The hair-trigger effect for a class of nonlocal nonlinear equations

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Abstract

We prove the hair-trigger effect for a class of nonlocal nonlinear evolution equations on $\mathbb{R}^d$ which have only two constant stationary solutions, 0 and $\theta > 0$. The effect consists in that the solution with an initial condition non identical to zero converges (when time goes to ∞) to θ locally uniformly in $\mathbb{R}^d$. We find also sufficient conditions for existence, uniqueness and comparison principle in the considered equations.

Keywords: hair-trigger effect, nonlocal diffusion, reaction-diffusion equation, front propagation, monostable equation, nonlocal nonlinearity, long-time behaviour, integral equation

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1 Introduction

The aim of the paper is to find sufficient conditions for the so-called 'hair-trigger' effect [2] in the following nonlinear nonlocal evolution equation on the Euclidean space $\mathbb{R}^d$, $d \geq 1$:

$$\frac{\partial u}{\partial t}(x,t) = \varkappa(a * u)(x,t) - mu(x,t) - u(x,t)(Gu)(x,t) \tag{1.1}$$

for $t > 0$, $x \in \mathbb{R}^d$, with an initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}^d$. Here $m, \varkappa > 0$, G is a mapping on a space of bounded on $\mathbb{R}^d$ functions, and $(a*u)(x,t)$ means the convolution (in x) between u and a nonnegative integrable probability kernel $a = a(x) \geq 0$ on $\mathbb{R}^d$; namely,

$$(a * u)(x,t) = \int_{\mathbb{R}^d} a(x-y)u(y,t)dy, \qquad \int_{\mathbb{R}^d} a(x)\, dx = 1. \tag{1.2}$$

We restrict ourselves to a monostable-type case, when (1.1) has two constant solutions $u \equiv 0$ and $u \equiv \theta > 0$ only. The hair-trigger effect means that, unless $u_0 \equiv 0$, the corresponding solution to (1.1) achieves an arbitrary chosen level between 0 and θ uniformly on an arbitrary chosen domain of $\mathbb{R}^d$ after a finite

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time. In other words, $u(x,t)$ converges, as $t \to \infty$, locally uniformly in $x \in \mathbb{R}^d$ to the positive stationary solution $u \equiv \theta$. The latter solution, therefore, is globally asymptotically stable in the sense of topology of local uniform convergence.

We interpret $u(x,t)$ as a density of a population at the point $x \in \mathbb{R}^d$ at the moment of time $t \geq 0$. The probability density $a(x)$ describes distribution of the birth of new individuals with constant intensity $\varkappa > 0$. Individuals in the population may also die either with the constant mortality rate $m > 0$ or because of the competition, described by the density dependent rate Gu, where G is an (in general, also nonlinear) operator on a space of bounded functions. The equation (1.1) can be also rewritten in a reaction-diffusion form, whose solution has another interpretation, see the discussion around (2.4) below.

The hair-trigger effect is an important tool in the study of the long-time behaviour of evolution equations. In particular, it allows to study the front propagation of the solutions to the equations [14, 16]. We prove the hair-trigger effect (Theorems 2.3–2.5), applying Weinberger's results for discrete dynamical systems [22]. We also prove the existence/uniqueness (Theorem 3.3) and the comparison principle (Theorem 4.1) for some generalizations of (1.1).

2 Assumptions and main results

We treat $u = u(x,t)$ as the local density of a system at the point $x \in \mathbb{R}^d$ and at the moment of time $t \in I$, where I is either a finite interval $[0,T]$, for some $T > 0$, or the whole $\mathbb{R}_+ := [0,\infty)$.

We assume that u_0 is a bounded function on $\mathbb{R}^d$. We will consider the following Banach spaces of real-valued functions on $\mathbb{R}^d$: the space $C_b(\mathbb{R}^d)$ of bounded continuous functions on $\mathbb{R}^d$ with sup-norm, the space $C_{ub}(\mathbb{R}^d)$ of bounded uniformly continuous functions on $\mathbb{R}^d$ with sup-norm, and the space $L^\infty(\mathbb{R}^d)$ of essentially bounded (with respect to the Lebesgue measure) functions on $\mathbb{R}^d$ with esssup-norm.

Let E be $C_{ub}(\mathbb{R}^d)$, $C_b(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$. We consider the space $C_b(I \to E)$ of continuous bounded functions on I (recall that I includes 0) with values in E equipped with the norm

$$\|u\|_{C_b(I\to E)} = \sup_{t\in I} \|u(\cdot,t)\|_E.$$

For $0 \leq T_1 \leq T_2$, we set

$$\mathcal{X}_{T_1,T_2} := C_b\bigl([T_1,T_2] \to E\bigr),$$

and let $\|\cdot\|_{T_1,T_2}$ denote the corresponding norm. We set also $\mathcal{X}_T := \mathcal{X}_{0,T}$, $\|\cdot\|_T := \|\cdot\|_{0,T}$, and

$$\mathcal{X}_\infty := C_b\bigl(\mathbb{R}_+ \to E\bigr)$$

with the corresponding norm $\|\cdot\|_\infty$. We will also omit the sub-index for the norm $\|\cdot\|_E$ in E, if it is clear whether we are working with sup- or esssup-norm.

We define also the set $\mathcal{X}_T^1$ of functions from $\mathcal{X}_T$, which are continuously differentiable on $(0,T]$ in the sense of the norm in E. A function $u \in \mathcal{X}_T^1$ which satisfies (1.1) is said to be a classical solution to (1.1); in particular, u will continuously (in the sense of the norm in E) depend on the initial condition u_0.

We will write $v \le w$, for $v, w \in E$, if $v(x) \le w(x)$, for all $x \in \mathbb{R}^d$ (a.e. $x \in \mathbb{R}^d$, in the case $E = L^\infty(\mathbb{R}^d)$). The upper index '+' will denote the cone of nonnegative functions in the corresponding spaces, namely,

$$E^+ := \{v \in E \mid v \ge 0\}, \qquad \mathcal{X}_\sharp^+ := \{u \in \mathcal{X}_\sharp \mid u \ge 0\},$$

where $\sharp$ is one of the sub-indexes above. Denote also, for an $r > 0$,

$$E_r^+ := \{v \in E : 0 \le v \le r\}.$$

We denote by $T_y : E \to E$, $y \in \mathbb{R}^d$, a translation operator, given by

$$(T_y v)(x) = v(x - y), \quad x \in \mathbb{R}^d. \tag{2.1}$$

We consider also the topology generated by the following semi-norms

$$\|v\|_\Lambda := \sup_{x\in\Lambda} |v(x)|, \qquad \Lambda \subset \mathbb{R}^d \text{ is a compact.} \tag{2.2}$$

In particular, $v_n \in E$ converges to $v \in E$ in the topology if and only if v_n converges to v locally uniformly, which we denote

$$v_n \xRightarrow{\text{loc}} v, \quad n \to \infty.$$

The latter convergence means that $1\!\!1_\Lambda v_n \to 1\!\!1_\Lambda v$ in E, where $1\!\!1_\Lambda$ is the indicator-function of the compact set $\Lambda \subset \mathbb{R}^d$.

In Section 3, we prove an existence and uniqueness result for a more general equation than (1.1); which can be read in the case of (1.1) as follows

Theorem 2.1. *Let $G : E \to E$ be such that $Gv \ge 0$, $v \in E^+$, and for some $\kappa > 0$,*

$$\|Gv - Gw\|_E \le e^{\kappa r}\|v - w\|_E, \quad v, w \in E_r^+, \ r > 0.$$

Then, for any $T > 0$ and $u_0 \in E^+$, there exists a unique nonnegative solution $u \in \mathcal{X}_T$ to (1.1).

To exclude the trivial case when $\|u(\cdot, t)\|_E$ converges to 0 uniformly in time, we assume that

$$\beta := \varkappa - m > 0. \tag{A1}$$

We suppose that there exist two constant solutions $u \equiv 0$ and $u \equiv \theta > 0$ to (1.1), more precisely,

$$\begin{gathered}\textit{there exists } \theta > 0 \textit{ such that}\\ 0 = G0 \le Gv \le G\theta = \beta, \quad v \in E_\theta^+.\end{gathered} \tag{A2}$$

We will also assume that G is (locally) Lipschitz continuous in E_θ^+, namely,

$$\begin{gathered}\textit{there exists } l_\theta > 0, \textit{ such that}\\ \|Gv - Gw\| \le l_\theta\|v - w\|, \quad v, w \in E_\theta^+.\end{gathered} \tag{A3}$$

We restrict ourselves to the case when the comparison principle for (1.1) holds. Namely, we assume that the right hand side (r.h.s. in the sequel) of (1.1) is a (quasi-)monotone operator:

$$\begin{gathered} \textit{for some } p \geq 0 \textit{ and for any } v, w \in E_\theta^+ \textit{ with } v \leq w, \\ \varkappa a * v - v\,Gv + pv \leq \varkappa a * w - w\,Gw + pw. \end{gathered} \tag{A4}$$

In Section 4, we also prove that the comparison principle holds for a more general equation than (1.1); which can be read in the case of (1.1) as follows

Theorem 2.2. *Let* (A1)–(A4) *hold. Let* $T > 0$ *be fixed. Suppose that* $u_1, u_2 \in \mathcal{X}_T^1$ *are such that, for all* $(x,t) \in \mathbb{R}^d \times (0,T]$,

$$\begin{gathered} \frac{\partial u_1}{\partial t} - \varkappa a * u_1 + mu_1 + u_1 G u_1 \leq \frac{\partial u_2}{\partial t} - \varkappa a * u_2 + mu_2 + u_2 G u_2, \\ u_1(x,t) \geq 0, \qquad u_2(x,t) \leq \theta, \\ 0 \leq u_1(x,0) \leq u_2(x,0) \leq \theta. \end{gathered}$$

Then $u_1(x,t) \leq u_2(x,t)$ *for all* $(x,t) \in \mathbb{R}^d \times [0,T]$.

Moreover, Proposition 4.2 states that if $0 \leq u_0 \leq \theta$, then under (A1)–(A4) the corresponding solution to (1.1) satisfies $0 \leq u \leq \theta$. As a result, one gets the comparison between any two solutions to (1.1) for all $t > 0$ having it for $t = 0$.

We assume that the kernel a is not degenerate at the origin, namely,

$$\textit{there exists } \rho, \delta > 0 \textit{ such that } a(x) \geq \rho \text{ for a.a. } x \in B_\delta(0). \tag{A5}$$

Here and below $B_r(x_0)$ denotes the ball with the radius $r > 0$ centred at the $x_0 \in \mathbb{R}^d$. In the case $x_0 = 0 \in \mathbb{R}^d$, we will just write $B_r := B_r(0)$.

Stability of the solution to (1.1) with respect to the initial condition in the topology of the locally uniform convergence requires continuity of G in the topology.

$$\begin{gathered} \textit{for any } v_n, v \in E_\theta^+, \textit{ such that } v_n \overset{\text{loc}}{\Longrightarrow} v,\ n \to \infty, \textit{ one has} \\ Gv_n \overset{\text{loc}}{\Longrightarrow} Gv,\ n \to \infty. \end{gathered} \tag{A6}$$

We will consider the translation invariant case only:

$$\begin{gathered} \textit{let } T_y,\ y \in \mathbb{R}^d, \textit{ be a translation operator, given by } (2.1), \textit{ then} \\ (T_y G v)(x) = (G T_y v)(x), \quad v \in E_\theta^+,\ x \in \mathbb{R}^d. \end{gathered} \tag{A7}$$

Under (A7), for any $r \equiv \mathit{const} \in (0,\theta)$, $Gr \equiv \mathit{const}$. In this case, we assume also that

$$Gr < \beta, \qquad r \in (0,\theta). \tag{A8}$$

In Section 5, we prove the 'hair-trigger' effect for the solutions to (1.1). For technical reasons, it will be done separately for kernels with and without the first moment. Namely, for the kernels which satisfy the condition

$$\int_{\mathbb{R}^d} |y| a(y)\, dy < \infty, \tag{A9}$$

we set

$$\mathfrak{m} := \int_{\mathbb{R}^d} x a(x)\, dx \in \mathbb{R}^d, \tag{2.3}$$

and assume, additionally to (A4), that

$$\begin{gathered}\text{there exist } p \geq 0,\ b \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),\ \delta > 0, \text{ such that}\\ a(x) - b(x) \geq \delta 1\!\!1_{B_\delta(0)}(x), \quad x \in \mathbb{R}^d,\\ w\, Gw \leq \varkappa b * w + pw, \quad w \in E_\theta^+.\end{gathered} \tag{A10}$$

Then we can prove the following

Theorem 2.3. *Let the conditions* (A1)–(A10) *hold. Let* $u_0 \in E_\theta^+$, $u_0 \not\equiv 0$, *and let* $u \in \mathcal{X}_\infty$ *be the corresponding solution to* (1.1). *Then, for* $\mathfrak{m}$ *defined by* (2.3) *and any compact set* $K \subset \mathbb{R}^d$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x\in K} u(x + t\mathfrak{m}, t) = \theta.$$

Remark 2.4. For a brevity of notations, in the case $E = L^\infty(\mathbb{R}^d)$, we will treat $u_0 \not\equiv 0$ as follows: there exist $\delta, \rho > 0$ and $x_0 \in \mathbb{R}^d$, such that $u_0(x) \geq \delta$ for a.a. $x \in B_\rho(x_0)$.

