Q], \mathbb{Q}_p)$ and so we conclude that $T\Omega(\mathbb{Q}(1)_L, \mathbb{Z}[Q])_p = T\Omega(L/K, 1)_p$ in $K_0(\mathbb{Z}_p[Q], \mathbb{Q}_p)$. Hence we obtain claims (iii) and (iv). \square

Remark 9.8. [BB10, Proposition 5.1] says that if M/\mathbb{Q} is a finite totally complex Galois extension such that $\text{Leo}(M, p)$ holds for *all* primes p then $\text{LTC}(M/\mathbb{Q}, 1)$ and $\text{ETNC}(M/\mathbb{Q}, 1)$ are equivalent (no rationality assumption is needed). By contrast, Proposition 9.7 (iii) & (iv) are useful in proving ‘prime-by-prime’ results in cases where $\text{Rat}(L/K)$ is known.

Remark 9.9. Let E/\mathbb{Q} be finite totally real Galois extension and let M be a totally complex quadratic extension of E . Then $\text{Leo}(M, p)$ holds if and only if $\text{Leo}(E, p)$ holds by [NSW08, Corollary 10.3.11]. Moreover, it is straightforward to see that M can be chosen such that M/\mathbb{Q} is Galois. Therefore if $L = E$ is totally real and $K = \mathbb{Q}$ in Proposition 9.7 then one only needs to assume that $\text{Leo}(E, p)$ holds in order to ensure that $(*)$ is satisfied. It seems plausible that $(*)$ could always be weakened to assuming $\text{Leo}(L, p)$, but this would require a careful generalisation of the proofs of [BB10], which the present authors have not done.

10. NEW EVIDENCE FOR THE LEADING TERM CONJECTURES

10.1. Groups G with the property that $\text{Perm}(G) = R_{\mathbb{C}}(G)$.

Theorem 10.1. *Let E/F be a finite Galois extension of totally real number fields with Galois group G such that $\text{Perm}(G) = R_{\mathbb{C}}(G)$ and E/\mathbb{Q} is Galois. Let p be an odd prime such that $\text{Leo}(E, p)$ holds and $\mu_p(E) = 0$.*

- (i) $\text{LTC}^{\text{tors}}(E/F, 0)$, $\text{LTC}^{\text{tors}}(E/F, 1)$ and $\text{LTC}_p(E/F, 1)$ all hold.
- (ii) If p satisfies any of the conditions of Theorem 9.2 then $\text{LTC}_p(E/F, 0)$ also holds.

Remark 10.2. Recall that the condition $\text{Perm}(G) = R_{\mathbb{C}}(G)$ was discussed in Remark 8.10. Note that Theorem 10.1 is only really interesting in the case that p divides $|G|$ (if $p \nmid |G|$

then, as is well-known to experts, Theorem 9.1 (ii) combined with Corollary 9.6 (ii) gives a more general and unconditional result).

Proof of Theorem 10.1. Since $\text{Perm}(G) = R_{\mathbb{C}}(G)$ implies that $\text{Char}_{\mathbb{Q}}(G) = R_{\mathbb{C}}(G)$, both $\text{LTC}^{\text{tors}}(E/F, 0)$ and $\text{LTC}^{\text{tors}}(E/F, 1)$ hold by Theorem 9.1 (ii). Moreover, $\text{ETNC}_p(E/F, 1)$ holds by Corollary 8.9 (iii). Hence Proposition 9.7 (iv) and Remark 9.9 together show that $\text{LTC}_p(E/F, 1)$ holds. The final claim now follows from Theorem 9.2. \square

Corollary 10.3. *Let m be a positive integer and let $G = (C_3)^m \rtimes C_2$ where C_2 acts on $(C_3)^m$ by inversion (in the case $m = 1$ we have $G \cong S_3$). Let E/F be a Galois extension of totally real number fields with $\text{Gal}(E/F) \cong G$ and such that E/\mathbb{Q} is Galois. If $\text{Leo}(E, 3)$ holds and $\mu_3(E) = 0$ then $\text{LTC}(E/F, 1)$ holds; if we further suppose either that 3 satisfies any of the conditions of Theorem 9.2 or that 3 splits completely in F/\mathbb{Q} , then $\text{LTC}(E/F, 0)$ also holds.*

