Accelerated front propagation for monostable equations with nonlocal diffusion

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April 21, 2017

Abstract

We describe acceleration of the pulled front propagation for solutions to a class of monostable nonlinear equations with a nonlocal diffusion in $\mathbb{R}^d$. We show that the acceleration takes place if either the diffusion kernel or the initial condition has ‘regular’ heavy tails in $\mathbb{R}^d$ (in particular, decays slower than exponentially). Under general assumptions which can be verified for particular models, we present sharp estimates for the time-space zone which separates the region of convergence to the unstable zero solution with the region of convergence to the stable positive constant solution. We show the variety of different possible rates of the propagation starting from a little bit faster than a linear one up to the exponential rate. We describe differences between the case when the initial condition decays in all directions at infinity with the case of the initial condition decreasing along all coordinate axes (in positive directions).

Keywords: nonlocal diffusion, reaction-diffusion equation, front propagation, acceleration, monostable equation, nonlocal nonlinearity, long-time behavior, integral equation

2010 Mathematics Subject Classification: 35B40, 35K57, 47G20, 45G10

Contents

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1 Introduction

1.1 Preliminaries and notations

We will deal with the equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = \varkappa(a*u)(x,t) - mu(x,t) - u(x,t)(Gu)(x,t), \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ := [0,\infty)$, $d \in \mathbb{N}$; $\varkappa, m > 0$; $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a initial condition; the function $a \in L^1(\mathbb{R}^d) := L^1(\mathbb{R}^d, dx)$ is a probability kernel, i.e. $a(x) \geq 0$ for almost all (a.a. in the sequel) $x \in \mathbb{R}^d$ and

$$\int_{\mathbb{R}^d} a(x)\,dx = 1; \tag{1.2}$$

the symbol $*$ stands for the convolution in x on $\mathbb{R}^d$, i.e.

$$(a*u)(x,t) := \int_{\mathbb{R}^d} a(x-y)u(y,t)\,dx;$$

and, finally, G is a mapping on functions which is acting in x, i.e. $(Gu)(x,t) := \bigl(Gu(\cdot,t)\bigr)(x)$.

The function $u(x,t)$ may be interpreted as the local density of an evolving in time system of entities which reproduce themselves, compete, and die. The reproduction appears according to the dispersion, which is realized via the fecundity rate $\varkappa$ and the density a of a probability dispersion distribution. The death may appear due the constant inner mortality $m > 0$ within the system, as well as due to the density dependent rate Gu, which describes a competition within the system (the conditions below will ensure that $Gu \geq 0$ under appropriate bounds on u). For references about underlying individual-based models, see Subsection 1.4; we discuss there also another interpretation for the equation (1.1) rewritten in the reaction-diffusion form (1.20).

We fix the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ on the Euclidean space $\mathbb{R}^d$, $d \geq 1$. All functions in the sequel are supposed to be $\mathcal{B}(\mathbb{R}^d)$-measurable; and all subsets of $\mathbb{R}^d$ are supposed to be from $\mathcal{B}(\mathbb{R}^d)$.

We will consider (1.1) in a space $E := L^\infty(\mathbb{R}^d, dx)$ of essentially bounded functions on $\mathbb{R}^d$ with ess sup-norm, with respect to (w.r.t. in the sequel) the Lebesgue measure dx.

By a solution to (1.1) on $\mathbb{R}_+$, we will understand the so-called classical solution, that is a mapping $u : \mathbb{R}_+ \to E$ which is continuous in $t \in \mathbb{R}_+$ and continuously differentiable (in the sense of the norm in E) in $t \in (0, \infty)$.

We will write $v \leq w$ for $v, w \in E$, if $v(x) \leq w(x)$ for a.a. $x \in \mathbb{R}^d$. We set

$$E^+ := \{v \in E \mid v \geq 0\}, \qquad E^+_r := \{v \in E : 0 \leq v \leq r\}$$

for an $r > 0$. For each $y \in \mathbb{R}^d$, we denote by $T_y : E \to E$ the translation operator, given by

$$(T_y v)(x) := v(x - y), \quad x \in \mathbb{R}^d. \tag{1.3}$$

Here and below we will just write $x \in \mathbb{R}^d$ omitting 'for a.a.' before this.

For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $|x|$ denote its Euclidean norm, and set

$$\langle x \rangle := \max_{1 \leq j \leq d} x_j \in \mathbb{R}, \tag{1.4}$$

$$\Delta(x) := \bigl\{y \in \mathbb{R}^d : y_j \geq x_j,\ 1 \leq j \leq d\bigr\}. \tag{1.5}$$

Let $B_R(x_0)$ denote the closed ball (w.r.t. the Euclidean norm in $\mathbb{R}^d$) with the center at $x_0 \in \mathbb{R}^d$ and the radius $R > 0$.

We consider also the topology on E generated by the semi-norms $\|v\|_\Lambda := \sup_{x\in\Lambda} |v(x)|$ for all $\Lambda \subset \mathbb{R}^d$ is a compact. In particular, $v_n \in E$ converges to $v \in E$ in this topology if and only if v_n converges to v locally uniformly, which we denote $v_n \Rightarrow$ loc v, $n \to \infty$. The latter convergence means that $1\!\!1_\Lambda v_n \to 1\!\!1_\Lambda v$ in E, where $1\!\!1_\Lambda$ is the indicator-function of a compact set $\Lambda \subset \mathbb{R}^d$.

1.2 Main assumptions and basic facts

Consider now our main assumptions. To exclude the trivial case when $\|u(\cdot, t)\|_E$ converges to 0 as $t \to \infty$, we assume that

$$\beta := \varkappa - m > 0. \tag{A1}$$

We suppose that there exist two constant solutions $u \equiv 0$ and $u \equiv \theta > 0$ to (1.1), more precisely,

$$\begin{gathered} \textit{there exists } \theta > 0, \textit{ such that} \\ 0 = G0 \leq Gv \leq G\theta = \beta, \quad v \in E^+_\theta. \end{gathered} \tag{A2}$$

We will also assume that G is (locally) Lipschitz continuous in E^+_θ, namely,

$$\begin{gathered} \textit{there exists } l_\theta > 0, \textit{ such that} \\ \|Gv - Gw\| \leq l_\theta \|v - w\|, \quad v, w \in E^+_\theta. \end{gathered} \tag{A3}$$

We restrict ourselves to the case when the comparison principle for (1.1) holds. Namely, we assume that the right hand side (r.h.s. in the sequel) of (1.1) is a (quasi-)monotone operator:

$$\begin{gathered} \textit{for some } p \geq 0 \textit{ and for any } v, w \in E^+_\theta \textit{ with } v \leq w, \\ \varkappa a * v - v\, Gv + pv \leq \varkappa a * w - w\, Gw + pw. \end{gathered} \tag{A4}$$

We assume that the kernel a is not degenerate at the origin, namely,

$$\text{there exists } \rho, \delta > 0 \text{ such that } a(x) \geq \rho \text{ for a.a. } x \in B_\delta(0). \tag{A5}$$

Stability of the solution to (1.1) with respect to the initial condition in the topology of the locally uniform convergence requires continuity of G in this topology. Namely, we assume that

$$\begin{gathered} \text{for any } v_n, v \in E_\theta^+, \text{ such that } v_n \overset{\text{loc}}{\rightrightarrows} v,\ n \to \infty, \text{ one has} \\ Gv_n \overset{\text{loc}}{\rightrightarrows} Gv,\ n \to \infty. \end{gathered} \tag{A6}$$

We will consider the translation invariant case only:

$$\begin{gathered} \text{let } T_y,\ y \in \mathbb{R}^d, \text{ be a translation operator, given by (1.3), then} \\ (T_y Gv)(x) = (GT_y v)(x), \quad v \in E_\theta^+,\ x \in \mathbb{R}^d. \end{gathered} \tag{A7}$$

Under (A7), for any $r \equiv const \in (0, \theta)$, $Gr \equiv const$. In this case, we assume also that

$$Gr < \beta, \qquad r \in (0, \theta). \tag{A8}$$

Finally, we will distinguish two cases. If the condition

$$\int_{\mathbb{R}^d} |y| a(y) dy < \infty \tag{A9}$$

holds, then we assume, additionally to (A4), that

$$\begin{gathered} \text{there exist } p \geq 0,\ b \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),\ \delta > 0, \text{ such that} \\ a(x) - b(x) \geq \delta 1\!\!1_{B_\delta(0)}(x), \quad x \in \mathbb{R}^d, \\ w\, Gw \leq \varkappa b * w + pw, \quad w \in E_\theta^+, \end{gathered} \tag{A10}$$

and also

$$\int_{\mathbb{R}^d} x a(x)\, dx = 0 \in \mathbb{R}^d. \tag{A11}$$

Otherwise, if (A9) does not hold, then we assume that,

$$\begin{gathered} \text{for each } n \in \mathbb{N}, \text{ there exist} \\ 0 \leq a_n \in L^1(\mathbb{R}^d), \quad \varkappa_n > 0, \quad G_n : E \to E, \quad \theta_n \in (0, \theta] \\ \text{which satisfy (A1)–(A10) instead of } a,\ \varkappa,\ G,\ \theta, \text{ correspondingly,} \\ \text{such that } \int_{\mathbb{R}^d} x a_n(x) dx = 0 \in \mathbb{R}^d,\ n \in \mathbb{N}, \quad \theta_n \to \theta,\ n \to \infty, \\ \varkappa_n a_n * w - wG_n w \leq \varkappa a * w - wGw, \quad w \in E_{\theta_n}^+. \end{gathered} \tag{A12}$$

Examples which fulfill the conditions above are considered in Subsection 1.4. The following results were shown in [26].

Theorem 1.1 ([26, Theorems 2.1, 2.2, Proposition 4.2]). *Let (A1)–(A4) hold. Let $u_0 \in E_\theta^+$. Then, for each $T > 0$, there exists a unique solution $u = u(x,t)$ to (1.1) for $t \in [0,T]$; and $0 \leq u(\cdot,t) \leq \theta$ for all $t > 0$. Moreover, if $v_0 \in E_\theta^+$, $v = v(x,t)$ is the corresponding solution to (1.1), and if $u_0 \leq v_0$, then $0 \leq u(\cdot,t) \leq v(\cdot,t) \leq \theta$ for all $t > 0$.*

Note also that if u_0 is (uniformly) continuous function on $\mathbb{R}^d$, then $u(\cdot,t)$ will be also (uniformly) continuous function on $\mathbb{R}^d$ for all $t>0$. The comparison between solutions in Theorem 1.1 was a part of a more general result, which we will also use.

Theorem 1.2 ([26, Theorems 2.2]). *Let* (A1)–(A4) *hold. Let* $T>0$ *be fixed. Suppose that* $u_1,u_2:[0,T]\to E$ *are continuous mappings, continuously differentiable in* $t\in(0,T]$*, and such that, for* $(x,t)\in\mathbb{R}^d\times(0,T]$,

$$\frac{\partial u_1}{\partial t}-\varkappa a*u_1+mu_1+u_1Gu_1\leq\frac{\partial u_2}{\partial t}-\varkappa a*u_2+mu_2+u_2Gu_2,$$
$$u_1(x,t)\geq 0,\qquad u_2(x,t)\leq\theta,$$
$$0\leq u_1(x,0)\leq u_2(x,0)\leq\theta.$$

Then $u_1(x,t)\leq u_2(x,t)$ *for* $(x,t)\in\mathbb{R}^d\times[0,T]$.

Theorem 1.3 ([26, Theorems 2.3, 2.5]). *Let either* (A1)–(A11) *hold or* (A12) *hold. Let* $u_0\in E_\theta^+$*,* $u_0\not\equiv 0$*, and let* $u=u(x,t)$ *be the corresponding solution to* (1.1)*. Then, for each compact set* $K\subset\mathbb{R}^d$,

$$\lim_{t\to\infty}\operatorname*{ess\,inf}_{x\in K}u(x,t)=\theta.$$

Remark 1.4. For a brevity of notations, we will treat $u_0\not\equiv 0$ as follows: there exist $\delta,\rho>0$ and $x_0\in\mathbb{R}^d$, such that $u_0(x)\geq\delta$ for a.a. $x\in B_\rho(x_0)$.

1.3 Description of the problem and results

The present paper is aimed to study the propagation of the front for a solution to the equation (1.1). To describe this notion, one needs two families $\{\mathscr{O}_t\}$ and $\{\mathscr{C}_t\}$ of subsets in $\mathbb{R}^d$, such that $\mathscr{O}_t\cap\mathscr{C}_t=\varnothing$ for all t, and $\mathscr{O}_t$, being unbounded, vanishes as $t\to\infty$, i.e. $\mathscr{O}_t\searrow\varnothing$, whereas $\mathscr{C}_t$, being non-necessary bounded, expands to $\mathbb{R}^d$, i.e. $\mathscr{C}_t\nearrow\mathbb{R}^d$, $t\to\infty$. Next, for an $x_t\in\mathscr{C}_t$, the values of $u(x_t,t)$ would converge (as $t\to\infty$) to θ, whereas, for a $y_t\in\mathscr{O}_t$, the values of $u(y_t,t)$ would converge to 0. Moreover, we are going to show that (under appropriate assumptions) these convergences hold uniformly in space, and the second one holds exponentially fast in time. Then, by the *front*, we will understand the set $\Gamma_t:=\mathbb{R}^d\setminus(\mathscr{O}_t\cup\mathscr{C}_t)$.

Naturally, we are interested to get the set Γ_t, in some sense, the 'narrowest' possible. We will do this as follows. We will choose an appropriate function $c:\mathbb{R}^d\to(0,\infty)$ and set

$$\Lambda(t,c):=\bigl\{x\in\mathbb{R}^d\bigm| c(x)\geq e^{-\beta t}\bigr\},\tag{1.6}$$

where $\beta>0$ is given by (A1). Two model examples for us will be

$$c(x)=b(|x|)\quad\text{and}\quad c(x)=\int_{\Delta(x)}b(|y|)\,dy,\qquad x\in\mathbb{R}^d,\tag{1.7}$$

where $\Delta(x)$ is given by (1.5) and $b:\mathbb{R}_+\to\mathbb{R}_+$ with $\int_{\mathbb{R}_+}b(s)s^{d-1}\,ds<\infty$. We will refer to them as to integrable and 'monotone' case, respectively. (In the 'monotone' case, evidently, c decays to 0 along all coordinate axes in $\mathbb{R}^d$.)

We set then, for a small enough $\varepsilon > 0$,

$$\Lambda_\varepsilon^\pm(t,c) := \Lambda\bigl((1\pm\varepsilon)t,c\bigr); \tag{1.8}$$

$$\mathscr{C}_t := \Lambda_\varepsilon^-(t,c), \qquad \mathscr{O}_t := \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c).$$

Note that $\Gamma_t = \mathbb{R}^d \setminus (\mathscr{O}_t \cup \mathscr{C}_t)$, in general, will also expand as $t \to \infty$ (moreover, in the accelerated case described below it will be even with necessity, see [34]). We are aimed to show that, for a small enough $\varepsilon > 0$,

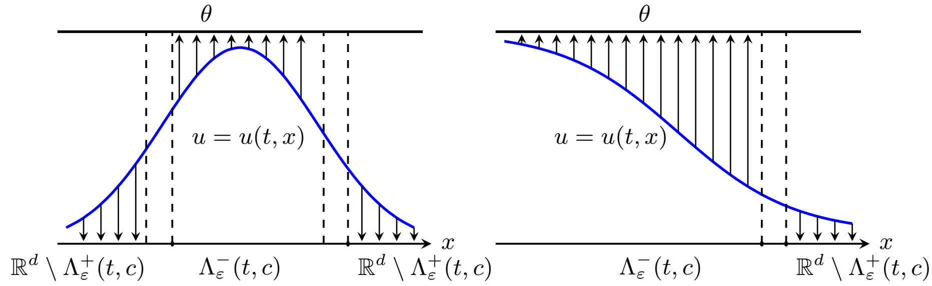
$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{x\in\Lambda_\varepsilon^-(t,c)} u(x,t) = \theta, \tag{1.9a}$$

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{x\notin\Lambda_\varepsilon^+(t,c)} u(x,t) = 0. \tag{1.9b}$$

We have shown in [26, Proposition 5.7], that if u_0 decays along a direction in $\mathbb{R}^d$, then the corresponding solution decays along this direction as well. Therefore, the choice between integrable and 'monotone' c in (1.7) will depend on the initial condition. Namely, if $\lim\limits_{|x|\to\infty} u_0(x) = 0$ (in particular, if $u_0(x) = q(|x|)$, $x \in \mathbb{R}^d$ for an appropriate $q : \mathbb{R}_+ \to \mathbb{R}_+$), then we will use an integrable c; in this case $\Lambda_\varepsilon^\pm(t,c)$ will be just a ball with a time-dependent radius. However, if

$$\lim_{\rho\to\infty} u_0((\rho,\ldots,\rho)) = 0, \qquad \lim_{\rho\to-\infty} u_0((\rho,\ldots,\rho)) \in (0,\theta]$$

(in particular, if $u_0(x) = \int_{\Delta(x)} p(y)\,dy$, $x \in \mathbb{R}^d$, for an integrable $p : \mathbb{R}^d \to \mathbb{R}_+$), then we will use a 'monotone' c with the correspondent unbounded $\Lambda_\varepsilon^\pm(t,c)$ (see Figure 1 for the case $d = 1$).



(a) The case of $u_0(\pm\infty) = 0$

(b) The case of monotone u_0 with $u_0(-\infty) \in (0,\theta]$, $u_0(\infty) = 0$

Figure 1: Sketch of the solution for $d = 1$.

The propagation of the front, described above, is said to have a constant speed (or just is linear in time), if $\Lambda(t,c) = t\,\Lambda(1,c)$. (Here and below $tA := \{tx : x \in A\}$ for an $A \subset \mathbb{R}^d$.) In particular, if $b(s) = e^{-\lambda s}$ and c is integrable, then $\Lambda(t,c)$ is the ball with the radius $\eta(t) = \frac{\beta}{\lambda}t$. See also the discussion in Subsection 1.5 below.

In contrast, the effect of an infinite speed of propagation, see [32, 34, 56], is called sometimes in literature an acceleration of the propagation, having in mind, for example, that the mentioned above radius $\eta(t)$ of the ball $\Lambda(t,c)$ (in the integrable case) is such that, cf. Lemma 2.15 below, $\frac{\eta(t)}{t} \to \infty$, $t \to \infty$.

To describe the class of the functions c for the accelerated case, we consider an appropriate sub-class $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ of regular sub-exponential functions on $\mathbb{R}$, which are, in particular, decreasing at ∞ and, for all $k>0$,

$$\lim_{s\to\infty} e^{ks}b(s)=\infty \tag{1.10}$$

see Definitions 2.4 and 2.10 for details. In particular, any function which is asymptotically proportional at ∞ to either of

$$(\log s)^{\nu}s^{-(d+\delta)}, \quad s^{\nu}\exp\bigl(-D(\log s)^{q}\bigr), \quad s^{\nu}\exp\bigl(-s^{\alpha}\bigr), \quad s^{\nu}\exp\Bigl(-\frac{s}{(\log s)^{\gamma}}\Bigr),$$

belongs to the class $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$, provided that $D,\delta>0$, $q,\gamma>1$, $\alpha\in(0,1)$, $\nu\in\mathbb{R}$; see Example 2.18.

We formulate now our main result for the model 'symmetric' case.

Theorem 1.5. *Let $b,q:\mathbb{R}_+\to\mathbb{R}_+$ be bounded functions such that, for some $M,\mu,r,\delta>0$,*

$$b(s)+q(s)\le\frac{M}{(1+s)^{d+\mu}} \quad \text{for a.a. } s\ge r, \tag{1.11}$$

and $q(s)\ge\delta$ for a.a. $s\in[0,\rho]$. Let either (A1)–(A11) hold or (A12) hold. Suppose that $a(x)=b(|x|)$, $x\in\mathbb{R}^d$. Let either of the following conditions holds

$$\sup_{s\in\mathbb{R}_+}\frac{q(s)}{b(s)}<\infty, \tag{1.12}$$

$$\sup_{s\in\mathbb{R}_+}\frac{b(s)}{q(s)}<\infty. \tag{1.13}$$

1) Let $q:\mathbb{R}\to[0,\theta]$ and

$$u_0(x)=q(|x|), \quad x\in\mathbb{R}^d. \tag{1.14}$$

Then

(a) if $b\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (1.12) holds, then (1.9) holds with $c=a$;

(b) if $q\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (1.13) holds, then (1.9) holds with $c=u_0$.