For the kernels which do not satisfy (A9), we will need an approximation of the solution from below. For this reason, we consider the following assumption

$$\begin{gathered}\text{let, for each } n \in \mathbb{N}, \text{ there exist}\\ 0 \leq a_n \in L^1(\mathbb{R}^d), \quad \varkappa_n > 0, \quad G_n : E \to E, \quad \theta_n \in (0, \theta]\\ \text{which satisfy (A1)–(A10) instead of } a,\ \varkappa,\ G,\ \theta, \text{ correspondingly,}\\ \text{such that, for some } \tilde{\mathfrak{m}} \in \mathbb{R}^d,\\ \int_{\mathbb{R}^d} x a_n(x) dx = \tilde{\mathfrak{m}},\ n \in \mathbb{N}, \quad \theta_n \to \theta,\ n \to \infty,\\ \varkappa_n a_n * w - wG_n w \leq \varkappa a * w - wGw, \quad w \in E_{\theta_n}^+.\end{gathered} \tag{A11}$$

Then the following counterpart of Theorem 2.3 holds.

Theorem 2.5. *Let the condition* (A11) *hold. Let* $u_0 \in E_\theta^+$, $u_0 \not\equiv 0$, *and let* $u \in \mathcal{X}_\infty$ *be the corresponding solution to* (1.1). *Then, for any compact set* $K \subset \mathbb{R}^d$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{x\in K} u(x + t\tilde{\mathfrak{m}}, t) = \theta.$$

In particular, if (A1)–(A10) hold and $\mathfrak{m} = 0 \in \mathbb{R}^d$ or if (A11) holds and $\tilde{\mathfrak{m}} = 0 \in \mathbb{R}^d$, then one gets the desired 'hair-trigger' effect. Note that, indeed, for a properly 'slanted' anisotropic kernel a with $\mathfrak{m} \neq 0 \in \mathbb{R}^d$, the solution to (1.1) may converge to 0 uniformly on any ball centred at the origin, whereas it will converge to θ on the 'time-moving' ball according to Theorems 2.3 or 2.5; see [13] for the corresponding result in the case of the Example 2 described below.

Examples

Note that the equation (1.1) may be rewritten in the reaction–diffusion form,

$$\frac{\partial u}{\partial t} = \varkappa(a * u - u) + Fu, \tag{2.4}$$

where $Fu := u(\beta - Gu)$, $u \in E_\theta^+$, provided that the operator

$$E_\theta^+ \ni u \mapsto \frac{F(u)}{u} \in E$$

is well defined. The solution u to the equation (2.4) may be interpreted as a density of a species which invades according to a nonlocal diffusion within the space $\mathbb{R}^d$ meeting a reaction F, see e.g. [11, 19, 20].

1. Reaction–diffusion equation with a local reaction

A particular example of (2.4), with $F(u) = f(u)$ for a function $f : \mathbb{R} \to \mathbb{R}$, was considered e.g. in [1, 3, 5–9, 16, 21, 23]. We assume (A1) and (A5) as before, whereas the assumptions (A2)–(A4), (A6)–(A8), (A10) are fulfilled if only

$$f \textit{ is Lipschitz continuous on } [0, \theta];$$

$$\lim_{r \to 0+} \frac{f(r)}{r} = \beta;$$

$$f(0) = f(\theta) = 0; \quad 0 < f(r) < \beta r, \; r \in (0, \theta).$$

If (A9) does not hold, then, to fulfil (A11), it is enough to take $\varkappa_n = \varkappa$, $a_n(x) := 1\!\!1_{\Lambda_n}(x)a(x)$, provided that $\Lambda_n \subset \mathbb{R}^d$ are such that $\Lambda_n \uparrow \mathbb{R}^d$ and $\int_{\Lambda_n} xa(x)dx = \tilde{\mathfrak{m}}$. In particular, if $a(-x) = a(x)$, $x \in \mathbb{R}^d$, one can take $\Lambda_n := B_n(0)$.

2. Spatial logistic equation: $Gu = \varkappa^- a^- * u$

Let $\varkappa^- > 0$ and $a^-(x)$ be a probability kernel. We consider $Gu = \varkappa^- a^- * u$, i.e. (1.1) has the form

$$\frac{\partial u}{\partial t} = \varkappa(a * u) - \varkappa^- u(a^- * u) - mu.$$

This equation first appeared, for the case $\varkappa a = \varkappa^- a^-$, $m = 0$, in [17, 18]; for the case $\varkappa a = \varkappa^- a^-$, $m > 0$ in [10], and for the different kernels in [4], where the so-called Bolker–Pacala model of spatial ecology was considered. The equation was rigorously derived from the Bolker–Pacala model in [15] for integrable u and in [12] for bounded u. The long-time behaviour of this equation was studied in [13, 14].

We assume (A1) and (A5) as before. Under (A1), we have in this case $\theta = \dfrac{\varkappa - m}{\varkappa^-} > 0$. Then the conditions (A2)–(A3), (A6)–(A8) are satisfied. The condition (A4) holds if and only if

$$\varkappa a(x) \geq (\varkappa - m)a^-(x), \qquad x \in \mathbb{R}^d. \tag{2.5}$$

Condition (A10) holds if we additionally assume that there exists $\delta > 0$, such that

$$\varkappa a(x) - (\varkappa - m)a^-(x) \geq \delta 1\!\!1_{B_\delta(0)}(x), \qquad x \in \mathbb{R}^d.$$

In this case we can put, in (A10), $b(x) = (\varkappa - m)a^-(x)$, $p = 0$.

If (A9) does not hold, then, to fulfil (A11), one can proceed as in the previous example. Namely, we define a_n as before, and we set $G_n u = \varkappa^- a_n^- * u$, where $a_n^-(x) := 1\!\!1_{\Lambda_n}(x)a^-(x)$, $x \in \mathbb{R}^d$.

3. The case $Gu = \varkappa^- a^- * u - g_1(a^- * u)$

Let $g(s) = \varkappa^- s - g_1(s)$, where $\varkappa^- > 0$, $g_1 : [0, \theta] \to \mathbb{R}_+$ is increasing and Lipschitz continuous, such that $g_1(s) = o(s)$, as $s \to 0$ and $\varkappa^- s \geq g_1(s)$, for $s \in (0, \theta)$. We define $Gv = g(a^- * v)$, where a^- is a probability density. Namely, we consider the following equation,

$$\frac{\partial u}{\partial t} = \varkappa(a * u) - \varkappa^- u(a^- * u) + u g_1(a^- * u) - mu.$$

As in the previous example, (A4) holds if and only if (2.5) holds. The rest of the assumptions can be characterized straightforward. Typical example is $g(s) = \beta(1 - (1 - s)^n)$, with $\theta = 1$. In this case the corresponding reaction is $F(u) = \beta u(1 - a^- * u)^n$.

3 Existence and uniqueness

We start with the following simple lemma

Lemma 3.1. *Let $a \in L^1(\mathbb{R}^d)$, $f \in E$. Then $a * f \in C_{ub}(\mathbb{R}^d)$. Moreover, if $v \in C_b(I \to E)$, $I \subset \mathbb{R}_+$, then $a * v \in C_b(I \to C_{ub}(\mathbb{R}^d))$.*

Proof. The convolution is a bounded function, as

$$|(a * f)(x)| \leq \|f\|_E \, \|a\|_{L^1(\mathbb{R}^d)}, \qquad a \in L^1(\mathbb{R}^d), f \in E. \tag{3.1}$$

Next, let $a_n \in C_0(\mathbb{R}^d)$, $n \in \mathbb{N}$, be such that $\|a - a_n\|_{L^1(\mathbb{R}^d)} \to 0$, $n \to \infty$. For any $n \geq 1$, the proof of that $a_n * f \in C_{ub}(\mathbb{R}^d)$ is straightforward. Next, by (3.1), $\|a * f - a_n * f\| \to 0$, $n \to \infty$. Hence $a * u$ is a uniform limit of uniformly continuous functions that fulfils the proof of the first statement. The second statement is followed from the first one and the inequality (3.1). □

Lemma 3.2. *Let $\{r_n\}_{n \in \mathbb{N}}$ be a sequence of numbers, such that $r_1 > 0$ and the following recurrence relation holds*

$$r_{n+1} = r_n + p e^{-q r_n}, \quad n \in \mathbb{N}, \tag{3.2}$$

where $p, q > 0$. Then the series $\sum\limits_{n \in \mathbb{N}} \dfrac{1}{r_n e^{q r_n}}$ is divergent.

Proof. By (3.2), r_n, $n \in \mathbb{N}$ is a positive increasing sequence. Passing to the limit in (3.2) when $n \to \infty$, one gets that $r_n \to \infty$, as $n \to \infty$. Hence, without loss of generality, one can assume that $b_n := e^{-qr_n} < (pq)^{-1}$, $n \in \mathbb{N}$. One can easily rewrite then (3.2) as follows:

$$b_{n+1} = b_n e^{-pqb_n}.$$

It is straightforward to check that

$$\frac{x}{1+pqx(e-1)} \le ye^{-pqy}, \quad 0 < x \le y \le \frac{1}{pq},$$

Therefore, if we set $c_1 := b_1$ and $c_{n+1} := \frac{c_n}{1+pq(e-1)c_n}$, $n \in \mathbb{N}$, we get $c_n \le b_n$, $n \in \mathbb{N}$. On the other hand, $\frac{1}{c_{n+1}} = \frac{1}{c_n} + pq(e-1)$, that leads to

$$\frac{1}{c_{n+1}} = \frac{1}{c_1} + n(e-1)pq, \quad n \in \mathbb{N}. \tag{3.3}$$

Therefore,

$$\sum_{n\in\mathbb{N}} \frac{1}{r_n e^{qr_n}} = \sum_{n\in\mathbb{N}} \frac{b_n}{-q\ln b_n} \ge \sum_{n\in\mathbb{N}} \frac{c_n}{-q\ln c_n} = \infty,$$

since, by (3.3),

$$\frac{c_n}{-\ln c_n} \sim \frac{1}{pq(e-1)n\ln n}, \quad n \to \infty.$$

The statement is proved. $\square$

Theorem 3.3. *Let $A, G : E \to E$ be such that $Gv \ge 0$, $v \in E^+$, and, for some $\kappa, \varkappa > 0$,*

$$\|Av - Aw\|_E \le \varkappa \|v - w\|_E, \qquad v, w \in E^+, \tag{3.4}$$

$$\|Gv - Gw\|_E \le e^{\kappa r} \|v - w\|_E, \qquad v, w \in E_r^+, \ r > 0. \tag{3.5}$$

Then, for any $T > 0$ and $u_0 \in E^+$, there exists a unique nonnegative solution $u \in \mathcal{X}_T$ to the equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = (Au)(x,t) - mu(x,t) - u(x,t)(Gu)(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{3.6}$$

where $t \in (0, T]$, $x \in \mathbb{R}^d$.

Proof. First, we note that, by (3.4),

$$\|Av\|_E \le \|A0\|_E + \varkappa \|v\|_E, \quad 0 \le v \in E. \tag{3.7}$$

We set

$$f_0 := \|A0\|_E.$$

Let $T > 0$ be arbitrary. Take any $0 \le v \in \mathcal{X}_T$. For any $\tau \in [0, T)$, consider the following linear equation in the space E on the interval $[\tau, T]$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = (Av)(x,t) - mu(x,t) - u(x,t)(Gv)(x,t), \quad t \in (\tau, T] \\ u(x,\tau) = u_\tau(x), \end{cases} \tag{3.8}$$

where $0 \le u_\tau \in E$, $\tau > 0$, and u_0 is the same as in (3.6). By assumptions on A and G, we have that $Av, Gv \in \mathcal{X}_T$ for all $v \in \mathcal{X}_T$. In the r.h.s. of (3.8), there is a time-dependent linear bounded operator (acting in u) in the space E whose coefficients are continuous on $[\tau, T]$. Therefore, there exists a unique solution to (3.8) in E on $[\tau, T]$, given by $u = \Phi_\tau v$ with

$$(\Phi_\tau v)(x,t) := (Bv)(x,\tau,t)u_\tau(x) + \int_\tau^t (Bv)(x,s,t)(Av)(x,s)\,ds, \tag{3.9}$$

for $x \in \mathbb{R}^d$, $t \in [\tau, T]$, where we set

$$(Bv)(x,s,t) := \exp\biggl(-\int_s^t \bigl(m + (Gv)(x,p)\bigr)dp\biggr), \tag{3.10}$$

for $x \in \mathbb{R}^d$, $t, s \in [\tau, T]$. Note that, in particular, $(\Phi_\tau v)(\cdot, t), (Bv)(\cdot, s, t) \in E$. Clearly, $(\Phi_\tau v)(x,t) \ge 0$ and, for any $\Upsilon \in (\tau, T]$,

$$\|\Phi_\tau v(\cdot,t)\| \le \|u_\tau\| + (f_0 + \varkappa \|v\|_{\tau,\Upsilon})(\Upsilon - \tau), \qquad t \in [\tau, \Upsilon], \tag{3.11}$$

where we used (3.7). Therefore, Φ_τ maps $\mathcal{X}^+_{\tau,\Upsilon}$ into itself, $\Upsilon \in (\tau, T]$.

