

# Heat kernel estimates on connected sums of parabolic manifolds

Alexander Grigor'yan\*  
Department of Mathematics  
University of Bielefeld  
33501 Bielefeld, Germany  
grigor@math.uni-bielefeld.de

Satoshi Ishiwata†  
Department of Mathematical Sciences  
Yamagata University  
Yamagata 990-8560, Japan  
ishiwata@sci.kj.yamagata-u.ac.jp

Laurent Saloff-Coste ‡  
Department of Mathematics  
Cornell University  
Ithaca, NY, 14853-4201, USA  
lsc@math.cornell.edu

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## Abstract

We obtain matching two sided estimates of the heat kernel on a connected sum of parabolic manifolds, each of them satisfying the Li-Yau estimate. The key result is the on-diagonal upper bound of the heat kernel at a central point. Contrary to the non-parabolic case (which was settled in [15]), the on-diagonal behavior of the heat kernel in our case is determined by the end with the *maximal* volume growth function. As examples, we give explicit heat kernel bounds on the connected sums  $\mathbb{R}^2 \# \mathbb{R}^2$  and  $\mathcal{R}^1 \# \mathbb{R}^2$  where  $\mathcal{R}^1 = \mathbb{R}_+ \times \mathbb{S}^1$ .

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# 1 Introduction

Let  $M$  be a Riemannian manifold. The heat kernel  $p(t, x, y)$  on  $M$  is the minimal positive fundamental solution of the heat equation  $\partial_t u = \Delta u$  on  $M$  where  $u = u(t, x)$ ,  $t > 0$ ,  $x \in M$  and  $\Delta$  is the (negative definite) Laplace-Beltrami operator on  $M$ . For example, in  $\mathbb{R}^n$  the heat kernel is given by the classical Gauss-Weierstrass formula

$$p(t, x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right).$$

The heat kernel is sensitive to the geometry of the underlying manifold  $M$ , which results in numerous applications of this notion in differential geometry. On the other hand, the heat kernel has a probabilistic meaning:  $p(t, x, y)$  is the transition density of Brownian motion  $(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M})$  on  $M$ . Namely, for any Borel set  $A \subset M$ , we have

$$\mathbb{P}_x(X_t \in A) = \int_A p(t, x, y) dy,$$

where  $\mathbb{P}_x(X_t \in A)$  is the probability that Brownian particle starting at the point  $x$  will be found in the set  $A$  in time  $t$ .

From now on let us assume that the manifold  $M$  is non-compact and geodesically complete. Dependence of the long time behavior of the heat kernel on the large scale geometry of  $M$  is an interesting and important problem that has been intensively studied during the past few decades by many authors (see, for example, [4], [10], [21] and references therein). In the case when the Ricci curvature of  $M$  is non-negative, P.Li and S.-T.Yau proved in their pioneering work [19] the following estimate, for all  $x, y \in M$  and  $t > 0$ :

$$p(t, x, y) \asymp \frac{C}{V(x, \sqrt{t})} \exp\left(-b \frac{d^2(x, y)}{t}\right), \tag{LY}$$

where the sign  $\asymp$  means that both  $\leq$  and  $\geq$  hold but with different values of positive constants  $C$  and  $b$ ,  $V(x, r)$  is the Riemannian volume of the geodesic ball of radius  $r$  centered at  $x \in M$ , and  $d(x, y)$  is the geodesic distance between the points  $x, y$ .

The estimate (LY) is satisfied also for the heat kernel of uniformly elliptic operators in divergence form in  $\mathbb{R}^n$  as was proved by Aronson [1]. It was proved by Fabes and Stroock [6], that the estimate (LY) is equivalent to the uniform parabolic Harnack inequality (see also [21]). Grigor'yan [7] and Saloff-Coste [20], [21] proved that (LY) is equivalent to the conjunction of the Poincaré inequality and the volume doubling property.

One of the simplest example of a manifold where (LY) fails is the hyperbolic space  $\mathbb{H}^n$ . A more interesting counterexample was constructed by Kuz'menko and Molchanov [18]: they showed that the connected sum  $\mathbb{R}^n \# \mathbb{R}^n$  of two copies of  $\mathbb{R}^n$ ,  $n \geq 3$ , admits a non-trivial bounded harmonic function, which implies that the Harnack inequality and, hence,

(*LY*) cannot be true. Benjamini, Chavel and Feldman [2] explained this phenomenon by a bottleneck-effect: if  $x$  and  $y$  belong to the different ends of the manifold  $\mathbb{R}^n \# \mathbb{R}^n$  and  $|x| \approx |y| \approx \sqrt{t} \rightarrow \infty$  then  $p(t, x, y) \ll t^{-n/2}$  where  $t^{-n/2}$  is predicted by the right hand side of (*LY*). This phenomenon is especially transparent from probabilistic viewpoint: Brownian particle can go from  $x$  to  $y$  only through the central part, which reduces drastically the transition density (see Fig. 1). A similar phenomenon was observed by B.Davies [5] on a model case of one-dimensional line complex.

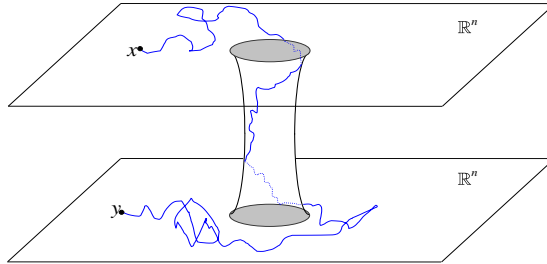


Figure 1: Brownian path goes from  $x$  to  $y$  via the bottleneck

Based on these early works, the first and the third authors of the present paper started a project on heat kernel bounds on connected sums of manifolds, provided each of them satisfies the Li-Yau estimate (*LY*). The results of this study are published in a series [11], [12], [13], [15], and [16]. In particular, they obtained in [15] matching upper and lower estimates of heat kernels on connected sums of manifolds when at least one of them is *non-parabolic*. Recall that a manifold  $M$  called *parabolic* if Brownian motion on  $M$  is recurrent, and *non-parabolic* otherwise. There are several equivalent definitions of parabolicity in different terms (see, for example, [9]).

In this paper we complement the results [15] by proving two-sided estimates of heat kernels on connected sums of *parabolic* manifolds. The detailed statements are given in the next section. We illustrate our results on the following two examples.

Consider first the manifold  $M = \mathcal{R}^1 \# \mathbb{R}^2$ , where  $\mathcal{R}^1 = \mathbb{R}_+ \times \mathbb{S}^1$  (see Fig. 2). For  $x \in M$ ,

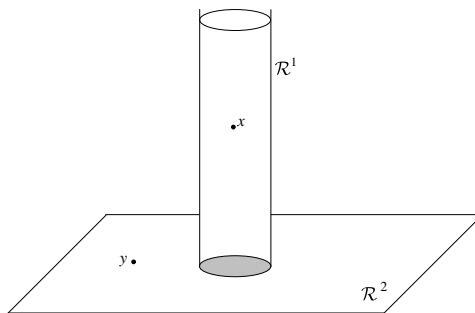


Figure 2: Connected sum  $\mathcal{R}^1 \# \mathbb{R}^2$

define  $|x| := d(x, K) + e$ , where  $K \subset M$  is the central part of  $M$ . Then we obtain that for

$x \in \mathcal{R}^1, y \in \mathbb{R}^2$  and  $t > 1$

$$p(t, x, y) \asymp \begin{cases} \frac{1}{t} e^{-b \frac{d^2(x,y)}{t}} & \text{if } |y| > \sqrt{t}, \\ \frac{1}{t} \left( 1 + \frac{|x|}{\sqrt{t}} \log \frac{e\sqrt{t}}{|y|} \right) & \text{if } |x|, |y| \leq \sqrt{t}, \\ \frac{1}{t} \log \frac{e\sqrt{t}}{|y|} & \text{if } |x| > \sqrt{t} \geq |y|. \end{cases}$$

In particular, if  $|x|, |y|$  are bounded and  $t \rightarrow \infty$ , then

$$p(t, x, y) \approx \frac{1}{t}.$$

If  $|x| \approx \sqrt{t} \rightarrow \infty$  and  $|y|$  remains bounded, then

$$p(t, x, y) \approx \frac{\log t}{t}.$$

Consider now the manifold  $M = \mathbb{R}^2 \# \mathbb{R}^2$ , or, equivalently, a catenoid (see Fig. 3).

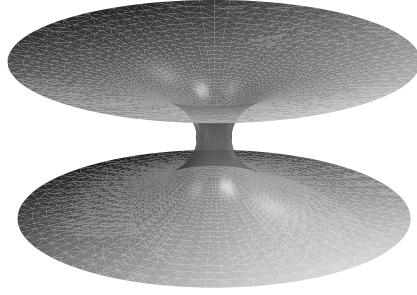


Figure 3: Catenoid

Then we have the following estimate for all  $x, y$  lying in different sheets and for  $t > 1$ :

$$p(t, x, y) \asymp \begin{cases} \frac{1}{t \log^2 t} (\log t + \log^2 \sqrt{t} - \log |x| \log |y|) & \text{if } |x|, |y| \leq \sqrt{t}, \\ \frac{1}{t \log t} \log \frac{e\sqrt{t}}{|y|} e^{-b \frac{d^2(x,y)}{t}} & \text{if } |y| \leq \sqrt{t} < |x|, \\ \frac{1}{t \log t} \log \frac{e\sqrt{t}}{|x|} e^{-b \frac{d^2(x,y)}{t}} & \text{if } |x| \leq \sqrt{t} < |y|, \\ \frac{1}{t} \left( \frac{1}{\log |x|} + \frac{1}{\log |y|} \right) e^{-b \frac{d^2(x,y)}{t}} & \text{if } |x|, |y| > \sqrt{t}. \end{cases}$$

In particular, if  $|x|, |y|$  are bounded and  $t \rightarrow \infty$ , then

$$p(t, x, y) \approx \frac{1}{t}.$$

If  $|x| \approx |y| \approx \sqrt{t} \rightarrow \infty$  then

$$p(t, x, y) \approx \frac{1}{t \log t}.$$

The heat kernel estimates on  $\mathbb{R}^2 \# \mathbb{R}^2$  was also obtained in [15] by an ad hoc method. In the present paper these estimates are part of our general Theorem 2.3. We also give further examples, in particular, the heat kernel estimates on  $\mathcal{R}^1 \# \mathcal{R}^1 \# \mathbb{R}^2$ .

In the next section we introduce necessary definitions and state our main results. In Section 3 we prove some auxiliary results about the integrated resolvent. In Section 4 we prove the main technical result of this paper – Theorem 2.1 about on-diagonal upper bound

of the heat kernel on the connected sum of parabolic manifolds. Finally, in Section 5 we use Theorem 2.1 and the gluing techniques from [15] to obtain full off-diagonal estimates of the heat kernels; they are stated in Theorems 2.3-2.5 and Corollaries 2.8 and 2.9.

NOTATION. Throughout this article, the letters  $c, C, b, \dots$  denote positive constants whose values may be different at different instances. When the value of a constant is significant, it will be explicitly stated. The notation  $f \approx g$  for two non-negative functions  $f, g$  means that there are two positive constants  $c_1, c_2$  such that  $c_1g \leq f \leq c_2g$  for the specified range of the arguments of  $f$  and  $g$ .

## 2 Statement of main results and examples

The main result will be stated in a more general setting of weighted manifolds that is explained below.

### 2.1 Weighted manifolds

Let  $M$  be a connected Riemannian manifold of dimension  $N$ . The Riemannian metric of  $M$  induces the geodesic distance  $d(x, y)$  between points  $x, y \in M$  and the Riemannian measure  $d\text{vol}$ . Given a smooth positive function  $\sigma$  on  $M$ , let  $\mu$  be the measure on  $M$  given by  $d\mu(x) = \sigma(x)d\text{vol}(x)$ . The pair  $(M, \mu)$  is called a *weighted manifold*. Any Riemannian manifold can be considered also as a weighted manifold with  $\sigma \equiv 1$ .