2) Let $\displaystyle\int_0^\infty q(s)s^{d-1}ds\in(0,\theta]$ and

$$u_0(x)=\int_{\Delta(x)}q(|y|)dy, \quad x\in\mathbb{R}^d. \tag{1.15}$$

Then

(a) if $b\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (1.12) holds, then (1.9) holds with

$$c(x):=\int_{\Delta(x)}a(y)dy, \quad x\in\mathbb{R}^d;$$

(b) if $q\in\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and (1.13) holds, then (1.9) holds with $c=u_0$.

Clearly, (1.14) and (1.11) imply that u_0 is integrable on $\mathbb{R}^d$, whereas (1.15) yields that u_0 is monotone along all coordinate axes.

In other words, c in (1.9) is either a or u_0, whichever decays slower, for an integrable u_0, and c is either $\int_{\Delta(x)} a$ or u_0, whichever decays slower, for a 'monotone' u_0. If both (1.12)–(1.13) hold, then (1.9) takes place with either of these functions c.

The proof of Theorem 1.5 (see Remark 2.25) follows from Propositions 2.21–2.24 below, which weaken the assumptions on u_0 and a. Roughly speaking, we allow that

$$b_+(|x|) \leq a(x) \leq b^+(|x|), \quad x \in \mathbb{R}^d, \tag{1.16}$$

and either

$$q_\circ(|x|) \leq u_0(x) \leq q^\circ(|x|), \quad x \in \mathbb{R}^d, \tag{1.17}$$

or

$$\int_{\Delta(x)} q_\circ(|y|)\, dy \leq u_0(x) \leq \int_{\Delta(x)} q^\circ(|y|)\, dy, \quad x \in \mathbb{R}^d, \tag{1.18}$$

provided that, for $s \to \infty$,

$$\log b_+(s) \sim \log b^+(s) \sim \log b(s), \qquad \log q_\circ(s) \sim \log q^\circ(s) \sim \log q(s), \tag{1.19}$$

where b, q are as the above, and (1.12)–(1.13) are properly modified.

In Examples 2.26–2.30, we rewrite (1.9) explicitly for different particular choices of a and u_0.

1.4 Reaction-diffusion form of the equation. Examples

Under assumptions (A1)–(A2), one can rewrite (1.1) in the so-called reaction-diffusion form

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \varkappa(a * u)(x,t) - \varkappa u(x,t) + (Fu)(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{1.20}$$

where

$$Fv := v(\beta - Gv), \quad v \in E, \tag{1.21}$$

and we will have then

$$F\theta = F0 = 0 \leq Fv \leq \beta v, \quad 0 \leq v \leq \theta. \tag{1.22}$$

The linear operator

$$Lu = \varkappa a * u - \varkappa u \tag{1.23}$$

in (1.20) is the generator of a nonlocal diffusion in $\mathbb{R}^d$, also known as the random walk in continuous time. The properties of solutions to the corresponding nonlocal evolution equation $\frac{\partial}{\partial t} u = Lu$ have been actively studied recently, see e.g. [3, 9, 27, 40] and the references therein.

The solution u to the equation (1.20) may be interpreted as a density of a species which invades according to a nonlocal diffusion within the space $\mathbb{R}^d$ meeting a reaction F, see e.g. [22, 47, 51].

In the recent decade, there is a growing interest to the study of nonlocal reaction-diffusion equations of the form (1.20) see e.g. [1, 5, 8, 13–17, 32, 38, 42, 48, 52, 55, 56] for the monostable case (1.22); for the origins of the topic see e.g. [4, 19, 49, 50, 54]. We will distinguish the following examples.

Example 1.6 (Reaction–diffusion equation with a local reaction)**.** A particular example of (1.20), with $F(u) = f(u)$ for a function $f : \mathbb{R} \to \mathbb{R}$, was considered e.g. in [1, 5, 8, 13–15, 17, 32, 52, 56]. Two classical examples are $f(s) = \nu s(\theta - s)$, cf. [28], and $f(s) = \nu s(\theta - s)^2$, cf. [39]; for some $\nu > 0$. We assume (A1) and (A5) as before, whereas the assumptions (A2)–(A4), (A6)–(A8), (A10) are fulfilled if

$$\begin{gathered} f \textit{ is Lipschitz continuous on } [0,\theta]; \\ \lim_{r\to 0+} \frac{f(r)}{r} = \beta; \qquad (1.24) \\ f(0) = f(\theta) = 0; \quad 0 < f(r) < \beta r, \ r \in (0,\theta). \end{gathered}$$

If (A9) does not hold, then, to fulfill (A12), it is enough to choose, for $m \in (0, \varkappa)$, a sequence of sets $\Lambda_n \subset \mathbb{R}^d$, $\Lambda_n \nearrow \mathbb{R}^d$, such that $\varkappa_n := \varkappa \int_{\Lambda_n} a(x)dx > m$, cf. (1.2), and define $a_n(x) := \bigl(\int_{\Lambda_n} a(x)dx\bigr)^{-1} 1\!\!1_{\Lambda_n}(x)a(x)$, $x \in \mathbb{R}^d$, provided that $\int_{\Lambda_n} xa(x)dx = 0$. In particular, if $a(-x) = a(x)$, $x \in \mathbb{R}^d$, then one can choose $\Lambda_n := B_n(0)$ (for big enough $n \in \mathbb{N}$, to ensure that $\varkappa_n > m$).

Note that the assumptions (1.24) imply, in particular, that

$$f(u) \le f'(0)u, \quad 0 \le u \le \theta. \qquad (1.25)$$

The importance of the assumption (1.25) for the front propagation described above was pointed out in e.g. [8, 16]; this condition can be weaken though, see [53]. Moreover, the condition (1.25) leads to the possibility to describe the front using the linearized version of the corresponding equations (1.1) and (1.20) about 0, that is just (1.32) below. We are going to show a similar result for a general F which fulfills (1.22). Note also that, for the classical Fisher–KPP-type equation, where (1.23) in (1.1) is replaced by the Laplace operator in $\mathbb{R}^d$, the condition (1.25) was used in e.g. [31, 54]. The condition $\beta = f'(0) > 0$ is also standard in the literature; the opposite degenerate case $f'(0) = 0$ leads to specific effects, see e.g. [2, 57].

Example 1.7 (Spatial logistic equation)**.** Let $\varkappa^- > 0$ and $a^-(x)$ be a probability kernel. We consider

$$Gu = \varkappa^- a^- * u. \qquad (1.26)$$

We assume (A1) and (A5) as before. Under (A1), we have then

$$\theta = \frac{\varkappa - m}{\varkappa^-} > 0.$$

One can rewrite then (1.1) in the form of (1.20) with the reaction given by

$$Fu = \varkappa^- u(\theta - a^- * u). \qquad (1.27)$$

This equation, for different relations between parameters, was considered in [6, 20, 24, 25, 30, 45, 46]. The underlying individual-based model was proposed in [7, 18], see also [20, 23, 30] regarding the derivation of the equation (1.1) in this case.

It is easy to check that the conditions (A2)–(A3), (A6)–(A8) are satisfied. The condition (A4) holds if and only if

$$\varkappa a(x) \geq (\varkappa - m)a^{-}(x), \qquad x \in \mathbb{R}^d. \tag{1.28}$$

Condition (A10) holds if we additionally assume that there exists $\delta > 0$, such that

$$\varkappa a(x) - (\varkappa - m)a^{-}(x) \geq \delta 1\!\!1_{B_\delta(0)}(x), \qquad x \in \mathbb{R}^d. \tag{1.29}$$

In this case we can put, in (A10), $b(x) = (\varkappa - m)a^{-}(x)$, $p = 0$. If (A9) does not hold, then, to fulfil (A12), one can proceed as in the previous example. Namely, we define a_n as before, and we set $G_n u = \varkappa^{-} a_n^{-} * u$, where $a_n^{-}(x) := 1\!\!1_{\Lambda_n}(x)a^{-}(x)$, $x \in \mathbb{R}^d$.

Example 1.8 (The case $Gu = \varkappa^{-}a^{-} * u - g_1(a^{-} * u)$)**.** Let $g(s) = \varkappa^{-}s - g_1(s)$, where $\varkappa^{-} > 0$, $g_1 : [0, \theta] \to \mathbb{R}_+$ is increasing and Lipschitz continuous, such that $g_1(s) = o(s)$, as $s \to 0$ and $\varkappa^{-}s \geq g_1(s)$, for $s \in (0, \theta)$. We define $Gv = g(a^{-} * v)$, where a^{-} is a probability density. Namely, we consider the following equation,

$$\frac{\partial u}{\partial t} = \varkappa(a * u) - \varkappa^{-}u(a^{-} * u) + ug_1(a^{-} * u) - mu.$$

As in the previous example, (A4) holds if and only if (1.28) holds. The rest of the assumptions can be characterized in a straightforward way. Typical example is $g(s) = \beta(1 - (1 - s)^n)$, with $\theta = 1$, $n \geq 2$. In this case the corresponding reaction is, cf. (1.27),

$$F(u) = \beta u(1 - a^{-} * u)^n, \quad n \geq 2. \tag{1.30}$$

Going back to the general G, note that one can start, conversely, from the nonlocal reaction-diffusion equation (1.20), where $\varkappa$ and a are as the above, and there exists $\theta > 0$, such that F fulfills (1.22). More precisely, we assume that $\frac{Fv}{v} \in E$ if only $v \in E$, and $\beta = \inf\{\gamma : Fv \leq \gamma v,\ 0 \leq v \leq \theta\} > 0$. Then one can rewrite (1.20) in the form (1.1) with $m := \varkappa - \beta > 0$ and $Gv := \beta - \frac{Fv}{v}$. Then (A1)–(A2) hold, provided that $\frac{Fv}{v} \to \beta$ as $v \to 0$, where both convergences are in the sense of the norm in E. Since $F0 = 0$, the convergence above with necessity implies that $F'(0)v = \beta v$, where $F'(0)$ is the Fréchet derivative taken over $\{v \geq 0\}$. Clearly, the reaction of the form (1.30) satisfies (1.22).

1.5 Overview of the existing results

Mollison [45, 46] was, probably, the first one who pointed out that, to have a propagation with a constant speed in a nonlocal reaction-diffusion equation, both the kernel a and the initial condition u_0 have to decay quick enough at infinity. Namely, he has considered the case $d = 1$, F given by (1.21) with G of the form (1.26), where $\varkappa^{-} = \varkappa$, $a^{-} = a$, $\beta = \varkappa$ (that corresponds to $m = 0$), and u_0 is a decreasing on the whole $\mathbb{R}$ initial condition. He has shown that the property of the corresponding propagation front to have an averaged constant speed is deeply related with the existence of a $\lambda > 0$, such that

$$\int_{\mathbb{R}} a(s)e^{\lambda s}ds < \infty, \qquad \sup_{s\in\mathbb{R}} u_0(s)e^{\lambda s} < \infty.$$

Note that such u_0 gave an unbounded set $\Lambda_\varepsilon^-(t,c) = (-\infty, \gamma t)$, cf. (1.6), (1.8) and Figure 1 (b), where γ was related to a speed of the propagation.

In [25, Theorems 5.9, 5.10, Corollary 5.11], we have considered, for a general $d \in \mathbb{N}$, the equation (1.1) with $m \in (0, \varkappa)$ and G given by (1.26) under the assumptions (1.28)–(1.29). There was proved that the corresponding front of the propagation has a constant speed, provided that $a \in L^\infty(\mathbb{R}^d)$ and

$$\begin{aligned} &\int_{\mathbb{R}^d} a(x)e^{\mu|x|}dx < \infty, &&\text{for some } \mu > 0,\\ &\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} u_0(x)e^{\lambda|x|} < \infty, &&\text{for any } \lambda > 0. \end{aligned} \tag{1.31}$$

In particular, here $\Lambda(t,c) = t\Lambda(1,c)$ appeared as a bounded convex set, cf. (1.6) and Figure 1 (a).

Consider also, regardless of (1.31), for a $u_0 \in E_\theta^+$, the following linear initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t} w(x,t) = \varkappa(a * w)(x,t) - mw(x,t), \\ w(x,0) = u_0(x), \end{cases} \tag{1.32}$$

whose solution is evidently given by

$$w(x,t) = e^{-mt}\big(e^{\varkappa tA}u_0\big)(x), \qquad Av := a * v, \quad v \in E,$$

as A is a bounded operator on E. Let (A1)–(A4) hold and u be the corresponding solution to (1.1). Then, by Theorem 1.1 and (A2), $(Gu)(\cdot,t) \geq 0$, $t \in \mathbb{R}_+$, hence by e.g. [37, Lemma 3.3.2],

$$u(x,t) \leq w(x,t), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R}_+. \tag{1.33}$$

It is crucial that the technique of [25] allows to prove the convergence (1.9b) for the solution w to (1.32) as well, provided that (1.31) holds, cf. Remark 4.9 below. As a result, because of (1.33), one gets that, for any G which satisfies (A1)–(A4), the conditions (1.31) imply (1.9b) for the solution to (1.1). In other words, the propagation of a solution to (1.1) is at most linear in time.

The conditions (1.31) are closed to the necessary ones, cf. [32, 56]. We have proved in [25, Theorem 5.21] that if a bit weaker form of (1.31) fails for a (roughly, if a is 'heavier' than any exponent at infinity), then the convergence (1.9a) holds with $\Lambda_\varepsilon^-(t,c)$ replaced by tK for an *arbitrary* compact set $K \subset \mathbb{R}^d$. Therefore, the propagation of the front is faster than linear.

The acceleration for the case $d = 1$ and the local operator $Fu = f(u)$, cf. Example 1.6, was known in mathematical biology, see e.g. the results and references in [33, 41, 43]. The first rigorous result in this direction was done by Garnier [32], who proved an analogue of (1.9) for $d = 1$ and a compactly supported initial condition u_0. However, in his approach, the set $\Lambda_\varepsilon^+(t,c) :=$

$\Lambda\bigl((1+\varepsilon)t,c\bigr)$ in (1.9b) was replaced by $\Lambda\bigl(\gamma t,c\bigr)$ with some (unknown) $\gamma>1$, i.e. the result was not sharp. Note that the technique in [32] was inspired by [35], where an acceleration was shown for the mentioned above classical KPP-equation with a slowly decaying initial condition; see also the recent paper [36]. Analogical results were obtained in [10, 12] for the equation of the type (1.20), where L, cf. (1.23), was replaced to a fractional Laplacian (in particular, the kernel a was singular and non-integrable); cf. also [11, 21, 44].

1.6 Structure of the paper

The paper is organized as follows. In Subsection 2.1, we describe the needed classes of heavy tailed functions on the real line (in particular, the class $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$, see Definition 2.10 and Example 2.18) and the corresponding classes of functions on $\mathbb{R}^d$ (see Definition 2.11). In Subsection 2.2, we formulate Theorems 2.19 and 2.20, which yield (1.9a) and (1.9b), correspondingly. Next, in Propositions 2.21–2.24, we present sufficient conditions on a and u_0 which ensure that (1.9a)–(1.9b) hold simultaneously. Finally, in Examples 2.26–2.30, we rewrite the sets $\Lambda_\varepsilon^\pm$ in (1.9) explicitly for several particular choices of a and u_0.

In Section 3, we prove various facts about classes of functions on $\mathbb{R}$ and $\mathbb{R}^d$ we consider. In particular, Proposition 3.13 shows why one can weaken the radially symmetric conditions up to (1.16)–(1.18), provided that (1.19) holds.

Subsection 4.1 is devoted to the proof of Theorems 2.19 that implies (1.9a). In Proposition 4.4, we consider an appropriate sub-solution, truncated by a small enough level $\lambda\in(0,\theta)$, to the linear problem (1.32); and then, using the continuity of G at $0\in E_\theta^+$ guaranteed by (A3), we show in Proposition 4.7 that it will be a sub-solution to (1.1) as well. Then, using the hair-trigger effect, see Theorem 1.3, we 'rise' the level λ arbitrary close to θ.

In Subsection 4.2, we prove Theorem 2.20 that implies (1.9b). Because of (1.33), it is enough to show (1.9b) for the corresponding solution to (1.32). In Proposition 4.12, we show that it will hold if only one can find a function ω such that, informally, $a*\omega$ and ω will have 'the same' behavior at 'infinity'. Using the technique developed in [27], we show that, for the case when, roughly speaking, a decays at 'infinity' slower than u_0, the ω can be constructed by a^α for an arbitrary $\alpha<1$ which is close enough to 1. The arbitrariness of the α and of the ε in (1.9b) allow us, in Theorem 4.21, to get rid of α. Note that, in the case when u_0 decays slower than a, one can just take $\omega=u_0$, see the proofs of Propositions 2.22–2.24 in Subsection 4.3.

2 Main results

2.1 Classes of functions

We define first several classes of functions on $\mathbb{R}$.

Definition 2.1. A function $b:\mathbb{R}\to\mathbb{R}_+$ is said to be

- *(right-side) long-tailed*, if there exists $\rho=\rho_b\geq 0$, such that $b(s)>0$ for all $s\geq\rho$; and, for any $\tau\geq 0$,

$$\lim_{s\to\infty}\frac{b(s+\tau)}{b(s)}=1; \tag{2.1}$$

– *(right-side) tail-decreasing*, if there exists $\rho = \rho_b \geq 0$, such that $b = b(s)$ is strictly decreasing on $[\rho, \infty)$ to 0. In particular, $b(s) > 0$, $s \geq \rho$;

– *(right-side) tail-log-convex*, if there exists $\rho = \rho_b > 0$, such that $b(s) > 0$, $s \geq \rho$, and the function $\log b$ is convex (and hence continuous) on (ρ, ∞).

Remark 2.2. By [29, Lemma 2.17], each long-tailed function b satisfies (1.10) for all $k > 0$.

Remark 2.3. By [27, Lemma 2.7], for each tail-log-convex function b and for each $\tau > 0$, the function $\dfrac{b(s+\tau)}{b(s)}$ in (2.1) is nondecreasing in $s \in [\rho, \infty)$.

Definition 2.4. Let $\mathcal{S}_{\mathrm{reg},d}$ be the set of all bounded functions $b : \mathbb{R} \to \mathbb{R}_+$ such that

1) b is tail-decreasing and tail-log-convex with the same $\rho = \rho_b > 1$, such that $b(\rho) \leq 1$ (without loss of generality); and

$$\int_{-\infty}^{\rho} b(s)\, ds + \int_{\rho}^{\infty} b(s) s^{d-1}\, ds < \infty \tag{2.2}$$

2) there exist $\delta = \delta_b \in (0,1)$ and an increasing function $h = h_b : (0, \infty) \to (0, \infty)$, with $h(s) < \dfrac{s}{2}$ and $\lim\limits_{s\to\infty} h(s) = \infty$, such that

$$\lim_{s\to\infty} \frac{b(s \pm h(s))}{b(s)} = 1, \tag{2.3}$$

$$\lim_{s\to\infty} b\bigl(h(s)\bigr) s^{1+\delta} = 0. \tag{2.4}$$

Remark 2.5. By [27, Proposition 2.16], a tail-decreasing function is long-tailed if and only if (2.3) holds.

Remark 2.6. By [29, Lemma 4.13], see also [27, Remark 3.3] for details, each $b \in \mathcal{S}_{\mathrm{reg},1} \subset \mathcal{S}_{\mathrm{reg},d}$, $d > 1$, is such that

$$(b * b)(s) = \int_{\mathbb{R}} b(s-\tau) b(\tau)\, d\tau \sim 2\biggl(\int_{\mathbb{R}} b(\tau) d\tau\biggr) b(s), \qquad s \to \infty.$$

Remark 2.7. By [27, Remark 3.4], for each $b \in \mathcal{S}_{\mathrm{reg},1}$ and for δ given by Definition 2.4, there exists $B = B(\delta) > 0$ and $s_\delta > 0$ such that

$$b(s) \leq \frac{B}{s^{1+\delta}}, \quad s \geq s_\delta. \tag{2.5}$$

Definition 2.8. Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ and, for some $\rho \geq 0$, $b_i(s) > 0$ for all $s \in [\rho, \infty)$, $i = 1, 2$. The functions b_1 and b_2 are said to be *(asymptotically) log-equivalent*, if

$$\log b_1(s) \sim \log b_2(s), \quad s \to \infty. \tag{2.6}$$

Remark 2.9. By [27, Proposition 3.11], if $b \in \mathcal{S}_{\mathrm{reg},d}$ and b_1 is an even bounded tail-decreasing and tail-log-convex function, which satisfies (2.2) and (2.3) with the same h, and if b and b_1 are log-equivalent, then (2.4) holds, i.e. $b_1 \in \mathcal{S}_{\mathrm{reg},d}$.