For any $T_2 > T_1 \ge 0$, we define

$$\mathcal{X}^+_{T_1,T_2}(r) := \bigl\{ v \in \mathcal{X}^+_{T_1,T_2} \bigm| \|v\|_{T_1,T_2} \le r \bigr\}, \quad r > 0. \tag{3.12}$$

Let now $0 \le \tau < \Upsilon \le T$, and take any $v, w \in \mathcal{X}^+_{\tau,\Upsilon}(r)$. By (3.9), one has, for any $x \in \mathbb{R}^d$, $t \in [\tau, \Upsilon]$,

$$\bigl|(\Phi_\tau v)(x,t) - (\Phi_\tau w)(x,t)\bigr| \le J_1 + J_2, \tag{3.13}$$

where

$$\begin{aligned} J_1 &:= \bigl|(Bv)(x,\tau,t) - (Bw)(x,\tau,t)\bigr| u_\tau(x), \\ J_2 &:= \int_\tau^t \bigl|(Bv)(x,s,t)(Av)(x,s) - (Bw)(x,s,t)(Aw)(x,s)\bigr|\,ds. \end{aligned}$$

Since $|e^{-a} - e^{-b}| \le |a - b|$, for any constants $a, b \ge 0$, one has, by (3.10), (3.1),

$$J_1 \le l(r)(\Upsilon - \tau)\|u_\tau\| \|v - w\|_{\tau,\Upsilon}. \tag{3.14}$$

Next, for any constants $a, b, p, q \ge 0$,

$$\bigl|pe^{-a} - qe^{-b}\bigr| \le e^{-a}|p - q| + q\max\bigl\{e^{-a}, e^{-b}\bigr\}|a - b|,$$

therefore, by (3.10), (3.1),

$$\begin{aligned} J_2 &\le \varkappa \int_\tau^t (Bv)(x,s,t)\,ds \|v - w\|_{\tau,\Upsilon} \\ &\quad + \int_\tau^t \max\bigl\{(Bv)(x,s,t), (Bw)(x,s,t)\bigr\}|(Aw)(x,s)|(t-s)l(r)\|v - w\|_{\tau,\Upsilon}\,ds \\ &\le \varkappa(\Upsilon - \tau)\|v - w\|_{\tau,\Upsilon} + l(r)(f_0 + \varkappa\|w\|_{\tau,\Upsilon})\|v - w\|_{\tau,\Upsilon} \int_\tau^t e^{-m(t-s)}(t-s)\,ds \\ &\le \Bigl(\varkappa + (f_0 + \varkappa\|w\|_{\tau,\Upsilon})\frac{l(r)}{me}\Bigr)(\Upsilon - \tau)\|v - w\|_{\tau,\Upsilon}, \end{aligned} \tag{3.15}$$

as $re^{-r} \leq e^{-1}$, $r \geq 0$.

Take any $\mu \geq \|u_\tau\|$. By (3.11)–(3.15), one has,

$$\big|(\Phi_\tau v)(x,t) - (\Phi_\tau w)(x,t)\big| \leq \Big(\mu l(r) + \varkappa + (f_0 + \varkappa r)\frac{l(r)}{me}\Big)(\Upsilon - \tau)\|v - w\|_{\tau,\Upsilon},$$
$$\big|(\Phi_\tau v)(x,t)\big| \leq \mu + (f_0 + \varkappa r)(\Upsilon - \tau).$$

Therefore, Φ_τ will be a contraction mapping on the set $\mathcal{X}^+_{\tau,\Upsilon}(r)$ if only

$$\Big(\mu l(r) + \varkappa + (f_0 + \varkappa r)\frac{l(r)}{me}\Big)(\Upsilon - \tau) < 1 \quad \text{and} \quad \mu + (f_0 + \varkappa r)(\Upsilon - \tau) \leq r.$$

If $\frac{f_0}{\varkappa} \leq r$, it is sufficient to show

$$\Big(\mu l(r) + \varkappa + 2\varkappa r\frac{l(r)}{me}\Big)(\Upsilon - \tau) < 1 \quad \text{and} \quad \mu + 2\varkappa r(\Upsilon - \tau) \leq r. \tag{3.16}$$

Take for $\alpha \in (0,1)$,

$$r := \mu + \frac{\alpha me}{l(\mu)}, \qquad \Upsilon := \tau + \frac{\alpha me}{2\varkappa r l(r)}. \tag{3.17}$$

Then, the second inequality in (3.16) holds, since l is monotonically increasing, namely,

$$\mu + 2\varkappa r(\Upsilon - \tau) = \mu + \frac{\alpha me}{l(r)} \leq \mu + \frac{\alpha me}{l(\mu)} = r.$$

Next

$$\Big(\mu l(r) + \varkappa + \frac{2\varkappa r l(r)}{me}\Big)(\Upsilon - \tau) = \frac{\alpha me\mu}{2\varkappa r} + \frac{\alpha me}{2rl(r)} + \alpha \leq \frac{\alpha me}{2\varkappa} + \frac{\alpha me}{2rl(\mu)} + \alpha.$$

In order to satisfy the second inequality in (3.16) it is sufficient to check,

$$\frac{\alpha me}{2\varkappa} + \frac{\alpha me}{2rl(\mu)} < 1 - \alpha,$$

but

$$rl(\mu) = \mu l(\mu) + \alpha me,$$

i.e. we need

$$\frac{\alpha me}{2(\mu l(\mu) + \alpha me)} + \frac{\alpha me}{2\varkappa} < 1 - \alpha. \tag{3.18}$$

Since g is increasing one can always choose $\alpha \in (0,1)$ and $\mu > 0$ that satisfy (3.18).

As a result, one gets that Φ_τ will be a contraction on the set $\mathcal{X}^+_{\tau,\Upsilon}(r)$ with Υ and r given by (3.17); the latter set naturally forms a complete metric space. Therefore, there exists a unique $u \in \mathcal{X}^+_{\tau,\Upsilon}(r)$ such that $\Phi_\tau u = u$. This u will be a solution to (3.6) on $[\tau, \Upsilon]$.

To fulfill the proof of the statement, one can do the following. Set $\tau := 0$, choose $r_0 > \max\{\|u_0\|, \frac{f_0}{\varkappa}\}$ and $\alpha \in (0,1)$ that satisfy (3.18) with $\mu = r_0$. One gets a solution u to (3.6) on $[0, \Upsilon_1]$ with $\|u\|_{\Upsilon_1} \leq r_0 + \frac{\alpha me}{l(r_0)} =: r_1$, $\Upsilon_1 = \frac{\alpha me}{2\varkappa r_1 l(r_1)}$.

Iterating this scheme, take sequentially, for each $n \in \mathbb{N}$, $\tau := \Upsilon_n$, $u_{\Upsilon_n}(x) := u(x, \Upsilon_n)$, $x \in \mathbb{R}^d$,

$$r_n := r_{n-1} + \frac{\alpha m e}{l(r_{n-1})} \geq \|u_{\Upsilon_n}\|.$$

Since $r_n > r_{n-1}$ and $l(r)$ is increasing, the same α as before will satisfy (3.18) with $\mu = r_n$ as well. Then, one gets a solution u to (3.6) on $[\Upsilon_n, \Upsilon_{n+1}]$ with initial condition u_{Υ_n}, where

$$\Upsilon_{n+1} := \Upsilon_n + \frac{\alpha m e}{2\varkappa r_n l(r_n)}, \tag{3.19}$$

and

$$\|u\|_{\Upsilon_n, \Upsilon_{n+1}} \leq r_n + \frac{\alpha m e}{l(r_n)} = r_{n+1}.$$

As a result, we will have a solution u to (3.6) on intervals $[0, \Upsilon_1]$, $[\Upsilon_1, \Upsilon_2]$, $\ldots$, $[\Upsilon_n, \Upsilon_{n+1}]$, $n \in \mathbb{N}$. By Lemma 3.1, the r.h.s. of (3.6), will be continuous on each of constructed time-intervals, therefore, one has that u is continuously differentiable on $(0, \Upsilon_{n+1}]$ and solves (1.1) there. By (3.19) and Lemma 3.2,

$$\Upsilon_{n+1} = \frac{\alpha m e}{2\varkappa} \sum_{j=0}^{n} \frac{1}{r_j l(r_j)} \to \infty, \quad n \to \infty,$$

therefore, one has a solution to (3.6) on any $[0, T]$, $T > 0$.

To prove uniqueness, suppose that $v \in \mathcal{X}_T$ is a solution to (3.6) on $[0, T]$, with $v(x, 0) \equiv u_0(x)$, $x \in \mathbb{R}^d$. Choose $r_0 > \|v\|_T \geq \|u_0\|$. Since $\{r_n\}_{n \geq 0}$ above is an increasing sequence, v will belong to each of sets $\mathcal{X}^+_{\Upsilon_n, \Upsilon_{n+1}}(r_{n+1})$, $n \geq 0$, $\Upsilon_0 := 0$, considered above. Then, being solution to (3.6) on each $[\Upsilon_n, \Upsilon_{n+1}]$, v will be a fixed point for Φ_{Υ_n}. By the uniqueness of such a point, v coincides with u on each $[\Upsilon_n, \Upsilon_{n+1}]$ and, thus, on the whole $[0, T]$. $\square$

Remark 3.4. Since $Av := \varkappa a * v$, $v \in E$, evidently satisfies (3.4), one gets Theorem 2.1.

4 Comparison principle

Let $H : E \to E$. For any $v \in E$, $r \in \mathbb{R}$, we set

$$(v \wedge r)(x) := \min\{v(x), r\}, \quad (v \vee r)(x) := \max\{v(x), r\}.$$

For any $u \in \mathcal{X}^1_T$, $r > 0$, we define

$$(\mathcal{F}_r u)(x, t) := \frac{\partial u}{\partial t}(x, t) - H(0 \vee u \wedge r)(x, t), \tag{4.1}$$

for all $t \in (0, T]$ and all $x \in \mathbb{R}^d$. Here and below we consider the left derivative at $t = T$ only.

Theorem 4.1. *Let $r \geq 0$ and $H : E \to E$ be such that, for some $p > 0$, the operator $H + p1\!\!1$ is monotone and Lipschitz continuous on E^+_r; namely,*

$$\|Hw - Hv\|_E \leq h(r)\|w - v\|_E, \qquad w, v \in E^+_r, \tag{4.2}$$

$$Hv + pv \leq Hw + pw, \qquad v \leq w, \ w, v \in E^+_r. \tag{4.3}$$

Let $T>0$ be fixed. Suppose that $u_1, u_2 \in \mathcal{X}_T^1$ are such that

$$0 \le u_1(x,t), \quad u_2(x,t) \le r, \qquad (x,t) \in \mathbb{R}^d \times (0,T], \tag{4.4}$$
$$(\mathcal{F}_r u_1)(x,t) \le (\mathcal{F}_r u_2)(x,t), \qquad (x,t) \in \mathbb{R}^d \times (0,T], \tag{4.5}$$
$$u_1(x,0) \le u_2(x,0), \qquad x \in \mathbb{R}^d. \tag{4.6}$$

Then $u_1(x,t) \le u_2(x,t)$ for all $(x,t) \in \mathbb{R}^d \times [0,T]$.