The Laplace operator  $\Delta$  of the weighted manifold  $(M, \mu)$  is defined by

$$\Delta = \frac{1}{\sigma} \text{div}(\sigma \nabla),$$

where  $\text{div}$  and  $\nabla$  are the divergence and the gradient of the Riemannian metric of  $M$ . It is easy to see that  $\Delta$  is the generator of the following Dirichlet form

$$D(f, f) = \int_M |\nabla f|^2 d\mu$$

in  $W^{1,2}(M, \mu)$ . The associated heat semigroup  $e^{t\Delta}$  has always a smooth positive kernel  $p(t, x, y)$  that is called the heat kernel of  $(M, \mu)$ . At the same time,  $p(t, x, y)$  is the minimal positive fundamental solution of the corresponding heat equation  $\partial_t u = \Delta u$  on  $M \times \mathbb{R}_+$  (see [10]). The heat kernel is also the transition probability density of Brownian motion  $(\{X_t\}, \{\mathbb{P}_x\})$  on  $M$  that is generated by  $\Delta$ .

A weighted manifold  $(M, \mu)$  is called *parabolic* if any positive superharmonic function on  $M$  is constant, and *non-parabolic* otherwise. The parabolicity is equivalent to each of the following properties, that can be regarded as equivalent definitions (see, for example, [9]):

1. There exists no positive fundamental solution of  $-\Delta$ .
2.  $\int_0^\infty p(t, x, y) dt = \infty$  for all/some  $x, y \in M$ .
3. Brownian motion on  $M$  is recurrent.

### 2.2 Notion of connected sum

Let  $(M, \mu)$  be a geodesically complete non-compact weighted manifold. Let  $K \subset M$  be a connected compact subset of  $M$  with non-empty interior and smooth boundary such that  $M \setminus K$  has  $k$  non-compact connected components  $E_1, \dots, E_k$ ; moreover, assume also that the

closures  $\overline{E}_i$  are disjoint. We refer to each  $E_i$  as an *end* of  $M$ . Clearly,  $\partial K$  is a disjoint union of  $\partial E_i$ ,  $i = 1, \dots, k$ .

Assume also that  $E_i$  is isometric to the exterior of a compact set  $K_i$  in another weighted manifold  $(M_i, \mu_i)$ . Then we refer to  $M$  as the connected sum of  $M_1, \dots, M_k$  and write

$$M = M_1 \# M_2 \# \dots \# M_k$$

(see Fig. 4).

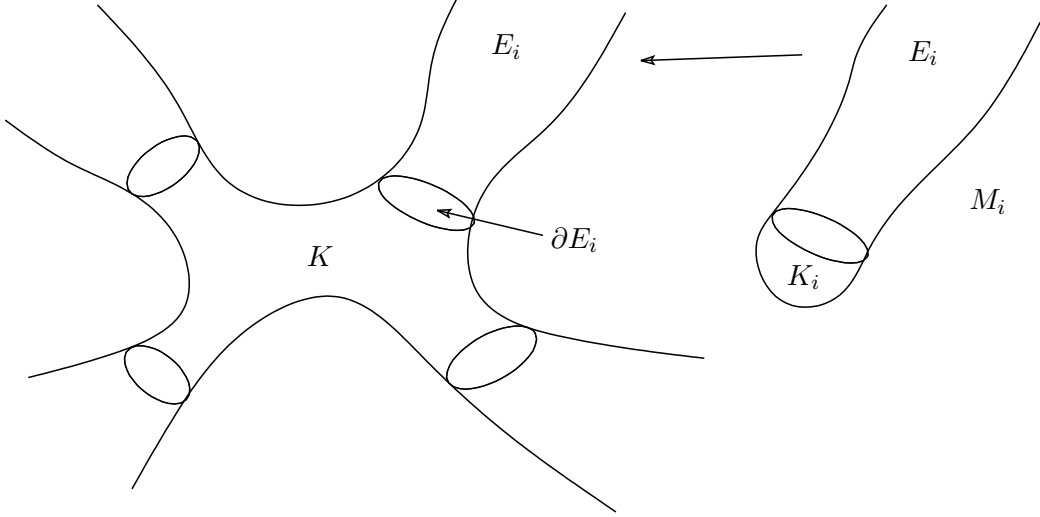


Figure 4: Connected sum  $M = M_1 \# M_2 \dots \# M_k$ .

Denote by  $d_i$  the geodesic distance on  $M_i$  and by  $B_i(x, r)$  the geodesic ball in  $M_i$  of radius  $r$  centered at  $x \in M_i$ . Set also  $V_i(x, r) = \mu_i(B_i(x, r))$ . Fix a reference point  $o_i \in K_i$  and set

$$V_i(r) = V_i(o_i, r).$$

In this paper we always assume that every manifold  $M_i$ ,  $i = 1, \dots, k$ , satisfies the following four conditions.

- (a) The heat kernel  $p_i(t, x, y)$  of  $(M_i, \mu_i)$  satisfies the Li-Yau estimate (*LY*), that is,

$$p_i(t, x, y) \asymp \frac{C}{V_i(x, \sqrt{t})} \exp\left(-b \frac{d_i^2(x, y)}{t}\right). \quad (2.1)$$

- (b)  $M_i$  is parabolic; under the standing assumption (2.1), the parabolicity of  $M_i$  is equivalent to

$$\int_0^\infty \frac{r dr}{V_i(r)} = \infty. \quad (2.2)$$

- (c)  $M_i$  has *relatively connected annuli*, that is, there exists a positive constant  $A > 1$  such that for any  $r > A^2$  and all  $x, y \in M_i$  with  $d_i(o_i, x) = d_i(o_i, y) = r$ , there exists a continuous path from  $x$  to  $y$  staying in  $B_i(o_i, Ar) \setminus B_i(o_i, A^{-1}r)$ . We denote this condition shortly by (*RCA*).

- (d)  $M_i$  is either *critical* or *subcritical*; here  $M_i$  is called critical if, for all large enough  $r$ ,

$$V_i(r) \approx r^2,$$

and subcritical if, for all large enough  $r$ ,

$$\int_1^r \frac{s ds}{V_i(s)} \leq \frac{Cr^2}{V_i(r)}. \quad (2.3)$$

For example, if  $V_i(r) \approx r^\alpha \log^\beta r$  for some  $0 < \alpha < 2$  and  $\beta \in \mathbb{R}$ , then  $M_i$  is subcritical. On the other hand, in the case  $V_i(r) \approx \frac{r^2}{\log^\beta r}$  with  $\beta > 0$  the manifold  $M_i$  is neither critical nor subcritical, although still parabolic.

Let us describe a class of manifolds satisfying all the hypotheses (a) – (d). For any  $0 < \alpha \leq 2$  consider a Riemannian *model manifold*  $\mathcal{R}^\alpha := (\mathbb{R}^2, g_\alpha)$ , where  $g_\alpha$  is a Riemannian metric on  $\mathbb{R}^2$  such that, in the polar coordinates  $(\rho, \theta)$ , it is given for  $\rho > 1$  by

$$g_\alpha = d\rho^2 + \rho^{2(\alpha-1)} d\theta^2.$$

For example, if  $\alpha = 2$  then  $g_2$  can be taken to be the Euclidean metric of  $\mathbb{R}^2$  so that in this case  $\mathcal{R}^2 = \mathbb{R}^2$ . If  $\alpha = 1$  then  $g_1 = d\rho^2 + d\theta^2$  so that the exterior domain  $\{\rho > 1\}$  of  $\mathcal{R}^1$  is isometric to the cylinder  $\mathbb{R}_+ \times \mathbb{S}$  (see Fig. 5).

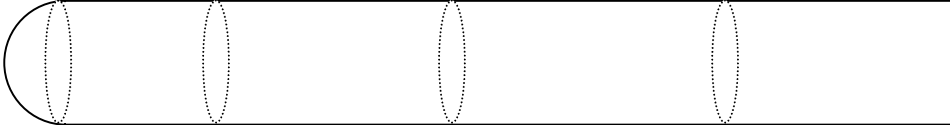


Figure 5: Model manifold  $\mathcal{R}^1$

For a general  $0 < \alpha < 2$ , the exterior domain  $\{\rho > 1\}$  of  $\mathcal{R}^\alpha$  is isometric to a certain surface of revolution in  $\mathbb{R}^3$ .

Observe that the volume function  $V(x, r)$  on  $\mathcal{R}^\alpha$  admits for  $r > 1$  the estimate

$$V(x, r) \approx \begin{cases} r^\alpha, & |x| < r \\ \min(r^2, r|x|^{\alpha-1}), & |x| \geq r \end{cases} \approx \frac{r^2}{1 + \frac{r}{(|x|+r)^{\alpha-1}}} \quad (2.4)$$

(see [14, Sec. 4.4]). In particular, if  $x = o$ , where  $o$  is the origin of  $\mathbb{R}^2$ , then

$$V(o, r) \approx r^\alpha. \quad (2.5)$$

By [14, Prop. 4.10],  $\mathcal{R}^\alpha$  satisfies the parabolic Harnack inequality and, hence, the Li-Yau estimate (LY). Obviously,  $\mathcal{R}^\alpha$  satisfies (2.2) and, hence,  $\mathcal{R}^\alpha$  is parabolic. It is easy to see that  $\mathcal{R}^\alpha$  satisfies (RCA). Note also that  $\mathcal{R}^\alpha$  is critical if  $\alpha = 2$  and subcritical if  $\alpha < 2$ . Hence,  $\mathcal{R}^\alpha$  satisfies all hypotheses (a) – (d).

One can make a similar family of examples also in class of weighted manifolds. Indeed, for any  $\alpha > 0$  consider in  $\mathbb{R}^2$  the following measure

$$d\mu_\alpha = \left(1 + |x|^2\right)^{\frac{\alpha}{2}-1} dx.$$

It is easy to see that  $(\mathbb{R}^2, \mu_\alpha)$  satisfies (2.5). The Li-Yau estimate on  $(\mathbb{R}^2, \mu_\alpha)$  holds by [14, Prop. 4.9]. Hence,  $(\mathbb{R}^2, \mu_\alpha)$  satisfies all the hypotheses (a) – (d) provided  $0 < \alpha \leq 2$ .

Returning to the general setting, let us mention that the hypotheses (a), (b), (c) are essential for our main result, whereas (d) is technical. Probably, the method of proof will work also without assuming (d) but, even if that is the case, the necessary computations will

become much more technical and complicated. So, we prefer to impose here the additional condition (d) to simplify the computational part of the proof, which even under (d) remains quite involved.

Observe also that the condition (b) follows from (d). Indeed, if the integral (2.2) converges then by (2.3)  $V_i(r) \leq Cr^2$ , which implies the divergence of the integral in (2.2). However, for the aforementioned reason, we state (b) independently of (d).

In fact, in the subcritical case we have

$$V_i(r) = o(r^2) \quad \text{as } r \rightarrow \infty, \quad (2.6)$$

as it follows from (2.2) and (2.3). Moreover, substituting (2.6) to the left hand side of (2.3), we obtain that, in the subcritical case,

$$V_i(r) = o\left(\frac{r^2}{\log r}\right) \quad \text{as } r \rightarrow \infty. \quad (2.7)$$

### 2.3 On-diagonal estimates

Denote by  $d(x, y)$  the geodesic distance between points  $x, y \in M$  and by  $V(x, r)$  the Riemannian volume of the geodesic ball on  $M$  of radius  $r$  centered at  $x \in M$ . Fix a reference point  $o \in K$  and set  $V(r) = V(o, r)$ . Set also

$$V_{\max}(r) = \max_{1 \leq i \leq k} V_i(r).$$

It is easy to see that, for all  $r > 0$ ,

$$V(r) \approx V_1(r) + V_2(r) + \cdots + V_k(r) \approx V_{\max}(r).$$

The first main result of this paper is as follows.