Definition 2.10. We define the set $\widetilde{\mathcal{S}}_{\mathrm{reg},d} \subset \mathcal{S}_{\mathrm{reg},d}$, $d \in \mathbb{N}$ as follows. Let $\widetilde{\mathcal{S}}_{\mathrm{reg},1} := \mathcal{S}_{\mathrm{reg},1}$, whereas, for $d > 1$, let $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ be the set of all functions $b \in \mathcal{S}_{\mathrm{reg},d}$, such that

– either, for some $\mu, M > 0$,

$$b(s) = \frac{M}{(1+s)^{d+\mu}}, \quad s \in \mathbb{R}_+, \tag{2.7}$$

– or, for all $\nu \geq 1$,

$$\lim_{s\to\infty} b(s)s^\nu = 0. \tag{2.8}$$

We introduce also several other classes of functions.

Definition 2.11. 1) Let $\mathcal{D}_d$ be the set of all bounded tail-decreasing functions $b : \mathbb{R} \to (0,\infty)$, such that (2.2) holds, and, for each $r > 0$, $\inf\limits_{|s|<r} b(s) > 0$.

2) Let $\mathcal{R}$ be the set of all bounded radially symmetric functions $c : \mathbb{R}^d \to (0,\infty)$, such that $c(x) = b(|x|)$, $x \in \mathbb{R}^d$ for some $b = b_c \in \mathcal{D}_d$. Note that, because of (2.2), $\mathcal{R} \subset L^1(\mathbb{R}^d)$.

3) Let $\mathcal{L} \subset \mathcal{R}$ be the set of all functions $c \in \mathcal{R}$, such that $b = b_c \in \mathcal{D}_d$ is tail-log-convex and long-tailed.

4) Let $\mathcal{I}$ be the set of all functions of the form

$$c(x) = \int_{\Delta(x)} p(y)dy, \qquad p \in \mathcal{R}, \tag{2.9}$$

where $\Delta(x)$ is given by (1.5).

5) Let $\mathcal{N} \subset \mathcal{I}$ be the set of all functions $c \in \mathcal{I}$ of the form (2.9) with $p \in \mathcal{L}$.

Definition 2.12. Let $b \in \mathcal{D}_d$. A function $c \in \mathcal{R} \cup \mathcal{I}$ given by

$$c(x) = \begin{cases} b(|x|),\ x \in \mathbb{R}^d, & \text{if } c \in \mathcal{R}, \\ \displaystyle\int_{\Delta(x)} b(|y|)dy,\ x \in \mathbb{R}^d, & \text{if } c \in \mathcal{I}, \end{cases} \tag{2.10}$$

is said to be *constructed by* the b.

Definition 2.13. For any $c \in \mathcal{R} \cup \mathcal{I}$, $t \geq 0$, and $\varepsilon \in (0,1)$, we define the sets

$$\Lambda_\varepsilon^\pm(t,c) := \bigl\{x \in \mathbb{R}^d : c(x)e^{\beta(1\pm\varepsilon)t} \geq 1\bigr\}. \tag{2.11}$$

Clearly, $\Lambda_\varepsilon^-(t,c) \subset \Lambda_\varepsilon^+(t,c)$.

Let $b : \mathbb{R} \to \mathbb{R}_+$ be a tail-decreasing function with, cf. Definition 2.1, $\rho > 1$ and $b(\rho) \leq 1$. Then, for $s \in \bigl(0, b(\rho)\bigr]$, there exists the inverse function $b^{-1} = b^{-1}(s)$, which is decreasing there. For arbitrary $\varepsilon \in (0,1)$ and $\beta > 0$, we set

$$\beta_\varepsilon^- := (1-\varepsilon)\beta > 0, \qquad \beta_\varepsilon^+ := (1+\varepsilon)\beta > 0; \tag{2.12}$$

$$t_{\rho,\varepsilon}^\pm = t_{\rho,\varepsilon}^\pm(b) := -\frac{1}{\beta_\varepsilon^\pm}\log b(\rho) \geq 0. \tag{2.13}$$

Since $\bigl(0, b(\rho)\bigr] \subset (0,1]$, one can define, for $t \geq t^-_{\rho,\varepsilon} > t^+_{\rho,\varepsilon}$, all the following functions

$$\eta(t) = \eta(t,b) := b^{-1}\bigl(e^{-\beta t}\bigr), \tag{2.14}$$

$$\eta^+_\varepsilon(t) = \eta^+_\varepsilon(t,b) := \eta\bigl((1+\varepsilon)t\bigr) = b^{-1}\bigl(e^{-\beta^+_\varepsilon t}\bigr), \tag{2.15}$$

$$\eta^-_\varepsilon(t) = \eta^-_\varepsilon(t,b) := \eta\bigl((1-\varepsilon)t\bigr) = b^{-1}\bigl(e^{-\beta^-_\varepsilon t}\bigr). \tag{2.16}$$

Clearly, all these functions are increasing to ∞, and

$$\eta^+_\varepsilon(t) \geq \eta(t) \geq \eta^-_\varepsilon(t) \geq \rho, \qquad t \geq t^-_{\rho,\varepsilon}. \tag{2.17}$$

Remark 2.14. Note that, if $c \in \mathcal{R}$ is constructed by a $b \in \mathcal{D}_d$, and, cf. Definition 2.1, $\rho \geq 0$ is such that b is decreasing on (ρ,∞) to 0 and $b(\rho) \leq 1$, then, for any $t \geq t^-_{\rho,\varepsilon}$, cf. (2.13), (1.8),

$$\Lambda^\pm_\varepsilon(t,c) = \bigl\{x \in \mathbb{R}^d : |x| \leq \eta^\pm_\varepsilon(t,b)\bigr\}. \tag{2.18}$$

Note also that, for a function $c \in \mathcal{I}$ constructed by some function $b \in \mathcal{D}_d$, we have, because of (2.9), that $\sup_{x\in\mathbb{R}^d} c(x) = \int_{\mathbb{R}^d} b(|y|)\,dy > 0$. As a result, in particular, for each $c \in \mathcal{R}\cup\mathcal{I}$ and $\varepsilon \in (0,1)$, $\Lambda^\pm_\varepsilon(t,c) \neq \varnothing$ for $t \geq \tau = \tau(\varepsilon,c) > 0$.

Lemma 2.15. *Let $b : \mathbb{R} \to \mathbb{R}_+$ be a tail-decreasing and long-tailed function. Then, for any $\varepsilon \in (0,1)$ and for any $c > 0$,*

$$\eta^-_\varepsilon(t) - ct \to \infty, \quad t \to \infty.$$

Proof. Since b is long-tailed, (1.10) holds for any $k > 0$. Therefore, one has

$$\exp\Bigl(\frac{\beta^-_\varepsilon}{c}\bigl(\eta^-_\varepsilon(t) - ct\bigr)\Bigr) = \exp\Bigl(\frac{\beta^-_\varepsilon}{c}\eta^-_\varepsilon(t)\Bigr) b(\eta^-_\varepsilon(t)) \to \infty, \quad t \to \infty. \qquad \square$$

Proposition 2.16. *Let $b_1, b_2 : \mathbb{R} \to \mathbb{R}_+$ be two tail-decreasing functions which are log-equivalent, i.e. (2.6) holds. Then, for any $0 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 1$, there exists $\tau = \tau(\varepsilon,\varepsilon_1,\varepsilon_2) > 0$, such that, for all $t \geq \tau$,*

$$\eta^-_{\varepsilon_2}(t,b_2) \leq \eta^-_\varepsilon(t,b_1) \leq \eta^-_{\varepsilon_1}(t,b_2) \leq \eta^+_{\varepsilon_1}(t,b_2) \leq \eta^+_\varepsilon(t,b_1) \leq \eta^+_{\varepsilon_2}(t,b_2). \tag{2.19}$$

Proposition 2.16 will be proven in Subsection 3.1 below.

Remark 2.17. By (2.18) and Proposition 2.16, we get the following result. Let $c^{(i)} \in \mathcal{R}$ be constructed by functions $b_i \in \mathcal{D}_d$, $i = 1,2$, which are log-equivalent. Then

$$\begin{aligned}\Lambda^-_{\varepsilon_2}(t,c^{(2)}) \subset \Lambda^-_\varepsilon(t,c^{(1)}) \subset \Lambda^-_{\varepsilon_1}(t,c^{(2)})&\\ \subset \Lambda^+_{\varepsilon_1}(t,c^{(2)}) \subset \Lambda^+_\varepsilon(t,c^{(1)}) &\subset \Lambda^+_{\varepsilon_2}(t,c^{(2)}),\end{aligned}$$

if only $0 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 1$ and $t \geq \tau(\varepsilon,\varepsilon_1,\varepsilon_2) > 0$. An analogue of this result for functions from $\mathcal{I}$ is given in Proposition 3.13 below.

Example 2.18. Consider examples of functions from $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ discussed in [27]. It is easy to see, cf. also [27, Theorem 3.5], that the property for an even bounded b to belong to $\widetilde{\mathcal{S}}_{\mathrm{reg},d}$ depends on its behavior at ∞ only, hence we consider the examples below for s bigger than some $s_0 > 0$ only. Next, because of Remark 2.9, we will 'classify' them according to the asymptotic of the function $l(s) := -\log b(s)$ when $s \to \infty$. For example, all functions $b_\nu(s) = s^\nu e^{-\sqrt{s}}$, $\nu \in \mathbb{R}$ belong to the same class. To find the corresponding $\eta_\varepsilon^\pm(t, b)$ for a $b \in \mathcal{S}_{\mathrm{reg},d}$, one has to solve the equation $b(s) = e^{-\beta(1\pm\varepsilon)t}$. In the example above, it cannot be done using elementary functions, unless $\nu = 0$. However, one can estimate then $\eta_\varepsilon^\pm(t, b_\nu)$ using Proposition 2.16 by the corresponding values of $\eta_\varepsilon^\pm(t, b_0)$ with an arbitrary exactness 'up to the ε'. In the table below, we present hence the form of $\eta(t, b)$ for this 'convenient' value of the parameter ν only.

$l(s) \sim -\log b(s)$ $s \to \infty$	$b(s) = b_\nu(s)$ $\nu \in \mathbb{R}$	$\eta(t, b) = \eta(t, b_0)$ $\nu = 0$
$D \log s,\ D > d$	$(\log s)^\nu \dfrac{1}{s^D}$	$\exp\Bigl(\dfrac{\beta}{D} t\Bigr)$
$D(\log s)^q,\ q > 1, D > 0$	$s^\nu \exp\bigl(-D(\log s)^q\bigr)$	$\exp\Bigl(\Bigl(\dfrac{\beta}{D} t\Bigr)^{\frac{1}{q}}\Bigr)$
$s^\alpha,\ \alpha \in (0, 1)$	$s^\nu e^{-s^\alpha}$	$(\beta t)^{\frac{1}{\alpha}}$
$\dfrac{s}{(\log s)^\alpha},\ \alpha > 1$	$s^\nu \exp\Bigl(-\dfrac{s}{(\log s)^\alpha}\Bigr)$	$\sim \beta t (\log t)^\alpha, \quad t \to \infty$

The last example only requires a justification (for the rest, the calculation of $\eta(t, b_0)$ is straightforward); for this see Remark 3.3. Regarding the restriction $\alpha > 1$ in the last example, see also [27, Remark 3.15].

2.2 Statements

In the following, we assume that $\beta > 0$ in (2.11) is given by (A1).

Theorem 2.19. *Let either* (A1)–(A11) *hold or* (A12) *hold. Let* $u_0 \in E_\theta^+$, $u_0 \not\equiv 0$, *cf. Remark 1.4; and let* u *be the corresponding solution to* (1.1). *Suppose also that there exists* $c \in \mathcal{L} \cup \mathcal{N}$, *such that*

$$(a * u_0)(x) \geq c(x), \quad x \in \mathbb{R}^d. \tag{2.20}$$

Then, for each $\varepsilon \in (0, 1)$,

$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{x \in \Lambda_\varepsilon^-(t,c)} u(x, t) = \theta. \tag{2.21}$$

Theorem 2.20. *Let* (A1)–(A4) *hold. Let* $u_0 \in E_\theta^+$, $\theta - u_0 \not\equiv 0$ *(cf. Remark 1.4); and let* u *be the corresponding solution to* (1.1). *Suppose that there exist functions* $b_1, b_2 \in \mathcal{D}_d$ *which are both log-equivalent to a function* $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$, *such that the following holds. Let* $c, c_2 \in \mathcal{R} \cup \mathcal{I}$ *be constructed by the functions* b *and* b_2, *correspondingly, cf.* (2.10). *Suppose that*

$$a(x) \leq b_1(|x|), \quad x \in \mathbb{R}^d, \tag{2.22}$$

$$u_0(x) \leq c_2(x), \quad x \in \mathbb{R}^d. \tag{2.23}$$

Then there exist $\nu > 1$, $\varepsilon_0 \in (0,1)$, *such that, for any* $\varepsilon \in (0,\varepsilon_0)$, *there exist* $C_\varepsilon = C_\varepsilon(u_0) > 0$ *and* $\tau_\varepsilon > 0$, *such that*

$$\operatorname*{ess\,sup}_{x\notin\Lambda_\varepsilon^+(t,c)} u(x,t) \leq C_\varepsilon e^{-\frac{\varepsilon\beta}{\nu}t}, \quad t \geq \tau_\varepsilon. \tag{2.24}$$

Theorems 2.19 and 2.20 will be proved in Subsections 4.1 and 4.2, correspondingly.

Consider now sufficient conditions on functions a and u_0 to get simultaneously (2.21) and (2.24).

Suppose that there exist constants $\mu, M, \delta, \rho, r, R_0 > 0$, a point $x_0 \in \mathbb{R}^d$, and bounded functions $b^+, b_+ : \mathbb{R} \to \mathbb{R}_+$, $v^\circ \in \mathcal{R} \cup \mathcal{I}$, $v_\circ : \mathbb{R}^d \to [0,\theta]$, such that

$$b_+(|x|) \leq a(x) \leq b^+(|x|), \qquad \text{for a.a. } x \in \mathbb{R}^d; \tag{B1}$$

$$b^+(s) \leq \frac{M}{(1+s)^{d+\mu}}, \qquad \text{for a.a. } s \geq r; \tag{B2}$$

$$b_+(s) \geq \delta, \qquad \text{for a.a. } s \in [0,\rho]; \tag{B3}$$

$$\theta \geq v^\circ(x) \geq u_0(x) \geq v_\circ(x), \qquad \text{for a.a. } x \in \mathbb{R}^d; \tag{B4}$$

$$v_\circ(x) \geq \delta, \qquad \text{for a.a. } x \in B_\rho(x_0). \tag{B5}$$

We will distinguish several cases below, in Propositions 2.21–2.24. From them, we get Theorem 1.5, taking $b_+ = b^+$, $v_\circ = v^\circ$ in (B1)–(B5), see Remark 2.25. The proofs of these Propositions are presented in Subsection 4.3.

In Propositions 2.21–2.24 below, we always assume that either (A1)–(A11) hold or (A12) holds; that (B1)–(B5) hold; that $u_0 \in E_\theta^+$, $u_0 \not\equiv 0$, $\theta - u_0 \not\equiv 0$ (cf. Remark 1.4); and that u is the corresponding solution to (1.1).

Case 1. $\lim\limits_{|x|\to\infty} u_0(x) = 0$.

Subcase 1.1. $\sup\limits_{x\in\mathbb{R}^d} \dfrac{u_0(x)}{a(x)} < \infty$.

Proposition 2.21. *Suppose that both* $b_+, b^+ \in \mathcal{D}_d$ *are log-equivalent to a function* $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. *Let* b_+ *is long-tailed and tail-log-convex, and let*

$$u_0(x) \leq b^+(|x|), \quad x \in \mathbb{R}^d. \tag{2.25}$$

Then

$$\begin{gathered}\textit{there exist } \nu > 1 \textit{ and } \varepsilon_0 \in (0,1)\textit{, such that, for each } \varepsilon \in (0,\varepsilon_0),\\ \textit{there exist } C_\varepsilon = C_\varepsilon(u_0) > 0 \textit{ and } \tau_\varepsilon > 0\textit{, such that}\\ \textit{both (2.21) and (2.24) hold}\end{gathered} \tag{2.26}$$

for $c(x) = b(|x|)$, $x \in \mathbb{R}^d$.

Subcase 1.2. $\sup\limits_{x\in\mathbb{R}^d} \dfrac{a(x)}{u_0(x)} < \infty$.

Proposition 2.22. *Let functions* $b_\circ, b^\circ \in \mathcal{D}_d$ *be such that* $v^\circ(x) = b^\circ(|x|)$, $v_\circ(x) = b_\circ(|x|)$, $x \in \mathbb{R}^d$, *and let both* $b_\circ$ *and* b° *be log-equivalent to a function* $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. *Suppose that* $b_\circ$ *is long-tailed and tail-log-convex, and*

$$a(x) \leq b^\circ(|x|), \quad x \in \mathbb{R}^d. \tag{2.27}$$

Then (2.26) *holds for* $c(x) = b(|x|)$, $x \in \mathbb{R}^d$.

Case 2. $\lim\limits_{\rho\to\infty} u_0((\rho,\ldots,\rho)) = 0$, $\lim\limits_{\rho\to-\infty} u_0((\rho,\ldots,\rho)) \in (0,\theta]$.

Subcase 2.1. $\sup\limits_{x\in\mathbb{R}^d} \dfrac{u_0(x)}{\int_{\Delta(x)} a(y)\,dy} < \infty$.

Proposition 2.23. *Suppose that both* $b_+, b^+ \in \mathcal{D}_d$ *are log-equivalent to a function* $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. *Let* b_+ *is long-tailed and tail-log-convex, and suppose that, for some* $\xi \in (0,\theta)$,

$$\zeta\, \mathbb{1}_{\mathbb{R}^d_-}(x) \leq u_0(x) \leq \int_{\Delta(x)} b^+(|y|)\,dy, \quad x \in \mathbb{R}^d, \tag{2.28}$$

where $\mathbb{R}_- := (-\infty, 0]$. *Then* (2.26) *holds for* $c(x) = \int_{\Delta(x)} b(|y|)\,dy$, $x \in \mathbb{R}^d$.

See also Remarks 4.25, 4.26 below.

Subcase 2.2. $\sup\limits_{x\in\mathbb{R}^d} \dfrac{\int_{\Delta(x)} a(y)\,dy}{u_0(x)} < \infty$.

Proposition 2.24. *Let functions* $b_\circ, b^\circ \in \mathcal{D}_d$ *be such that* $v^\circ(x) = \int_{\Delta(x)} b^\circ(|y|)\,dy$, $v_\circ(x) = \int_{\Delta(x)} b_\circ(|y|)\,dy$, $x \in \mathbb{R}^d$, *and let both* $b_\circ$ *and* b° *be log-equivalent to a function* $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. *Suppose that* $b_\circ$ *is long-tailed and tail-log-convex, and* (2.27) *holds. Then* (2.26) *holds for* $c(x) = \int_{\Delta(x)} b(|y|)\,dy$, $x \in \mathbb{R}^d$.

Remark 2.25. Propositions 2.21–2.24 immediately imply Theorem 1.5. The only observation, which might to be mentioned to see this, is that $q(s) \geq \delta$, $s \in [0,\rho]$, yields

$$\int_{\Delta(x)} q(|y|)dy \geq \mathrm{const} \cdot \mathbb{1}_{\mathbb{R}^d_-}(x), \quad x \in \mathbb{R}^d.$$

Consider now several examples. In all of them we assume that either (A1)–(A11) hold or (A12) holds, and also that u_0 is separated from 0 in a neighborhood of the origin.

Example 2.26. Let, for some $\mu > 0$, $\nu \geq 0$, $r > 1$, one of the following two pairs of conditions hold, for a.a. $|x| \geq r$,

$$\begin{gathered}(\log|x|)^{-\nu}(1+|x|)^{-d-\mu} \leq a(x) \leq (\log|x|)^{\nu}(1+|x|)^{-d-\mu},\\ u_0(x) \leq (\log|x|)^{\nu}(1+|x|)^{-d-\mu},\end{gathered} \tag{2.29}$$

or

$$
\begin{aligned}
(\log|x|)^{-\nu}(1+|x|)^{-d-\mu} \leq u_0(x) \leq (\log|x|)^{\nu}(1+|x|)^{-d-\mu},\\
a(x) \leq (\log|x|)^{\nu}(1+|x|)^{-d-\mu}.
\end{aligned}
\tag{2.30}
$$

Then, for (2.29), we just apply Proposition 2.21 with $b(s) = (1+s)^{-(d+\mu)}$; for (2.30), we apply Proposition 2.22 with the same b.