Proof. Define, cf. (4.5), the following function

$$\phi_r(x,t) := (\mathcal{F}_r u_2)(x,t) - (\mathcal{F}_r u_1)(x,t) \ge 0, \tag{4.7}$$

for $(x,t) \in \mathbb{R}^d \times [0,T]$. For $w \in \mathcal{X}_T$, consider the mapping

$$\begin{aligned}\Theta(t,w) :&= Kw + e^{Kt}\big(H\big(0 \vee (e^{-Kt}w + u_1) \wedge r\big) - H(u_1 \wedge r)\big)\\ &\quad + e^{Kt}\phi_r(x,t).\end{aligned} \tag{4.8}$$

We have, for $w \ge 0$,

$$0 \le u_1 \wedge r \le (e^{-Kt}w + u_1) \wedge r \le r.$$

Since, for any $x,y,z \ge 0$, $x \ge y$,

$$0 \le x \wedge z - y \wedge z \le x - y,$$

one has, by (4.3), (4.7), that $0 \le w \in \mathcal{X}_T$ yields

$$\Theta(t,w) \ge (K-p)w + e^{Kt}\phi_r(x,t) \ge 0, \tag{4.9}$$

if only $K \ge p$ that we will assume in the following. Next, by (4.2), $w \in \mathcal{X}_T$ implies, for all $t \in [0,T]$,

$$\|\Theta(t,w)\|_T \le (K + h(r))\|w\|_T + e^{Kt}\|\phi_r\|_T. \tag{4.10}$$

Therefore, since $u_1, u_2 \in \mathcal{X}_T^1$ implies, by (4.1), (4.7), that $\phi_r \in \mathcal{X}_T$, one gets that $\Theta(t,w) \in \mathcal{X}_T$.

Define also the function

$$v(x,t) := e^{Kt}(u_2(x,t) - u_1(x,t)), \quad x \in \mathbb{R}^d,\ t \in [0,T]. \tag{4.11}$$

Clearly, $v \in \mathcal{X}_T^1$, and it is straightforward to check that

$$\Theta(t, v(s,t)) = \frac{\partial}{\partial t} v(x,t), \tag{4.12}$$

for all $x \in \mathbb{R}^d$, $t \in (0,T]$. Therefore, v solves the following integral equation in E:

$$\begin{cases} v(x,t) = v(x,0) + \displaystyle\int_0^t \Theta(s, v(x,s))ds, & (x,t) \in \mathbb{R}^d \times (0,T],\\ v(x,0) = u_2(x,0) - u_1(x,0), & x \in \mathbb{R}^d, \end{cases} \tag{4.13}$$

where $v(x,0) \ge 0$ by (4.6).

Consider also another integral equation in E:

$$\tilde{v}(x,t) = (\Psi\tilde{v})(x,t) \tag{4.14}$$

where

$$(\Psi w)(x,t) := v(x,0) + \int_0^t \max\bigl\{\Theta(s, w(x,s)), 0\bigr\}\, ds, \qquad w \in \mathcal{X}_T. \tag{4.15}$$

If we take $\tilde{T} < T$ such that the following inequality holds

$$q_1 := 2r(K + h(r)) + e^{KT}\|\phi_r\|_T \le \frac{r}{\tilde{T}},$$

then, $w \in \mathcal{X}^+_{\tilde{T}}(2r)$ yields $\Psi w \in \mathcal{X}^+_{\tilde{T}}(2r)$, where, cf. (3.12), $\mathcal{X}^+_{\tilde{T}}(2r) := \mathcal{X}^+_{0,\tilde{T}}(2r)$. Let $w_1, w_2 \in \mathcal{X}_{\tilde{T}}(2r)$; by (4.2), (4.8), we have, for all $(t,x) \in [0,\tilde{T}] \times \mathbb{R}^d$,

$$|\Theta(t, w_1) - \Theta(t, w_2)| \le (K + h(r))\|w_1 - w_2\|_{\tilde{T}} =: q_2\|w_1 - w_2\|_{\tilde{T}}. \tag{4.16}$$

Therefore, using the elementary inequality $|\max\{a,0\} - \max\{b,0\}| \le |a-b|$, $a, b \in \mathbb{R}$, we obtain from (4.15), that

$$\|\Psi w_1 - \Psi w_2\|_{\tilde{T}} \le q_2\tilde{T}\|w_2 - w_1\|_{\tilde{T}}.$$

Therefore, for $\tilde{T} < \max\{\frac{r}{2q_1}, \frac{1}{q_2}\}$, Ψ is a contraction on $\mathcal{X}^+_{\tilde{T}}(2r)$. Thus, there exists a unique solution $\tilde{v}$ to (4.14) on $[0,\tilde{T}]$. By (4.14), (4.15),

$$\tilde{v}(x,t) \ge v(x,0) \ge 0, \qquad x \in \mathbb{R}^d,\ t \in [0,\tilde{T}]. \tag{4.17}$$

By the considerations above, $w \in \mathcal{X}^+_T$ yields $\Theta(s, w(x,s)) \in \mathcal{X}^+_T$. Hence $\tilde{v}$ is a solution to (4.13) on $[0,\tilde{T}]$ as well. Namely,

$$\tilde{v}(x,t) = v(x,0) + \int_0^t \Theta(s, \tilde{v}(x,s))\, ds =: \Xi(\tilde{v})(x,t). \tag{4.18}$$

By the same arguments as the above, Ξ is a contraction on $\mathcal{X}_{\tilde{T}}(2r)$, for the same $\tilde{T}$. We deduce that $v = \tilde{v}$ on $\mathbb{R}^d \times [0,\tilde{T}]$. Then, by (4.17), $v(x,t) \ge 0$ on $\mathbb{R}^d \times [0,\tilde{T}]$, that yields

$$0 \le u_1(x,\tilde{T}) \le u_2(x,\tilde{T}) \le r,\ x \in \mathbb{R}^d.$$

In the same way, the proof can be extended on $[\tilde{T}, 2\tilde{T}], [2\tilde{T}, 3\tilde{T}], \ldots$, keeping the same q_1 and q_2, and, therefore, on the whole $[0,T]$. Then $v(x,t) \ge 0$ on $\mathbb{R}^d \times [0,T]$, that yields the statement. □

Clearly, Theorem 4.1 in the case $r = \theta$, $Hv = \varkappa a * v - vGv - mv$, $v \in E$, implies Theorem 2.2. To use the latter statement, one needs to know that a solution to (1.1) remains in E^+_θ for all times provided that $u_0 \in E^+_\theta$. Consider the corresponding statement.

Proposition 4.2. *Let* (A1)–(A4) *hold and* $0 \le u_0 \le \theta$. *Then there exists a unique solution* $u \in \mathcal{X}^+_\infty$ *to* (1.1), *and* $0 \le u(x,t) \le \theta$ *for any* $x \in \mathbb{R}^d$, $t \ge 0$.

Proof. We set $Hv := \varkappa a * v - vGv - mv$ for $v \in E_\theta^+$, and $Hv := H(0 \vee v \wedge \theta)$ for $v \in E \setminus E_\theta^+$. Prove, first, that H is (globally) Lipschitz continuous on E. Indeed, for any $x, y \in \mathbb{R}$,

$$|x \wedge \theta - y \wedge \theta| = \frac{1}{2}\big|(x + \theta - |x - \theta|) - (y + \theta - |y - \theta|)\big| \leq |x - y|$$

and, similarly, $|x \vee 0 - y \vee 0| \leq |x - y|$. Therefore, denoting $v_\theta := 0 \vee v \wedge \theta$ for $v \in E$, one gets that $\|v_\theta - w_\theta\| \leq \|v - w\|$ for $w, v \in E$, and hence

$$\begin{aligned}\|Hw - Hv\|_E &\leq (\varkappa + m + \sup_{v \in E} \|G(0 \vee v \wedge \theta)\|_E + \theta l_\theta)\|w - v\|_E \\ &= (2\varkappa + \theta l_\theta)\|w - v\|_E.\end{aligned}$$

As a result, for any $T > 0$, the initial value problem

$$\frac{\partial \tilde{u}}{\partial t}(x, t) = H(\tilde{u})(x, t), \quad \tilde{u}(x, 0) = u_0(x), \quad x \in \mathbb{R}^d,\ t \in (0, T],$$

has a unique classical solution $\tilde{u} \in \mathcal{X}_T$, i.e., for $\mathcal{F}_\theta$ defined by (4.1), $\mathcal{F}_\theta \tilde{u} \equiv 0$.

Note that, for any $r \geq \theta$, $v \in E_r^+$ implies $Hv = H(v \wedge \theta)$. In particular, applying this for $v = 0 \vee \tilde{u} \wedge r$, one gets

$$\mathcal{F}_r \tilde{u} = \mathcal{F}_\theta \tilde{u} \equiv 0. \tag{4.19}$$

Moreover, by (A4), there exists $p \geq 0$, such that, for any $r \geq \theta$, $v, w \in E_r^+$, $v \leq w$,

$$p(w - v) + Hw - Hv \geq p(w \wedge \theta - v \wedge \theta) + H(w \wedge \theta) - H(v \wedge \theta) \geq 0.$$

Assume that $\|\tilde{u}\|_T > \theta$. Then, by the arguments above and (4.19), we may apply Theorem 4.1 for the case $r = \|\tilde{u}\|_T$, $u_1 \equiv 0$, $u_2 = \tilde{u}$ (note that, evidently, $\mathcal{F}_r 0 = 0$). It yields $\tilde{u} \geq 0$. Next, similarly, we can apply Theorem 4.1 for the case $r = \theta$, $u_1 = \tilde{u}$, $u_2 \equiv \theta$ (since $\mathcal{F}_\theta \theta = 0$). It implies then that $\tilde{u} \leq \theta$, that contradicts the assumption, therefore, $\|\tilde{u}\|_T \leq \theta$. Apply once more Theorem 4.1 for the case $r = \theta$, $u_1 \equiv 0$, $u_2 = \tilde{u}$, then $\tilde{u} \geq 0$. As a result, the function $\tilde{u} = 0 \vee \tilde{u} \wedge \theta$ solves (1.1).

Choose an arbitrary extension of G on E^+ such that (3.5) holds. By Theorem 2.1, there exists a unique solution $u \in \mathcal{X}_\infty^+$ to (1.1). Hence $0 \leq u = \tilde{u} \leq \theta$. The proof is fulfilled. □

5 The hair-trigger effect: proofs of Theorems 2.3–2.5

Proposition 5.1. *Let the conditions of Theorem 3.3 hold. Suppose, additionally, that A and G are continuous on E^+ in the topology generated by (2.2), i.e. for any $v_n, v \in E^+$ with $v_n \overset{\mathrm{loc}}{\Longrightarrow} v$, one has*

$$Av_n \overset{\mathrm{loc}}{\Longrightarrow} Av, \quad Gv_n \overset{\mathrm{loc}}{\Longrightarrow} Gv, \qquad n \to \infty.$$

Let $T > 0$ be fixed and, for some $\rho > 0$, $\{u(\cdot, 0), u_n(\cdot, 0) : n \in \mathbb{N}\} \subset E_\rho^+$ be the initial conditions to (3.6), and let $\{u(\cdot, t), u_n(\cdot, t) : n \in \mathbb{N}\}$ be the corresponding solutions to (3.6) on $[0, T]$. Assume that $u_n(\cdot, 0) \overset{\mathrm{loc}}{\Longrightarrow} u(\cdot, 0)$, $n \to \infty$. Then $u_n(\cdot, t) \overset{\mathrm{loc}}{\Longrightarrow} u(\cdot, t)$, $n \to \infty$ uniformly in $t \in [0, T]$.

Proof. By the proof of Theorem 3.3, there exist $0 = \tau_0 < \tau_1 < \ldots < \tau_N = T$ and $\rho = r_0 \le r_1 \le \ldots \le r_N =: r$, such that the following holds. Let, for any $\tau = \tau_k$, $\Upsilon = \tau_{k+1}$, $0 \le k \le N-1$, the mapping Φ_τ be defined by (3.9) for $t \in [\tau, \Upsilon]$, with $u_\tau(x) = u(x,\tau)$, $x \in \mathbb{R}^d$; and, for each $n \in \mathbb{N}$, we set

$$(\Phi_{\tau,n} v)(x,t) := (Bv)(x,\tau,t)u_{\tau,n}(x) + \int_\tau^t (Bv)(x,s,t)(Av)(x,s)\,ds, \qquad (5.1)$$

where $u_{\tau,n}(x) = u_n(x,\tau)$, $x \in \mathbb{R}^d$. Then $v \in \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$, $\{u_\tau, u_{\tau,n} : n \in \mathbb{N}\} \subset E_{r_k}$ implies $\{\Phi_\tau v, \Phi_{\tau,n} v : n \in \mathbb{N}\} \subset \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$.