Proof. First note that every character in $\text{Irr}_{\mathbb{C}}(G)$ is lifted from either a linear character of $G/N \cong C_2$ or an irreducible degree 2 character of a copy of $C_3 \rtimes C_2 \cong S_3$ (one for each of the $(3^m - 1)/2$ quotients of $(C_3)^m$ of order 3). Since $\text{Perm}(C_2) = \text{Irr}_{\mathbb{C}}(C_2)$ and $\text{Perm}(S_3) = \text{Irr}_{\mathbb{C}}(S_3)$, this shows that $\text{Perm}(G) = \text{Irr}_{\mathbb{C}}(G)$. Thus we can apply Theorem 10.1 with $p = 3$. If 3 does not split completely in F/\mathbb{Q} , the desired result now follows from Corollary 9.6 (ii) and the fact that $K_0(\mathbb{Z}_2[G], \mathbb{Q}_2)_{\text{tors}}$ is trivial (see [JN16, Lemma 3.10]); if 3 splits completely in F/\mathbb{Q} we also use Theorem 9.3. \square

Corollary 10.4. *Let G be a finite group. There exist infinitely many Galois extensions of totally real number fields E/F with $\text{Gal}(E/F) \cong G$ such that, if $\text{Leo}(E, p)$ holds and $\mu_p(E) = 0$ for all odd prime divisors p of $|G|$, then for $r \in \{0, 1\}$ both $\text{LTC}^{\text{tors}}(E/F, r)$ and $\text{LTC}_{\text{odd}}(E/F, r)$ hold (in fact, if $|G|$ is odd then $\text{LTC}(E/F, r)$ holds).*

Proof. By Cayley's theorem, there exists a positive integer n such that G embeds into S_n , the symmetric group of degree n . Moreover, $\text{Perm}(S_n) = R_{\mathbb{C}}(S_n)$ and there are infinitely many totally real and tamely ramified Galois extensions E/\mathbb{Q} with $\text{Gal}(E/\mathbb{Q}) \cong S_n$ (see [KM01, Proposition 2], for example). Fix such an extension E/\mathbb{Q} . By Theorem 10.1, $\text{LTC}^{\text{tors}}(E/\mathbb{Q}, 0)$ and $\text{LTC}^{\text{tors}}(E/\mathbb{Q}, 1)$ both hold and $\text{LTC}_p(E/\mathbb{Q}, 1)$ holds for all odd prime divisors p of $|G|$; moreover, since E/\mathbb{Q} is tamely ramified, $\text{LTC}_p(E/\mathbb{Q}, 0)$ also holds for all such primes. Thus $\text{LTC}_{\text{odd}}(E/\mathbb{Q}, 0)$ and $\text{LTC}_{\text{odd}}(E/\mathbb{Q}, 1)$ both hold by Corollary 9.6 (ii). By construction, there is a sub-extension E/F with $\text{Gal}(E/F) \cong G$ and the functoriality properties of the LTCs with respect to restriction show that for $r \in \{0, 1\}$ both $\text{LTC}^{\text{tors}}(E/F, r)$ and $\text{LTC}_{\text{odd}}(E/F, r)$ hold. If $|G|$ is odd, then $\text{LTC}_2(E/F, r)$ also holds by Corollary 9.6 (ii). \square

10.2. The group of affine transformations. Let q be a prime power and let \mathbb{F}_q be the finite field with q elements. The group $\text{Aff}(q)$ of affine transformations on \mathbb{F}_q is the group of transformations of the form $x \mapsto ax + b$ with $a \in \mathbb{F}_q^{\times}$ and $b \in \mathbb{F}_q$. Thus $\text{Aff}(q)$ is isomorphic to the semidirect product $\mathbb{F}_q \rtimes \mathbb{F}_q^{\times}$ with the natural action. Note that in particular $\text{Aff}(3) \cong S_3$ and $\text{Aff}(4) \cong A_4$.

Theorem 10.5. *Let E/F be a finite Galois extension of totally real number fields with Galois group $G \cong \text{Aff}(q)$ for some prime power q . Let N denote the commutator subgroup of G and suppose that E^N/\mathbb{Q} is abelian (in particular, this is the case when $F = \mathbb{Q}$.) Let p be a prime and suppose that $\text{Leo}(E, p)$ holds. Then the p -adic Stark conjecture at $s = 1$ holds for every $\rho \in R_{\mathbb{C}_p}(G)$.*

Proof. The result for linear characters in $R_{\mathbb{C}_p}(G)$ follows from Theorem 6.1. We now identify G with $\text{Aff}(q)$ and note that G is a Frobenius group with Frobenius kernel $N = \{x \mapsto x + b \mid b \in \mathbb{F}_q\}$ and Frobenius complement $H = \{x \mapsto ax \mid a \in \mathbb{F}_q^\times\}$ (see [CR81, §14A] for background on Frobenius groups). Let $\psi \in \text{Irr}_{\mathbb{C}_p}(N)$ with $\psi \neq 1_N$. Then the induced character $\tau := \text{ind}_N^G \psi$ is of degree $|G/N| = q - 1$ and [CR81, (14.4)] shows that $\tau \in \text{Irr}_{\mathbb{C}_p}(G)$. Since there are $q - 1$ linear characters in $\text{Irr}_{\mathbb{C}_p}(G)$ and $(q - 1) + (q - 1)^2 = q(q - 1) = |G|$, we conclude that τ is in fact the unique non-linear character in $\text{Irr}_{\mathbb{C}_p}(G)$. By Frobenius reciprocity ([CR81, (10.9)]), Mackey's subgroup theorem ([CR81, (10.13)]), and the fact that N is abelian, we have

$$\langle \tau, \text{ind}_H^G 1_H \rangle_G = \langle \psi, \text{res}_N^G \text{ind}_H^G 1_H \rangle_N = \langle \psi, \text{ind}_{\{e\}}^N 1_{\{e\}} \rangle_N = 1.$$