**Theorem 2.1** *Let  $M = M_1 \# \cdots \# M_k$  be a connected sum of non-compact complete manifolds  $M_1, \dots, M_k$ . Assume that each  $M_i$  is parabolic and satisfies (LY) and (RCA). We also assume that each  $M_i$  is either critical or subcritical. Then we have*

$$p(t, o, o) \approx \frac{1}{V_{\max}(\sqrt{t})} \approx \frac{1}{V(\sqrt{t})}, \quad (2.8)$$

for all  $t > 0$ .

Let us mention for comparison the following result of [15]: if all manifolds  $M_i$  are non-parabolic and satisfy (LY) and (RCA), then the heat kernel on  $M = M_1 \# \cdots \# M_k$  satisfies

$$p(t, o, o) \approx \frac{1}{V_{\min}(\sqrt{t})}, \quad (2.9)$$

where

$$V_{\min}(r) := \min_{1 \leq i \leq k} V_i(r).$$

The proof of the upper bound in (2.9), that is, of the inequality

$$p(t, o, o) \leq \frac{C}{V_{\min}(\sqrt{t})}, \quad (2.10)$$

goes as follows. By [8, Prop. 5.2], the upper bound in (LY) on  $M_i$  is equivalent to a certain Faber-Krahn type inequality on  $M_i$ . Using a technique for merging of such inequalities,

developed in [16, Thm. 3.5], one obtains a similar Faber-Krahn inequality on  $M$ , which then implies the heat kernel upper bound (2.10) by [8, Thm. 5.2] (see [16, Thm. 4.5] and [15, Cor. 4.7] for the details). The reason for appearing of  $V_{\min}$  in (2.10) is that the Faber-Krahn inequality on  $M$  cannot be stronger than that of each end  $M_i$  and, hence, is determined by the end with the smallest function  $V_i(r)$ .

The proof of the lower bound in (2.9), that is, of the inequality

$$p(t, o, o) \geq \frac{c}{V_{\min}(\sqrt{t})} \quad (2.11)$$

uses the comparison

$$p(t, x, y) \geq p_{E_i}(t, x, y)$$

on each end  $E_i$ , where  $p_{E_i}(t, x, y)$  is the Dirichlet heat kernel on  $E_i$  vanishing on  $\partial E_i$ . By [12, Thm 3.1], non-parabolicity of  $M_i$  and (LY) imply that, away from  $\partial E_i$ ,

$$p_{E_i}(t, x, y) \geq cp_i(Ct, x, y). \quad (2.12)$$

It follows that, for any  $i = 1, \dots, k$ ,

$$p(t, o, o) \geq \frac{c}{V_i(\sqrt{t})},$$

which is equivalent to (2.11).

In the present setting, when all the manifolds  $M_i$  are parabolic, both arguments described above work but give non-optimal results. For example, one obtains as above the upper bound (2.10), which in general is weaker than the upper in (2.8). As far as the lower bound is concerned, the estimate (2.12) fails in the parabolic case and has to be replaced by a weaker one (cf. [12, Thm 4.9]), which does not yield an optimal lower bound for  $p(t, o, o)$ . This explains why we have to develop entirely new method for obtaining optimal bounds for  $p(t, o, o)$  in the case when all manifolds  $M_i$  are parabolic. The most significant part of the estimate (2.8) is the upper bound

$$p(t, o, o) \leq \frac{C}{V_{\max}(\sqrt{t})}. \quad (2.13)$$

The proof of (2.13) is the main achievement of the present paper. We use for that a new method involving the *integrated resolvent*

$$\gamma_\lambda(x) = \int_K \int_0^\infty e^{-t\lambda} p(t, x, y) dt d\mu(y)$$

defined for  $\lambda > 0$ . The parabolicity of  $M$  implies that  $\gamma_\lambda(x) \rightarrow \infty$  as  $\lambda \rightarrow 0$ , and the rate of increase of  $\gamma_\lambda(x)$  as  $\lambda \rightarrow 0$  is related to the rate of decay of  $p(t, o, o)$  as  $t \rightarrow \infty$ . In fact, the integrated resolvent  $\gamma_\lambda$  on the connected sum  $M$  satisfies a certain integral equation involving as coefficients the Laplace transforms of the exit probabilities at each end. This allows to estimate the rate of growth of  $\gamma_\lambda$  as  $\lambda \rightarrow 0$  and then to recover the upper bound (2.13) in the subcritical case. In the critical case one has to use instead  $\partial_\lambda \gamma_\lambda$ .

Since  $V_{\max}(r) \approx V(o, r)$  and  $V(o, r)$  satisfies the volume doubling property, the upper bound (2.13) implies automatically a matching lower bound of  $p(t, o, o)$  by [3, Thm. 7.2] (see Section 4.3 for the details).

**Remark 2.2** Kasahara and Kotani recently obtained in [17, Example 6.1] the same on-diagonal heat kernel estimates for a connected sum of two Bessel processes on the half line  $[0, \infty)$  by using the Stieltjes transforms.

## 2.4 Off-diagonal estimates

In order to state the estimates for  $p(t, x, y)$  for arbitrary  $x, y \in M$ , we need some notation. For any  $x \in M$  set

$$|x| := d(x, K) + e.$$

For all  $x \in M$  and for all  $t > 2$ , define the following functions:

$$D(x, t) := \begin{cases} 1, & \text{if } |x| > \sqrt{t} \text{ and } x \in E_i, \\ \frac{|x|^2 V_i(\sqrt{t})}{t V_i(|x|)}, & \text{if } |x| \leq \sqrt{t} \text{ and } x \in E_i, \\ 0, & \text{if } x \in K, \end{cases} \quad (2.14)$$

$$U(x, t) := \begin{cases} \frac{1}{\log|x|}, & \text{if } |x| > \sqrt{t} \\ \frac{1}{\log \sqrt{t}} \log \frac{e\sqrt{t}}{|x|}, & \text{if } |x| \leq \sqrt{t}, \end{cases} \quad (2.15)$$

$$W(x, t) := \begin{cases} 1, & \text{if } |x| > \sqrt{t} \\ \frac{\log|x|}{\log \sqrt{t}}, & \text{if } |x| \leq \sqrt{t}. \end{cases} \quad (2.16)$$

It is clear that  $U(x, t) \leq 1$ ,  $U(x, t) \nearrow 1$  as  $t \rightarrow \infty$ , and  $W(x, t) \leq 1$  and  $W(x, t) \searrow 0$  as  $t \rightarrow \infty$ . It is also useful to observe that

$$1 \leq U(x, t) + W(x, t) \leq 2. \quad (2.17)$$

If  $V_i(r)$  is either critical or subcritical, then it is possible to show that  $D(x, t)$  is bounded.

The next three theorems constitute our second main result. It is obtained by combining Theorem 2.1 with several results from [12], [13] and [15].

In the first theorem we consider the case when  $x$  and  $y$  lie at different ends.

**Theorem 2.3** *In the setting of Theorem 2.1, the following estimates are true for all  $x \in E_i$ ,  $y \in E_j$  with  $i \neq j$  and  $t > t_0$ , where  $t_0$  is large enough.*

(i) *If all the manifolds  $M_l$ ,  $l = 1, \dots, k$ , are subcritical then*

$$p(t, x, y) \asymp \frac{C}{V_{\max}(\sqrt{t})} e^{-b \frac{d^2(x, y)}{t}}. \quad (2.18)$$

(ii) *Suppose that at least one of the manifolds  $M_l$ ,  $l = 1, \dots, k$ , is critical.*

(ii)<sub>1</sub> *If both of  $M_i$  and  $M_j$  are subcritical, then*

$$p(t, x, y) \asymp \frac{C}{t} (1 + (D(x, t) + D(y, t)) \log t) e^{-b \frac{d^2(x, y)}{t}}. \quad (2.19)$$

(ii)<sub>2</sub> *If both of  $M_i$  and  $M_j$  are critical, then*

$$p(t, x, y) \asymp \frac{C}{t} (U(x, t)U(y, t) + W(x, t)U(y, t) + U(x, t)W(y, t)) e^{-b \frac{d^2(x, y)}{t}}. \quad (2.20)$$

(ii)<sub>3</sub> *If  $M_i$  is subcritical and  $M_j$  is critical, then*

$$p(t, x, y) \asymp \frac{C}{t} (1 + D(x, t)U(y, t) \log t) e^{-b \frac{d^2(x, y)}{t}}. \quad (2.21)$$

The next two theorems cover the case when  $x, y$  lie at the same end.

**Theorem 2.4** *In the setting of Theorem 2.1, assume that  $x, y \in E_i$  and  $t > t_0$ .*

(a) *If  $\sqrt{t} \leq \min(|x|, |y|)$  then*

$$p(t, x, y) \asymp \frac{C}{V_i(x, \sqrt{t})} e^{-b \frac{d^2(x, y)}{t}}. \quad (2.22)$$

(b) *Moreover, if  $V_i(r) \approx V_{\max}(r)$  for all large  $r$ , then (2.22) holds for all  $t > t_0$ . In particular, this is the case when  $M_i$  is critical.*

Estimate (2.22) means that, for a restricted time, Brownian motion on each end does not see the other ends, which is natural to expect. Note that the same phenomenon holds also in the case when all  $M_i$  are non-parabolic.

The second claim of Theorem 2.4 means that, on the maximal end, Brownian motion does not see the other ends for all times. It is interesting to observe that in the case when all  $M_i$  are non-parabolic, a similar statement holds for the minimal end.

**Theorem 2.5** *In the setting of Theorem 2.1, assume that  $M_i$  is subcritical,  $x, y \in E_i$  and  $t > t_0$ . If  $\sqrt{t} \geq \min(|x|, |y|)$  then the following is true.*

(i) *If all the manifolds  $M_l$ ,  $l = 1, \dots, k$ , are subcritical, then*

$$p(t, x, y) \asymp C \left( \frac{D(x, t)D(y, t)}{V_i(\sqrt{t})} + \frac{1}{V_{\max}(\sqrt{t})} \right) e^{-b \frac{d^2(x, y)}{t}}. \quad (2.23)$$

(ii) *If at least one of the manifolds  $M_l$ ,  $l = 1, \dots, k$ , is critical then*

$$p(t, x, y) \asymp C \left( \frac{D(x, t)D(y, t)}{V_i(\sqrt{t})} + \frac{1}{t} (1 + (D(x, t) + D(y, t)) \log t) \right) e^{-b \frac{d^2(x, y)}{t}}. \quad (2.24)$$

**Remark 2.6** All the estimates of Theorems 2.3-2.5 can be extended to all  $x, y \in M$  including also a possibility  $x \in K$  or  $y \in K$ . This follows from the local Harnack inequality for the heat kernel  $p(t, x, y)$  and from a careful analysis of the estimates. The latter shows that in all cases when  $|x|$  (or  $|y|$ ) remains bounded, the terms containing  $D(x, t)$  are dominated by others and, hence, can be eliminated, which is equivalent to setting  $D(x, t) = 0$  as in (2.14). A graphical summary of the estimates of Theorems 2.3-2.5 can be found at the following location:

<https://www.math.uni-bielefeld.de/~grigor/tables.pdf>

**Remark 2.7** By [15, Lemma 5.9], for all  $x, y \in M$  and  $0 < t \leq t_0$ , the heat kernel on  $M$  satisfies the Li-Yau estimate (LY) with constants depending on  $t_0$ . For this result it suffices to assume that each end  $M_i$  satisfies the Li-Yau estimate. Hence, in Theorems 2.3-2.5 we do not worry about the estimates for  $t \leq t_0$ .