Prove that if, for some $\{w, w_n : n \in \mathbb{N}\} \subset \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$, we have $w_n(\cdot,t) \overset{\mathrm{loc}}{\Longrightarrow} w(\cdot,t)$, $n \to \infty$, uniformly in $t \in [\tau, \Upsilon]$, then

$$\Phi_{\tau,n} w_n(\cdot,t) \overset{\mathrm{loc}}{\Longrightarrow} \Phi_\tau w(\cdot,t), \quad n \to \infty, \qquad (5.2)$$

uniformly in $t \in [\tau, \Upsilon]$. Indeed, applying the inequalities,

$$|e^{-a} - e^{-b}| \le |a-b|, \qquad \bigl|pe^{-a} - qe^{-b}\bigr| \le |p-q| + q|a-b|,$$

for $a, b, p, q \ge 0$, we get, for any bounded $\Lambda \subset \mathbb{R}^d$,

$$\begin{aligned}
&1\!\!1_\Lambda(x)\bigl|(\Phi_{\tau,n} w_n)(x,t) - (\Phi_\tau w)(x,t)\bigr| \\
&\le 1\!\!1_\Lambda(x)\bigl|(\Phi_{\tau,n} w_n)(x,t) - (\Phi_{\tau,n} w)(x,t)\bigr| + 1\!\!1_\Lambda(x)\bigl|(\Phi_{\tau,n} w)(x,t) - (\Phi_\tau w)(x,t)\bigr| \\
&\le 1\!\!1_\Lambda(x)\bigl|u_{\tau,n}(x) - u_\tau(x)\bigr| + r_k \int_\tau^t 1\!\!1_\Lambda(x)\bigl|(Gw_n)(x,p) - (Gw)(x,p)\bigr|\,dp \\
&\quad + \int_\tau^t 1\!\!1_\Lambda(x)\bigl|(Aw_n)(x,s) - (Aw)(x,s)\bigr|\,ds \\
&\quad + \int_\tau^t 1\!\!1_\Lambda(x)\bigl|(Aw)(x,s)\bigr| \int_s^t \bigl|(Gw_n)(x,p) - (Gw)(x,p)\bigr|\,dp\,ds \\
&\le \bigl\|u_{\tau,n} - u_\tau\bigr\|_\Lambda + r_k \int_\tau^\Upsilon \bigl\|(Gw_n)(\cdot,p) - (Gw)(\cdot,p)\bigr\|_\Lambda dp \\
&\quad + \int_\tau^\Upsilon \bigl\|(Aw_n)(\cdot,s) - (Aw)(\cdot,s)\bigr\|_\Lambda\,ds \\
&\quad + (\|A(0)\|_E + \varkappa r) \int_\tau^\Upsilon \int_s^\Upsilon \bigl\|(Gw_n)(\cdot,p) - (Gw)(\cdot,p)\bigr\|_\Lambda\,dp\,ds.
\end{aligned}$$

Hence (5.2) holds. Iterating this scheme, one gets that, for each $m \in \mathbb{N}$, $v \in \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$,

$$(\Phi_{\tau,n})^m v(\cdot,t) \overset{\mathrm{loc}}{\Longrightarrow} (\Phi_\tau)^m v, \quad n \to \infty, \qquad (5.3)$$

uniformly in $t \in [\tau, \Upsilon]$. Therefore, for any bounded $\Lambda \subset \mathbb{R}^d$,

$$\begin{aligned}
&\bigl|1\!\!1_\Lambda(x)(u_n(x,t) - u(x,t))\bigr| \\
&\le \bigl|1\!\!1_\Lambda(x)\bigl(u_n(x,t) - (\Phi_{\tau,n})^m v(x,t)\bigr)\bigr| + \bigl|1\!\!1_\Lambda(x)\bigl((\Phi_{\tau,n})^m v(x,t) - (\Phi_\tau)^m v(x,t)\bigr)\bigr| \\
&\quad + \bigl|1\!\!1_\Lambda(x)\bigl(u(x,t) - (\Phi_\tau)^m v(x,t)\bigr)\bigr| \\
&\le \bigl\|u_n - (\Phi_{\tau,n})^m v\|_{\tau,\Upsilon} + \sup_{t \in [\tau,\Upsilon]} \bigl\|\bigl((\Phi_{\tau,n})^m v(\cdot,t) - (\Phi_\tau)^m v(\cdot,t)\bigr)\bigr\|_\Lambda \\
&\quad + \bigl\|u - (\Phi_\tau)^m v\|_{\tau,\Upsilon},
\end{aligned}$$

for any $m \in \mathbb{N}$. Passing m to ∞, one gets then the statement by (5.3). $\square$

Proposition 5.2. *Let $0 \le u_0 \in C_{ub}(\mathbb{R}^d)$, and suppose that $u \in \mathcal{X}_\infty$ is the corresponding classical solution to* (1.1)*. Suppose also, that there exists $C > 0$, such that*

$$0 \le u(x,t) \le C, \quad x \in \mathbb{R}^d, \ t \ge 0,$$

and $g_C := \sup\limits_{v \in E_C^+} |Gv| < \infty$. Then $u \in C_{ub}(\mathbb{R}^d \times \mathbb{R}_+)$ and, moreover, $\|u(\cdot,t)\| \in C_{ub}(\mathbb{R}_+)$. In particular, these inclusions hold if we assume (A1)–(A4)*.*

Proof. Being classical solution to (1.1), u satisfies the integral equation

$$u(x,t) = u_0(x) + \int_0^t \bigl(\varkappa(a*u)(x,s) - mu(x,s) - u(x,s)(Gu)(x,s)\bigr)\,ds.$$

Hence for any $x, y \in \mathbb{R}^d$, $0 \le \tau < t$, one has

$$\bigl|u(x,t) - u(y,\tau)\bigr| \le (2\varkappa C + 2mC + Cg_C)(t-\tau),$$

that fulfils the proof of the first inclusion. Then, the second one follows from the inequality $\bigl|\|u(\cdot,t)\| - \|u(\cdot,\tau)\|\bigr| \le \|u(\cdot,t) - u(\cdot,\tau)\|$, $t, \tau \in \mathbb{R}_+$. Finally, if the conditions (A1)–(A4) hold, then, by Proposition 4.2, one gets that the solution u exists and satisfies the conditions above if only $C := \theta$. Moreover, (A3) implies that, for any $v \in E_\theta^+$,

$$\|Gv\| \le \|G0\| + l_\theta \|v\| \le \|G0\| + \theta l_\theta < \infty,$$

that fulfils the proof. □

The maximum principle is a 'standard counterpart' of the comparison principle, see e.g. [6]. We will use in the sequel that, under some additional assumptions, the solutions to (1.1) are strictly positive; this is a quite common feature of linear parabolic equations, however, in general, it may fail for nonlinear ones. Consider the corresponding statement.

Proposition 5.3. *Let $E = C_b(\mathbb{R}^d)$. Let* (A1)–(A5) *hold with $G : E \to E$, such that $Gl \not\equiv \beta$, for $l \in (0,\theta)$. (In particular, the latter holds, if we assume, additionally,* (A7)–(A8)*.) Let $u_0 \in E_\theta^+$, $u_0 \not\equiv \theta$, $u_0 \not\equiv 0$, be the initial condition to* (1.1) *and $u \in \mathcal{X}_\infty$ be the corresponding solution. Then*

$$u(x,t) > \inf_{\substack{y \in \mathbb{R}^d \\ s > 0}} u(y,s) \ge 0, \qquad x \in \mathbb{R}^d, t > 0.$$

Proof. By Proposition 4.2, $0 \le u(x,t) \le \theta$, $x \in \mathbb{R}^d$, $t \ge 0$. We denote

$$(L_a u)(x,t) = \varkappa(a*u)(x,t) - \varkappa u(x,t). \tag{5.4}$$

Then, by (A2),

$$\frac{\partial u}{\partial t}(x,t) - (L_a u)(x,t) = u(x,t)(\beta - (Gu)(x,t)) \ge 0. \tag{5.5}$$

Prove that, under (5.5), u cannot attain its infimum on $\mathbb{R}^d \times (0,\infty)$ without being a constant. Indeed, suppose that, for some $x_0 \in \mathbb{R}^d$, $t_0 > 0$,

$$u(x_0,t_0) \le u(x,t), \quad x \in \mathbb{R}^d, t > 0. \tag{5.6}$$

Then, clearly,

$$\frac{\partial u}{\partial t}(x_0, t_0) = 0, \tag{5.7}$$

and (5.5) yields $(L_a u)(x_0, t_0) \leq 0$. On the other hand, (5.4) and (5.6) imply $(L_a u)(x_0, t_0) \geq 0$. Therefore,

$$\int_{\mathbb{R}^d} a(x_0 - y)(u(y, t_0) - u(x_0, t_0))\, dy = 0. \tag{5.8}$$

Then, by (A5), for all $y \in B_\delta(x_0)$,

$$u(y, t_0) = u(x_0, t_0). \tag{5.9}$$

By the same arguments, for an arbitrary $x_1 \in \partial B_\delta(x_0)$, we obtain (5.9), for all $y \in B_\delta(x_1)$. Hence, (5.9) holds on $B_{2\delta}(x_0)$, and so on. As a result, (5.9) holds, for all $y \in \mathbb{R}^d$, thus $u(\cdot, t_0)$ is a constant, i.e.

$$u(x, t_0) = u(x_0, t_0) =: l_0 \in [0, \theta], \quad x \in \mathbb{R}^d.$$

Then, considering (1.1) at (x_0, t_0), and taking into account (5.7), one gets

$$0 = u(x_0, t_0)\bigl(\beta - (Gu)(x_0, t_0)\bigr) = l_0(\beta - Gl_0).$$

By the assumption, the latter equality is possible if only $l_0 \in \{0, \theta\}$, i.e. either $u(\cdot, t_0) \equiv 0$ or $u(\cdot, t_0) \equiv \theta$. By (5.6), $u(x_0, t_0) = \theta \geq \sup_{y \in \mathbb{R}^d, s > 0} u(y, s)$ implies $u \equiv \theta$, that contradicts $u_0 \not\equiv \theta$. Hence $u(x, t_0) = u(x_0, t_0) = 0$, $x \in \mathbb{R}^d$. Then, by Theorem 3.3, $u(x, t) = 0$, $x \in \mathbb{R}^d$, $t \geq t_0$. And now one can consider the reverse time in (1.1) starting from $t = t_0$. Namely, we set $w(x, t) := u(x, t_0 - t)$, $t \in [0, t_0]$, $x \in \mathbb{R}^d$. Then $w(x, 0) = u(x, t_0) = 0$, $x \in \mathbb{R}^d$, and

$$\frac{\partial w}{\partial t}(x, t) = w(x, t)(Gw)(x, t) - \varkappa^+(a^+ * w)(x, t) + mw(x, t). \tag{5.10}$$

Prove that the equation (5.10) with the initial condition $w(\cdot, 0) \equiv 0$ has a unique classical solution $w \equiv 0$ in $\mathcal{X}_{t_0}$. Indeed, let $w \in \mathcal{X}_{t_0}$ solve (5.10). Suppose that the set

$$K := \bigl\{ t \in [0, t_0] \bigm| \|w(\cdot, t)\|_E > 0 \bigr\}$$

is not empty, i.e. $w \not\equiv 0$. We define then $T := \inf K$. In particular, $\|w(\cdot, t)\|_E = 0$ for $t \in [0, T)$ (note that the latter interval might be empty if $T = 0$). Since the function $\tau \mapsto \|w(\cdot, \tau)\|_E$ is continuous, we have that $\|w(\cdot, T)\|_E = 0$ as well. Therefore, $T = t_0$ would contradict the assumption $K \neq \emptyset$; hence $T < t_0$. Consider now the equation (5.10) for $t \in [T, t_0]$ with the initial value $w(\cdot, T) \equiv 0$. It is straightforward to check that the assumptions on G imply that, for any $r > 0$, there exists $\Delta T > 0$, such that $T + \Delta T < t_0$ and the mapping

$$\Psi(w)(x, t) = \int_T^{T+t} w(x, s)(Gw)(x, s) - \varkappa(a * w)(x, s) + mw(x, s) ds.$$

is a contraction on $\mathcal{X}_{\Delta T}(r)$. Therefore, by the uniqueness arguments, $w(\cdot, t) \equiv 0$ for $t \in [T, T + \Delta T]$ that contradicts the choice of T. Therefore, $K = \emptyset$, i.e. $w(\cdot, t) \equiv 0$ for all $t \in [0, t_0]$, in particular, $u(\cdot, 0) = w(\cdot, t_0) \equiv 0$, that contradicts $u_0 \not\equiv 0$. Thus, the initial assumption was wrong, and (5.6) can not hold. The proof is fulfilled. □

In the sequel, it will be useful to consider the solution to (1.1) as a nonlinear transformation of the initial condition.

Definition 5.4. For a fixed $t > 0$, define the mapping Q_t on E^+ by

$$(Q_t f)(x) := u(x,t), \quad x \in \mathbb{R}^d, f \in E^+, \tag{5.11}$$

where $u(x,t)$ is the solution to (1.1) with the initial condition $u(x,0) = f(x)$.

Let us collect several properties of Q_t needed below.