Hence τ can be expressed as \mathbb{Z} -linear combination of $\text{ind}_H^G 1_H$ and linear characters in $\text{Irr}_{\mathbb{C}_p}(G)$. We have already shown that the p -adic Stark conjecture at $s = 1$ holds for all linear characters in $\text{Irr}_{\mathbb{C}_p}(G)$; by Corollary 5.2, it also holds for $\text{ind}_H^G 1_H$ since $\text{Leo}(E^H, p)$ holds. Thus the p -adic Stark conjecture at $s = 1$ holds for τ and therefore for all $\rho \in R_{\mathbb{C}_p}(G)$ by Remark 4.12. \square

Corollary 10.6. *Let E/\mathbb{Q} be a totally real Galois extension with Galois group $G \cong \text{Aff}(p^m)$ for some odd prime p and some positive integer m . If $\text{Leo}(E, p)$ holds then $\text{LTC}(E/\mathbb{Q}, 1)$ holds; if we further suppose that p satisfies any of the conditions of Theorem 9.2, then $\text{LTC}(E/\mathbb{Q}, 0)$ also holds.*

Proof. Initially, we do not assume that E is totally real, that p is odd or that $\text{Leo}(E, p)$ holds. Let $\ell \neq p$ be a prime. We observe that $\text{ETNC}^{\text{tors}}(E/\mathbb{Q}, 0)$ and $\text{ETNC}_\ell(E/\mathbb{Q}, 0)$ both hold by [JN16, Theorem 4.6]. Then Proposition 9.7 (ii) implies that $\text{LTC}^{\text{tors}}(E/\mathbb{Q}, 0)$ and $\text{LTC}_\ell(E/\mathbb{Q}, 0)$ hold as well. The validity of $\text{LTC}^{\text{tors}}(E/\mathbb{Q}, 1)$ follows from Corollary 9.6 (i). As the group ring $\mathbb{Z}_\ell[G]$ is N -hybrid in the sense of [JN16, Definition 2.5] by [JN16, Example 2.16], we conclude that $\text{LTC}_\ell(E/\mathbb{Q}, 1)$ holds by [JN16, Theorem 5.12].

Now assume that p is odd, E is totally real and that $\text{Leo}(E, p)$ holds. By Theorem 10.5 the p -adic Stark conjecture at $s = 1$ holds for every $\rho \in R_{\mathbb{C}_p}(G)$. Moreover, $\mu_p(E) = 0$ by Remark 8.5. Thus $\text{ETNC}_p(E/\mathbb{Q}, 1)$ holds by Theorem 8.1 (iii). Hence Proposition 9.7 (iv) and Remark 9.9 together show that $\text{LTC}_p(E/F, 1)$ holds. The claim final claim now follows from Theorem 9.2. \square

Remark 10.7. The first paragraph of the proof of Corollary 10.6 shows unconditionally that for any Galois extension E/\mathbb{Q} with $\text{Gal}(E/\mathbb{Q}) \cong \text{Aff}(p^m)$ for some prime p and some $m \geq 1$, $\text{LTC}^{\text{tors}}(E/\mathbb{Q}, 1)$ holds and $\text{LTC}_\ell(E/\mathbb{Q}, 1)$ holds for all primes $\ell \neq p$. Moreover, Corollary 10.6 can be generalised to the case of extensions E/F as described in the statement of Theorem 10.5, subject to the further hypothesis that E/\mathbb{Q} is Galois.