In both cases, we will get, see Example 2.18, that there exists $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$
\lim_{t\to\infty} \operatorname*{ess\,inf}_{|x|\leq \exp\left(\frac{\beta(1-\varepsilon)t}{d+\mu}\right)} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{ess\,sup}_{|x|\geq \exp\left(\frac{\beta(1+\varepsilon)t}{d+\mu}\right)} u(x,t) = 0, \tag{2.31}
$$

Example 2.27. Let now $d = 1$ and, for some $r, \mu, M > 0$, $\zeta \in (0, \theta)$,

$$
\begin{aligned}
a(x) = M(1+|x|)^{-1-\mu}, \quad |x| > r,\\
\zeta \mathbb{1}_{\mathbb{R}_-}(x) \leq u_0(x) \leq \zeta \int_x^\infty a(y)dy \leq \theta, \quad x \in \mathbb{R}, \zeta \in (0,\theta),
\end{aligned}
$$

and u_0 is decreasing on $\mathbb{R}$. Then the front is described via the function

$$
\int_x^\infty a(y)dy = \frac{M}{\mu+1} x^{-\mu},
$$

if x is big enough. Therefore, by Propositions 2.16 and 2.23,

$$
\lim_{t\to\infty} \operatorname*{ess\,inf}_{x\leq \exp\left(\frac{\beta(1-\varepsilon)t}{\mu}\right)} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{ess\,sup}_{x\geq \exp\left(\frac{\beta(1+\varepsilon)t}{\mu}\right)} u(x,t) = 0,
$$

i.e. the motion of the front goes a bit faster than in (2.31) with $d = 1$. The same difference of the front propagations for integrable and monotone initial conditions, in the case of the fractional Laplacian, was observed in [10].

Example 2.28. Let, for some $\nu \geq 0$, $r > 1$, and $\alpha \in (0,1)$, one of the following two pairs of conditions hold, for a.a. $|x| \geq r$,

$$
\begin{aligned}
(1+|x|)^{-\nu}\exp(-|x|^\alpha) \leq a(x) \leq (1+|x|)^{\nu}\exp(-|x|^\alpha),\\
u_0(x) \leq (1+|x|)^{\nu}\exp(-|x|^\alpha),
\end{aligned}
$$

or

$$
\begin{aligned}
(1+|x|)^{-\nu}\exp(-|x|^\alpha) \leq u_0(x) \leq (1+|x|)^{\nu}\exp(-|x|^\alpha),\\
a(x) \leq (1+|x|)^{\nu}\exp(-|x|^\alpha),
\end{aligned}
$$

Then, the same arguments as in Example 2.26 related to the function $b(s) = \exp(-s^\alpha)$ will imply that, cf. Example 2.18, there exists $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$
\lim_{t\to\infty} \operatorname*{ess\,inf}_{|x|\leq \left(\beta(1-\varepsilon)t\right)^{\frac{1}{\alpha}}} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{ess\,sup}_{|x|\geq \left(\beta(1+\varepsilon)t\right)^{\frac{1}{\alpha}}} u(x,t) = 0, \tag{2.32}
$$

Example 2.29. Let now $d = 1$ and, for some $r, M > 0$, $\zeta \in (0,\theta)$, $\alpha \in (0,1)$, one of the following two pairs of conditions hold

$$\begin{aligned} a(x) &= M\exp(-|x|^\alpha), \quad |x| > r, \\ \zeta 1\!\!1_{\mathbb{R}_-}(x) \le u_0(x) &\le \zeta \int_x^\infty a(y)dy \le \theta, \quad x \in \mathbb{R}, \zeta \in (0,\theta), \end{aligned} \tag{2.33}$$

or

$$\begin{aligned} u_0(x) &= M\exp(-x^\alpha), \quad x > r, \\ a(x) &\le M|x|^{\alpha-1}\exp(-|x|^\alpha), \quad |x| > r, \end{aligned} \tag{2.34}$$

and, in both cases, u_0 is decreasing on $\mathbb{R}$. Then, for (2.33), by Proposition 2.23, the front is described via the function $M\int_x^\infty \exp(-y^\alpha)dy$ for big enough x. Moreover, by Remark 4.22 below, since $\exp(-y^\alpha)$ is log-equivalent to $y^{\alpha-1}\exp(-y^\alpha)$, the estimates of the front may be described by

$$M\int_x^\infty y^{\alpha-1}\exp(-y^\alpha)dy = \frac{M}{\alpha}\exp(-x^\alpha). \tag{2.35}$$

Therefore,

$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{x \le (\beta(1-\varepsilon)t)^{\frac{1}{\alpha}}} u(x,t) = \theta, \qquad \lim_{t\to\infty} \operatorname*{ess\,sup}_{x \ge (\beta(1+\varepsilon)t)^{\frac{1}{\alpha}}} u(x,t) = 0, \tag{2.36}$$

i.e., in contrast to Example 2.27, the estimate for the front appeared the same as in (2.32) with $d = 1$.

For (2.34), we have from (2.35) that $u_0(x) = \int_x^\infty b(y)dy$, where b is also logarithmically equivalent to $\exp(-x^\alpha)$. Therefore, by Proposition 2.24, one gets (2.36) as well.

In the last example, one shows that an 'intermediate' front propagation is possible as well.

Example 2.30. Let, for some $M, P, r, \alpha > 0$ and for all $|x| > r$

$$\begin{aligned} a(x) &= M\exp\bigl(-(\alpha\log|x|)^2\bigr), \\ u_0(x) &\le P\exp\bigl(-(\alpha\log|x|)^2\bigr). \end{aligned}$$

Then, by Proposition 2.21 and Example 2.18, one gets

$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{|x| \le \exp\left(\sqrt{\alpha\beta(1-\varepsilon)t}\right)} u(x,t) = \theta,$$

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{|x| \ge \exp\left(\sqrt{\alpha\beta(1+\varepsilon)t}\right)} u(x,t) = 0.$$

Similarly, using Example 2.18, one can construct a and u_0, such that the front will be described approximately by $\beta(1 \pm \varepsilon)t(\log t)^\gamma$ for any $\gamma > 1$, that demonstrates slower motion than that in (2.32).

3 Technical tools

3.1 Technical tools on $\mathbb{R}$

The function $\eta(t)$, defined by (2.14), is increasing; then, one gets that, for any $0<\varepsilon_1<\varepsilon_2<1$, one has (cf. 2.13)

$$\eta^-_{\varepsilon_2}(t)\leq\eta^-_{\varepsilon_1}(t)\leq\eta^+_{\varepsilon_1}(t)\leq\eta^+_{\varepsilon_2}(t),\qquad t\geq t^-_{\rho,\varepsilon_2}>t^-_{\rho,\varepsilon_1}. \tag{3.1}$$

The following simple lemma shows that the latter inequalities hold for different big enough times as well.

Lemma 3.1. *Let $b:\mathbb{R}\to\mathbb{R}_+$ be a tail-decreasing function. For any $0<\varepsilon_1<\varepsilon_2<1$ and for any $t_1,t_2\geq t^-_{\rho,\varepsilon_2}>t^-_{\rho,\varepsilon_1}$, there exists $\tau=\tau(t_1,t_2,\varepsilon_1,\varepsilon_2)\geq 0$, such that, for all $t\geq\tau$,*

$$\eta^-_{\varepsilon_2}(t_2+t)\leq\eta^-_{\varepsilon_1}(t_1+t)\leq\eta^+_{\varepsilon_1}(t_1+t)\leq\eta^+_{\varepsilon_2}(t_2+t). \tag{3.2}$$

Proof. By (2.17), all expressions in (3.2) are not smaller than ρ. Since b is decreasing on $[\rho,\infty)$, we have from (2.15), (2.16), that (3.2) is equivalent to

$$e^{-\beta^-_{\varepsilon_2}(t+t_2)}\geq e^{-\beta^-_{\varepsilon_1}(t+t_1)}\geq e^{-\beta^+_{\varepsilon_1}(t+t_1)}\geq e^{-\beta^+_{\varepsilon_2}(t+t_2)},$$

that always holds if only, cf. (2.12),

$$t\geq\frac{1}{\varepsilon_2-\varepsilon_1}\max\bigl\{0,t_2(1-\varepsilon_2)-t_1(1-\varepsilon_1),t_1(1+\varepsilon_1)-t_2(1+\varepsilon_2)\bigr\}\geq 0.$$

The statement is proved. □

Moreover, $\eta(t,b)$ is 'increasing' in function b as well. Namely, one has the following result.

Lemma 3.2. *Let $b_1,b_2:\mathbb{R}\to\mathbb{R}_+$ be two tail-decreasing functions, such that, for some $\rho\geq\max\{\rho_{b_1},\rho_{b_2}\}$ (cf. Definition 2.1), $0<b_1(s)\leq b_2(s)$ for $s\geq\rho$. Then $\eta^\pm_\varepsilon(t,b_1)\leq\eta^\pm_\varepsilon(t,b_2)$ for any $\varepsilon\in(0,1)$ and $t\geq-\frac{1}{\beta^-_\varepsilon}\log b_1(\rho)$.*

Proof. Set $\rho_i=\rho_{b_i}$, $i=1,2$. By the tail-decreasing property of b_1,b_2, we have

$$\begin{aligned}-\frac{1}{\beta^-_\varepsilon}\log b_1(\rho)&\geq\max\bigl\{-\frac{1}{\beta^-_\varepsilon}\log b_2(\rho),-\frac{1}{\beta^-_\varepsilon}\log b_1(\rho_1)\bigr\}\\&\geq\max\bigl\{-\frac{1}{\beta^-_\varepsilon}\log b_2(\rho_2),-\frac{1}{\beta^-_\varepsilon}\log b_1(\rho_1)\bigr\}=\max\bigl\{t^-_{\varepsilon,\rho_1}(b_1),t^-_{\varepsilon,\rho_2}(b_2)\bigr\}.\end{aligned}$$

Next,

$$\begin{aligned}\eta^\pm_\varepsilon(t,b_2)&=b_2^{-1}(e^{-\beta^\pm_\varepsilon t})=b_2^{-1}\bigl(b_1(\eta^\pm_\varepsilon(t,b_1))\bigr)\\&\geq b_2^{-1}\bigl(b_2(\eta^\pm_\varepsilon(t,b_1))\bigr)=\eta^\pm_\varepsilon(t,b_1),\end{aligned}$$

where we used that b_2^{-1} decreases and $b_1(s)\leq b_2(s)$ for $s=\eta^\pm_\varepsilon(t,b_1)\geq\rho$. □

Consider now the proof of Proposition 2.16.

Proof of Proposition 2.16. Let $\rho_0 > 0$ be such that b_1 and b_2 are both positive and decreasing to 0 on $[\rho_0, \infty)$ and $b_i(\rho_0) < 1$, $i = 1, 2$. Let $0 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 1$ be fixed.

Consider functions $g_i(s) := -\log b_i(s)$, $s \in \mathbb{R}$, $i = 1, 2$. By (2.6), for a $\delta = \delta(\varepsilon, \varepsilon_1, \varepsilon_2) \in (0, 1)$ which will be specify later, there exists $\rho_\delta > \rho_0$ such that

$$(1-\delta) g_2(s) \le g_1(s) \le (1+\delta) g_2(s), \quad s > \rho_\delta. \tag{3.3}$$

By (3.1), (2.13), all expressions in (2.19) are bigger than $\min\bigl\{\eta^-_{\varepsilon_2}(t, b_1), \eta^-_{\varepsilon_2}(t, b_2)\bigr\}$, provided that $t > \frac{1}{\beta^-_{\varepsilon_2}} \max\bigl\{-\log b_1(\rho_0), -\log b_2(\rho_0)\bigr\}$. Then, since $\eta^\pm_\varepsilon(t)$ are increasing to ∞, there exists $\rho = \rho(\varepsilon_2) > \rho_\delta > \rho_0$ and $\tau = \tau(\rho, \rho_\delta) = \tau(\varepsilon, \varepsilon_1, \varepsilon_2) > 0$, such that all expressions in (2.19) are bigger than ρ, if only $t > \tau$.

Since the functions g_i, $i = 1, 2$ are increasing to ∞ on $[\rho, \infty)$, we have, by (2.15), (3.3),

$$\begin{aligned} &\exp\bigl\{-(1+\delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr)\bigr\} \\ &\qquad \le \exp\bigl\{-g_1\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr)\bigr\} = b_1\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr) = \exp(-\beta^\pm_\varepsilon t) \\ &\qquad\qquad \le \exp\bigl\{-(1-\delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr)\bigr\}, \end{aligned}$$

for all $t > \tau$. Then, by (2.12), we have

$$(1-\delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr) < (1 \pm \varepsilon)\beta t < (1+\delta) g_2\bigl(\eta^\pm_\varepsilon(t, b_1)\bigr), \quad t > \tau.$$

Hence, for $t > \tau$,

$$\begin{aligned} \frac{1+\varepsilon}{1+\delta}\beta t < g_2\bigl(\eta^+_\varepsilon(t, b_1)\bigr) < \frac{1+\varepsilon}{1-\delta}\beta t, \\ \frac{1-\varepsilon}{1+\delta}\beta t < g_2\bigl(\eta^-_\varepsilon(t, b_1)\bigr) < \frac{1-\varepsilon}{1-\delta}\beta t. \end{aligned} \tag{3.4}$$

It is straightforward to verify that the inequality $\varepsilon_1 < \varepsilon < \varepsilon_2$ implies

$$\begin{aligned} 1+\varepsilon_1 < \frac{1+\varepsilon}{1+\delta} < \frac{1+\varepsilon}{1-\delta} < 1+\varepsilon_2, \\ 1-\varepsilon_2 < \frac{1-\varepsilon}{1+\delta} < \frac{1-\varepsilon}{1-\delta} < 1-\varepsilon_1, \end{aligned}$$

if only we choose δ such that

$$0 < \delta < \min\biggl\{\frac{\varepsilon_2 - \varepsilon}{1+\varepsilon_2}, \frac{\varepsilon - \varepsilon_1}{1+\varepsilon_1}\biggr\}.$$

Then, we get from (3.4)

$$\begin{aligned} g_2\bigl(\eta^+_{\varepsilon_1}(t, b_2)\bigr) < g_2\bigl(\eta^+_\varepsilon(t, b_1)\bigr) < g_2\bigl(\eta^+_{\varepsilon_2}(t, b_2)\bigr), \\ g_2\bigl(\eta^-_{\varepsilon_2}(t, b_2)\bigr) < g_2\bigl(\eta^-_\varepsilon(t, b_1)\bigr) < g_2\bigl(\eta^-_{\varepsilon_1}(t, b_2)\bigr), \end{aligned}$$

for $t > \tau$. Since g_2 is increasing, we obtain the statement. ☐

Remark 3.3. Explain the asymptotic of $\eta(t, g)$, $t \to \infty$ for the function

$$g(s) = \exp\Bigl(-\frac{s}{(\log s)^\alpha}\Bigr), \quad s > s_0,$$

cf. Example 2.18. To find $\eta(t,g)$, one has to solve the equation $g(s)=e^{-\beta t}$, i.e. $s(\log s)^{-\alpha}=\beta t$. Making substitution $s=e^\tau$, one easily gets

$$-\frac{\tau}{\alpha}e^{-\frac{\tau}{\alpha}}=-\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}}.$$

Since $s>e^\alpha$ implies $-\frac{\tau}{\alpha}<-1$ and assuming t big enough, to ensure that $-\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}}>-\frac{1}{e}$, one has that the solution to the latter equation can be given in terms of the negative real branch W_{-1} of Lambert W-function, that is the function such that $W_{-1}(\nu)\exp(W_{-1}(\nu))=\nu$, $W_{-1}(\nu)<-1$, $\nu\in(-e^{-1},0)$. Namely, one gets $-\frac{\tau}{\alpha}=W_{-1}\bigl(-\alpha^{-1}(\beta t)^{-\frac{1}{\alpha}}\bigr)$, and, therefore

$$\eta(t,g)=\exp\biggl(-\alpha W_{-1}\Bigl(-\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}}\Bigr)\biggr).$$

However, $\exp(-W_{-1}(\nu))=\nu^{-1}W_{-1}(\nu)$, therefore,

$$\exp(-\alpha W_{-1}(\nu))=(-\nu)^{-\alpha}(-W_{-1}(\nu))^{\alpha},$$

i.e.

$$\eta(t,g)=\alpha^\alpha\beta t\biggl(-W_{-1}\Bigl(-\frac{1}{\alpha(\beta t)^{\frac{1}{\alpha}}}\Bigr)\biggr)^{\alpha},\quad t>\frac{1}{\beta}\Bigl(\frac{e}{\alpha}\Bigr)^{\alpha}.$$

To get a feeling about the behavior of $\eta(t,g)$ for large t, note that $W_{-1}(\nu)\sim\log(-\nu)$, $\nu\to 0-$. As a result,

$$\eta(t,g)\sim\beta t(\log t)^{\alpha},\quad t\to\infty.$$

3.2 Technical tools on $\mathbb{R}^d$

We will use also the following classes of functions, cf. Definition 2.11.

Definition 3.4. 1) Let $\mathcal{M}$ be the set of all bounded functions $c:\mathbb{R}^d\to(0,\infty)$ which satisfy the following monotonicity property: for an arbitrary $x\in\mathbb{R}^d$ and for any $1\leq j\leq d$, the function

$$\mathbb{R}\ni s\mapsto c(x+se_j)\in\mathbb{R}_+ \tag{3.5}$$

is strictly decreasing on $\mathbb{R}$, converges to 0 as $s\to\infty$, and there exists $c_-\in(0,\infty)$, such that, for any $x\in\mathbb{R}^d$,

$$\lim_{s\to-\infty}c\bigl(x+(s,\ldots,s)\bigr)=c_-.$$

Clearly, $\mathcal{I}\subset\mathcal{M}$, and if $c\in\mathcal{I}$ has the form (2.9), then $c_-=\int_{\mathbb{R}^d}p(y)\,dy\in(0,\infty)$.

2) Let $\widetilde{\mathcal{D}}_d\subset\mathcal{D}_d$ denote the set of all functions $b\in\mathcal{D}_d$ which are strictly decreasing to 0 on the whole $\mathbb{R}_+$.

3) Let $\widetilde{\mathcal{R}}\subset\mathcal{R}$, $\widetilde{\mathcal{L}}\subset\mathcal{L}$, $I\subset\mathcal{I}$, $\widetilde{\mathcal{N}}\subset\mathcal{N}$ denote the corresponding subclasses of functions from $\mathcal{R}\cup\mathcal{I}$ constructed by functions from $\widetilde{\mathcal{D}}_d$.

Remark 3.5. Note that the sets $\Lambda_\varepsilon^\pm(t,c)$ are well-defined and, for big enough t, non-empty for $c\in\mathcal{M}$ as well, cf. Remark 2.14. Next, for $c\in\mathcal{R}\cup\mathcal{M}$, $x\in\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c)$ is equivalent to $c(x)<e^{-\beta(1\pm\varepsilon)t}$. In particular,

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c)}c(x)=0.$$

On the other hand, for $c\in\mathcal{R}$, we have that $\lim\limits_{|x|\to\infty}c(x)=0$. Moreover, for $c\in\mathcal{I}$, we have, by (1.5) and (2.9), cf. (1.4),

$$\lim_{\langle x\rangle\to\infty}c(x)=\lim_{\langle x\rangle\to\infty}\int_{\Delta(x)}b(|y|)dy=0.$$

Proposition 3.6. *Let $c^{(i)}\in\mathcal{R}\cup\mathcal{I}$ be constructed by $b_i\in\mathcal{D}_d$, $i=1,2$. Suppose that there exists $\rho>0$, such that $b_1(s)\leq b_2(s)$ for all $s\geq\rho$. Then, for any $\varepsilon>0$ there exists $\tau=\tau(\varepsilon,b_1,b_2)>0$, such that $\Lambda_\varepsilon^\pm(t,c^{(1)})\subset\Lambda_\varepsilon^\pm(t,c^{(2)})$ for all $t\geq\tau$. In particular, if $b_1(s)=b_2(s)$ for all $s\geq\rho$, then $\Lambda_\varepsilon^\pm(t,c^{(1)})=\Lambda_\varepsilon^\pm(t,c^{(2)})$ for all $t\geq\tau$.*

Remark 3.7. Here and below, we will mean that if $f,g\in\mathcal{R}\cup\mathcal{I}$, then either $f,g\in\mathcal{R}$ or $f,g\in\mathcal{I}$.

Proof of Proposition 3.6. For $c^{(i)}\in\mathcal{R}$, $i=1,2$, the statement follows from (2.18) and Lemma 3.2. Let now $c^{(i)}\in\mathcal{I}$, $i=1,2$. Note that, for any $c\in\mathcal{M}$, the inequality $c(x)<e^{-\beta(1\pm\varepsilon)t}$ for some big enough t, is equivalent to the existence of some $\rho_t>0$ such that $\langle x\rangle\geq\rho_t$, where, recall, $\langle x\rangle$ is given by (1.4). Then the inequality

$$|y|\geq\langle y\rangle\geq\langle x\rangle,\qquad y\in\Delta(x),\ x\in\mathbb{R}^d,\tag{3.6}$$

shows that $x\in\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c)$ implies $|y|\geq\rho_t$, $y\in\Delta(x)$.