If  $V_i(r)$  is a power function for each  $i = 1, \dots, k$ , then we can simplify the heat kernel estimates of Theorems 2.3-2.5 as follows. In the next statement  $x, y$  lie at different ends.

**Corollary 2.8** *Suppose that  $V_i(r) \approx r^{\alpha_i}$  for all  $i = 1, \dots, k$  and  $r \geq 1$ , where  $0 < \alpha_i \leq 2$ .*

(i) *Assume that  $0 < \alpha_i < 2$  for all  $i = 1, \dots, k$  and set*

$$\alpha = \max_{1 \leq i \leq k} \alpha_i.$$

*Then, for all  $x, y$  lying at different ends and for all  $t > 2$ , we have*

$$p(t, x, y) \asymp \frac{C}{t^{\alpha/2}} e^{-b \frac{d^2(x, y)}{t}}.$$

(ii) Assume that  $\alpha_l = 2$  for some  $1 \leq l \leq k$ . Then the following estimates hold for  $i \neq j$ ,  $x \in E_i$ ,  $y \in E_j$ ,  $t > 2$ .

(ii)<sub>1</sub> Let  $\alpha_i < 2$  and  $\alpha_j < 2$ . If  $\min(|x|, |y|) \geq \sqrt{t}$  then

$$p(t, x, y) \asymp \frac{C \log t}{t} e^{-b \frac{d^2(x, y)}{t}},$$

and if  $\min(|x|, |y|) \leq \sqrt{t}$  then

$$p(t, x, y) \asymp \frac{C}{t} \left( 1 + \log t \left[ \left( \frac{|x|}{\sqrt{t}} \right)^{2-\alpha_i} + \left( \frac{|y|}{\sqrt{t}} \right)^{2-\alpha_j} \right] \right).$$

(ii)<sub>2</sub> If  $\alpha_i = \alpha_j = 2$  then

$$p(t, x, y) \asymp \frac{C}{t} \left( U(x, t) U(y, t) + U(x, t) \frac{\log |y|}{\log |y| + \log t} + U(y, t) \frac{\log |x|}{\log |x| + \log t} \right) e^{-b \frac{d^2(x, y)}{t}}.$$

Consequently, if  $|x|, |y| \geq \sqrt{t}$  then

$$p(t, x, y) \asymp \frac{C}{t} \left( \frac{1}{\log |x|} + \frac{1}{\log |y|} \right) e^{-b \frac{d^2(x, y)}{t}}, \quad (2.25)$$

if  $|x|, |y| \leq \sqrt{t}$  then

$$p(t, x, y) \asymp \frac{C}{t \log^2 t} \left( \log t + \log^2 \sqrt{t} - \log |x| \log |y| \right), \quad (2.26)$$

and if  $|x| \geq \sqrt{t} \geq |y|$  then

$$p(t, x, y) \asymp \frac{C}{t \log t} \log \frac{e\sqrt{t}}{|y|} e^{-b \frac{d^2(x, y)}{t}}. \quad (2.27)$$

Similarly, if  $|y| \geq \sqrt{t} \geq |x|$  then

$$p(t, x, y) \asymp \frac{C}{t \log t} \log \frac{e\sqrt{t}}{|x|} e^{-b \frac{d^2(x, y)}{t}}. \quad (2.28)$$

(ii)<sub>3</sub> If  $\alpha_i < 2$  and  $\alpha_j = 2$  then

$$p(t, x, y) \asymp \frac{C}{t} \left( 1 + \left( \frac{|x|}{|x| + \sqrt{t}} \right)^{2-\alpha_i} U(y, t) \log t \right) e^{-b \frac{d^2(x, y)}{t}}. \quad (2.29)$$

Consequently, if  $|y| \geq \sqrt{t}$  then

$$p(t, x, y) \asymp \frac{C}{t} e^{-b \frac{d^2(x, y)}{t}},$$

if  $|x|, |y| \leq \sqrt{t}$  then

$$p(t, x, y) \asymp \frac{C}{t} \left( 1 + \left( \frac{|x|}{\sqrt{t}} \right)^{2-\alpha_i} \log \frac{e\sqrt{t}}{|y|} \right),$$

and if  $|x| \geq \sqrt{t} \geq |y|$  then

$$p(t, x, y) \asymp \frac{C}{t} \log \frac{e\sqrt{t}}{|y|} e^{-b \frac{d^2(x, y)}{t}}.$$

**Proof.** All the estimates of Corollary 2.8 follow immediately from those of Theorem 2.3 and the definitions of functions  $D$  and  $W$ . In the case  $(ii)_2$ , in the range  $|x|, |y| \leq \sqrt{t}$ , Theorem 2.3 gives the estimate

$$p(t, x, y) \asymp \frac{C}{t \log^2 \sqrt{t}} \left( \log \frac{e\sqrt{t}}{|x|} \log \frac{e\sqrt{t}}{|y|} + \log |y| \log \frac{e\sqrt{t}}{|x|} + \log |x| \log \frac{e\sqrt{t}}{|y|} \right).$$

Since the sum in the brackets is equal to

$$\left( \log |x| + \log \frac{e\sqrt{t}}{|x|} \right) \left( \log |y| + \log \frac{e\sqrt{t}}{|y|} \right) - \log |x| \log |y| = \left( 1 + \log \sqrt{t} \right)^2 - \log |x| \log |y|,$$

we obtain (2.26). ■

Let us state some consequences of Theorems 2.3-2.5 in the general setting, but under some specific restrictions of the variables  $x, y, t$ .

**Corollary 2.9** *Under the hypotheses of Theorems 2.3-2.5, we have the following estimates.*

(a) (Long time regime) For fixed  $x, y \in M$  and  $t \rightarrow \infty$ ,

$$p(t, x, y) \approx \frac{1}{V_{\max}(\sqrt{t})}. \quad (2.30)$$

(b) (Medium time regime) Let  $x \in E_i$  and  $y \in E_j$  with  $i \neq j$ . If  $|x| \approx |y| \approx \sqrt{t}$  then in the cases (i) and  $(ii)_3$  we have (2.30), in the case  $(ii)_1$  we have

$$p(t, x, y) \approx \frac{\log t}{t}, \quad (2.31)$$

and in the case  $(ii)_2$

$$p(t, x, y) \approx \frac{1}{t \log t}. \quad (2.32)$$

**Proof.** (a) The estimate (2.30) follows easily from Theorem 2.1 by using a local Harnack inequality. However, we show here how it follows from Theorems 2.3, 2.5. Observe that, for a fixed  $x \in E_i$  and large  $t$  we have

$$D(x, t) \approx \frac{V_i(\sqrt{t})}{t}, \quad U(x, t) \approx 1, \quad W(x, t) \approx \frac{1}{\log t}. \quad (2.33)$$

Assume that  $x \in E_i, y \in E_j$  and consider the cases (i),  $(ii)_1, (ii)_2$  and  $(ii)_3$  as in Theorem 2.3.

Case (i). Using (2.18), (2.23), (2.33) and  $V_i(x, \sqrt{t}) \approx V_i(\sqrt{t})$  as  $t \rightarrow \infty$  we obtain

$$p(t, x, y) \approx \frac{V_i(\sqrt{t})}{t^2} \delta_{ij} + \frac{1}{V_{\max}(\sqrt{t})} \approx \frac{1}{V_{\max}(\sqrt{t})},$$

where we have also used that  $V_j(r) V_{\max}(r) = o(r^4)$ .

Case  $(ii)_1$ . By (2.19), (2.24) and (2.33) we have

$$\begin{aligned} p(t, x, y) &\approx \frac{V_i(\sqrt{t})}{t^2} \delta_{ij} + \frac{1}{t} \left\{ 1 + \left( \frac{V_i(\sqrt{t})}{t} + \frac{V_j(\sqrt{t})}{t} \right) \log t \right\} \\ &\approx \frac{1}{t} \approx \frac{1}{V_{\max}(\sqrt{t})}, \end{aligned}$$

because of  $V_{\max}(r) \approx r^2$  and (2.7).

Case  $(ii)_2$ . If  $i \neq j$  then by (2.20) and (2.33)

$$p(t, x, y) \approx \frac{1}{t} \left( 1 + \frac{1}{\log t} \right) \approx \frac{1}{t} \approx \frac{1}{V_{\max}(\sqrt{t})}.$$

If  $i = j$  then (2.30) follows trivially from (2.22).

Case  $(ii)_3$ . In this case necessarily  $i \neq j$ , and we obtain by (2.21)

$$p(t, x, y) \approx \frac{1}{t} \left\{ 1 + \frac{V_i(\sqrt{t})}{t} \log t \right\} \approx \frac{1}{t} \approx \frac{1}{V_{\max}(\sqrt{t})}.$$

(b) In the case  $|x| \approx |y| \approx \sqrt{t}$  we have  $d^2(x, y) \approx t$  and

$$D(x, t) \approx 1, \quad U(x, t) \approx \frac{1}{\log t}, \quad W(x, t) \approx 1.$$

Then the required estimates follow directly from those stated in Theorem 2.3. ■

Let us observe the following. In the medium time regime, that is, when  $x$  and  $y$  lie at different ends and  $|x| \approx |y| \approx \sqrt{t}$ , we have by (b): in the cases (i) and  $(ii)_3$

$$p(t, x, y) \approx \frac{1}{V_{\max}(\sqrt{t})},$$

that is,  $p(t, x, y)$  behaves itself as in the long time regime, whereas in the case  $(ii)_1$

$$p(t, x, y) \approx \frac{\log t}{t} \gg \frac{1}{V_{\max}(\sqrt{t})},$$

and in the case  $(ii)_2$

$$p(t, x, y) \approx \frac{1}{t \log t} \ll \frac{1}{V_{\max}(\sqrt{t})}.$$

Hence, we observe in the case  $(ii)_2$  the *bottleneck effect*: the heat kernel value  $\frac{1}{t \log t}$  in the medium time regime is significantly *smaller* than that of long time regime  $\frac{1}{t}$ . For example, this case happens for  $M = \mathbb{R}^2 \# \mathbb{R}^2$  (see Fig. 1). A similar bottleneck effect was observed in [15] for  $M = \mathbb{R}^n \# \mathbb{R}^n$  with  $n \geq 3$ : the heat kernel of  $M$  in the long time regime is comparable to  $\frac{1}{t^{n/2}}$  whereas in the medium time regime – to  $\frac{1}{t^{n-1}}$ . In the case  $n = 2$  the bottleneck effect is quantitatively weaker as the distinction between the two regimes is determined by  $\log t$  in contrast to the power of  $t$  in the case  $n \geq 3$ .

On the contrary, in the case  $(ii)_1$  we observe an interesting *anti-bottleneck effect*: the heat kernel value  $\frac{\log t}{t}$  in the medium time regime is significantly *larger* than that of the long time regime  $\frac{1}{t}$ . This effect occurs only when there are at least three ends, one of them being critical and two – subcritical. For example, this is the case for  $M = \mathcal{R}^1 \# \mathcal{R}^1 \# \mathcal{R}^2$  (see Fig. 6).

## 2.5 Examples

We present here heat kernel bounds on some specific examples using Theorems 2.3-2.5 and Corollary 2.8.