Proposition 5.5. *Let* (A1)–(A8) *hold. Then, for any fixed $t > 0$, the mapping $Q := Q_t : E^+ \to E^+$ satisfies the following properties*

(Q1) *$Q : E^+_\theta \to E^+_\theta$;*

(Q2) *let T_y, $y \in \mathbb{R}^d$, be a translation operator, given by* (2.1)*, then*

$$(QT_y f)(x) = (T_y Q f)(x), \quad x, y \in \mathbb{R}^d, \ f \in E^+_\theta; \tag{5.12}$$

(Q3) *$Q0 = 0$, $Q\theta = \theta$, and $Qr > r$, for any constant $r \in (0, \theta)$;*

(Q4) *if $f, g \in E^+_\theta$ and $f \le g$, then $Qf \le Qg$;*

(Q5) *if $f_n \overset{\mathrm{loc}}{\Longrightarrow} f$, then $(Qf_n)(x) \to (Qf)(x)$ for (a.a.) $x \in \mathbb{R}^d$.*

Proof. The property (Q1) follows from Proposition 4.2. To prove (Q2) we note that, by (A7), $T_y G = G T_y$, and $T_y(a * u) = a * (T_y u)$, then, by (3.10), $B(T_y v) = T_y(Bv)$, therefore, by (3.9), if $\tau = 0$ and $u_\tau = T_y f$, then $\Phi_\tau T_y = T_y \Phi$, where $\Phi = \Phi(f)$ is given by (3.9) with f in place of u_τ only. As a result, $\Phi^n_\tau T_y = T_y \Phi^n$ hence, for $v \in \mathcal{X}^+_{\tau,\Upsilon}(r_{k+1})$ (c.f. notations of Theorem 3.3), we have, for $t \in [\tau, \Upsilon]$, $x \in \mathbb{R}^d$,

$$(Q_t T_y f)(x) = \lim_{n\to\infty} (\Phi^n_\tau (T_y f) v)(x,t) = \lim_{n\to\infty} (T_y \Phi^n_\tau (f) v)(x,t) = (T_y Q_t f)(x),$$

and one can continue the same considerations on the next time-interval.

By (Q2), $u_0(x) \equiv r \in (0, \theta)$ yields $u(\cdot, t) = \mathrm{const}$, $t \ge 0$. Then, by (A2) and (A8), for any $t \ge 0$, we have

$$Qr = u(t) = r + \int_0^t u(s)(\beta - (Gu)(s))ds > 0.$$

Hence the property (Q3) holds. The property (Q4) holds by Theorem 4.1. The property (Q5) is a weaker version of Proposition 5.1. □

Let S^{d-1} denote a unit sphere in $\mathbb{R}^d$ centred at the origin:

$$S^{d-1} = \bigl\{ x \in \mathbb{R}^d \bigm| |x| = 1 \bigr\}; \tag{5.13}$$

in particular, $S^0 = \{-1, 1\}$.

Definition 5.6. A function $f \in E$ is said to be increasing (decreasing, constant) along the vector $\xi \in S^{d-1}$ if, for a.a. $x \in \mathbb{R}^d$, the function $f(x+s\xi) = (T_{-s\xi} f)(x)$ is increasing (decreasing, constant) in $s \in \mathbb{R}$, respectively.

Proposition 5.7. *Let* (A1)–(A8) *hold. Let* $u_0 \in E_\theta^+$ *be the initial condition for the equation* (1.1) *which is increasing (decreasing, constant) along a vector* $\xi \in S^{d-1}$*; and* $u(\cdot,t) \in E_\theta^+$, $t \geq 0$, *be the corresponding solution (cf. Proposition 4.2). Then, for any* $t > 0$, $u(\cdot,t)$ *is increasing (decreasing, constant, respectively) along the* ξ.

Proof. Let u_0 be decreasing along a $\xi \in S^{d-1}$. Take any $s_1 \leq s_2$ and consider two initial conditions to (1.1): $u_0^i(x) = u_0(x + s_i\xi) = (T_{-s_i\xi}u_0)(x)$, $i = 1, 2$ (c.f. (2.1)). Since u_0 is decreasing, $u_0^1(x) \geq u_0^2(x)$, $x \in \mathbb{R}^d$. Then, by Proposition 5.5,

$$T_{-s_1\xi}Q_t u_0 = Q_t T_{-s_1\xi}u_0 = Q_t u_0^1 \geq Q_t u_0^2 = Q_t T_{-s_2\xi}u_0 = T_{-s_2\xi}Q_t u_0,$$

that proves the statement. The cases of a decreasing u_0 can be considered in the same way. The constant function along a vector is decreasing and decreasing simultaneously. □

To prove the hair-trigger effect (Theorems 2.3–2.5), we will follow the abstract scheme proposed in [22]. Note that all statements there were considered in the space $E = C_b(\mathbb{R}^d)$.

Consider the set N_θ of all nonincreasing functions $\varphi \in C(\mathbb{R})$, such that $\varphi(s) = 0$, $s \geq 0$, and

$$\varphi(-\infty) := \lim_{s\to-\infty} \varphi(s) \in (0, \theta).$$

It is easily seen that $N_\theta \subset E_\theta^+$, for $E = C_b(\mathbb{R}^d)$.

For arbitrary $s \in \mathbb{R}$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, define the following mapping $V_{s,c,\xi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}^d)$

$$(V_{s,c,\xi}g)(x) = g(x \cdot \xi + s + c), \quad x \in \mathbb{R}^d. \tag{5.14}$$

Fix an arbitrary $\varphi \in N_\theta$. For $t > 0$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, consider the mapping $R_{t,c,\xi} :\ L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, given by

$$(R_{t,c,\xi}g)(s) = \max\bigl\{\varphi(s), (Q_t(V_{s,c,\xi}g))(0)\bigr\}, \quad s \in \mathbb{R}, \tag{5.15}$$

where $Q_t : E \to E$ is a mapping which satisfies the conditions (Q1)–(Q5) in Proposition 5.5 (in particular, one can consider Q_t given by (5.11) provided that (A1)–(A8) hold). Consider now the following sequence of functions

$$f_{n+1}(s) = (R_{t,c,\xi}f_n)(s), \quad f_0(s) = \varphi(s), \qquad s \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}. \tag{5.16}$$

By Proposition 5.5 and [22, Lemma 5.1], $\varphi \in E_\theta^+$ with $E = C_b(\mathbb{R}^d)$ implies $f_n \in E_\theta^+$ and $f_{n+1}(s) \geq f_n(s)$, $s \in \mathbb{R}$, $n \in \mathbb{N}$; hence one can define the following limit

$$f_{t,c,\xi}(s) := \lim_{n\to\infty} f_n(s), \quad s \in \mathbb{R}. \tag{5.17}$$

Also, by [22, Lemma 5.1], for fixed $\xi \in S^{d-1}$, $t > 0$, $n \in \mathbb{N}$, the functions $f_n(s)$ and $f_{t,c,\xi}(s)$ are nonincreasing in s and in c; moreover, $f_{t,c,\xi}(s)$ is a lower semicontinuous function of s, c, ξ, as a result, this function is continuous from the right in s and in c. Note also, that $0 \leq f_{t,c,\xi} \leq \theta$. Then, for any c, ξ, one can define the limiting value

$$f_{t,c,\xi}(\infty) := \lim_{s\to\infty} f_{t,c,\xi}(s).$$

Next, for any $t > 0$, $\xi \in S^{d-1}$, we define

$$c_t^*(\xi) = \sup\{c \mid f_{t,c,\xi}(\infty) = \theta\} \in \mathbb{R} \cup \{-\infty, \infty\},$$

where, as usual, $\sup \emptyset := -\infty$. By [22, Propositions 5.1, 5.2], one has

$$f_{t,c,\xi}(\infty) = \begin{cases} \theta, & c < c_t^*(\xi), \\ 0, & c \geq c_t^*(\xi), \end{cases} \tag{5.18}$$

cf. also [22, Lemma 5.5]; moreover, $c_t^*(\xi)$ is a lower semicontinuous function of ξ. It is crucial that, by [22, Lemma 5.4], neither $f_{t,c,\xi}(\infty)$ nor $c_t^*(\xi)$ depends on the choice of $\varphi \in N_\theta$. Note that the monotonicity of $f_{t,c,\xi}(s)$ in s and (5.18) imply that, for $c < c_t^*(\xi)$, $f_{t,c,\xi}(s) = \theta$, $s \in \mathbb{R}$.

Define

$$\Upsilon_t := \bigl\{x \in \mathbb{R}^d \bigm| x \cdot \xi \leq c_t^*(\xi), \xi \in S^{d-1}\bigr\}, \quad t > 0. \tag{5.19}$$

We will need the following Weinberger's result:

Lemma 5.8 (cf. [22, Theorem 6.2])**.** *Let $E = C_b(\mathbb{R}^d)$ and $v_0 \in E_\theta^+$. Let, for some fixed $t > 0$, $Q = Q_t : E \to E$ be a mapping which satisfies the conditions (Q1)–(Q5) in Proposition 5.5, and Υ_t be defined by* (5.19)*. Suppose that*

$$\operatorname{int}(\Upsilon_t) \neq \emptyset. \tag{5.20}$$

Then, for any compact set $\mathscr{C}_t \subset \operatorname{int}(\Upsilon_t)$ and for any $\sigma \in (0, \theta)$, one can choose a radius $r_\sigma = r_\sigma(Q_t, \mathscr{C}_t) > 0$, such that, for any fixed $x_0 \in \mathbb{R}^d$,

$$v_0(x) \geq \sigma, \quad x \in B_{r_\sigma}(x_0), \tag{5.21}$$

implies

$$\lim_{n\to\infty} \min_{x \in n\mathscr{C}_t} Q_t^n v_0(x) = \theta. \tag{5.22}$$

Remark 5.9. There is no loss of generality if we assume that (5.21) holds for $x_0 = 0$ only. Indeed, for any $x_0 \in \mathbb{R}^d$, $\mathscr{C}_t \subset \operatorname{int}(\Upsilon_t)$, there exist $N = N(x_0, \mathscr{C}_t)$, $\tilde{\mathscr{C}}_t \subset \operatorname{int}(\Upsilon_t)$, such that, for all $n \geq N$, one gets $x_0 + n\mathscr{C}_t \subset n\tilde{\mathscr{C}}_t$. Therefore, we have

$$\begin{aligned} \theta &\geq \lim_{n\to\infty} \min_{x \in n\mathscr{C}_t} (Q_t^n T_{-x_0} u_0)(x) = \lim_{n\to\infty} \min_{x \in n\mathscr{C}_t} (T_{-x_0} Q_t^n u_0)(x) \\ &= \lim_{n\to\infty} \min_{x \in x_0 + n\mathscr{C}_t} (Q_t^n u_0)(x) \geq \lim_{n\to\infty} \min_{x \in n\tilde{\mathscr{C}}_t} (Q_t^n u_0)(x) = \theta. \end{aligned}$$

Here and below, for $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, $s \in \mathbb{R}$, we denote

$$x + A := \{x + y \mid y \in A\} \subset \mathbb{R}^d, \quad sA := \{sy \mid y \in A\} \subset \mathbb{R}^d.$$

Proposition 5.10. *Let* (A1)–(A8) *hold and $u_0 \in C_{ub}(\mathbb{R}^d)$. Let Q_t, $t > 0$, be given by* (5.11)*, and let the corresponding Υ_t, $t > 0$, be given by* (5.19)*. Suppose that, for some compact $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$, there exists $\mathfrak{n} \in \operatorname{int}(\mathscr{C})$, such that*

$$\frac{1}{j}\mathfrak{n} \in \operatorname{int}(\Upsilon_{\frac{1}{j}}), \qquad j \in \mathbb{N}. \tag{5.23}$$

Let $\sigma \in (0,\theta)$ and $r_\sigma = r_\sigma(Q_1,\mathscr{C})$ be chosen according to Lemma 5.8. Suppose that

$$u_0(x) \geq \sigma, \qquad x \in B_{r_\sigma(Q_1,\mathscr{C})}. \tag{5.24}$$

Then, for the corresponding solution u to (1.1) and for any compact $K \subset \mathbb{R}^d$, the following limit holds

$$\min_{x\in K} u(x+t\mathfrak{n},t) \to \theta, \qquad t\to\infty. \tag{5.25}$$

Proof. First, we note that, by Proposition 5.5, the conditions (Q1)–(Q5) hold for all $Q = Q_t$, $t > 0$. We denote $K_1 := -\mathfrak{n} + \mathscr{C}$. Because of (5.24), one can apply Lemma 5.8 for $t = 1$ and $v_0(x) := u_0(x)$, $x \in \mathbb{R}^d$. Namely, since $Q_1^n v_0(y) = Q_1^n u_0(y) = u(y,n)$, $y \in \mathbb{R}^d$, one gets, by (5.22),

$$\min_{x\in nK_1} u(x+n\mathfrak{n},n) = \min_{y\in n\mathscr{C}} u(y,n) \to \theta, \quad n\to\infty. \tag{5.26}$$