Next, we have to prove that $\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c^{(2)})\subset\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c^{(1)})$ for big enough t. Let $\rho_0\geq\rho$ be such that $b_1(\rho_0)\leq b_2(\rho_0)\leq1$. Choose $\tau>0$ such that $t\geq\tau$ implies that $\langle x\rangle\geq\rho_0$ for all $x\in\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c^{(2)})$. By the above, for all $y\in\Delta(x)$, we will have $|y|\geq\rho_0$, and hence $b_1(|y|)\leq b_2(|y|)$, $y\in\Delta(x)$. Thus $c^{(1)}(x)\leq c^{(2)}(x)$ and hence $x\in\mathbb{R}^d\setminus\Lambda_\varepsilon^\pm(t,c^{(1)})$. If the tails of b_1 and b_2 are equal, then one can interchange these functions in the above, and get the second statement. □

Corollary 3.8. *Let $c\in\mathcal{R}\cup\mathcal{I}$. Then there exists $c_1,c_2\in\widetilde{\mathcal{R}}\cup\widetilde{\mathcal{I}}$, such that $c_1(x)\leq c(x)\leq c_2(x)$, $x\in\mathbb{R}^d$, and, for each $\varepsilon\in(0,1)$, there exists $\tau=\tau(\varepsilon,c,c_1,c_2)>0$, such that $\Lambda_\varepsilon^\pm(t,c)=\Lambda_\varepsilon^\pm(t,c_1)=\Lambda_\varepsilon^\pm(t,c_2)$ for all $t\geq\tau$.*

Proof. Let c be constructed by a $b\in\mathcal{D}_d$. By Definitions 2.11 and 2.1, there exists $\rho>0$ such that b is decreasing on (ρ,∞) to 0 and, for come $D_1,D_2>0$, $D_1\leq b(s)\leq D_2$ for $s\in[0,\rho]$. Choose $\rho'\geq\rho$, such that $b(\rho')\leq D_1$. Set $b_1(s)=b_2(s)=b(s)$ for $s>\rho'$ and define b_1 on $[0,\rho']$ as an arbitrary decreasing function with $b_1(0)\leq D_1$. Similarly, we define b_2 on $[0,\rho']$ as an arbitrary decreasing bounded function with $b_2(\rho')\geq D_2$. As a result, $b_1(s)\leq b(s)\leq b_2(s)$, $s\in\mathbb{R}_+$. Let $c_1,c_2\in\widetilde{\mathcal{R}}\cup\widetilde{\mathcal{I}}$ be constructed by $b_1,b_2\in\widetilde{\mathcal{D}}_d$, such that either $c_1,c_2,c\in\widetilde{\mathcal{R}}$ or $c_1,c_2,c\in\widetilde{\mathcal{I}}$. Then in both cases, evidently, $c_1(x)\leq c(x)\leq c_2(x)$, $x\in\mathbb{R}^d$. The rest of the proof follows from Proposition 3.6. □

Definition 3.9. Let $b \in \widetilde{\mathcal{D}}_d$ and $\alpha \in (0,1)$ be such that $b^\alpha \in \widetilde{\mathcal{D}}_d$. Let $c \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ be constructed by b. Then we denote by $c_\alpha \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ the function constructed by b^α, cf. Definition 2.12. In other words, for all $x \in \mathbb{R}^d$,

$$c_\alpha(x) := \begin{cases} c(x)^\alpha = b(|x|)^\alpha, & \text{if } c \in \widetilde{\mathcal{R}}, \\ \displaystyle\int_{\Delta(x)} b(|y|)^\alpha dy, & \text{if } c \in \widetilde{\mathcal{I}}. \end{cases} \tag{3.7}$$

In particular, $c_1 = c$. Clearly, $c \in \widetilde{\mathcal{R}}$ implies $c_\alpha \in \widetilde{\mathcal{R}}$, whereas $c \in \widetilde{\mathcal{I}}$ implies $c_\alpha \in \widetilde{\mathcal{I}}$.

Remark 3.10. It is easy to see that, if $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$ for some $\alpha_0 \in (0,1)$, then $b^\alpha \in \widetilde{\mathcal{D}}_d$ for all $\alpha \in [\alpha_0, 1]$.

Proposition 3.11. *For any $\alpha_0 \in \bigl(\frac{3}{4}, 1\bigr)$ there exists $\varepsilon_0 = \varepsilon_0(\alpha_0) \in (0,1)$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\alpha = \alpha(\varepsilon) \in (\alpha_0, 1)$ such that the following holds. For any $b \in \widetilde{\mathcal{D}}_d$ such that $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$, let $c, c_\alpha \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ be constructed by b and b^α, correspondingly. Then there exists $\tau = \tau(\varepsilon, b) > 0$, such that, for any $t \geq \tau$,*

$$\Lambda_\varepsilon^-(t, c_\alpha) \subset \Lambda_{\frac{\varepsilon}{2}}^-(t, c), \tag{3.8}$$

$$\Lambda_{\frac{\varepsilon}{2}}^+(t, c_\alpha) \subset \Lambda_\varepsilon^+(t, c). \tag{3.9}$$

Remark 3.12. For any $b \in \widetilde{\mathcal{D}}_d$, there exists $\rho > 0$, such that $b(\rho) \leq 1$. Then, for any $s \geq \rho$, one gets that $b(s)^\alpha \geq b(s)$. Therefore, for big enough $t > 0$, the inclusions $\Lambda_{\frac{\varepsilon}{2}}^\pm(t, c) \subset \Lambda_{\frac{\varepsilon}{2}}^\pm(t, c_\alpha)$ follow from Proposition 3.6.

Proof of Proposition 3.11. We will prove (3.9). The proof of (3.8) is fully analogous. Consider two cases separately.

1) For a $c \in \widetilde{\mathcal{R}}$. Since $\alpha_0 \in \bigl(\frac{3}{4}, 1\bigr)$, one can define

$$\varepsilon_0 := \frac{1 - \alpha_0}{\alpha_0 - \frac{1}{2}} \in (0,1).$$

Take an arbitrary $\varepsilon \in (0, \varepsilon_0)$, then one easily has that

$$\alpha := \frac{1 + \frac{\varepsilon}{2}}{1 + \varepsilon} \in (\alpha_0, 1). \tag{3.10}$$

Take an arbitrary $b \in \widetilde{\mathcal{D}}_d$ such that $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$, and let $c \in \widetilde{\mathcal{R}}$ be constructed by b. Prove that then there is an equality in (3.9). Indeed, by (2.18), the equality in (3.9) is just equivalent to

$$\eta_{\frac{\varepsilon}{2}}^+(t, b^\alpha) = \eta_\varepsilon^+(t, b), \quad t \geq \tau := t_{\rho,\varepsilon}^-.$$

To prove the latter equality, apply $\log b^\alpha = \alpha \log b$ to both its parts:

$$-\Bigl(1 + \frac{\varepsilon}{2}\Bigr)\beta t = -\alpha(1 + \varepsilon)\beta t,$$

that is equivalent to (3.10).

2) For a $c \in \widetilde{\mathcal{I}}$. Prove the following inequality, which is equivalent to (3.10),

$$\mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c) \subset \mathbb{R}^d \setminus \Lambda_{\frac{\varepsilon}{2}}^+(t,c_\alpha), \quad t \geq \tau. \tag{3.11}$$

Recall that the inclusion $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c)$ is equivalent to

$$c(x) = \int_{\Delta(x)} b(|y|)dy < e^{-\beta(1+\varepsilon)t}. \tag{3.12}$$

We will use Hölder's inequality to estimate $c_\alpha(x)$. It is easy to see that the function

$$f(\alpha) := \alpha - \sqrt{\alpha(1-\alpha)} : \bigl(\tfrac{1}{2},1\bigr) \to (0,1)$$

is increasing. We set $p := p(\alpha) := \frac{1}{f(\alpha)} > 1$ and $q := q(\alpha) := \frac{1}{1-f(\alpha)} > 1$. Then $\frac{1}{p} + \frac{1}{q} = 1$ and, by (3.12), we have

$$\begin{aligned} c_\alpha(x) &= \int_{\Delta(x)} b(|y|)^{f(\alpha)+(\alpha-f(\alpha))}dy \\ &\leq \biggl(\int_{\Delta(x)} b(|y|)^{f(\alpha)p}dy\biggr)^{\frac{1}{p}} \biggl(\int_{\Delta(x)} b(|y|)^{(\alpha-f(\alpha))q}dy\biggr)^{\frac{1}{q}} \\ &< e^{-\beta(1+\varepsilon)f(\alpha)t}\biggl(\int_{\Delta(x)} b(|y|)^{(\alpha-f(\alpha))q}dy\biggr)^{\frac{1}{q}} \end{aligned} \tag{3.13}$$

The inclusion $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$ means that (2.2) holds with b replaced by b^{α_0}. Therefore, to get the finiteness of the latter integral in (3.13), it is enough to have there α such that $\alpha_0 < g(\alpha) < 1$, where

$$g(\alpha) := (\alpha - f(\alpha))q(\alpha) = \frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{1-\alpha}}, \quad \alpha \in \bigl(\tfrac{1}{2},1\bigr).$$

It is easy to see that $g : \bigl(\frac{1}{2},1\bigr) \to \bigl(\frac{1}{2},1\bigr)$ is increasing and $g(\alpha) < \alpha$, $\alpha \in \bigl(\frac{1}{2},1\bigr)$. Note also that $g\bigl(\frac{9}{10}\bigr) = \frac{3}{4}$. As a result, for the given $\alpha_0 \in \bigl(\frac{3}{4},1\bigr)$, there exists a unique $\alpha_1 \in \bigl(\frac{9}{10},1\bigr)$, such that $\alpha_0 = g(\alpha_1) < \alpha_1$. Hence, for any $\alpha \in (\alpha_1,1) \subset (\alpha_0,1)$, one gets $g(\alpha) > g(\alpha_1) = \alpha_0$, and then $\int_{\mathbb{R}^d} b(|y|)^{g(\alpha)}dy < \infty$; in particular, the latter integral in (3.13) is finite.

Next, the function $h(\varepsilon) = \dfrac{1+\frac{\varepsilon}{2}}{1+\varepsilon} : (0,1) \to \bigl(\frac{3}{4},1\bigr)$ is decreasing; cf. (3.10). Therefore, there exists a unique $\varepsilon_0 \in (0,1)$, such that $h(\varepsilon_0) = \alpha_1$; then we have $h : (0,\varepsilon_0) \to (\alpha_1,1)$. Take and fix now an arbitrary $\varepsilon \in (0,\varepsilon_0)$. Since,

$$f : (\alpha_1,1) \to \bigl(f(\alpha_1),1\bigr) \subset (\alpha_1,1) = (h(\varepsilon_0),1)$$

is increasing (we used here that $f(\alpha) < \alpha$), there exists a unique $\alpha = \alpha(\varepsilon) \in (\alpha_1,1)$, such that

$$f(\alpha) = h(\varepsilon) = \frac{1+\frac{\varepsilon}{2}}{1+\varepsilon}. \tag{3.14}$$

Therefore, after $\varepsilon_0, \varepsilon, \alpha$ are chosen, we take an arbitrary $b \in \widetilde{\mathcal{D}}_d$ such that $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$, and let $c \in \widetilde{\mathcal{I}}$ be constructed by b. For this α, by the above,

$\int_{\mathbb{R}^d} b(|y|)^{g(\alpha)} dy < \infty$; therefore, there exists $r > 0$, such that, for all $x \in \mathbb{R}^d$ with $\langle x \rangle > r$,

$$\int_{\Delta(x)} b(|y|)^{g(\alpha)} dy \leq 1.$$

The latter inequality together with (3.14) and (3.13) implies that

$$c_\alpha(x) \leq e^{-\beta(1+\frac{\varepsilon}{2})t}, \tag{3.15}$$

provided that $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c)$ (i.e. (3.12) holds) and $\langle x \rangle > r$. In (3.12), $\langle x \rangle \to \infty$ if and only if $t \to \infty$; cf. Remark 3.5. Therefore, there exists $\tau = \tau(r) = \tau(\varepsilon, b) > 0$, such that $t \geq \tau$ in (3.12) implies $\langle x \rangle \geq r$. As a result, for any $t \geq \tau$ and any $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,c)$, one gets (3.15), that means that $x \in \mathbb{R}^d \setminus \Lambda_{\frac{\varepsilon}{2}}^+(t, c_\alpha)$; i.e. (3.11) holds. □

Proposition 3.13. *Let $b_1, b_2 \in \mathcal{D}_d$ be log-equivalent functions such that, for some $\alpha_0 \in \left(\frac{3}{4}, 1\right)$, $b_i^{\alpha_0} \in \mathcal{D}_d$, $i = 1,2$. Let $c^{(i)} \in \mathcal{R} \cup \mathcal{I}$ be constructed by b_i, $i = 1,2$. Then there exists $\varepsilon_0 = \varepsilon_0(\alpha_0) \in (0,1)$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\tau = \tau(\varepsilon) > 0$, such that, for any $t \geq \tau$,*

$$\Lambda_\varepsilon^-(t, c^{(1)}) \subset \Lambda_{\frac{\varepsilon}{2}}^-(t, c^{(2)}), \tag{3.16}$$

$$\Lambda_{\frac{\varepsilon}{2}}^+(t, c^{(1)}) \subset \Lambda_\varepsilon^+(t, c^{(2)}). \tag{3.17}$$

Proof. We assume first that $b_i \in \widetilde{\mathcal{D}}_d$ and hence $c^{(i)} \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$, $i = 1,2$. Let ε_0 by given by Proposition 3.11. Take an arbitrary $\varepsilon \in (0, \varepsilon_0)$ and consider $\alpha = \alpha(\varepsilon) \in (\alpha_0, 1)$ also given by Proposition 3.11. Let $\rho_0 > 0$ be such that $b_i(\rho_0) \leq 1$, $i = 1,2$. Set $\delta := 1 - \alpha \in (0, 1 - \alpha_0)$. By (2.6), there exists $\rho_\alpha \geq \rho_0$, such that

$$1 - \delta < \frac{-\log b_1(s)}{-\log b_2(s)} < 1 + \delta, \quad s > \rho_\alpha,$$

in particular,

$$b_1(s) < b_2(s)^\alpha, \quad s > \rho_\alpha. \tag{3.18}$$

By Remark 3.10, $b_2^\alpha \in \widetilde{\mathcal{D}}_d$, and hence, by (3.18) and Proposition 3.6, applying to b_1 and b_2^α, one gets

$$\Lambda_{\frac{\varepsilon}{2}}^+(t, c^{(1)}) \subset \Lambda_{\frac{\varepsilon}{2}}^+(t, c_\alpha^{(2)}).$$

The latter inequality together with (3.9) for $c = c^{(2)}$ imply (3.17).

Next, by (3.18), $b_3(s) := b_1(s)^{\frac{1}{\alpha}} < b_2(s)$, if only $s > \rho_\alpha$. From here we have that $b_3 \in \widetilde{\mathcal{D}}_d$ and, moreover, by Proposition 3.6, applying to b_3 and b_2,

$$\Lambda_{\frac{\varepsilon}{2}}^-(t, c^{(3)}) \subset \Lambda_{\frac{\varepsilon}{2}}^-(t, c^{(2)}),$$

where $c^{(3)} \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ in constructed by b_3, cf. Remark 3.7. The latter inequality together with (3.8) for $c = c^{(3)}$ imply (3.16).

Let now $b_i \in \mathcal{D}_d$, $i = 1,2$ be arbitrary. Then, by the proof of Corollary 3.8, there exist $\tilde{c}^{(i)} \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ constructed by $\tilde{b}_i \in \widetilde{\mathcal{D}}_d$, such that $b_i(s) = \tilde{b}_i(s)$ for big enough s. Then $\tilde{b}_1$ and $\tilde{b}_2$ are log-equivalent. Applying the previous considerations to $\tilde{b}_i$, $i = 1,2$, we get (3.16) and (3.17), with $c^{(i)}$ replaced by $\tilde{c}^{(i)}$, $i = 1,2$, for big enough t. Then, by Corollary 3.8, one gets the statement. □

Functions from the class $\mathcal{L}$ introduced in Definition 2.11 are analogous to long-tailed functions in the case of $\mathbb{R}^d$, $d > 1$ because of the following lemma.

Lemma 3.14 ([27, Lemma 4.4])**.** *Let $c \in \mathcal{R}$ be constructed by a long-tailed function $b \in \mathcal{D}_d$ (in particular, let $c \in \mathcal{L}$). Then, for any $r > 0$,*

$$\lim_{|x|\to\infty} \sup_{|y|\leq r} \left| \frac{c(x+y)}{c(x)} - 1 \right| = 0. \tag{3.19}$$

The next lemma shows the corresponding results for functions from $\mathcal{N}$.

Lemma 3.15. *Let $c \in \mathcal{I}$ be constructed by a long-tailed function $b \in \mathcal{D}_d$ (in particular, let $c \in \mathcal{N}$). Then*

$$\lim_{\langle x\rangle\to\infty} \frac{c(x+h)}{c(x)} = 1, \quad h \in \mathbb{R}^d_+. \tag{3.20}$$

Proof. Let b be decreasing on (ρ, ∞) for some $\rho > 0$. Fix an arbitrary $h \in \mathbb{R}^d_+$, $h \neq 0$, and take any $R > \langle h\rangle$. Note that, for any $x, y \in \mathbb{R}^d$, such that $y_j \in [x_j, x_j + R]$, $1 \leq j \leq d$, one has $\bigl||y| - |x|\bigr| \leq |y - x| \leq R\sqrt{d}$. Assume now that $\langle x\rangle \geq \rho + R\sqrt{d}$ (that implies $|x| \geq \rho + R\sqrt{d}$, by (3.6)). Then, for any y as above,

$$\left| \frac{b(|y|)}{b(|x|)} - 1 \right| \leq \sup_{|\tau|\leq R\sqrt{d}} \left| \frac{b(|x|+\tau)}{b(|x|)} - 1 \right| \to 0, \quad |x| \to \infty,$$

because of e.g. [29, formula (2.18)]; cf. also [27, Remark 2.2].

Therefore, for any $\varepsilon \in (0, 1)$, there exists $r = r(\varepsilon, R) > \rho + R\sqrt{d}$, such that, for all $x \in \mathbb{R}^d$ with $\langle x\rangle \geq r$ (that, again, implies $|x| \geq r$), one has

$$1 - \varepsilon \leq \frac{b(|y|)}{b(|x|)} \leq 1 + \varepsilon, \quad y_j \in [x_j, x_j + R], \ 1 \leq j \leq d.$$

As a result,

$$\begin{aligned} 1 - \frac{c(x+h)}{c(x)} &= \frac{\displaystyle\int_{x_1}^{x_1+h_1} \cdots \int_{x_d}^{x_d+h_d} b(|y|)\,dy}{\displaystyle\int_{x_1}^{\infty} \cdots \int_{x_d}^{\infty} b(|y|)\,dy} \leq \frac{\displaystyle\int_{x_1}^{x_1+\langle h\rangle} \cdots \int_{x_d}^{x_d+\langle h\rangle} \frac{b(|y|)}{b(|x|)}\,dy}{\displaystyle\int_{x_1}^{x_1+R} \cdots \int_{x_d}^{x_d+R} \frac{b(|y|)}{b(|x|)}\,dy} \\ &\leq \frac{1+\varepsilon}{1-\varepsilon} \frac{\langle h\rangle^d}{R^d} < \varepsilon, \end{aligned}$$

provided that $R = R(\langle h\rangle, \varepsilon) > \langle h\rangle$ is chosen big enough. The statement is proved. ☐

Remark 3.16. Note that the previous result remains true if $c \in \mathcal{M}$ is defined by (2.9) with $\Delta(x)$ replaced by $\Delta(x + x_0)$ for a fixed $x_0 \in \mathbb{R}^d$.

The following proposition gives a sufficient condition for (2.20); the result is a generalisation of [29, Theorem 4.2].