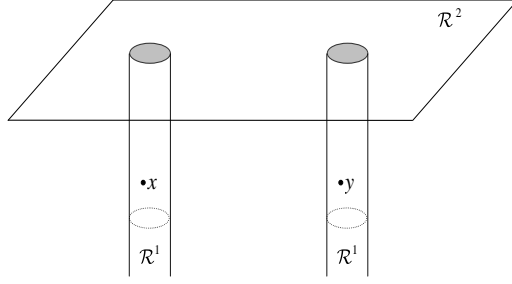


Figure 6: Connected sum  $\mathcal{R}^1 \# \mathcal{R}^1 \# \mathcal{R}^2$

**Example 2.10 (Heat kernel on  $\mathcal{R}^{\alpha_1} \# \mathcal{R}^{\alpha_2}$ )** Let us write down the heat kernel bounds on the connected sum

$$M = M_1 \# M_2 = \mathcal{R}^{\alpha_1} \# \mathcal{R}^{\alpha_2},$$

where  $1 \leq \alpha_1 \leq \alpha_2 < 2$ . In this case both  $M_1$  and  $M_2$  are subcritical so that Theorem 2.3(i), Theorem 2.4 and Theorem 2.5(i) apply. Observe that

$$D(x, t) = \begin{cases} 1, & \text{if } |x| > \sqrt{t}, \\ \left(\frac{|x|}{\sqrt{t}}\right)^{2-\alpha_i}, & \text{if } |x| \leq \sqrt{t}, \end{cases} \quad (2.34)$$

and

$$V_{\max}(r) \approx r^{\alpha_2}, \quad r > 1.$$

In the case  $x \in E_1$  and  $y \in E_2$ , we obtain by (2.18) or by Corollary 2.8(i),

$$p(t, x, y) \asymp \frac{C}{t^{\alpha_2/2}} e^{-b \frac{d^2(x, y)}{t}}.$$

Assume now that  $x, y \in E_1$ . If  $|x|, |y| > \sqrt{t}$ , then by (2.22) we have

$$p(t, x, y) \asymp \frac{C}{V_1(x, \sqrt{t})} e^{-b \frac{d^2(x, y)}{t}}.$$

If  $|x|, |y| \leq \sqrt{t}$  then by (2.23) and (2.34) we obtain

$$p(t, x, y) \approx \frac{1}{t^{\alpha_1/2}} \left(\frac{|x||y|}{t}\right)^{2-\alpha_1} + \frac{1}{t^{\alpha_2/2}}. \quad (2.35)$$

In particular, in the long time regime  $t \rightarrow \infty$  we obtain

$$p(t, x, y) \approx \frac{1}{t^{\alpha_2/2}},$$

which, of course, matches (2.30). Assume now that  $|x| > \sqrt{t} \geq |y|$ . Substituting (2.34) into (2.23), we obtain

$$p(t, x, y) \asymp C \left( \frac{1}{t^{\alpha_1/2}} \left(\frac{|y|}{\sqrt{t}}\right)^{2-\alpha_1} + \frac{1}{t^{\alpha_2/2}} \right) e^{-b \frac{d^2(x, y)}{t}}.$$

A similar estimate holds in the case  $|y| > \sqrt{t} \geq |x|$ .

Finally, if  $x, y \in E_2$  then we have by Theorem 2.4 that for all  $t > 1$

$$p(t, x, y) \asymp \frac{C}{V_2(x, \sqrt{t})} e^{-b \frac{d^2(x, y)}{t}}.$$

**Example 2.11 (Heat kernel on  $\mathcal{R}^1 \# \mathcal{R}^2$ )** Consider  $M = M_1 \# M_2 = \mathcal{R}^1 \# \mathcal{R}^2$  (see Fig. 2). Suppose that  $x \in E_1, y \in E_2$ . Then by Theorem 2.3(ii)<sub>3</sub> or by the estimate (2.29) of Corollary 2.8

$$p(t, x, y) \asymp \frac{C}{t} \left( 1 + \frac{|x|}{|x| + \sqrt{t}} U(y, t) \log t \right) e^{-b \frac{d^2(x, y)}{t}}.$$

Using (2.15) we obtain: if  $|y| > \sqrt{t}$ , then

$$p(t, x, y) \asymp \frac{C}{t} e^{-b \frac{d^2(x, y)}{t}};$$

if  $|x|, |y| \leq \sqrt{t}$ , then

$$p(t, x, y) \asymp \frac{C}{t} \left( 1 + \frac{|x|}{\sqrt{t}} \log \frac{e\sqrt{t}}{|y|} \right),$$

and if  $|x| > \sqrt{t} \geq |y|$ , then

$$p(t, x, y) \asymp \frac{C}{t} \log \frac{e\sqrt{t}}{|y|} e^{-b \frac{d^2(x, y)}{t}}.$$

Assume that  $x, y \in E_1$ . If  $\min(|x|, |y|) \leq \sqrt{t}$ , then we obtain by (2.24) and (2.34)

$$p(t, x, y) \approx \frac{1}{t} \left( 1 + \frac{|x||y|}{\sqrt{t}} + \frac{|x| + |y|}{\sqrt{t}} \log t \right) e^{-b \frac{d^2(x, y)}{t}}.$$

In particular, if  $|x| > \sqrt{t} \geq |y|$ , we obtain

$$p(t, x, y) \asymp \frac{C}{t} (|y| + \log t) e^{-b \frac{d^2(x, y)}{t}}.$$

Similar estimate follows when  $|y| > \sqrt{t} \geq |x|$ . If  $\min(|x|, |y|) > \sqrt{t}$ , we obtain by Theorem 2.4

$$p(t, x, y) \asymp \frac{C}{\sqrt{t}} e^{-b \frac{d^2(x, y)}{t}}.$$

In the case  $x, y \in E_2$ , we obtain by Theorem 2.4

$$p(t, x, y) \asymp \frac{C}{t} e^{-b \frac{d^2(x, y)}{t}}. \tag{2.36}$$

**Example 2.12 (Heat kernel on  $\mathbb{R}^2 \# \mathbb{R}^2$ )** Suppose that  $x \in E_1$  and  $y \in E_2$ . If  $|x|, |y| \leq \sqrt{t}$ , then by (2.20), or by (2.26)

$$p(t, x, y) \approx \frac{1}{t \log^2 t} \left( \log t + \log^2 \sqrt{t} - \log |x| \log |y| \right).$$

In particular, in the long time regime  $|x| \approx |y| \approx 1$  we obtain

$$p(t, x, y) \approx \frac{1}{t},$$

and in the medium time regime  $|x| \approx |y| \approx \sqrt{t}$  we have

$$p(t, x, y) \approx \frac{1}{t \log t},$$

which means a mild bottleneck-effect on  $\mathbb{R}^2 \# \mathbb{R}^2$ .

If  $|x|, |y| \geq \sqrt{t}$  then the heat kernel on  $\mathbb{R}^2 \# \mathbb{R}^2$  satisfies (2.25), that is,

$$p(t, x, y) \asymp \frac{C}{t} \left( \frac{1}{\log |x|} + \frac{1}{\log |y|} \right) e^{-b \frac{d^2(x, y)}{t}}.$$

The cases  $|x| > \sqrt{t} \geq |y|$  and  $|y| > \sqrt{t} \geq |x|$  are covered by (2.27) and (2.28), respectively.

If  $x, y \in E_1$  or  $x, y \in E_2$  then  $p(t, x, y)$  satisfies (2.36) by Theorem 2.4.

**Example 2.13 (Heat kernel on  $\mathcal{R}^1 \# \mathcal{R}^1 \# \mathcal{R}^2$ )** Let  $M = M_1 \# M_2 \# M_3 = \mathcal{R}^1 \# \mathcal{R}^1 \# \mathcal{R}^2$  (see Fig. 6). If  $x$  and  $y$  are at the same end, or  $x \in \mathcal{R}^1$  and  $y \in \mathcal{R}^2$ , then the heat kernel  $p(t, x, y)$  satisfies same estimates as in the above case  $\mathcal{R}^1 \# \mathcal{R}^2$ .

Assume now that  $x \in E_1$  and  $y \in E_2$ . Then by Corollary 2.8(ii)<sub>1</sub> we obtain the following estimates: if  $\min(|x|, |y|) \leq \sqrt{t}$  then

$$p(t, x, y) \approx \frac{1}{t} \left( 1 + \frac{\log t}{\sqrt{t}} (|x| + |y|) \right),$$

and if  $\min(|x|, |y|) > \sqrt{t}$ , then

$$p(t, x, y) \asymp \frac{\log t}{t} e^{-b \frac{d^2(x, y)}{t}}.$$

In particular, if  $|x| \approx |y| \approx \sqrt{t}$ , then

$$p(t, x, y) \approx \frac{\log t}{t}.$$

### 3 Some auxiliary estimates

In this section we prove some auxiliary results to be used in the proof of Theorem 2.1.

Let  $(M, \mu)$  be a geodesically complete non-compact weighted manifold (we do not even assume parabolicity of  $M$  unless it is clearly stated). For any open set  $\Omega \subset M$ , denote by  $p_\Omega(t, x, y)$  the Dirichlet heat kernel in  $\Omega$ . Assume from now on that  $\Omega$  has smooth boundary. Then  $p_\Omega(t, x, y) = 0$  whenever  $x$  or  $y$  belongs to  $\partial\Omega$ . Denote also by  $P_t^\Omega$  the associated heat semigroup. Denote as before by  $(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M})$  Brownian motion on  $M$ . Let  $\tau_\Omega$  be the first exit time of  $X_t$  from  $\Omega$ , that is,

$$\tau_\Omega = \inf \{t > 0 : X_t \notin \Omega\}.$$

Then, for any bounded continuous function  $f$  on  $M$ ,

$$P_t^\Omega f(x) = \mathbb{E}_x (f(X_t) 1_{\{\tau_\Omega > t\}}). \quad (3.1)$$

### 3.1 Integrated resolvent

The resolvent operator  $G_\lambda^\Omega$  is defined for any  $\lambda > 0$  as an operator on non-negative measurable functions  $f$  on  $\Omega$  by

$$G_\lambda^\Omega f(x) = \int_0^\infty e^{-\lambda t} P_t^\Omega f dt.$$

Clearly,  $G_\lambda^\Omega$  is a linear operator that preserves non-negativity. Note that by definition  $G_\lambda^\Omega f$  vanishes in  $\Omega^c$ . If  $\Omega = M$  then we write  $G_\lambda \equiv G_\lambda^M$ . Clearly,  $G_\lambda^\Omega$  is an integral operator whose kernel

$$g_\lambda^\Omega(x, y) = \int_0^\infty e^{-\lambda t} p_\Omega(t, x, y) dt$$

is called the *resolvent kernel*. In general,  $G_\lambda^\Omega f$  may take value  $+\infty$ . However, if  $f$  is bounded and continuous then the function  $u = G_\lambda^\Omega f$  is finite and, moreover, is the minimal non-negative solution of the equation  $\Delta u - \lambda u = -f$  (see [10]). It follows from (3.1) that

$$G_\lambda^\Omega f(x) = \mathbb{E}_x \left( \int_0^{\tau_\Omega} f(X_t) e^{-\lambda t} dt \right). \quad (3.2)$$

If in addition  $\Omega$  is precompact then the function  $u = G_\lambda^\Omega f$  solves the Dirichlet problem

$$\begin{cases} \Delta u - \lambda u = -f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For the proof of Theorem 2.1 we need the notion of *integrated resolvent*. Fix a compact set  $K \subset M$  with non-empty interior  $\overset{\circ}{K}$  such that  $K$  is the closure of  $\overset{\circ}{K}$  and the boundary  $\partial K$  is smooth. Fix also once and for all a reference point  $o \in K$ .