Next, by (5.23), $0 \in -\frac{1}{2}\mathfrak{n} + \mathrm{int}(\Upsilon_{\frac{1}{2}})$. Choose now any compact $K_2 \subset -\frac{1}{2}\mathfrak{n} + \mathrm{int}(\Upsilon_{\frac{1}{2}})$, such that $0 \in \mathrm{int}(K_2)$. By Lemma 5.8 for $t = \frac{1}{2}$ and $\mathscr{C}_{\frac{1}{2}} := K_2 + \frac{1}{2}\mathfrak{n} \subset \mathrm{int}(\Upsilon_{\frac{1}{2}})$ there exists a radius $r_\sigma(Q_{\frac{1}{2}},\mathscr{C}_{\frac{1}{2}}) > 0$. By (5.26), there exists $N_1 \geq 1$, such that, for all $n \geq N_1$,

$$B_{r_\sigma(Q_{\frac{1}{2}},\mathscr{C}_{\frac{1}{2}})}(0) \cup K \subset nK_1, \qquad u(x+n\mathfrak{n},n) \geq \sigma, \quad x\in nK_1. \tag{5.27}$$

Set $S_1 := N_1$; by the latter inclusion and (5.27), one can apply Lemma 5.8 for $v_0(x) := u(x+S_1\mathfrak{n},S_1)$, $x \in \mathbb{R}^d$. Then

$$Q_{\frac{1}{2}}^n v_0(y) = u\Big(y+S_1\mathfrak{n}, S_1+\frac{n}{2}\Big), \quad y\in\mathbb{R}^d,$$

and hence

$$\begin{aligned}&\min_{x\in nK_2} u\Big(x+\Big(S_1+\frac{n}{2}\Big)\mathfrak{n}, S_1+\frac{n}{2}\Big)\\ &= \min_{y\in n\mathscr{C}_{\frac{1}{2}}} u\Big(y+S_1\mathfrak{n}, S_1+\frac{n}{2}\Big) \to \theta, \quad n\to\infty.\end{aligned} \tag{5.28}$$

Similarly, choose a compact $K_3 \subset -\frac{1}{3}\mathfrak{n}+\mathrm{int}(\Upsilon_{\frac{1}{3}})$ with $0 \in \mathrm{int}(K_3)$, and consider Lemma 5.8 with $t = \frac{1}{3}$ and $\mathscr{C}_{\frac{1}{3}} := K_3 + \frac{1}{3}\mathfrak{n} \subset \mathrm{int}(\Upsilon_{\frac{1}{3}})$. Then, there exists a radius $r_\sigma(Q_{\frac{1}{3}},\mathscr{C}_{\frac{1}{3}}) > 0$, and, by (5.28), there exists $N_2 \geq 2$ such that for all $n \geq N_2$,

$$B_{r_\sigma(Q_{\frac{1}{3}},\mathscr{C}_{\frac{1}{3}})} \cup K \subset nK_2$$

and

$$u\Big(x+\Big(S_1+\frac{n}{2}\Big)\mathfrak{n}, S_1+\frac{n}{2}\Big) \geq \sigma, \quad x\in nK_2.$$

Set $S_2 := S_1 + \frac{N_2}{2} = N_1 + \frac{N_2}{2} \geq 2$ and apply Lemma 5.8 with $v_0(x) := u(x+S_2\mathfrak{n},S_2)$, $x \in \mathbb{R}^d$. We have

$$\min_{x\in nK_3} u\Big(x+\Big(S_2+\frac{n}{3}\Big)\mathfrak{n}, S_2+\frac{n}{3}\Big) = \min_{x\in n\mathscr{C}_{\frac{1}{3}}} u\Big(x+S_2\mathfrak{n}, S_2+\frac{n}{3}\Big) \to \theta, \quad n\to\infty.$$

By induction, for any $K_j \subset -\frac{1}{j}\mathfrak{n}+\mathrm{int}(\Upsilon_{\frac{1}{j}})$, $j \geq 3$, with $0 \in \mathrm{int}(K_j)$, one can set $\mathscr{C}_{\frac{1}{j}} := K_j + \frac{1}{j}\mathfrak{n} \subset \mathrm{int}(\Upsilon_{\frac{1}{j}})$ and choose $N_{j-1} \geq j-1$ such that for all $n \geq N_{j-1}$,

$$B_{r_\sigma(Q_{\frac{1}{j}},\mathscr{C}_{\frac{1}{j}})} \cup K \subset nK_{j-1},$$
$$u\Big(x+\Big(S_{j-2}+\frac{n}{j-1}\Big)\mathfrak{n}, S_{j-2}+\frac{n}{j-1}\Big) \geq \sigma, \quad x \in nK_{j-1}.$$

Set

$$S_{j-1} := S_{j-2} + \frac{N_{j-1}}{j-1} = N_1 + \frac{N_2}{2} + \ldots + \frac{N_{j-1}}{j-1} \geq j-1.$$

Then, by Lemma 5.8, similarly to the above,

$$\min_{x \in nK_j} u\Big(x + \Big(S_{j-1} + \frac{n}{j}\Big)\mathfrak{n}, S_{j-1} + \frac{n}{j}\Big) \to \theta, \quad n \to \infty. \tag{5.29}$$

Suppose that (5.25) does not hold. Then, for some $\varepsilon > 0$, there exist sequences $x_m \in K$, $m \in \mathbb{N}$, and $t_m \to \infty$ as $m \to \infty$, such that

$$u(x_m + t_m\mathfrak{n}, t_m) \leq \theta - \varepsilon. \tag{5.30}$$

Since $u_0 \in C_{ub}(\mathbb{R}^d)$, then by Proposition 5.2, $u \in C_{ub}(\mathbb{R}^d \times \mathbb{R}_+)$. Thus there exists $\delta = \delta(\varepsilon)$, such that

$$|u(x,t) - u(y,s)| < \frac{\varepsilon}{2}, \qquad |x-y| < \delta, \ |t-s| < \delta.$$

We choose $j \in \mathbb{N}$ such that $\max\{1, |\mathfrak{n}|\} < \delta j$. By (5.29), there exists $N'_j > N_{j-1}$, such that, for all $n \geq N'_j$, we have that $K \subset nK_j$ and

$$\min_{x \in K} u\Big(x + \Big(S_{j-1} + \frac{n}{j}\Big)\mathfrak{n}, S_{j-1} + \frac{n}{j}\Big) > \theta - \frac{\varepsilon}{4}.$$

Choose m, such that $t_m \geq S_{j-1} + \dfrac{N'_j}{j}$. Let n_m be the entire part of $j(t_m - S_{j-1})$. Then $n_m \geq N'_j$ and, for $q_m := S_{j-1} + \dfrac{n_m}{j}$, we easily get that

$$\max\{1, |\mathfrak{n}|\}\, |t_m - q_m| < \delta.$$

Therefore,

$$\begin{aligned}
u(x_m + t_m\mathfrak{n}, t_m) &\geq \min_{x \in K} u(x + t_m\mathfrak{n}, t_m) \\
&\geq \min_{x \in K} u(x + q_m\mathfrak{n}, q_m) - \max_{x \in K} \big|u(x + q_m\mathfrak{n}, q_m) - u(x + t_m\mathfrak{n}, t_m)\big| \\
&\geq \min_{x \in K} u(x + q_m\mathfrak{n}, q_m) - \frac{\varepsilon}{2} > \theta - \frac{3}{4}\varepsilon,
\end{aligned}$$

that contradicts (5.30). Therefore (5.25) holds and the proof is fulfilled. $\square$

We are going to find now sufficient conditions that fulfil the additional assumptions of Proposition 5.10. We start with the following lemma.

Lemma 5.11. *Let $b \in L^1(\mathbb{R} \to \mathbb{R}_+)$ be such that*

$$\int_{\mathbb{R}} b(s)\,ds = 1, \qquad \int_{\mathbb{R}} |s| b(s) ds < \infty,$$

and let $v \in L^\infty(\mathbb{R} \to \mathbb{R}_+)$ be a nonincreasing function. Then the following limit holds

$$\lim_{r\to\infty} \int_{-r}^{r} \big((b * v)(s) - v(s)\big) ds = \big(v(-\infty) - v(\infty)\big) \int_{\mathbb{R}} s\, b(s)\, ds. \tag{5.31}$$

Proof. For $r > 0$ and $\rho := \dfrac{r}{2}$, we have, by Fubini's theorem,

$$\int_{-r}^{r} (b * v)(s) ds - \int_{-r}^{r} v(s) ds = \int_{-\infty}^{\infty} b(y) \int_{-r}^{r} v(s-y) ds dy - \int_{-r}^{r} v(s) ds$$
$$= \int_{-\infty}^{\infty} b(y) W_r(y)\, dy = I_1(r) + I_2(r),$$

where

$$W_r(y) := \int_{-r-y}^{r-y} v(s) ds - \int_{-r}^{r} v(s) ds, \quad y \in \mathbb{R},$$
$$I_1(r) := \int_{|y|\le\rho} b(y) W_r(y)\, dy, \qquad I_2(r) := \int_{|y|>\rho} b(y) W_r(y)\, dy.$$

Clearly,

$$\sup_{|y|\le\rho} b(y)|W_r(y)| \le 2\|v\|_\infty |y| b(y) \in L^1(\mathbb{R}).$$

Next, because of the monotonicity of v, we have,

$$y v(-r) \le \int_{-r-y}^{-r} v(s) ds \le y v(-r-y). \tag{5.32}$$

Since $-r - y < -\frac{r}{2}$ for $|y| \le \rho = \frac{r}{2}$, we have that

$$\mathbb{1}_{|y|\le\rho} \int_{-r-y}^{-r} v(s) ds \to v(-\infty) y, \qquad r \to \infty;$$

and, similarly, $\mathbb{1}_{|y|\le\rho} \int_{r-y}^{r} v(s) ds \to v(\infty) y$. Therefore, by the dominated convergence theorem,

$$\lim_{r\to\infty} I_1(r) = \big(v(-\infty) - v(\infty)\big) \int_{|y|\le\rho} y b(y) dy. \tag{5.33}$$

On the other hand,

$$|I_2(r)| \le 2v(-\infty)\, r \int_{|y|>\rho} b(y) dy \le 4v(-\infty)\, \frac{r}{\rho} \int_{|y|>\rho} b(y)|y| dy \to 0, \tag{5.34}$$

as $r \to \infty$. Combining (5.33) and (5.34), one gets the statement. $\square$

Proposition 5.12. *Let* (A1)–(A9) *hold. Let* Υ_t*,* $t>0$*, be defined by* (5.19)*, and* $\mathfrak{m}$ *be defined by* (2.3)*. Then*

$$\varkappa\, t\, \mathfrak{m} \in \operatorname{int}(\Upsilon_t). \tag{5.35}$$

Proof. Fix $t>0$. For a $\xi \in S^{d-1}$, we set

$$c := \varkappa t \int_{\mathbb{R}^d} y\cdot\xi a(y)dy = \varkappa t\, \mathfrak{m}\cdot\xi \in \mathbb{R}. \tag{5.36}$$

Let $f_{t,c,\xi}$ be defined by (5.17). By the definition of Υ_t and (5.18), we have that if $f_{t,c,\xi}(\infty)=\theta$ for all $\xi\in S^{d-1}$, then (5.35) holds. Suppose, in contrast, that, for some $\xi\in S^{d-1}$, $f_{t,c,\xi}(\infty)=0$. Fix such a ξ, consider the corresponding c according to (5.36), and denote $f:=f_{t,c,\xi}$. Note that, by [22, Lemma 5.2] and the discussion thereafter, $f(-\infty)=\theta$.