10.3. Further specific Galois extensions.

Theorem 10.8. *Let m be a positive integer and let $G = (C_3)^m \rtimes C_2$ where C_2 acts on $(C_3)^m$ by inversion (in the case $m = 1$ we have $G \cong S_3$) or let $G = D_{12}$ (the dihedral group of order 12). Let E/\mathbb{Q} be a totally real Galois extension with $\text{Gal}(E/\mathbb{Q}) \cong G$. If $\text{Leo}(E, 3)$ holds then both $\text{LTC}(E/\mathbb{Q}, 0)$ and $\text{LTC}(E/\mathbb{Q}, 1)$ hold.*

Remark 10.9. For a particular extension E/\mathbb{Q} satisfying the hypotheses of Theorem 10.8 it is straightforward to verify $\text{Leo}(E, 3)$ using either the algorithm of Buchmann and Sands [BS87] or by checking that the 3-adic regulator is non-zero using a computational algebra system such as Magma [BCP97]. In this particular setting, this is a simpler way

of verifying $\text{LTC}(E/\mathbb{Q}, 0)$ and $\text{LTC}(E/\mathbb{Q}, 1)$ than the algorithms of Janssen [Jan10] and Debeerst [Deb11], respectively. Similar remarks also apply to Corollary 10.6.

Proof of Theorem 10.8. In each case, E is a Galois 3-extension of a quadratic or bi-quadratic extension of \mathbb{Q} and so $\mu_3(E) = 0$ by Remark 8.5. If $G = (C_3)^m \rtimes C_2$ then $\text{LTC}(E/\mathbb{Q}, 0)$ and $\text{LTC}(E/\mathbb{Q}, 1)$ both hold by Corollary 10.3. Now suppose that $G = D_{12}$. Since $D_{12} \cong S_3 \times C_2$ we see that $\text{Perm}(D_{12}) = R_{\mathbb{C}}(D_{12})$ and so $\text{LTC}^{\text{tors}}(E/\mathbb{Q}, 0)$, $\text{LTC}^{\text{tors}}(E/\mathbb{Q}, 1)$ and $\text{LTC}_3(E/\mathbb{Q}, 1)$ all hold by Theorem 10.1 (i). Moreover, by [JN16, Example 3.13] the group ring $\mathbb{Z}_2[D_{12}]$ is weakly N -hybrid where N is the Sylow 3-subgroup of D_{12} , meaning that the map

$$(10.1) \quad \mathfrak{q}_{D_{12}/N}^{D_{12}} : K_0(\mathbb{Z}_2[D_{12}], \mathbb{Q}_2)_{\text{tors}} \longrightarrow K_0(\mathbb{Z}_2[D_{12}/N], \mathbb{Q}_2)_{\text{tors}}$$

is injective. Thus $\text{LTC}_2(E/\mathbb{Q}, 1)$ holds by Theorem 9.1 (i) and the functoriality properties of the LTCs with respect to quotients. Therefore $\text{LTC}(E/\mathbb{Q}, 1)$ holds by Corollary 9.6 (ii) and so $\text{LTC}(E/\mathbb{Q}, 0)$ also holds by Theorem 9.2 (iii). \square

Corollary 10.10. *Let L/\mathbb{Q} be a Galois extension with $\text{Gal}(L/\mathbb{Q}) \cong S_4$ or $S_4 \times C_2$. If $E := L^{V_4}$ is totally real and $\text{Leo}(E, 3)$ holds then $\text{LTC}_{\text{odd}}(L/\mathbb{Q}, 0)$ and $\text{LTC}_{\text{odd}}(L/\mathbb{Q}, 1)$ both hold.*

Proof. Let $G = S_4$ or $S_4 \times C_2$ and let $r \in \{0, 1\}$. Since $\text{Perm}(G) = R_{\mathbb{C}}(G)$, we have that $\text{LTC}^{\text{tors}}(L/\mathbb{Q}, r)$ holds by Theorem 9.1 (ii). Since G/V_4 is isomorphic to either S_3 or $S_3 \times C_2 \cong D_{12}$, Theorem 10.8 shows that $\text{LTC}(E/\mathbb{Q}, r)$ holds. Moreover, the group ring $\mathbb{Z}_3[S_4]$ is ‘ V_4 -hybrid’ by [JN16, Example 2.18] and so $\mathbb{Z}_3[S_4 \times C_2]$ is also V_4 -hybrid by [JN16, Lemma 2.9]. Hence

$$\mathfrak{q}_{G/V_4}^G : K_0(\mathbb{Z}_3[G], \mathbb{Q}_3)_{\text{tors}} \longrightarrow K_0(\mathbb{Z}_3[G/V_4], \mathbb{Q}_3)_{\text{tors}}$$

is injective by [JN16, Proposition 3.8]. Thus by the functoriality properties of the LTCs with respect to quotient maps, $\text{LTC}_3(L/\mathbb{Q}, r)$ also holds. Therefore $\text{LTC}_{\text{odd}}(L/\mathbb{Q}, r)$ holds by Corollary 9.6. \square

Remark 10.11. It is possible to have L totally complex and E totally real in Corollary 10.10 (consider the Galois closure of $\mathbf{x}^4 + \mathbf{x} + 2$). Moreover, it is interesting to compare Theorem 10.8 and Corollary 10.10 to [JN16, Theorem 4.18] and to Example 8.11.

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