For any  $\lambda > 0$ , define the function  $\gamma_\lambda$  on  $M$  by

$$\gamma_\lambda(x) := G_\lambda 1_K(x) = \int_K g_\lambda(x, z) d\mu(z) = \int_K \int_0^\infty e^{-\lambda t} p(t, x, z) dz dt. \quad (3.3)$$

The function  $\gamma_\lambda$  is called the integrated resolvent. Set also

$$\dot{\gamma}_\lambda = G_\lambda \gamma_\lambda. \quad (3.4)$$

It follows from the resolvent equation  $G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta$  that

$$\dot{\gamma}_\lambda = -\frac{\partial}{\partial \lambda} \gamma_\lambda = \int_K \int_0^\infty t e^{-\lambda t} p(t, x, z) dz dt. \quad (3.5)$$

**Lemma 3.1** (i) *If there exist positive constants  $C, \lambda_0$  and a function  $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that, for some  $x \in K$ ,*

$$\gamma_\lambda(x) \leq \frac{C}{\lambda F(\frac{1}{\sqrt{\lambda}})} \quad \text{for all } \lambda \in (0, \lambda_0], \quad (3.6)$$

*then there exist positive constants  $C', t_0$  such that*

$$p(t, o, o) \leq \frac{C'}{F(\sqrt{t})} \quad \text{for all } t \geq t_0. \quad (3.7)$$

(ii) If there exist positive constants  $C, \lambda_0$  such that, for some  $x \in K$ ,

$$\dot{\gamma}_\lambda(x) \leq \frac{C}{\lambda} \quad \text{for all } \lambda \in (0, \lambda_0], \quad (3.8)$$

then there exist positive constants  $C', t_0$  such that

$$p(t, o, o) \leq \frac{C'}{t} \quad \text{for all } t \geq t_0.$$

**Proof.** (i) Set  $\delta = (\text{diam}K)^2$ . By the local Harnack inequality, there exist positive constants  $c_1, c_2$  such that, for all  $x, z \in K$  and  $s > 2c_2\delta$ ,

$$p(s, x, z) \geq c_1 p(s - c_2\delta, o, o), \quad (3.9)$$

which implies by (3.3), for all  $x \in K$ ,

$$\gamma_\lambda(x) \geq c_1 \text{vol}(K) \int_{2c_2\delta}^{\infty} e^{-\lambda s} p(s - c_2\delta, o, o) ds.$$

Using the monotonicity of  $p(s, o, o)$  with respect to  $s$  (see [10, Exercises 7.22]), we obtain, for  $t \geq 4c_2\delta$ ,

$$\begin{aligned} \gamma_\lambda(x) &\geq c_1 \text{vol}(K) \int_{t/2}^t e^{-\lambda s} p(s - c_2\delta, o, o) ds \\ &\geq c_1 \text{vol}(K) \int_{t/2}^t e^{-\lambda s} p(t, o, o) ds \geq c t e^{-\lambda t} p(t, o, o). \end{aligned} \quad (3.10)$$

Set  $t_0 := \max\{4c_2\delta, \lambda_0^{-1}\}$ . For any  $t \geq t_0$  and using (3.6) and (3.10) with  $\lambda = t^{-1}$ , we obtain

$$\frac{C}{\lambda F(\frac{1}{\sqrt{\lambda}})} \geq c t e^{-1} p(t, o, o),$$

which implies

$$p(t, o, o) \leq \frac{C}{F(\sqrt{t})}.$$

(ii) Arguing as in (i) and using (3.9) and (3.5), we obtain, for  $t \geq 4c_2\delta$  and  $x \in K$ ,

$$\begin{aligned} \dot{\gamma}_\lambda(x) &= \int_0^\infty \int_K s e^{-\lambda s} p(s, x, z) ds d\mu(z) \\ &\geq c_1 \text{vol}(K) \int_{t/2}^t s e^{-\lambda s} p(t, o, o) ds \geq c t^2 e^{-\lambda t} p(t, o, o). \end{aligned} \quad (3.11)$$

Assuming  $t \geq t_0 := \max\{4c_2\delta, \lambda_0^{-1}\}$  and using (3.8) and (3.11) with  $\lambda = t^{-1}$ , we obtain

$$\frac{C}{\lambda} \geq c t^2 e^{-1} p(t, o, o),$$

which implies

$$p(t, o, o) \leq \frac{C}{t}.$$

■

**Remark 3.2** Lemma 3.1 will be used in the proof of Theorem 2.1 in Section 4.2 as follows. In the case when all the ends are subcritical, we will prove the following upper bound for the integrated resolvent:

$$\sup_{\partial K} \gamma_\lambda \leq \frac{C}{\lambda V_{\max}(\frac{1}{\sqrt{\lambda}})}, \quad (3.12)$$

which then implies by Lemma 3.1(i) the desired upper bound

$$p(t, o, o) \leq \frac{C}{V_{\max}(\sqrt{t})}.$$

However, in the case when one of the ends is critical, we obtain instead of (3.12) a weaker inequality

$$\sup_{\partial K} \gamma_\lambda \leq C \log \frac{1}{\lambda}, \quad (3.13)$$

which yields

$$p(t, o, o) \leq C \frac{\log t}{t}$$

instead of the desired estimate

$$p(t, o, o) \leq \frac{C}{t}. \quad (3.14)$$

In order to be able to prove the latter, we will use the second part of Lemma 3.1. Namely, we will prove that in the critical case

$$\sup_{\partial K} \dot{\gamma}_\lambda \leq \frac{C}{\lambda}, \quad (3.15)$$

which then will imply (3.14) by Lemma 3.1(ii).

Note that the estimate (3.13) of  $\gamma_\lambda$  is already optimal as it is matched by the estimate (3.15) of  $\dot{\gamma}_\lambda = -\frac{\partial}{\partial \lambda} \gamma_\lambda$ . However, the function  $\gamma_\lambda$  alone does not allow to recover an optimal estimate of the heat kernel, while its  $\lambda$ -derivative  $\dot{\gamma}_\lambda$  does.

## 3.2 Comparison principles

Fix an open set  $\Omega \subset M$  and  $\lambda > 0$ . We say that a function  $u$  is  $\lambda$ -harmonic in  $\Omega$  if it satisfies in  $\Omega$  the equation  $\Delta u - \lambda u = 0$ . A function  $u$  is called  $\lambda$ -superharmonic if  $\Delta u - \lambda u \leq 0$ . We will frequently use the following minimum principle: if  $\Omega$  is precompact,  $u \in C(\overline{\Omega})$  is  $\lambda$ -superharmonic in  $\Omega$  and  $u \geq 0$  on  $\partial\Omega$  then  $u \geq 0$  in  $\Omega$ . It implies the comparison principle: if  $u, v \in C(\overline{\Omega})$ ,  $u$  is  $\lambda$ -superharmonic in  $\Omega$  and  $v$  is  $\lambda$ -harmonic in  $\Omega$  then

$$u \geq v \text{ on } \partial\Omega \Rightarrow u \geq v \text{ in } \Omega. \quad (3.16)$$

Let now  $\Omega$  be an exterior domain, that is,  $\Omega = F^c$  where  $F$  is a compact subset of  $M$ . Let  $v \in C(\overline{\Omega})$  be non-negative and  $\lambda$ -harmonic in  $\Omega$ . We say that  $v$  is *minimal* in  $\Omega$  if there exists an exhaustion  $\{U_k\}$  of  $M$  by precompact open sets  $U_k \supset F$  and a sequence  $\{v_k\}$  of functions  $v_k \in C(\overline{U_k \setminus F})$  that are non-negative and  $\lambda$ -harmonic in  $U_k \setminus F$  and such that  $v_k|_{\partial U_k} = 0$  and  $v_k \uparrow v$  in  $\overline{\Omega}$ . Then the following modification of the comparison principle holds in  $\Omega$ : if  $u, v \in C(\overline{\Omega})$ ,  $u$  is non-negative  $\lambda$ -superharmonic in  $\Omega$  and  $v$  is non-negative minimal  $\lambda$ -harmonic in  $\Omega$  then (3.16) is satisfied. Indeed, by the comparison principle in  $U_k \setminus F$  we obtain  $u \geq v_k$  whence the claim follows.

We are left to mention that, for any non-negative bounded function  $f$  with compact support, the function  $G_\lambda f$  is non-negative, minimal,  $\lambda$ -harmonic outside  $\text{supp } f$ , since  $G_\lambda^{U_k} f \uparrow G_\lambda f$ .

### 3.3 Functions $\Phi_\lambda^\Omega$ and $\Psi_\lambda^\Omega$

In any open set  $\Omega \subset M$ , consider a function

$$\Phi_\lambda^\Omega := \lambda G_\lambda^\Omega 1 = \int_0^\infty \lambda e^{-\lambda t} P_t^\Omega 1 dt. \quad (3.17)$$

Since  $0 \leq P_t^\Omega 1 \leq 1$ , we see that

$$0 \leq \Phi_\lambda^\Omega \leq 1. \quad (3.18)$$

It follows from (3.1) that

$$\Phi_\lambda^\Omega(x) = \int_0^\infty \lambda e^{-\lambda t} \mathbb{P}_x(\tau_\Omega > t) dt. \quad (3.19)$$

Let  $A$  be a precompact open subset of  $M$  with smooth boundary and let  $K \subset A$ . Set

$$\gamma_\lambda^A(x) := G_\lambda^A 1_K(x) = \int_K g_\lambda^A(x, z) d\mu(z) = \int_K \int_0^\infty e^{-\lambda t} p_A(t, x, z) dt d\mu(z). \quad (3.20)$$

**Lemma 3.3** (a) *The following inequality holds in  $A$ :*

$$\gamma_\lambda - \gamma_\lambda^A \leq (\sup_{\partial A} \gamma_\lambda) (1 - \Phi_\lambda^A). \quad (3.21)$$

(b) *The following inequality holds in  $K^c$ :*

$$\gamma_\lambda \leq (\sup_{\partial K} \gamma_\lambda) (1 - \Phi_\lambda^{K^c}). \quad (3.22)$$

**Proof.** (a) By (3.17), the function  $\Phi_\lambda^A$  satisfies

$$\begin{cases} \Delta \Phi_\lambda^A - \lambda \Phi_\lambda^A = -\lambda & \text{in } A \\ \Phi_\lambda^A = 0 & \text{on } \partial A. \end{cases}$$

It follows that the function  $u := 1 - \Phi_\lambda^A$  solves the boundary value problem

$$\begin{cases} \Delta u - \lambda u = 0 & \text{in } A \\ u = 1 & \text{on } \partial A. \end{cases}$$

Note that  $\gamma_\lambda - \gamma_\lambda^A = G_\lambda 1_K - G_\lambda^A 1_K$  is  $\lambda$ -harmonic in  $A$  and is equal to  $\gamma_\lambda$  on  $\partial A$ , which implies by the comparison principle in  $A$  that

$$\gamma_\lambda - \gamma_\lambda^A \leq (\sup_{\partial A} \gamma_\lambda) u \quad \text{in } A,$$

which proves (3.21).

(b) Set  $\Omega = K^c$ . As in (a), the function  $u := 1 - \Phi_\lambda^\Omega$  solves the following boundary value problem:

$$\begin{cases} \Delta u - \lambda u = 0 & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega \end{cases}$$

The function  $\gamma_\lambda = G_\lambda 1_K$  is non-negative,  $\lambda$ -harmonic, and minimal in  $\Omega$ . On  $\partial\Omega = \partial K$  we have

$$\gamma_\lambda \leq \sup_{\partial K} \gamma_\lambda = (\sup_{\partial K} \gamma_\lambda) u. \quad (3.23)$$

Since  $u$  is non-negative and  $\lambda$ -harmonic in  $\Omega$ , it follows by the comparison principle in  $\Omega$  that (3.23) holds also in  $\Omega$ , which proves (3.22). ■

Set

$$\Psi_\lambda^\Omega := G_\lambda^\Omega (1 - \Phi_\lambda^\Omega) \quad (3.24)$$

and observe that  $\Psi_\lambda^\Omega \geq 0$  by (3.18).