We set $u_0(x):=f(x\cdot\xi)$, $x\in\mathbb{R}^d$, and consider the corresponding solution u to (1.1). Then, by (5.14), we evidently have

$$(V_{s,c,\xi}f)(x) = (T_{-(c+s)\xi}u_0)(x), \quad x\in\mathbb{R}^d.$$

Next, as it was mentioned above, the functions f_n and $f=f_{t,c,\xi}$ in (5.17) are monotone, hence the limit in (5.17) is locally uniform. Therefore, passing n to ∞ in (5.16), we will get from (5.15) and Proposition 5.5, that

$$\begin{aligned} f(s) &= \max\bigl\{\varphi(s), (Q_t(V_{s,c,\xi}f))(0)\bigr\} = \max\bigl\{\varphi(s), (T_{-(c+s)\xi}Q_t u_0)(0)\bigr\} \\ &= \max\bigl\{\varphi(s), u\bigl((c+s)\xi,t\bigr)\bigr\}. \end{aligned} \tag{5.37}$$

Since f is nonincreasing on $\mathbb{R}$, u_0 is nonincreasing along ξ, cf. Definition 5.6; then, by Proposition 5.7, u also has the same property. As a result, the function

$$v(s) := u\bigl((c+s)\xi,t\bigr), \quad s\in\mathbb{R} \tag{5.38}$$

is nonincreasing on $\mathbb{R}$. Next, by our assumptions, $f(-\infty)=\theta>\varphi(-\infty)$ and $f(\infty)=0$; therefore, we get from (5.37), that

$$\lim_{s\to\infty} v(s)=0, \qquad \lim_{s\to-\infty} v(s)=\theta. \tag{5.39}$$

Next, (5.37) implies that, for each $s\in\mathbb{R}$, cf. (2.4),

$$\begin{aligned} u_0(s\xi) &\geq u\bigl((c+s)\xi,t\bigr) \\ &= u_0\bigl((c+s)\xi\bigr) + \int_0^t \varkappa\Bigl((a*u)\bigl((c+s)\xi,\tau\bigr) - u\bigl((c+s)\xi,\tau\bigr)\Bigr)d\tau \\ &\quad + \int_0^t u\bigl((c+s)\xi,\tau\bigr)\Bigl(\beta-(Gu)\bigl((c+s)\xi,\tau\bigr)\Bigr)d\tau. \end{aligned}$$

Therefore, for $r>c$,

$$\begin{aligned} 0 \geq{}& \varkappa\int_{-r}^{r}\int_0^t \Bigl((a*u)\bigl((c+s)\xi,\tau\bigr) - u\bigl((c+s)\xi,\tau\bigr)\Bigr)d\tau\, ds \\ &+ \int_{-r}^{r}\int_0^t u\bigl((c+s)\xi,\tau\bigr)\Bigl(\beta-(Gu)\bigl((c+s)\xi,\tau\bigr)\Bigr)d\tau\, ds \\ &+ \int_{-r}^{r}\Bigl(u_0\bigl((c+s)\xi\bigr)-u_0(s\xi)\Bigr)\, ds =: S_1(r)+S_2(r)+S_3(r). \end{aligned} \tag{5.40}$$

Note that u_0 is constant along any $\eta \in S^{d-1}$ orthogonal to ξ, cf. Definition 5.6; and, by Proposition 5.7, u has the same property. Namely, for each $s \in \mathbb{R}$ and $\eta \in S^{d-1}$ orthogonal to ξ,

$$u(x,t) = u(x + s\eta, t), \qquad t \geq 0,\ x \in \mathbb{R}^d. \tag{5.41}$$

For $d \geq 2$, choose any $\{\eta_1,\ \eta_2,\ \ldots,\ \eta_{d-1}\} \subset S^{d-1}$ which form a complement of $\xi \in S^{d-1}$ to an orthonormal basis in $\mathbb{R}^d$. Then

$$\begin{aligned}
&(a * u)(s\xi, t) = \int_{\mathbb{R}^d} a(y) u(s\xi - y, t) dy \\
&= \int_{\mathbb{R}^d} a\biggl(\sum_{j=1}^{d-1} y_j \eta_j + y_d \xi\biggr) u\biggl(-\sum_{j=1}^{d-1} y_j \eta_j + (s - y_d)\xi, t\biggr)\, dy_1 \ldots dy_d \\
&= \int_{\mathbb{R}} \biggl(\int_{\mathbb{R}^{d-1}} a\biggl(\sum_{j=1}^{d-1} y_j \eta_j + y_d \xi\biggr)\, dy_1 \ldots dy_{d-1}\biggr) u\bigl((s - y_d)\xi, t\bigr)\, dy_d,
\end{aligned} \tag{5.42}$$

where we used (5.41) with $\eta = -\sum\limits_{j=1}^{d-1} y_j \eta_j$, which is orthogonal to the ξ. Therefore, one can set

$$\check{a}(s) := \begin{cases} \displaystyle\int_{\mathbb{R}^{d-1}} a\biggl(\sum_{j=1}^{d-1} y_j \eta_j + s\xi\biggr)\, dy_1 \ldots dy_{d-1}, & d \geq 2, \\ a^{\pm}(s\xi), & d = 1 \end{cases} \tag{5.43}$$

for $s \in \mathbb{R}$. We also denote $\check{u}(s,t) := u(s\xi, t)$, $s \in \mathbb{R}$. Then one can continue (5.42), as follows: $(a * u)(s\xi, t) = (\check{a} * \check{u})(s,t)$, where the convolution in the r.h.s. is in $s \in \mathbb{R}$. Since $\int_{\mathbb{R}} \check{a}(s) ds = \int_{\mathbb{R}^d} a(y) dy = 1$ and (A9) yields

$$\int_{\mathbb{R}} |s| \check{a}(s) ds = \int_{\mathbb{R}^d} |y \cdot \xi| a(y) dy < \infty,$$

we may apply Lemma 5.11 with $b = \check{a}$ and v given by (5.38). Then, by (5.31), (5.39) and the dominated convergence theorem, we have

$$\begin{aligned}
S_1(r) &= \varkappa \int_0^t \int_{-r}^r \bigl((\check{a} * \check{u})(s + c, \tau) - \check{u}(s + c, \tau)\bigr) ds d\tau \\
&\to \varkappa t\theta \int_{\mathbb{R}} s\check{a}(s) ds = \varkappa\theta t \int_{\mathbb{R}^d} y \cdot \xi a(y) dy = \varkappa\theta t\, \xi \cdot \mathfrak{m},
\end{aligned} \tag{5.44}$$

as $r \to \infty$. Next, by (5.39), (5.36), we have, cf. (5.32),

$$S_3(r) = \int_r^{r+c} u_0(s\xi) ds - \int_{-r}^{-r+c} u_0(s\xi) ds \to -\theta c = -\theta\varkappa t\, \xi \cdot \mathfrak{m}, \tag{5.45}$$

as $r \to \infty$. Therefore, combining (5.40), (5.44), (5.45) with the inequality $u(\beta - Gu) \geq 0$, we deduce that

$$\int_{-\infty}^{\infty} \int_0^t u\bigl((c + s)\xi, \tau\bigr)\Bigl(\beta - (Gu)\bigl((c + s)\xi, \tau\bigr)\Bigr) d\tau\, ds = \lim_{r \to \infty} S_2(r) = 0. \tag{5.46}$$

Let $w_0 \in C_b(\mathbb{R}^d)$ be such that $0 \leq w_0 \leq u_0$ and $w_0 \not\equiv 0$. The by Theorem 4.1 and Proposition 5.3, we have

$$u(x,\tau) \geq w(x,\tau) > 0, \quad x \in \mathbb{R}^d,\ \tau > 0.$$

Hence (5.46) is possible if and only if $(Gu)\big(s\xi,\tau\big) = \beta$ for (a.a.) $s \in \mathbb{R}$ and all $\tau \in [0,t]$; note that $u(\cdot,\tau)$ is continuous in $\tau \geq 0$ and G is continuous on E_θ^+ because of (A3). In particular, $(Gu_0)\big(s\xi\big) = \beta$, $s \in \mathbb{R}$. Then we have by (A7) that, for any $p > 0$,

$$(GT_{-p\xi}u_0)(s\xi) = (T_{-p\xi}Gu_0)(s\xi) = (Gu_0)((s+p)\xi) = \beta, \quad s \in \mathbb{R}. \tag{5.47}$$

Since, $(T_{-p\xi}u_0)(x) = f(x\cdot\xi + p)$, $x \in \mathbb{R}^d$, and $f(\infty) = 0$, we have that $T_{-p\xi}u_0 \overset{\mathrm{loc}}{\Longrightarrow} 0$, as $p \to \infty$. Then, by (A6), (A2) we get that $GT_{-p\xi}u_0 \overset{\mathrm{loc}}{\Longrightarrow} G0 = 0$, as $p \to \infty$, that contradicts (5.47). The proof is fulfilled. □

Therefore, under assumptions (A1)–(A9), one has that (5.20) holds for all $T > 0$ and, moreover, (5.23) holds for $\mathfrak{n} = \mathfrak{m}$ given by (2.3). Now, we are going to get rid of the condition (5.24).

Proposition 5.13. *Let* (A1)–(A9) *hold and* $\mathfrak{m}$ *be given by* (2.3)*. Then there exists* $\alpha_0 > 0$*, such that, for all* $\alpha \in (0,\alpha_0)$*, there exists* $q_0 = q_0(\alpha) \in (0,\theta)$*, such that there exists* $T = T(\alpha,q_0) > 0$*, such that, for all* $q \in (0,q_0)$*, the function*

$$w(x,t) = q\exp\biggl(-\frac{|x - t\mathfrak{m}|^2}{\alpha t}\biggr), \quad x \in \mathbb{R}^d, t > T, \tag{5.48}$$

is a sub-solution to (1.1) *on* $t > T$*; i.e., cf.* (4.1),

$$(\mathcal{F}_\theta w)(x,t) := \frac{\partial w(x,t)}{\partial t} - \varkappa(a*w)(x,t) + mw(x,t) + w(x,t)(Gw)(x,t) \leq 0$$

for all $x \in \mathbb{R}^d$, $t > T$.

Proof. By (A2), (A3), for each $0 < q_0 < \min\{\theta, \frac{\beta}{2l_\theta}\}$ (where, recall, $\beta = \varkappa - m$), we have that $v \in E_{q_0}^+$ yields $0 \leq Gv \leq \frac{\beta}{2}$. Then, for each $q \in (0,q_0)$,

$$\mathcal{F}_\theta w \leq \frac{\partial w}{\partial t} - \varkappa a * w + \Bigl(m + \frac{\beta}{2}\Bigr)w, \tag{5.49}$$

and $m + \frac{\beta}{2} < \varkappa$. Now, to show that the r.h.s. of (5.49) can be made nonpositive by an appropriate choice of q, α, T, we can proceed exactly as in the proof of [13, Proposition 5.19] (just replacing in the latter J_q on $\varkappa a$ and m on $m + \frac{\beta}{2}$). □

Proposition 5.14. *Let* (A1)–(A10) *hold. Then, there exists* $t_1 > 0$*, such that, for any* $t > t_1$ *and for any* $\tau > 0$*, there exists* $q_1 = q_1(t,\tau) > 0$*, such that the following holds. If* $u_0 \in E_\theta^+$ *is such that there exist* $\eta > 0$*,* $r > 0$*,* $x_0 \in \mathbb{R}^d$ *with* $u_0(x) \geq \eta$*,* $x \in B_r(x_0)$ *and* $u \in \mathcal{X}_\infty$ *is the corresponding solution to* (1.1)*, then*

$$u(x,t) \geq q_1\exp\Bigl(-\frac{|x - x_0|^2}{\tau}\Bigr), \quad x \in \mathbb{R}^d. \tag{5.50}$$

Proof. We choose p and b as in (A10). Then one can rewrite (1.1) as follows

$$\frac{\partial}{\partial t}u(x,t) = \varkappa((a-b)*u)(x,t) - (m+p)u(x,t) + f(x,t),$$

where, for all $x \in \mathbb{R}^d$ and $t \geq 0$,

$$f(x,t) := \varkappa(b*u)(x,t) - u(x,t)(Gu)(x,t) + pu(x,t) \geq 0,$$

because of (A10). The rest of the proof repeats [13, Proposition 5.20]. $\square$

Now we are finally ready to proof Theorems 2.3–2.5.

Proof of Theorem 2.3. As it was mentioned above, one can get the statement, combining Propositions 5.10 and 5.12, provided that (5.24) holds. To get rid of the latter assumption, one can literally follow the proof of [13, Theorem 5.10] using the results of Propositions 5.13 and 5.14. $\square$

Proof of Theorem 2.5. Without loss of generality we can assume that $\theta - \theta_n \leq \frac{\theta}{2}$, $n \in \mathbb{N}$. Consider $v_0 \in E^+_{\theta/2} \cap C^\infty(\mathbb{R}^d)$, such that for some $x_0 \in \mathbb{R}^d$, $\delta \in (0, \frac{\theta}{2})$,

$$\delta 1\!\!1_{B_\delta(x_0)}(x) \leq v_0(x) \leq u_0(x), \qquad x \in \mathbb{R}^d.$$

Let $u_n(x,0) = v_0(x)$ and $u_n \in \mathcal{X}_\infty$ solves the following equation

$$\mathcal{F}^{(n)}_{\theta_n} u_n := \frac{\partial u_n}{\partial t} - \varkappa_n a_n * u_n + u_n G_n u_n + m u_n = 0. \tag{5.51}$$

Therefore by (A11) we obtain,

$$\mathcal{F}^{(n)}_{\theta_n} u_n = 0 = \mathcal{F}_\theta u \leq \mathcal{F}^{(n)}_{\theta_n} u. \tag{5.52}$$

Hence by Theorem 4.1 applied to $\mathcal{F}^{(n)}_{\theta_n}$, we obtain

$$0 \leq u_n(x,t) \leq u(x,t) \leq \theta.$$

Applying Theorem 2.3 to the equation (5.51), we have

$$\theta = \lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \lim_{t\to\infty} \operatorname*{essinf}_{x\in K} u_n(x + t\mathfrak{m}, t) \leq \lim_{t\to\infty} \operatorname*{essinf}_{x\in K} u(x + t\mathfrak{m}, t) \leq \theta,$$

that fulfils the proof. $\square$

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