Proposition 3.17. *Let* $f \in L^1(\mathbb{R}^d \to \mathbb{R}_+)$ *and* $c : \mathbb{R}^d \to (0,\infty)$ *be a bounded function, such that* (3.19) *holds (e.g.* $c \in \mathcal{L}$*). Then*

$$\liminf_{|x|\to\infty} \frac{(c * f)(x)}{c(x)} \geq \int_{\mathbb{R}^d} f(y)\, dy. \tag{3.21}$$

Moreover, there exists $D > 0$ *such that*

$$(c * f)(x) \geq Dc(x), \quad x \in \mathbb{R}^d.$$

Proof. For any $r > 0$, we have

$$\begin{aligned}\frac{(c * f)(x)}{c(x)} &\geq \int_{|y|\leq r} \frac{c(x-y)}{c(x)} f(y)\, dy\\ &\geq \biggl(1 - \sup_{|y|\leq r}\biggl|\frac{c(x-y)}{c(x)} - 1\biggr|\biggr) \int_{|y|\leq r} f(y)\, dy.\end{aligned}$$

Take an arbitrary $\delta \in (0,1)$ and choose $r = r(\delta) > 0$ such that $\int_{|y|\leq r} f(y)\, dy > (1-\delta)\int_{\mathbb{R}^d} f(y)\, dy$. Next, by (3.19), there exists $\rho = \rho(r) = \rho(\delta) \geq r$, such that $\sup\limits_{|y|\leq r}\bigl|\frac{c(x-y)}{c(x)} - 1\bigr| < \delta$, for all $|x| \geq \rho$. As a result, for any $\delta \in (0,1)$, there exists $\rho = \rho(\delta) > 0$, such that

$$\frac{(c * f)(x)}{c(x)} > (1-\delta)^2 \int_{\mathbb{R}^d} f(y)\, dy, \quad |x| \geq \rho,$$

that yields (3.21). Finally, by e.g. [25, Lemma 2.1], $c * f$ is a continuous function on $B_\rho(0)$; then, it is easy to see that $c(x) > 0$, $x \in \mathbb{R}^d$ implies that $(c*f)(x) > 0$, $x \in \mathbb{R}^d$. Hence the boundedness of c yields $\inf\limits_{|x|\leq\rho} \frac{(c*f)(x)}{c(x)} > 0$, that fulfilled the statement. $\square$

4 Proofs

4.1 Proof of Theorem 2.19

Let $\beta \in (0,\infty)$ be defined by (A1). Let $c \in \widetilde{\mathcal{R}} \cup \mathcal{M}$, $\varepsilon \in (0,1)$, and $\Lambda_\varepsilon^-(t,c)$ be given by (2.11).

For any $\lambda > 0$, we define the function

$$\begin{aligned} g(x,t) = g_{c,\varepsilon,\lambda}(x,t) &= \lambda \min\bigl\{1, c(x)e^{\beta_\varepsilon^- t}\bigr\} &(4.1)\\ &= \lambda 1\!\!1_{\Lambda_\varepsilon^-(t,c)}(x) + \lambda c(x) e^{\beta_\varepsilon^- t} 1\!\!1_{\mathbb{R}^d \setminus \Lambda_\varepsilon^-(t,c)}(x), \quad x \in \mathbb{R}^d, t \geq 0. &(4.2)\end{aligned}$$

Lemma 4.1. *Let* $c \in \widetilde{\mathcal{L}}$ *be given by a (long-tailed and tail-log-convex) function* $b \in \widetilde{\mathcal{D}}_d$*, and* $\rho > 0$ *be such that* $b(\rho) \leq 1$*. Define, for any* $\lambda > 0$*,* $\varepsilon \in (0,1)$*,*

$$f(s,t) := \lambda 1\!\!1_{s \leq \eta_\varepsilon^-(t)} + \lambda e^{\beta_\varepsilon^- t} b(s)\, 1\!\!1_{s > \eta_\varepsilon^-(t)} \in [0,\lambda], \quad s \in \mathbb{R}_+,\ t \geq t_{\rho,\varepsilon}^-, \tag{4.3}$$

i.e. $g(x,t) = f(|x|,t)$*, where* g *is given by* (4.2)*. Then, for any* $\tau > 0$*,*

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}_+} \biggl|\frac{f(s+\tau,t)}{f(s,t)} - 1\biggr| = 0. \tag{4.4}$$

Proof. Take an arbitrary $\varepsilon \in (0,1)$. For an arbitrary fixed $\tau \in \mathbb{R}_+$, choose $t_0 \geq t^-_{\rho,\varepsilon}$, such that $\eta^-_\varepsilon(t_0) \geq \tau$. Then, for any $t \geq t_0$, the function $F_{\tau,t}(s) := \frac{f(s+\tau,t)}{f(s,t)}$ takes the following values. For $0 \leq s \leq \eta^-_\varepsilon(t) - \tau$, one has $F_{\tau,t}(s) = 1$. For $\eta^-_\varepsilon(t) - \tau < s \leq \eta^-_\varepsilon(t)$, we have $F_{\tau,t}(s) = e^{\beta^-_\varepsilon t} b(s+\tau)$ and, since b is decreasing on $[\eta^-_\varepsilon(t), \infty)$, one gets

$$\frac{b(\eta^-_\varepsilon(t)+\tau)}{b(\eta^-_\varepsilon(t))} = e^{\beta^-_\varepsilon t} b(\eta^-_\varepsilon(t)+\tau) \leq e^{\beta^-_\varepsilon t} b(s+\tau) \leq e^{\beta^-_\varepsilon t} b(\eta^-_\varepsilon(t)) = 1.$$

Finally, for $s > \eta^-_\varepsilon(t)$, we have, $F_{\tau,t}(s) = \frac{b(s+\tau)}{b(s)} \leq 1$ (since b is decreasing) and, by Remark 2.3,

$$\frac{b(s+\tau)}{b(s)} \geq \frac{b(\eta^-_\varepsilon(t)+\tau)}{b(\eta^-_\varepsilon(t))}.$$

As a result, for all $s \in \mathbb{R}_+$,

$$0 \leq 1 - F_{\tau,t}(s) \leq 1\!\!1_{\{s > \eta^-_\varepsilon(t) - \tau\}}(s) \biggl(1 - \frac{b(\eta^-_\varepsilon(t)+\tau)}{b(\eta^-_\varepsilon(t))}\biggr), \tag{4.5}$$

that implies the statement because of (2.1). □

Lemma 4.2. *Let $c \in \widetilde{\mathcal{N}}$ and g be given by* (4.2)*. Then, for any $h \in \mathbb{R}^d_+$,*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \left| \frac{g(x+h,t)}{g(x,t)} - 1 \right| = 0. \tag{4.6}$$

Proof. Take an arbitrary $x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d_+$. By the monotonicity of functions (3.5), we have $c(x+h) \leq c(x)$. Next, it is easy to see that $x \in \mathbb{R}^d \setminus \Lambda^-_\varepsilon(t,c)$ implies $x+h \in \mathbb{R}^d \setminus \Lambda^-_\varepsilon(t,c)$, and hence

$$\frac{g(x+h,t)}{g(x,t)} = \frac{c(x+h)}{c(x)} \leq 1.$$

Let $x \in \Lambda^-_\varepsilon(t,c)$. If $x+h \in \Lambda^-_\varepsilon(t,c)$, then $\frac{g(x+h,t)}{g(x,t)} = 1$. Let now h be such that $x+h \in \mathbb{R}^d \setminus \Lambda^-_\varepsilon(t,c)$. Then

$$\frac{g(x+h,t)}{g(x,t)} = e^{\beta^-_\varepsilon t} c(x+h) \leq 1.$$

Moreover, since $x \in \Lambda^-_\varepsilon(t,c)$ implies $c(x)e^{\beta^-_\varepsilon t} \geq 1$, one has for such x, h the following estimate

$$0 \leq 1 - \frac{g(x+h,t)}{g(x,t)} \leq 1 - \frac{c(x+h)}{c(x)}.$$

As a result,

$$\left| \frac{g(x+h,t)}{g(x,t)} - 1 \right| = 1 - \frac{g(x+h,t)}{g(x,t)} \leq \sup_{y: c(y+h) < e^{-\beta^-_\varepsilon t}} \biggl(1 - \frac{c(y+h)}{c(y)}\biggr).$$

Because of (3.20), for the chosen $h \in \mathbb{R}^d_+$ and for an arbitrary $\delta > 0$, there exists $\rho = \rho(\delta, h) > 0$, such that $\sup\limits_{1 \le j \le d} y_j > \rho$ implies

$$0 \le 1 - \frac{c(y+h)}{c(y)} \le \delta.$$

Choose now $t_0 = t_0(\rho, \varepsilon, h) = t_0(\delta, \varepsilon, h)$, such that $c\bigl((\rho, \ldots, \rho) + h\bigr) > e^{-\beta_\varepsilon^- t_0}$. Prove that then, for any $t \ge t_0$, the inequality $c(y+h) \le e^{-\beta_\varepsilon^- t}$ implies $\sup\limits_{1 \le j \le d} y_j > \rho$. Indeed, on the contrary, suppose that, for some $t \ge t_0$, the inequality $c(y+h) \le e^{-\beta_\varepsilon^- t}$ holds, however, $\sup\limits_{1 \le j \le d} y_j \le \rho$. The latter yields

$$e^{-\beta_\varepsilon^- t} \ge c(y+h) \ge c\bigl((\rho, \ldots, \rho) + h\bigr) > e^{-\beta_\varepsilon^- t_0},$$

that contradicts to that $t \ge t_0$. As a result, for all $x \in \mathbb{R}^d$ and $t > t_0$,

$$\left| \frac{g(x+h,t)}{g(x,t)} - 1 \right| \le \sup_{y:\, \sup\limits_{1 \le j \le d} y_j > \rho} \left(1 - \frac{c(y+h)}{c(y)} \right) < \delta,$$

that implies the statement. □

Definition 4.3. Let (A1) hold. A function $v : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *sub-solution* to (1.32) on $[\tau, \infty)$ for some $\tau \ge 0$, if

$$(\mathcal{F}_m v)(x,t) := \frac{\partial}{\partial t} v(x,t) - \varkappa (a * v)(x,t) + m v(x,t) \le 0 \tag{4.7}$$

for a.a. $x \in \mathbb{R}^d$ and for all $t \in [\tau, \infty)$.

Proposition 4.4. *Let* (A1) *hold and* $c \in \widetilde{\mathcal{L}} \cup \widetilde{\mathcal{N}}$. *Then, for any* $\varepsilon \in (0,1)$ *and for any* $\lambda > 0$, *there exists* $\tau_0 = \tau_0(\varepsilon) > 0$, *such that the function* $g = g(x,t)$, *given by* (4.2), *is a sub-solution to* (1.32) *on* $[\tau_0, \infty)$.

Proof. Take an arbitrary $\varepsilon \in (0,1)$. One has

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= \lambda \beta_\varepsilon^- e^{\beta_\varepsilon^- t} b(|x|)\, \mathbb{1}_{|x| > \eta_\varepsilon^-(t)} \\ &= \beta(1-\varepsilon) g(x,t) \mathbb{1}_{|x| > \eta_\varepsilon^-(t)} \le \beta(1-\varepsilon) g(x,t). \end{aligned} \tag{4.8}$$

Therefore, by (4.7), (4.8),

$$-(\mathcal{F}_m g) \ge \varkappa a * g - m g - \beta(1-\varepsilon) g = \varkappa a * g - \varkappa g + \beta \varepsilon g. \tag{4.9}$$

To find now an appropriate bound from below for $Lg = \varkappa a * g - \varkappa g$, cf. (1.23), consider two cases separately.

1. Let $c \in \widetilde{\mathcal{L}}$, $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Let $\rho \ge 0$ be such that $b(\rho) \le 1$, and $t^-_{\rho,\varepsilon}$ and $\eta_\varepsilon^-(t) = \eta_\varepsilon^-(t,b)$ be given by (2.13) and (2.16), correspondingly. Let $\lambda \in [0, \lambda_0]$. Since f given by (4.3) is decreasing in its first coordinate, we have

$$\begin{aligned} \varkappa (a * g)(x,t) &= \varkappa \int_{\mathbb{R}^d} a(-y) g(x+y,t)\, dy = \varkappa \int_{\mathbb{R}^d} a(-y) f(|x+y|,t)\, dy \\ &\ge \varkappa \int_{\mathbb{R}^d} a(-y) f(|x|+|y|,t)\, dy = \varkappa \int_{\mathbb{R}^d} a(y) f(|x|+|y|,t)\, dy \\ &= \varkappa g(x,t) \int_{\mathbb{R}^d} a(y) \frac{f(|x|+|y|,t)}{f(|x|,t)}\, dy, \end{aligned} \tag{4.10}$$

for a.a. $x \in \mathbb{R}^d$. Note that, by (4.5),

$$0 < \frac{f(|x|+|y|,t)}{f(|x|,t)} \leq 1, \quad x,y \in \mathbb{R}^d,\ t \in \mathbb{R}_+. \tag{4.11}$$

By (4.10), (4.11), (1.2), we have, cf. (4.9),

$$\varkappa(a*g)(x,t) - \varkappa g(x,t) \geq -\varkappa g(x,t) \int_{\mathbb{R}^d} a(y) \left| \frac{f(|x|+|y|,t)}{f(|x|,t)} - 1 \right| dy$$

Next, by (4.4), (4.11), and the dominated convergence theorem, one gets

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} a(y) \sup_{x\in\mathbb{R}^d} \left| \frac{f(|x|+|y|,t)}{f(|x|,t)} - 1 \right| dy = 0.$$

Therefore, for any $\delta \in (0,1)$ (small enough later), there exists a $\tau_0 \geq t^-_{\rho,\varepsilon}$, such that, for all $t \geq \tau_0$ and for a.a. $x \in \mathbb{R}^d$,

$$\varkappa(a*g)(x,t) - \varkappa g(x,t) \geq -\varkappa\delta g(x,t).$$

As a result, by (4.9),

$$-\mathcal{F}_m g \geq -\varkappa\delta g + \beta\varepsilon g \geq 0,$$

if only $\delta < \dfrac{\beta\varepsilon}{\varkappa}$. The proof, for $c \in \widetilde{\mathcal{R}}$, is fulfilled.

2. Let $c \in \widetilde{\mathcal{N}}$. Denote, for any $y \in \mathbb{R}^d$,

$$y^+ := \bigl(|y_1|, \ldots, |y_d|\bigr) \in \mathbb{R}^d_+.$$

Since the function c is decreasing along all basis directions (i.e. the functions (3.5) are all decreasing, $j = 1, \ldots, d$), we easily get that the function g given by (4.2) has the same property (in x). Therefore, since $y_j \leq y_j^+$, $j = 1, \ldots, d$, one gets

$$g(x+y,t) \geq g(x+y^+,t).$$

Therefore, we will have, instead of (4.10),

$$\begin{aligned} \varkappa(a*g)(x,t) &= \varkappa \int_{\mathbb{R}^d} a(-y) g(x+y,t)\,dy \geq \varkappa \int_{\mathbb{R}^d} a(-y) g(x+y^+,t)\,dy \\ &= \varkappa g(x,t) \int_{\mathbb{R}^d} a(y) \left(\frac{g(x+y^+,t)}{g(x,t)} - 1 \right) dy + \varkappa g(x,t) \int_{\mathbb{R}^d} a(y)\,dy \end{aligned}$$

Taking into account (4.6) for $h = y^+$, the rest of the proof is fully analogous to the first part. □

Definition 4.5. A function $w : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *sub-solution* to (1.1) on $[\tau, \infty)$ for some $\tau \geq 0$, if $(\mathcal{F}w)(x,t) \leq 0$ for a.a. $x \in \mathbb{R}^d$ and for all $t \in [\tau, \infty)$, where $\mathcal{F}$ is given by, cf. (4.7),

$$(\mathcal{F}u)(x,t) := \frac{\partial}{\partial t} u(x,t) - \varkappa(a*u)(x,t) + m u(x,t) + u(x,t)(Gu)(x,t). \tag{4.12}$$

The proof of the following statement follows directly from Theorem 1.2.

Proposition 4.6. *Let* (A1)–(A4) *hold. Let* $0 \le u \le \theta$ *be a solution to* (1.1), *and* $g : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a sub-solution to* (1.1) *on* $[\tau, \infty)$ *for some* $\tau \ge 0$. *Suppose that, for some* $t_0, t_1 \ge \tau$, *we have* $u(x, t_0) \ge g(x, t_1)$ *for a.a.* $x \in \mathbb{R}^d$. *Then, for all* $t \ge 0$,

$$u(x, t + t_0) \ge g(x, t + t_1), \quad x \in \mathbb{R}^d.$$

We are going to find now, using the continuity of G at 0 (on E^+, cf. (A3)) and Proposition 4.4, sufficient conditions to have (4.2) as a sub-solution to (1.1) as well.

Proposition 4.7. *Let* (A1)–(A4) *hold and* $c \in \widetilde{\mathcal{L}} \cup \widetilde{\mathcal{N}}$. *Then, for any* $\varepsilon \in (0, 1)$, *there exist* $\lambda_0 = \lambda_0(\varepsilon) > 0$ *and* $\tau_0 = \tau_0(\varepsilon) > 0$, *such that, for any* $\lambda \in [0, \lambda_0]$, *the function* $g = g(x, t)$, *given by* (4.2), *is a sub-solution to* (1.1) *on* $[\tau_0, \infty)$.

Proof. Take an arbitrary $\varepsilon \in (0, 1)$. For any $\delta \in (0, \varepsilon\beta)$, one has that $m + \delta < m + \beta = \varkappa$; hence one can apply Proposition 4.4 to the equation (1.32) with m replaced by $m + \delta$. More precisely, we choose $\varepsilon_1 \in (0, 1)$ to ensure that

$$(\varkappa - (m + \delta))(1 - \varepsilon_1) = (\varkappa - m)(1 - \varepsilon), \tag{4.13}$$

namely, $\varepsilon_1 := \dfrac{\beta\varepsilon - \delta}{\beta - \delta}$. Then, by (4.13) and Proposition 4.4, there exists $\tau_0 = \tau_0(\varepsilon_1) = \tau_0(\varepsilon)$, such that

$$-\mathcal{F}_{m+\delta} g(x, t) \ge 0, \quad t \ge \tau_0, \tag{4.14}$$

where $\mathcal{F}_{m+\delta}$ and g are given by (4.7) and (4.1), correspondingly.

Next, by (A1)–(A3), there exists $\lambda_0 = \lambda_0(\delta) = \lambda_0(\varepsilon) > 0$, such that $0 \le v \le \lambda_0$, $v \in E$, implies

$$0 \le Gv < \delta. \tag{4.15}$$

Clearly, (4.1) yields that $0 \le g(x, t) \le \lambda$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$. Then, by (4.12), (4.7), (4.14), (4.15) we have, for any $\lambda \in [0, \lambda_0]$ and for any $t \ge \tau_0$,

$$-\mathcal{F}g = -\mathcal{F}_m g - gGg = -\mathcal{F}_{m+\delta} g + \delta g - gGg \ge 0.$$

The statement is proved. □

Now we are ready to prove Theorem 2.19.

Proof of Theorem 2.19. Recall that, by Theorem 1.1, $0 \le u_0 \le \theta$ implies $0 \le u(\cdot, t) \le \theta$ for $t > 0$; and then, by (A2), $Gu \le \beta$. Rewrite (1.1) in the form (1.20) with F given by (1.21), then, by (1.22), $Fu \ge 0$. Therefore, for all $t > 0$ and a.a. $x \in \mathbb{R}^d$,

$$\begin{aligned} u(x, t) &= e^{-\varkappa t} u_0(x) + \varkappa \int_0^t e^{-\varkappa(t-s)} (a * u)(x, s)\, ds \\ &\quad + \int_0^t e^{-\varkappa(t-s)} (Fu)(x, s)\, ds \\ &\ge e^{-\varkappa t} u_0(x) + \varkappa \int_0^t e^{-\varkappa(t-s)} (a * u)(x, s)\, ds. \end{aligned}$$

The same inequality for $u(x,s)$ implies

$$\begin{aligned} u(x,t) &\geq \varkappa \int_0^t e^{-\varkappa(t-s)}(a*u)(x,s)ds \\ &\geq \varkappa \int_0^t e^{-\varkappa(t-s)}e^{-\varkappa s}(a*u_0)(x)ds \\ &= \varkappa t e^{-\varkappa t}(a*u_0)(x) \geq \varkappa t e^{-\varkappa t}c(x), \end{aligned} \tag{4.16}$$

for all $t \geq 0$ and a.a. $x \in \mathbb{R}^d$, because of (2.20).