**Lemma 3.4** *Assume that  $M$  is parabolic. Then we have the following identity for all  $x \in \Omega$ :*

$$\Psi_\lambda^\Omega(x) = \int_0^\infty t e^{-\lambda t} \partial_t \mathbb{P}_x(\tau_\Omega \leq t) dt. \quad (3.25)$$

**Proof.** Integrating by parts in (3.19) together with the parabolicity of  $M$ , we obtain

$$\begin{aligned} \Phi_\lambda^\Omega(x) &= - \int_0^\infty \mathbb{P}_x(\tau_\Omega > t) d e^{-\lambda t} = 1 + \int_0^\infty e^{-\lambda t} \partial_t \mathbb{P}_x(\tau_\Omega > t) \\ &= 1 - \int_0^\infty e^{-\lambda t} \partial_t \mathbb{P}_x(\tau_\Omega \leq t). \end{aligned} \quad (3.26)$$

On the other hand, we have

$$\begin{aligned} \Psi_\lambda^\Omega &= G_\lambda^\Omega 1 - G_\lambda^\Omega \Phi_\lambda^\Omega = G_\lambda^\Omega 1 - \lambda G_\lambda^\Omega G_\lambda^\Omega 1 \\ &= G_\lambda^\Omega 1 + \lambda \frac{\partial}{\partial \lambda} G_\lambda^\Omega 1 = \frac{\partial}{\partial \lambda} (\lambda G_\lambda^\Omega 1) = \frac{\partial}{\partial \lambda} \Phi_\lambda^\Omega. \end{aligned}$$

Hence, differentiating (3.26) in  $\lambda$ , we obtain (3.25). ■

### 3.4 Some local estimates

Recall that, for any open set  $A$  containing  $K$ , we have defined

$$\gamma_\lambda^A(x) = G_\lambda^A 1_K(x) = \int_K \int_0^\infty e^{-\lambda t} p_A(t, x, z) dt d\mu(z).$$

Set also

$$\dot{\gamma}_\lambda^A(x) := G_\lambda^A \dot{\gamma}_\lambda^A(x) = - \frac{\partial}{\partial \lambda} \gamma_\lambda^A(x) = \int_K \int_0^\infty t e^{-\lambda t} p_A(t, x, z) dt d\mu(z). \quad (3.27)$$

Note that  $\gamma_\lambda^A$  and  $\dot{\gamma}_\lambda^A$  vanish outside  $A$ . Note also that  $\gamma_\lambda = \gamma_\lambda^M$  and  $\dot{\gamma}_\lambda = \dot{\gamma}_\lambda^M$ .

In what follows we fix a precompact open set  $A \supset K$  with smooth boundary.

**Lemma 3.5** *There exists a positive constant  $C = C(A)$  such that, for all  $\lambda > 0$ ,*

$$\sup_A \gamma_\lambda^A \leq C, \quad (3.28)$$

$$\sup_A \dot{\gamma}_\lambda^A \leq C^2, \quad (3.29)$$

and

$$\sup_A \Psi_\lambda^A \leq C. \quad (3.30)$$

**Proof.** It follows from (3.20) that

$$\gamma_\lambda^A(x) \leq \int_A \int_0^\infty p_A(t, x, z) dt d\mu(z) = \int_A g^A(x, z) d\mu(z),$$

where  $g^A = g_0^A$  is the Green function of  $\Delta$  in  $A$ . The function

$$u(x) = \int_A g^A(x, z) d\mu(z)$$

solves the following boundary value problem

$$\begin{cases} \Delta u = -1 & \text{in } A, \\ u = 0 & \text{on } \partial A, \end{cases}$$

which implies that  $u(x)$  is bounded. Hence, (3.28) holds with  $C = \sup u$ .

By (3.27) we have

$$\hat{\gamma}_\lambda^A(x) = \int_A g_\lambda^A(x, z) \gamma_\lambda^A(z) d\mu(z),$$

which implies by (3.28), for any  $x \in A$ ,

$$\hat{\gamma}_\lambda^A(x) \leq \sup_A \gamma_\lambda^A \int_A g^A(x, z) d\mu(z) \leq C \sup u = C^2,$$

which proves (3.29).

Finally, it follows from (3.24) that

$$\Psi_\lambda^A(x) \leq G_\lambda^A 1(x) = \int_A g^A(x, z) d\mu(z) \leq C,$$

which proves (3.30). ■

### 3.5 Global estimates of $\Phi_\lambda^\Omega$ and $\Psi_\lambda^\Omega$

So far we have used a compact set  $K$  and a precompact open set  $A \supset K$ . We have also assume that  $K$  and  $A$  have smooth boundaries.

In the next Lemma we estimate  $\inf_{\partial A} \Phi_\lambda^{K^c}$  from below using additional geometric assumptions. Denote by  $K_\epsilon$  the  $\epsilon$ -neighborhood of  $K$ . We will assume in addition that  $K_\epsilon \subset A$  for some large enough  $\epsilon$  specified below.

**Lemma 3.6** *Let  $M$  be a geodesically complete, non-compact parabolic manifold satisfying (LY), (RCA). Fix a reference point  $o \in K$  and set  $V(r) = V(o, r)$ . Assume in addition that  $K_\epsilon \subset A$  for sufficiently large  $\epsilon = \epsilon(K) > 0$ . Then there exists a constant  $c > 0$  such that*

$$\inf_{\partial A} \Phi_\lambda^{K^c} \geq c \int_{(\text{diam}A)^2}^{\infty} (1 - e^{-\lambda s}) \frac{1}{V(\sqrt{s})H(\sqrt{s})^2} ds, \quad (3.31)$$

where

$$H(r) := 1 + \left( \int_1^r \frac{s}{V(s)} ds \right)_+. \quad (3.32)$$

In addition, we have:

(i) if  $V(r)$  is subcritical then, for  $0 < \lambda \leq \frac{1}{(\text{diam}A)^2}$ ,

$$\inf_{\partial A} \Phi_\lambda^{K^c} \geq c\lambda V\left(\frac{1}{\sqrt{\lambda}}\right). \quad (3.33)$$

(ii) If  $V(r)$  is critical then, for  $0 < \lambda \leq \frac{1}{(\text{diam}A)^2}$ ,

$$\inf_{\partial A} \Phi_\lambda^{K^c} \geq \frac{c}{\log \frac{1}{\lambda}}. \quad (3.34)$$

**Proof.** Denote  $\Omega = K^c$ . By [12, Theorem 4.9 and (4.23)], if  $\epsilon$  is big enough then, for all  $a, y$  outside  $K_{\epsilon/2}$  and for all  $s > 0$ , the following estimate holds:

$$p_\Omega(s, x, y) \geq C \frac{D(s, x, y)}{V(x, \sqrt{s})} \exp\left(-c \frac{d^2(x, y)}{s}\right), \quad (3.35)$$

where

$$D(s, x, y) = \frac{H(|x|)H(|y|)}{(H(|x|) + H(\sqrt{s}))(H(|y|) + H(\sqrt{s}))}.$$

By [13, (3.29)], we have, for any  $x \notin K_\epsilon$ ,

$$\mathbb{P}_x(\tau_\Omega > t) \geq c \int_t^\infty \inf_{y \in K_\epsilon \setminus K_{\epsilon/2}} p_\Omega(s, x, y) ds,$$

where  $c = c(K, \epsilon) > 0$ , which implies by (3.19)

$$\begin{aligned} \Phi_\lambda^\Omega(x) &\geq c \int_0^\infty \lambda e^{-\lambda t} \left( \int_t^\infty \inf_{y \in K_\epsilon \setminus K_{\epsilon/2}} p_\Omega(s, x, y) ds \right) dt \\ &= c \int_0^\infty \left( \int_0^s \lambda e^{-\lambda t} \inf_{y \in K_\epsilon \setminus K_{\epsilon/2}} p_\Omega(s, x, y) dt \right) ds \\ &= c \int_0^\infty (1 - e^{-\lambda s}) \inf_{y \in K_\epsilon \setminus K_{\epsilon/2}} p_\Omega(s, x, y) ds. \end{aligned} \quad (3.36)$$

Assume that  $x \in \partial A$ . Since  $y \in K_\epsilon$ , we see that  $d(x, y) \leq \text{diam}A$ . Also,  $|x|, |y|$  are bounded by  $\text{diam}A + e$ . It follows from (3.35) that if  $s \geq (\text{diam}A)^2$  then

$$p_\Omega(s, x, y) \geq \frac{c}{V(\sqrt{s})H(\sqrt{s})^2}.$$

Substituting into (3.36) yields (3.31).

In the case (i), when  $V$  is subcritical, we obtain from (3.32)

$$H(r) \approx \frac{r^2}{V(r)}. \quad (3.37)$$

Substituting into (3.31), we obtain, for  $0 < \lambda \leq \frac{1}{(\text{diam}A)^2}$ ,

$$\inf_{\partial A} \Phi_\lambda^\Omega \geq c \int_{1/\lambda}^\infty (1 - e^{-\lambda s}) \frac{V(\sqrt{s})}{s^2} ds \geq c\lambda V\left(\frac{1}{\sqrt{\lambda}}\right),$$

which proves (3.33).

In the case (ii), when  $V$  is critical, we have

$$H(r) \approx \log r, \quad (3.38)$$

which implies, for  $0 < \lambda \leq \frac{1}{(\text{diam}A)^2}$ ,

$$\begin{aligned} \inf_{\partial A} \Phi_\lambda^\Omega &\geq c \int_{1/\lambda}^\infty (1 - e^{-\lambda s}) \frac{ds}{s \log^2 s} \\ &\geq c(1 - e^{-1}) \int_{1/\lambda}^\infty \frac{d \log s}{\log^2 s} \\ &= c(1 - e^{-1}) \frac{1}{\log \frac{1}{\lambda}}, \end{aligned}$$

which proves (3.34). ■

**Lemma 3.7** *Let  $M$  be a geodesically complete, non-compact parabolic manifold satisfying (LY), (RCA). Assume in addition that  $K_\epsilon \subset A$  for sufficiently large  $\epsilon = \epsilon(K) > 0$ . Assume also that  $V(r) := V(o, r)$  is either critical or subcritical. Then there exists a constant  $C > 0$  such that, for small enough  $\lambda > 0$ ,*

$$\sup_{\partial A} \Psi_\lambda^{K^c} \leq \frac{C}{\lambda \log^2 \frac{1}{\lambda}}. \quad (3.39)$$

**Proof.** Set  $\Omega = K^c$ . Fix  $a \in \partial A$  and set

$$T = \frac{1}{\lambda \log^2 \frac{1}{\lambda}}.$$

In the identity (3.25) for  $\Psi_\lambda^\Omega$ , let us decompose the integration into two intervals:  $[0, T]$  and  $[T, \infty)$ . For the first interval, we have by integration by parts

$$\int_0^T te^{-\lambda t} \partial_t \mathbb{P}_a(\tau_\Omega \leq t) dt = Te^{-\lambda T} \mathbb{P}_a(\tau_\Omega \leq T) - \int_0^T e^{-\lambda t} (1 - \lambda t) \mathbb{P}_a(\tau_\Omega \leq t) dt.$$

Assume that  $\lambda < e$  so that  $\log^2 \frac{1}{\lambda} > 1$  and, hence,  $\lambda T < 1$ . It follows that  $1 - \lambda t \geq 0$  on  $[0, T]$  and, therefore, the integral in the right hand side of the above identity is non-negative. It follows that

$$\int_0^T te^{-\lambda t} \partial_t \mathbb{P}_a(\tau_\Omega \leq t) dt \leq T,$$

which matches the required estimate (3.39).