Fix an arbitrary $\varepsilon \in (0,1)$. Take any $\delta \in (0,\varepsilon)$ and consider $\lambda_0 = \lambda_0(\delta) > 0$ and $\tau_0 = \tau_0(\delta) > 0$, both given by Proposition 4.7. Set now

$$\lambda := \min\bigl\{\lambda_0, \varkappa\tau_0 e^{-(\varkappa+\beta_\delta^-)\tau_0}\bigr\}.$$

Then, by (4.16) and (4.1), we have, for a.a. $x \in \mathbb{R}^d$,

$$u(x,\tau_0) \geq \lambda e^{\beta_\delta^- \tau_0} c(x) \geq \lambda \min\bigl\{e^{\beta_\delta^- \tau_0}c(x), 1\bigr\} = g_{c,\delta,\lambda}(x,\tau_0).$$

Therefore, by Propositions 4.7 and 4.6, one gets, for any $\tau \geq 0$,

$$u(x,\tau_0+\tau) \geq g_{c,\delta,\lambda}(x,\tau_0+\tau), \quad \text{for a.a. } x \in \mathbb{R}^d.$$

As a result,

$$u(x,\tau_0+\tau) \geq \lambda \quad \text{for a.a. } x \in \Lambda_\delta^-(\tau_0+\tau,c), \tau \geq 0. \tag{4.17}$$

First, we suppose that $c \in \widetilde{\mathcal{L}} \cup \widetilde{\mathcal{N}}$. We will distinguish two cases.

1. Let $c \in \widetilde{\mathcal{L}}$, $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Fix $\tau \geq 0$. Since (2.18) holds, we have that the set

$$\begin{aligned} \widetilde{\Lambda} :&= \{y \in \mathbb{R}^d : B_1(y) \subset \Lambda_\delta^-(\tau_0+\tau,c)\} \\ &= \bigl\{y \in \mathbb{R}^d : B_1(y) \subset B_{\eta_\delta^-(\tau_0+\tau,b)}(0)\bigr\} \end{aligned}$$

is nothing but $B_{\eta_\delta^-(\tau_0+\tau,b)-1}(0)$ and, moreover,

$$\Lambda_\delta^-(\tau_0+\tau,c) = \bigcup_{y\in\widetilde{\Lambda}} B_1(y). \tag{4.18}$$

Take and fix now an arbitrary $y \in \widetilde{\Lambda}$, i.e. $|y| \leq \eta_\delta^-(\tau_0+\tau) - 1$. Then, by (4.17),

$$u(x,\tau_0+\tau) \geq \lambda 1\!\!1_{B_1(y)}(x) \quad \text{for a.a. } x \in \mathbb{R}^d.$$

Consider now equation (1.1) with the initial condition $v_0(x) = u(x,\tau_0+\tau)$, $x \in \mathbb{R}^d$; let $v(x,t)$ be the corresponding solution to (1.1). By the uniqueness in Theorem 1.1, $v(x,t) = u(x,\tau_0+\tau+t)$, $t \in \mathbb{R}_+$.

Take an arbitrary $\mu \in (0,\theta)$. Apply Theorem 1.3 to the solution v and $K = B_1(y)$; then there exists $t_\mu \geq 1$, such that $v(x,t) \geq \mu$ for a.a. $x \in B_1(y)$. As a result,

$$u(x,\tau_0+t_\mu+\tau) \geq \mu, \tag{4.19}$$

for all $\tau \geq 0$ and a.a. $x \in B_1(y)$. Stress that t_μ does not depend on a y with $|y| \leq \eta_\delta^-(\tau_0+\tau) - 1$. As a result, by (4.18) for any $\delta \in (0,1)$ and $\mu \in (0,\theta)$, there exist $\lambda_0 = \lambda_0(\delta) > 0$, $\tau_0 = \tau_0(\delta) > 0$, and $t_\mu \geq 1$, such that, for all $\tau \geq 0$ and for a.a. x with $|x| \leq \eta_\delta^-(\tau_0+\tau)$, the inequality (4.19) holds.

Apply now Lemma 3.1 for $\varepsilon_2 := \varepsilon > \delta =: \varepsilon_1$, $t_1 = \tau_0$, $t_2 = \tau_0 + t_\mu$. One gets that there exists $\tau_1 \geq 0$, such that, for all $\tau \geq \tau_1$,

$$\eta_\varepsilon^-(\tau + \tau_0 + t_\mu) \leq \eta_\delta^-(\tau+\tau_0),$$

i.e. (4.19) holds for all $\tau \geq \tau_1$ and a.a. x with $|x| \leq \eta_\varepsilon^-(\tau+\tau_0+t_\mu)$. Since $\mu \in (0,\theta)$ was arbitrary, the latter fact yields (2.21).

2. Let now $c \in \widetilde{\mathcal{N}}$. Consider the norm on $\mathbb{R}^d$ given by

$$|x|_\infty := |(x_1,\ldots,x_d)|_\infty := \max_{1\leq j\leq d} |x_j|.$$

Let $\widetilde{B}_{\frac{1}{2}}(x)$ denote the ball with the center at an $x \in \mathbb{R}^d$ and the radius $\frac{1}{2}$ w.r.t. the $|\cdot|_\infty$-norm. Then, clearly,

$$\widetilde{B}_{\frac{1}{2}}(x) = \bigtimes_{j=1}^{d} \Bigl[x_j - \frac{1}{2}, x_j + \frac{1}{2}\Bigr] = \bigtimes_{j=1}^{d} \bigl[y_j - 1, y_j\bigr] =: C_1(y),$$

where $y_j = x_j + \frac{1}{2}$, $1 \leq j \leq d$. Stress that, by (2.11), if $c \in \widetilde{\mathcal{N}} \subset \mathcal{M}$, i.e. the functions (3.5) are decreasing on $\mathbb{R}$, then $y \in \Lambda_\delta^-(\tau_0+\tau,c)$ implies that

$$C_1(y) \subset \Lambda_\delta^-(\tau_0+\tau,c).$$

Therefore, cf. (4.18),

$$\Lambda_\delta^-(\tau_0+\tau,c) = \bigcup_{y\in\Lambda_\delta^-(\tau_0+\tau,c)} C_1(y).$$

Hence, one can just repeat the previous proof, applying Theorem 1.3 to the solution v and $K = C_1(y)$ with $y \in \Lambda_\delta^-(\tau_0+\tau,c)$.

Let now $c \in \mathcal{L}\cup\mathcal{N} \subset \mathcal{R}\cup\mathcal{I}$. By Corollary 3.8, there exists $\tilde{c} \in \widetilde{\mathcal{R}}\cup\widetilde{\mathcal{I}}$ such that $c(x) \geq \tilde{c}(x)$, $x \in \mathbb{R}^d$, and, for each $\varepsilon \in (0,1)$, $\Lambda_\varepsilon^-(t,c) = \Lambda_\varepsilon^-(t,\tilde{c})$ for $t > \tilde{\tau}(\varepsilon)$. Moreover, by the proof of Corollary 3.8, one can easily get that $\tilde{c} \in \widetilde{\mathcal{L}}\cup\widetilde{\mathcal{N}}$ then. Hence (2.20) implies $(a * u_0)(x) \geq \tilde{c}(x)$, $x \in \mathbb{R}^d$, and, therefore, one can apply the previous considerations and get (2.21) with c replaced by $\tilde{c}$, that yields the statement by the above. □

4.2 Proof of Theorem 2.20

First, we note that, because of the inequality (1.33), which requires (A1)–(A4) only, it is enough to show (2.24) for u replaced by w; recall that the latter function is the solution to (1.32).

For a function $\widetilde{\omega} : \mathbb{R}^d \to (0,+\infty)$, we define, for any $f : \mathbb{R}^d \to \mathbb{R}$,

$$\|f\|_{\widetilde{\omega}} := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{\widetilde{\omega}(x)} \in [0,\infty]. \tag{4.20}$$

If $\widetilde{\omega}(x) = b(|x|)$, $x \in \mathbb{R}^d$, for a function $b : \mathbb{R}_+ \to (0,\infty)$, we will use the notation $\|f\|_b := \|f\|_{\widetilde{\omega}}$.

Proposition 4.8 (cf. [25, Propostion 5.2])**.** *Let a function* $\widetilde{\omega} : \mathbb{R}^d \to (0,+\infty)$ *be such that* $a * \widetilde{\omega}$ *is well-defined (for example, let* $\widetilde{\omega}$ *be bounded) and, for some* $\gamma \in (0,\infty)$*,*

$$\frac{(a * \widetilde{\omega})(x)}{\widetilde{\omega}(x)} \leq \gamma, \quad x \in \mathbb{R}^d. \tag{4.21}$$

Let $0 \leq u_0 \in L^\infty(\mathbb{R}^d)$ *and* $\|u_0\|_{\widetilde{\omega}} < \infty$*; let* $w = w(x,t)$ *be the corresponding solution to* (1.32)*. Then*

$$\|w(\cdot,t)\|_{\widetilde{\omega}} \leq \|u_0\|_{\widetilde{\omega}} e^{(\varkappa(\gamma-1)+\beta)t}, \quad t \geq 0. \tag{4.22}$$

Proof. For any $f : \mathbb{R}^d \to \mathbb{R}$, with $\|f\|_{\widetilde{\omega}} < \infty$, we have

$$\left| \frac{(a * f)(x)}{\widetilde{\omega}(x)} \right| \leq \int_{\mathbb{R}^d} \frac{a(y)\widetilde{\omega}(x-y)}{\widetilde{\omega}(x)} \frac{|f(x-y)|}{\widetilde{\omega}(x-y)} dy \leq \frac{a * \widetilde{\omega}(x)}{\widetilde{\omega}(x)} \|f\|_{\widetilde{\omega}}. \tag{4.23}$$

The solution w to the linear equation (1.32) with a bounded, in E operator in the r.h.s., exists and is unique on the whole $\mathbb{R}_+$. Therefore, for any $0 \leq \tau < \Upsilon$, we have that

$$w(x,t) = (\Phi_\tau w)(x,t), \quad t \in [\tau, \Upsilon],$$

where, for any $t \in [\tau, \Upsilon]$,

$$(\Phi_\tau v)(x,t) := e^{-\varkappa(t-\tau)} u_\tau(x) + \int_\tau^t e^{-\varkappa(t-s)} \bigl(\varkappa (a * v)(x,s) + \beta v(x,s) \bigr)\, ds.$$

Let $\|v\|_{\tau,\Upsilon} := \sup\limits_{t \in [\tau,\Upsilon]} \|v(\cdot,t)\|$, where the latter norm is the norm in E. Then, one easily gets, that $\|\Phi_\tau v\|_{\tau,\Upsilon} < \infty$ and

$$\|\Phi_\tau v_1 - \Phi_\tau v_2\|_{\tau,\Upsilon} \leq (\varkappa + \beta) \|v_1 - v_2\|_{\tau,\Upsilon} (\Upsilon - \tau).$$

Therefore, Φ_τ is contraction mapping on the set of such v, provided that $\Upsilon - \tau < \frac{1}{\varkappa+\beta}$. Fixing any $\delta \in \bigl(0, \frac{1}{\varkappa+\beta}\bigr)$, one gets that w is the limit of $(\Phi_\tau)^n v$, $n \to \infty$, for any v, on time intervals $[0,\delta]$, $[\delta, 2\delta]$, and so on.

Suppose that, for some $\tau = (N-1)\delta$, $N \in \mathbb{N}$, that $\|u_\tau\|_{\widetilde{\omega}} \leq \|u_0\|_{\widetilde{\omega}} e^{pt}$, for

$$p := \varkappa(\gamma - 1) + \beta.$$

Take any $0 \leq v(\cdot,t) \in E$, $t \in [\tau, \Upsilon]$, $\Upsilon := \tau + \delta$, such that

$$\|v(\cdot,t)\|_{\widetilde{\omega}} \leq \|u_0\|_{\widetilde{\omega}} e^{pt}, \quad t \in [\tau, \Upsilon]. \tag{4.24}$$

We will check the following inequality

$$\|(\Phi_\tau v)(\cdot,t)\|_{\widetilde{\omega}} \leq \|u_0\|_{\widetilde{\omega}} e^{pt}, \quad t \in [\tau, \Upsilon].$$

By (4.23)–(4.24), one gets, for $t \in [\tau, \Upsilon]$,

$$\begin{aligned}
0 &\leq \frac{(\Phi_\tau v)(x,t)}{\widetilde{\omega}(x)} \\
&\leq e^{-\varkappa(t-\tau)} \frac{u_\tau(x)}{\widetilde{\omega}(x)} + \varkappa \int_\tau^t e^{-\varkappa(t-s)} \frac{(a * v)(x,s)}{\widetilde{\omega}(x)} ds + \beta \int_\tau^t e^{-\varkappa(t-s)} \frac{v(x,s)}{\widetilde{\omega}(x)} ds \\
&\leq \|u_0\|_{\widetilde{\omega}} e^{-\varkappa(t-\tau)} e^{p\tau} + \|u_0\|_{\widetilde{\omega}} \left(\varkappa \frac{(a * \widetilde{\omega})(x)}{\widetilde{\omega}(x)} + \beta \right) \int_\tau^t e^{-\varkappa(t-s)} e^{ps} ds \\
&= \|u_0\|_{\widetilde{\omega}} e^{-\varkappa t} e^{(p+\varkappa)\tau} + \frac{\varkappa\gamma + \beta}{p + \varkappa} \|u_0\|_{\widetilde{\omega}} e^{-\varkappa t} \bigl(e^{(p+\varkappa)t} - e^{(p+\varkappa)\tau} \bigr) = \|u_0\|_{\widetilde{\omega}} e^{pt}.
\end{aligned}$$

Since w is the limiting function for the sequence $\Phi_\tau^n v$, $n \in \mathbb{N}$, one gets the statement. $\square$

Remark 4.9. In [25, Propostion 5.2], we considered, for an arbitrary $\lambda > 0$ and a unit vector, $\xi \in \mathbb{R}^d$, the function $\widetilde{\omega}(x) = e^{-\lambda x\cdot\xi}$ (here $x\cdot\xi$ stands for the scalar product in $\mathbb{R}^d$). Then, clearly, $\frac{(a^+ * \widetilde{\omega})(x)}{\widetilde{\omega}(x)} \equiv \int_{\mathbb{R}^d} a^+(y) e^{\lambda y\cdot\xi}\,dy =: \gamma$, provided that the latter integral is finite (that was the crucial assumption to get the constant speed of the front in [25]). Note that then [25, Proposition 4.18] and (4.22) implies that $w(x,t) \leq \alpha_\xi e^{\beta_\xi t - \lambda_\xi x\cdot\xi}$, $x \in \mathbb{R}^d$, $t \geq 0$ for some $\alpha_\xi, \lambda_\xi > 0$, $\beta_\xi \in \mathbb{R}$.

Proposition 4.10 ([27, Proposition 2.21])**.** *Let a function* $\omega : \mathbb{R}^d \to (0,+\infty)$ *be such that, for any* $\lambda > 0$*,*

$$\Omega_\lambda := \Omega_\lambda(\omega) := \bigl\{x \in \mathbb{R}^d : \omega(x) < \lambda\bigr\} \neq \varnothing. \tag{4.25}$$

Suppose further that

$$\eta := \limsup_{\lambda\to 0+} \sup_{x\in\Omega_\lambda} \frac{(a*\omega)(x)}{\omega(x)} \in (0,\infty). \tag{4.26}$$

Then, for any $\delta \in (0,1)$*, there exists* $\lambda = \lambda(\delta,\omega) \in (0,1)$*, such that* (4.21) *holds, with*

$$\widetilde{\omega}(x) := \omega_\lambda(x) := \min\bigl\{\lambda, \omega(x)\bigr\}, \quad x \in \mathbb{R}^d, \tag{4.27}$$

and $\gamma := \max\{1, (1+\delta)\eta\}$*.*

Remark 4.11. It is easy to check that any function $\omega \in \widetilde{\mathcal{R}} \cup \mathcal{M}$ satisfies (4.25) and, moreover,

$$\Omega_\lambda \subset \Omega_{\lambda'}, \quad 0 < \lambda < \lambda'. \tag{4.28}$$

Proposition 4.12. *Let* $\omega \in \widetilde{\mathcal{R}} \cup \mathcal{M}$ *be such that, cf.* (4.26)*,*

$$\limsup_{\lambda\to 0+} \sup_{x\in\Omega_\lambda} \frac{(a*\omega)(x)}{\omega(x)} \leq 1. \tag{4.29}$$

Let $0 \leq u_0 \in L^\infty(\mathbb{R}^d)$ *be such that* $\|u_0\|_\omega < \infty$*, cf.* (4.20)*, and let* $w = w(x,t)$ *be the corresponding solution to* (1.32)*. Then, for any* $\varepsilon \in (0,1)$*, there exist* $A_\varepsilon > 0$ *and* $t_0 = t_0(\varepsilon) > 0$*, such that, cf.* (2.11)*,*

$$\operatorname*{ess\,sup}_{x\notin\Lambda_\varepsilon^+(t,\omega)} w(x,t) \leq \bigl(A_\varepsilon + \|u_0\|_\omega\bigr) e^{-\frac{\varepsilon\beta}{2}t}, \quad t \geq t_0. \tag{4.30}$$

Proof. Take an arbitrary $\varepsilon \in (0,1)$ and let $\delta = \delta(\varepsilon) \in (0,1)$ be chosen later. By Remark 4.11 and Proposition 4.10, there exists $\lambda = \lambda(\delta,\omega) = \lambda(\varepsilon,\omega) \in (0,1)$, such that (4.21) holds, with $\widetilde{\omega}$ given by (4.27) and $\gamma = 1 + \delta$. Set $\|u_0\|_\infty := \|u_0\|_{L^\infty(\mathbb{R}^d)}$. Note that

$$\frac{u_0(x)}{\omega_\lambda(x)} \leq \frac{\theta}{\lambda} 1\!\!1_{\mathbb{R}^d\setminus\Omega_\lambda}(x) + \frac{u_0(x)}{\omega(x)} 1\!\!1_{\Omega_\lambda}(x) \leq \frac{\|u_0\|_\infty}{\lambda} + \|u_0\|_\omega < \infty, \tag{4.31}$$

and one can apply Proposition 4.8. Namely, setting $A_\varepsilon := \frac{\|u_0\|_\infty}{\lambda} > 0$, one gets from (4.31), (4.22) that, for a.a. $x \in \Omega_\lambda$ and for all $t \geq 0$,

$$\begin{aligned} w(x,t) &\leq \|u_0\|_{\omega_\lambda} e^{(\varkappa\delta+\beta)t}\omega_\lambda(x) \\ &\leq \bigl(A_\varepsilon + \|u_0\|_\omega\bigr)e^{(\varkappa\delta+\beta)t}\omega(x). \end{aligned} \tag{4.32}$$

By (2.11) and (4.25),

$$\mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,\omega) = \Omega_{e^{-\beta_\varepsilon^+ t}}, \quad t > 0. \tag{4.33}$$

Set $t_0 = t_0(\varepsilon) := -\frac{1}{\beta_\varepsilon^+}\log\lambda > 0$. By (4.28), one gets from (4.33) that, for any $t \geq t_0$,

$$\mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,\omega) \subset \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t_0,\omega) = \Omega_\lambda.$$

Hence, by (4.32), (4.33), for a.a. $x \in \mathbb{R}^d \setminus \Lambda_\varepsilon^+(t,\omega)$, one gets

$$\begin{aligned} w(x,t) &\leq \bigl(A_\varepsilon + \|u_0\|_\omega\bigr)e^{(\varkappa\delta+\beta)t}\omega(x) \\ &\leq \bigl(A_\varepsilon + \|u_0\|_\omega\bigr)e^{(\varkappa\delta+\beta)t}e^{-\beta_\varepsilon^+ t}, \end{aligned}$$

and

$$\varkappa\delta + \beta - \beta_\varepsilon^+ = \varkappa\delta + \beta - \beta(1+\varepsilon) = \varkappa\delta - \beta\varepsilon = -\frac{\varepsilon\beta}{2},$$

if only we set from the very beginning $\delta := \frac{\varepsilon\beta}{2\varkappa}$. The statement is proved. □

Remark 4.13. It is easy to see from the proof above, that the denominator 2 in the right-hand side of (4.30) can be changed on $1+\nu$, for an arbitrary $\nu \in (0,1)$; then $t_0 = t_0(\varepsilon,\nu)$.

Because of (1.33), we immediately get the following

Corollary 4.14. *Let the assumptions* (A1)–(A4) *hold. Then the statement of Proposition 4.12 remains true for $w(x,t)$ replaced by $u(x,t)$ which is the solution to* (1.1).

Proposition 4.15. *Let $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. Then there exists $\alpha_1 \in (0,1)$, such that, for all $\alpha \in [\alpha_1,1]$, $b^\alpha \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$.*

Proof. Let $d > 1$ and $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d} \subset \mathcal{S}_{\mathrm{reg},d}$. If b is given by (2.7), then, for any $\alpha' \in \bigl(\frac{d}{d+\mu},1\bigr)$,

$$\int_0^\infty b(s)^{\alpha'} s^{d-1}\,ds < \infty. \tag{4.34}$$

If b is such that, for all $\nu \geq 1$, (2.8) holds, then, evidently, (4.34) holds for all $\alpha' \in (0,1)$. For $d = 1$ and $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},1} = \mathcal{S}_{\mathrm{reg},1}$, by Remark 2.7, (4.34) holds if only $\alpha' \in \bigl(\frac{1}{1+\delta},1\bigr)$, where δ is given by (2.5).

Then, by [27, Theorem 3.5], there exists $\alpha_1 \in (\alpha',1)$, such that, for all $\alpha \in [\alpha_0,1]$, $b^\alpha \in \mathcal{S}_{\mathrm{reg},d}$. It is easy to see that, in each of the cases above, $b^\alpha \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$ as well. □

Proposition 4.16 ([27, Propositions 4.5, 4.9])**.** *Let* (2.22) *hold with $b_1 \in \widetilde{\mathcal{D}}_d$ which is log-equivalent to a function $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. Then there exists $\alpha_1 \in (0,1)$, such that, for all $\alpha \in (\alpha_1,1)$, the function $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$, satisfies* (4.29).

Now we are going to find examples of $\varpi \in \widetilde{\mathcal{I}}$ such that (4.29) holds. We start with the following definition.

Definition 4.17. Let $p \in \widetilde{\mathcal{R}}$ be constructed by a function $b \in \widetilde{\mathcal{D}}_d$, i.e. $p(x) = b(|x|)$, $x \in \mathbb{R}^d$. For any $\lambda \in \bigl(0, b(0)\bigr)$, we set

$$\Theta_\lambda(p) := \bigl\{x \in \mathbb{R}^d : \Delta(x) \subset \Omega_\lambda(p)\bigr\}, \tag{4.35}$$

where $\Delta(x)$ is given by (1.5).

Remark 4.18. Let $\rho_{\lambda,b} > 0$ be the unique number such that $b\bigl(\rho_{\lambda,b}\bigr) = \lambda$. Then, evidently, $\Omega_\lambda(p) = \bigl\{x \in \mathbb{R}^d : |x| > \rho_{\lambda,b}\bigr\}$. Therefore, by (3.6), one gets, for any $x \in \mathbb{R}^d$ with $\langle x\rangle > \rho_{\lambda,b}$ and for any $y \in \Delta(x)$, that $|y| > \rho_{\lambda,b}$, and hence $y \in \Omega_\lambda(p)$. As a result,

$$\bigl\{x \in \mathbb{R}^d : \langle x\rangle > \rho_{\lambda,b}\bigr\} \subset \Theta_\lambda(p);$$

in particular, the latter set is not empty.

Proposition 4.19. *Let $p \in \widetilde{\mathcal{R}}$ be constructed by a function $b \in \widetilde{\mathcal{D}}_d$. Suppose that* (4.29) *holds with $\varpi = p$ and $\Omega_\lambda = \Omega_\lambda(p)$. Let $c \in \widetilde{\mathcal{I}}$ be given by* (2.9). *Then the following analogue to* (4.29) *holds:*

$$\limsup_{\lambda \to 0+} \sup_{x \in \Theta_\lambda(p)} \frac{(a * c)(x)}{c(x)} \leq 1. \tag{4.36}$$

Proof. Take an arbitrary $\delta \in (0,1)$. By (4.29) with $\varpi = p$, there exists $\lambda_0 = \lambda_0(\delta)$, such that, for all $\lambda \in (0, \lambda_0)$, we have

$$\frac{(a * p)(x)}{p(x)} \leq 1 + \delta, \quad x \in \Omega_\lambda(p). \tag{4.37}$$

Next, for any $x \in \mathbb{R}^d$, one gets

$$\begin{aligned}(a * c)(x) &= \int_{\mathbb{R}^d} a(x-y) \int_{\Delta(y)} p(z)\,dz\,dy \\ &= \int_{\mathbb{R}^d} a(x-y) \int_{\Delta(x)} p(z-(x-y))\,dz\,dy = \int_{\Delta(x)} (a * p)(z)\,dz\end{aligned}$$

As a result, by (4.37) and (4.35), we have that, for any $x \in \Theta_\lambda(p)$,

$$\frac{(a * c)(x)}{c(x)} = \frac{\displaystyle\int_{\Delta(x)} \frac{(a*p)(z)}{p(z)} p(z)\,dz}{c(x)} \leq 1 + \delta.$$

Since the latter holds for any $\lambda \in (0, \lambda_0)$, one gets the statement. $\square$

To get from (4.36) the inequality (4.29) with $\varpi = c$ and $\Omega_\lambda = \Omega_\lambda(c)$, consider the following lemma.

Lemma 4.20. *Let $p \in \widetilde{\mathcal{R}}$ be constructed by a long-tailed function $b \in \widetilde{\mathcal{D}}_d$ (for example, let $p \in \widetilde{\mathcal{L}}$). Let $c \in \widetilde{\mathcal{I}}$ be given by* (2.9). *Then there exists $\lambda_1 > 0$, such that, for all $\lambda \in (0, \lambda_1)$,*

$$\Omega_\lambda(c) \subset \Theta_\lambda(p).$$

Proof. By Lemma 3.14, we have that (3.19) holds with c replaced by p. As a result, for any $\varepsilon > 0$ and $r > 0$, there exists $R = R(\varepsilon, r) > 0$, such that

$$p(x+y) \geq (1-\varepsilon)p(x), \quad |y| \leq r, \ |x| \geq R.$$

Therefore, $x \in \Omega_\lambda(c)$ with $|x| \geq R$ implies that

$$\begin{aligned}\lambda &\geq \int_{x_1}^{x_1+\frac{r}{\sqrt{d}}} \cdots \int_{x_d}^{x_d+\frac{r}{\sqrt{d}}} b\Bigl(\sqrt{y_1^2+\ldots+y_d^2}\Bigr)\, dy_1 \ldots dy_d \\ &\geq \frac{r^d}{d^{\frac{d}{2}}} p\Bigl(x + \Bigl(\frac{r}{\sqrt{d}}, \ldots, \frac{r}{\sqrt{d}}\Bigr)\Bigr) \geq \frac{r^d}{d^{\frac{d}{2}}}(1-\varepsilon)p(x).\end{aligned}$$

Choose now $\varepsilon = \frac{1}{2}$ and $r = 2^{\frac{1}{d}}\sqrt{d} > 0$, and consider the corresponding R. Since $\lambda \downarrow 0$ if and only if $\langle x \rangle \to \infty$, there exists $\lambda_1 > 0$ such that, for all $\lambda \in (0, \lambda_1)$, the inclusion $x \in \Omega_\lambda(c)$ implies $\langle x \rangle > R$ and hence $|x| > R$. Moreover, for any $y \in \Delta(x)$, we have that $y \in \Omega_\lambda(c)$, by the monotonicity of c in each of variables; and, by (3.6), $\langle x \rangle > R$ implies $|y| > R$. As a result, for any $y \in \Delta(x)$ (including $y = x$), we have that $p(y) \leq \lambda$, i.e $\Delta(x) \subset \Omega_\lambda(p)$. Then, by (4.35), $x \in \Theta_\lambda(p)$, that proves the statement. □

Theorem 4.21. *Let $b : \mathbb{R} \to (0, \infty)$ be an even long-tailed function, such that, for some $\alpha_0 \in \bigl(\frac{3}{4}, 1\bigr)$, $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$; and, for each $\alpha \in (\alpha_0, 1)$, the inequality (4.29) holds with $\omega(x) = b(|x|)^\alpha$, $x \in \mathbb{R}^d$. Let $\tilde{b} \in \widetilde{\mathcal{D}}_d$ be log-equivalent to b, cf. Definition 2.8. Let $c, \tilde{c} \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ be constructed by the functions b and $\tilde{b}$, correspondingly, cf. (2.10). Suppose that $0 \leq u_0 \in L^\infty(\mathbb{R}^d)$ and $\|u_0\|_{\tilde{c}} < \infty$; and let $w = w(x,t)$ be the corresponding solution to (1.32). Then there exists $\varepsilon_0 \in (0,1)$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, there exist $C_\varepsilon = C_\varepsilon(u_0) > 0$ and $\tau = \tau(\varepsilon) > 0$, such that*

$$\operatorname*{ess\,sup}_{x \notin \Lambda_\varepsilon^+(t,c)} w(x,t) \leq C_\varepsilon e^{-\frac{\varepsilon\beta}{4}t}, \quad t \geq \tau. \tag{4.38}$$

Proof. Let $\alpha_0 \in \bigl(\frac{3}{4}, 1\bigr)$ be given. Note that by Remark 3.10, $b^{\alpha_0} \in \widetilde{\mathcal{D}}_d$ implies $b \in \widetilde{\mathcal{D}}_d$. Let $\varepsilon_0 = \varepsilon_0(\alpha_0)$ by given by Proposition 3.11. Take an arbitrary $\varepsilon \in (0, \varepsilon_0)$ and consider $\alpha = \alpha(\varepsilon) \in (\alpha_0, 1)$ also given by Proposition 3.11. Since $\log b(s) \sim \log \tilde{b}(s)$, $s \to \infty$, there exists $\rho = \rho(\alpha) = \rho(\varepsilon) > 0$, such that

$$\begin{gathered}-\log \tilde{b}(s) \geq -\alpha \log b(s) > 0, \quad s > \rho, \\ \tilde{b}(s) \leq b(s)^\alpha, \quad s > \rho.\end{gathered}$$

Since both $\tilde{b}$ and b are decreasing and separated from 0 on $[0, \rho]$, there exists $B > 0$, such that $\tilde{b}(s) \leq Bb(s)^\alpha$, $s \in \mathbb{R}_+$. Let $c_\alpha \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ be given by (3.7), and, recall, $\tilde{c} \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ is constructed by $\tilde{b}$, cf. (2.10). Then, clearly, $\tilde{c}(x) \leq Bc_\alpha(x)$, $x \in \mathbb{R}^d$. As a result,

$$\|u_0\|_{c_\alpha} \leq \frac{1}{B}\|u_0\|_{\tilde{c}} < \infty.$$

By the assumed, (4.29) holds for $\omega = p^\alpha$, where $p(x) = b(|x|)$, $x \in \mathbb{R}^d$. Therefore, for $c \in \widetilde{\mathcal{R}}$, one gets that (4.29) holds for $\omega = c_\alpha \in \widetilde{\mathcal{R}}$, cf. Definition 3.9. Let now $c \in \widetilde{\mathcal{I}}$. Since b is long-tailed, the function b^α is long-tailed as well. Then, one can use Lemma 4.20 with p replaced by b^α; one gets then, for some $\lambda_1 > 0$,

$$\Omega_\lambda(c_\alpha) \subset \Theta_\lambda(p^\alpha), \quad \lambda \in (0, \lambda_1).$$

Therefore, Proposition 4.19 implies that (4.29) holds for $\omega = c_\alpha \in \widetilde{\mathcal{I}}$.

As a result, one can use now Proposition 4.12 with $\omega = c_\alpha \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ and ε replaced by $\frac{\varepsilon}{2}$. Namely, there exist $A_\varepsilon > 0$ and $t_0 = t_0(\varepsilon) > 0$, such that

$$\operatorname*{ess\,sup}_{x \notin \Lambda^+_{\frac{\varepsilon}{2}}(t, c_\alpha)} w(x,t) \le \bigl(A_\varepsilon + B^{-1}\|u_0\|_{\tilde{c}}\bigr) e^{-\frac{\varepsilon\beta}{4} t}, \quad t \ge t_0. \tag{4.39}$$

On the other hand, by Proposition 3.11, there exists $\tau = \tau(\varepsilon) > 0$, such that (3.9) holds, i.e.

$$\mathbb{R}^d \setminus \Lambda^+_\varepsilon(t, c) \subset \mathbb{R}^d \setminus \Lambda^+_{\frac{\varepsilon}{2}}(t, c_\alpha), \quad t \ge \tau. \tag{4.40}$$

Combining (4.39) and (4.40), one gets (4.38). □

Remark 4.22. By Proposition 3.13, one can get (4.38) for any $c \in \mathcal{R} \cup \mathcal{I}$ constructed by a function $b_1 \in \mathcal{D}_d$ which is equivalent to the b.

Remark 4.23. Using a bit more cumbersome expressions for ε_0 and $\alpha = \alpha(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$ in the proof of Proposition 3.11, one can obtain (3.9) with $\frac{\varepsilon}{2}$ replaced by any $\varepsilon' \in (0, \varepsilon)$. As a result, we may apply Proposition 4.12 inside the proof of Theorem 4.21 with ε replaced by $\frac{\varepsilon}{1+\nu'}$ for any $\nu' \in (0,1)$. Combining this observation with Remark 4.13, one can get, as a result, (4.38), where, in the denominator of the right-hand side, the number 4 will be replaced by $1 + \nu''$ for an arbitrary $\nu'' \in (0,1)$, by a redefining of $\tau = \tau(\varepsilon, \nu'')$.

Remark 4.24. For $c \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$, the condition $\|u_0\|_c < \infty$ separates, in some sense, the cases of 'decreasing' and 'symmetric' initial conditions. Namely, if, for example, $u_0 \in \mathcal{M}$, then the inequality $\|u_0\|_c < \infty$ is impossible for any $c \in \widetilde{\mathcal{R}}$, cf. (4.20); and hence c must be from $\widetilde{\mathcal{I}}$.

Finally, one can prove Theorem 2.20.

Proof of Theorem 2.20. Let $b_1, b_2 \in \mathcal{D}_d$ and $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$ satisfy the conditions of Theorem 2.20, and let $c, c_2 \in \mathcal{R} \cup \mathcal{I}$ be constructed by b and b_2 correspondingly. By Corollary 3.8, there exists $\tilde{c} \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$, such that $c_2(x) \le \tilde{c}(x)$, $x \in \mathbb{R}^d$. Then (2.23) yields $\|u_0\|_{\tilde{c}} < \infty$. Let $\tilde{c}$ be constructed by a $\tilde{b} \in \widetilde{\mathcal{D}}_d$. By the proof of Corollary 3.8, $\tilde{b}(s) = b_2(s)$ for $s > \rho$. Similarly, there exists $b_3, b_4 \in \widetilde{\mathcal{D}}_d$, such that $b_3(s) = b_1(s)$, $b_4(s) = b(s)$ for $s > \rho$ (without loss of generality), and $b_1(s) \le b_3(s)$, $b(s) \le b_4(s)$ for $s \in [0, \rho]$. In particular, the functions $\tilde{b}$, b_3 and b_4 are log-equivalent, and, by e.g. [27, Theorem 3.5], $b_4 \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. By Propositions 4.15 (with b replaced by b_4) and 4.16 (with b replaced by b_4 and b_1 replaced by $b_3 \in \widetilde{\mathcal{D}}_d$), there exists $\alpha_1 \in (0,1)$, such that, for all $\alpha \in [\alpha_1, 1]$, $b_4^\alpha \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$ and, for all $\alpha \in (\alpha_1, 1)$, the function $\omega(x) = b_4(|x|)^\alpha$, $x \in \mathbb{R}^d$, satisfies (4.29). Choose any $\alpha_0 \in \bigl(\max\{\alpha_1, \frac{3}{4}\}, 1\bigr)$. Then all assumptions of Theorem 4.21 are fulfilled for $\tilde{c}, \tilde{b}$ as the above, and for c replaced by the function $c_4 \in \widetilde{\mathcal{R}} \cup \widetilde{\mathcal{I}}$ constructed by b_4. Hence one gets (4.38) for c replaced by c_4. By Corollary 3.8, redefining $\tau = \tau(\varepsilon)$ if needed, we will have (4.38) for c. Therefore, by (1.33), we obtain (2.24) with $\nu = 4$. □

4.3 Proofs of Propositions 2.21–2.24

Recall that, in Propositions 2.21–2.24, we always assume that either (A1)–(A11) hold or (A12) holds; that (B1)–(B5) hold; and that $u_0 \in E_\theta^+$, $u_0 \not\equiv 0$, cf. Remark 1.4, and u is the corresponding solution to (1.1).

Proof of Proposition 2.21. Let $\varepsilon_0 \in (0,1)$ be chosen later. Take an arbitrary $\varepsilon \in (0,\varepsilon_0)$.

Let $c_+ \in \mathcal{L} \subset L^1(\mathbb{R}^d)$ be constructed by $b_+ \in \mathcal{D}_d$. Note that (2.25) yields $u_0 \in L_1(\mathbb{R}^d)$. Therefore, one can apply Proposition 3.17 with $c = c_+ > 0$ and $f = u_0$; namely, there exists $D > 0$, such that $a * u_0 \geq c_+ * u_0 \geq Dc_+ \in \mathcal{L}$. Then, by Theorem 2.19, the convergence (2.21) holds, with ε replaced by $\frac{\varepsilon}{2} < \varepsilon_0$ and c replaced by Dc_+. Since the functions Db_+ and b are also log-equivalent, one can apply Proposition 3.13 with $b_1 = b$ and $b_2 = Db_+$, to get inclusion $\Lambda_\varepsilon^-(t,c) \subset \Lambda_{\frac{\varepsilon}{2}}^-(t,Dc_+)$. As a result, (2.21) holds, with $c(x) = b(|x|)$, $x \in \mathbb{R}^d$. Note that we had not any restrictions on ε_0 here.

Since $b^+ \in \mathcal{D}_d$, we can apply then Theorem 2.20 with $b_1 = b_2 = b^+$ and the given $b \in \widetilde{\mathcal{S}}_{\mathrm{reg},d}$. Indeed, (B1) implies (2.22), and, for the $c_2 \in \mathcal{R}$ constructed by b^+, (2.25) is just (2.23). Therefore, (2.24) holds. ☐

Proof of Proposition 2.22. The proof of (2.21) is essentially the same as that for Proposition 2.21, with only the difference that we will apply now Proposition 3.17 for $c = v_\circ > 0$ and $f = a \in L^1(\mathbb{R}^d)$. Next, since $b^\circ \in \mathcal{D}_d$ and (B4) holds, we can apply Theorem 2.20 with $b_1 = b_2 = b^\circ$. ☐

Proof of Proposition 2.23. Let $\varepsilon_0 \in (0,1)$ be chosen later. Take an arbitrary $\varepsilon \in (0,\varepsilon_0)$. By (B1) and (2.28), we have

$$\begin{aligned}(a * u_0)(x) &\geq \zeta \int_{\mathbb{R}^d} b_+(|y|)\mathbb{1}_{\mathbb{R}^d_-}(x-y)\,dy\\ &= \zeta \int_{\Delta(x)} b_+(|y|)\,dy =: \tilde{c}(x), \quad x \in \mathbb{R}^d.\end{aligned}$$

Since b_+ is long-tailed and tail-log-convex, one gets that $\tilde{c} \in \mathcal{N}$. Therefore, one can apply Theorem 2.19 to get (2.21) with c replaced by $\tilde{c}$ and ε replaced by $\frac{\varepsilon}{2}$. Since the functions b and ζb_+ are log-equivalent, one can apply Proposition 3.13 with $c^{(1)}(x) = c(x) := \int_{\Delta(x)} b(|y|)dy$ and $c^{(2)}(x) = \tilde{c}(x)$, $x \in \mathbb{R}^d$; and then (3.16) leads to (2.21) for this c.

To get (2.24) we will need just to repeat all corresponding arguments from the proof of Proposition 2.21 with only the difference that Theorem 2.20 will be applied now for functions from $\mathcal{I}$. ☐

Remark 4.25. Using [26, Proposition 5.5 (Q2)] and modifying accordingly the proof of Theorem 2.19, one can replace $\mathbb{R}^d_-$ in (2.28) by $\bigtimes_{j=1}^{d}(-\infty,\overline{y}_j]$, for an arbitrary fixed $\overline{y} \in \mathbb{R}^d$.

Remark 4.26. If, additionally, $u_0(x) = \int_{\Delta(x)} p(y)dy$, $x \in \mathbb{R}^d$ for some $p \in L^1(\mathbb{R}^d)$, then, evidently,

$$\sup_{x\in\mathbb{R}^d} \frac{p(x)}{a(x)} < \infty \quad \Longrightarrow \quad \sup_{x\in\mathbb{R}^d} \frac{u_0(x)}{\displaystyle\int_{\Delta(x)} a(y)dy} < \infty.$$

Proof of Proposition 2.24. First, we apply Proposition 3.17 with $f = a$ and c replaced by $v_\circ \in \mathcal{N}$. Then, similarly to the proof of Proposition 2.23, we may apply Theorem 2.19 to get (2.21) with c replaced by $v_\circ$ and ε replaced by $\frac{\varepsilon}{2}$, and, by using the log-equivalence between b and $b_\circ$ and Proposition 3.13, we will get (2.21) for the required c.

To get (2.24), one can use the same arguments as in the proof of Proposition 2.22. ☐

Acknowledgements

Financial supports by the DFG through CRC 701, Research Group "Stochastic Dynamics: Mathematical Theory and Applications", and by the European Commission under the project STREVCOMS PIRSES-2013-612669 are gratefully acknowledged.

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