Let us estimate the integral (3.25) over  $[T, \infty)$ . By [13, Remark 4.3], if  $\epsilon$  is large enough then, for all  $a \in \partial A \subset \Omega$  and for all  $t \geq t_0$  (where  $t_0$  depends on  $\text{diam } A$ ), we have

$$\partial_t \mathbb{P}_a(\tau_\Omega \leq t) \leq \frac{C}{V(\sqrt{t}) H^2(\sqrt{t})}, \quad (3.40)$$

where  $H$  is defined by (3.32). Assuming that  $\lambda$  is so small that  $T > t_0$  and using (3.40), we obtain

$$\int_T^\infty te^{-\lambda t} \partial_t \mathbb{P}_a(\tau_\Omega \leq t) dt \leq C \int_T^\infty \frac{te^{-\lambda t} dt}{V(\sqrt{t}) H^2(\sqrt{t})}. \quad (3.41)$$

Consider first the case when  $V(r)$  is critical, that is,  $V(r) \approx r^2$ . Then  $H(r) \approx \log r$  and we obtain

$$\int_T^\infty te^{-\lambda t} \partial_t \mathbb{P}_a(\tau_\Omega \leq t) dt \leq C \int_T^\infty \frac{e^{-\lambda t} dt}{\log^2 t} \leq \frac{C}{\log^2 T} \int_0^\infty e^{-\lambda t} dt = \frac{C}{\lambda \log^2 T}.$$

Taking  $\lambda > 0$  sufficiently small so that  $\log^2 \frac{1}{\lambda} \leq \frac{1}{\sqrt{\lambda}}$ , we obtain  $T \geq \frac{1}{\sqrt{\lambda}}$  and  $\log T \geq \frac{1}{2} \log \frac{1}{\lambda}$ , whence

$$\int_T^\infty te^{-\lambda t} \partial_t \mathbb{P}_a(\tau_\Omega \leq t) dt \leq 4CT,$$

which proved (3.39) in the critical case.

Assume now that  $V(r)$  is subcritical. Then, for  $r > 2$ , we have

$$\frac{r^2}{V(r)} \leq 3 \int_{r/2}^r \frac{tdt}{V(t)} \leq 3H(r).$$

Substituting into (3.41), we obtain

$$\int_T^\infty te^{-\lambda t} \partial_t \mathbb{P}_a(\tau_\Omega \leq t) dt \leq C \int_T^\infty \frac{e^{-\lambda t} dt}{H(\sqrt{t})} \leq \frac{C}{\lambda H(\sqrt{T})} \leq \frac{CV(\sqrt{T})}{\lambda T},$$

where in the last inequality we have used (3.37). In order to prove that the right hand side is bounded by  $CT$ , it suffices to verify that

$$V(\sqrt{T}) \leq C\lambda T^2.$$

Since  $\log \frac{1}{\lambda} \approx \log T$  and, hence,  $\lambda \approx \frac{1}{T \log^2 T}$ , it suffices to prove that

$$V(\sqrt{T}) \leq \frac{CT}{\log^2 T}$$

for large enough  $T$ . Putting  $T = r^2$ , this inequality is equivalent to

$$\log^2 r \leq C \frac{r^2}{V(r)}. \quad (3.42)$$

Since  $M$  is subcritical, there exists a constant  $b > 0$  such that, for large enough  $r$ ,

$$b \leq \int_1^r \frac{tdt}{V(t)} \leq C \frac{r^2}{V(r)}. \quad (3.43)$$

Since

$$\int_1^r \frac{tdt}{V(t)} = \int_1^r \frac{t^2}{V(t)} d \log t, \quad (3.44)$$

substituting (3.43) into the right hand side of (3.44), we obtain

$$\log r = \int_1^r d \log t \leq \int_1^r \frac{C}{b} \frac{t^2}{V(t)} d \log t = \int_1^r \frac{C}{b} \frac{tdt}{V(t)} \leq \frac{C^2}{b} \frac{r^2}{V(r)}.$$

Substituting this into (3.44) again, we obtain for large  $r > 0$ ,

$$\log^2 r = 2 \int_1^r \log t d \log t \leq 2 \int_1^r \frac{C^2}{b} \frac{t^2}{V(t)} d \log t \leq \frac{2C^3}{b} \frac{r^2}{V(r)},$$

whence (3.42) follows. ■

## 4 On-diagonal estimates at center

In this section we prove Theorem 2.1. In order to obtain the upper bound of  $p(t, o, o)$  on  $M = M_1 \# \dots \# M_k$ , we use the integrated resolvent introduced in the previous section. This idea of using the resolvent on a connected sum goes back to Woess [22, p. 96] where it was used in the setting of connected sums of graphs. Implementation in the present case of manifolds requires much more technique, though.

### 4.1 Estimates of integrated resolvent on connected sums

From now on let  $M = M_1 \# M_2 \# \dots \# M_k$  be a connected sum of parabolic manifolds  $M_1, \dots, M_k$  with a central part  $K$ . Let  $A$  be a connected, precompact open subset of  $M$  with smooth boundary and such that  $K \subset A$ . In fact, we will need that  $K_\epsilon \subset A$  for large enough  $\epsilon$ . Set

$$\partial A_i := \partial A \cap E_i, \quad 1 \leq i \leq k$$

so that  $\partial A = \sqcup_i \partial A_i$  (see Fig. 7).

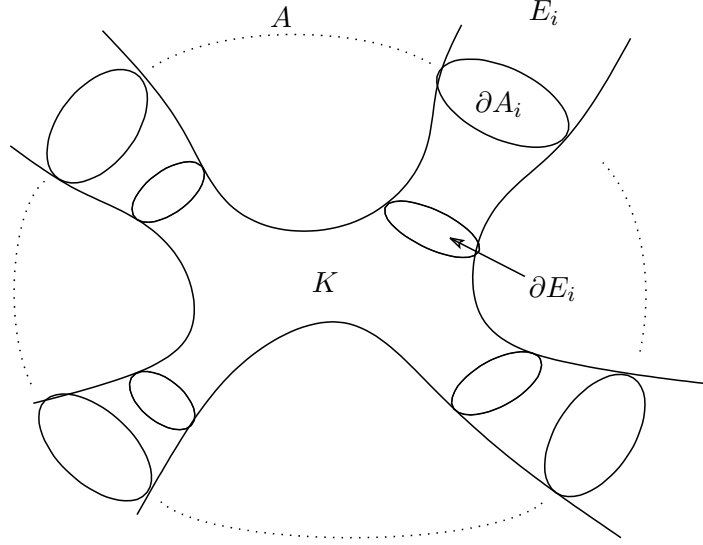


Figure 7: Sets  $K$  and  $A$  in the connected sum  $M$ .

**Lemma 4.1** *There is a constant  $h = h(A, K) > 0$  such that, for any  $\lambda > 0$ ,*

$$h(\sup_{\partial K} \gamma_\lambda) \sum_{i=1}^k \inf_{\partial A_i} \Phi_\lambda^{E_i} \leq \sup_{\partial K} \gamma_\lambda^A. \quad (4.1)$$

**Proof.** As it follows from (3.3) and (3.20) the function

$$u := \gamma_\lambda - \gamma_\lambda^A = G_\lambda 1_K - G_\lambda^A 1_K$$

is  $\lambda$ -harmonic in  $A$ . Consider the function  $h_i$  in  $A$  that solves the Dirichlet problem

$$\begin{cases} \Delta h_i = 0 & \text{in } A \\ h_i = 1_{\partial A_i} & \text{on } \partial A. \end{cases}$$

Since on  $\partial A_i$  we have

$$u \leq \sup_{\partial A_i} \gamma_\lambda = (\sup_{\partial A_i} \gamma_\lambda) h_i,$$

it follows that on  $\partial A$

$$u \leq \sum_{i=1}^n (\sup_{\partial A_i} \gamma_\lambda) h_i. \quad (4.2)$$

Since  $h_i$  is  $\lambda$ -superharmonic in  $A$ , we conclude by the comparison principle in  $A$  that (4.2) holds in  $A$ . Let us also observe that on  $\partial A$

$$\sum_{i=1}^k h_i = 1, \quad (4.3)$$

which implies then that (4.3) holds in  $A$ .

Since in  $E_i$  we have  $\Phi_\lambda^{K^c} = \Phi_\lambda^{E_i}$ , we obtain by Lemma 3.3(b) that in  $E_i$

$$\gamma_\lambda \leq (\sup_{\partial K} \gamma_\lambda) (1 - \Phi_\lambda^{E_i}),$$

which implies

$$\sup_{\partial A_i} \gamma_\lambda \leq (\sup_{\partial K} \gamma_\lambda) \sup_{\partial A_i} (1 - \Phi_\lambda^{E_i}) = (\sup_{\partial K} \gamma_\lambda) (1 - \inf_{\partial A_i} \Phi_\lambda^{E_i}).$$

Substituting into (4.2) and recalling the definition of  $u$ , we obtain that on  $A$

$$\gamma_\lambda \leq \gamma_\lambda^A + (\sup_{\partial K} \gamma_\lambda) \sum_{i=1}^k (1 - \inf_{\partial A_i} \Phi_\lambda^{E_i}) h_i. \quad (4.4)$$

Let  $x \in \partial K$  be a point where  $\gamma_\lambda$  attains its maximum on  $\partial K$ . Considering (4.4) at this point  $x$  we obtain

$$\gamma_\lambda(x) \leq \gamma_\lambda^A(x) + \gamma_\lambda(x) \sum_{i=1}^k (1 - \inf_{\partial A_i} \Phi_\lambda^{E_i}) h_i(x),$$

whence by (4.3)

$$\gamma_\lambda(x) \sum_{i=1}^k (\inf_{\partial A_i} \Phi_\lambda^{E_i}) h_i(x) \leq \gamma_\lambda^A(x).$$

This implies (4.1) with  $h := \min_i \inf_{\partial K} h_i > 0$ . ■

**Lemma 4.2** *There exists a constant  $h = h(A, K) > 0$  such that*

$$h(\sup_{\partial K} \dot{\gamma}_\lambda) \sum_{i=1}^k \inf_{\partial A_i} \Phi_\lambda^{E_i} \leq \sup_{\partial K} \dot{\gamma}_\lambda^A + (\sup_{\partial K} \gamma_\lambda) \left( \sup_{\partial K} \Psi_\lambda^A + \sum_{i=1}^k \sup_{\partial A_i} \Psi_\lambda^{E_i} \right). \quad (4.5)$$

**Proof.** By (3.4) and (3.27), the function

$$v := \dot{\gamma}_\lambda - \dot{\gamma}_\lambda^A = G_\lambda \gamma_\lambda - G_\lambda^A \gamma_\lambda^A$$

solves in  $A$  the following boundary value problem:

$$\begin{cases} \Delta v - \lambda v = -(\gamma_\lambda - \gamma_\lambda^A) & \text{in } A \\ v = \dot{\gamma}_\lambda & \text{on } \partial A. \end{cases}$$

Consider also function  $w$  that solves the problem

$$\begin{cases} \Delta w - \lambda w = 0 & \text{in } A \\ w = \dot{\gamma}_\lambda & \text{on } \partial A. \end{cases}$$

Then we have

$$v = G_\lambda^A (\gamma_\lambda - \gamma_\lambda^A) + w. \quad (4.6)$$

Using the estimate (3.21) of Lemma 3.3(a) and (3.24), we obtain that in  $A$

$$G_\lambda^A (\gamma_\lambda - \gamma_\lambda^A) \leq (\sup_{\partial A} \gamma_\lambda) G_\lambda^A (1 - \Phi_\lambda^A) = (\sup_{\partial A} \gamma_\lambda) \Psi_\lambda^A. \quad (4.7)$$

Observe that

$$\gamma_\lambda \leq \sup_{\partial K} \gamma_\lambda \text{ in } K^c$$

because the constant function  $\sup_{\partial K} \gamma_\lambda$  is  $\lambda$ -superharmonic in  $K^c$ , while  $\gamma_\lambda$  is minimal  $\lambda$ -harmonic that is bounded by  $\sup_{\partial K} \gamma_\lambda$  on  $\partial K^c$ . Hence, we obtain from (4.7) that

$$G_\lambda^A (\gamma_\lambda - \gamma_\lambda^A) \leq (\sup_{\partial K} \gamma_\lambda) \Psi_\lambda^A \text{ in } A. \quad (4.8)$$

In order to estimate  $w$ , let us represent this function in the form

$$w = \sum_{i=1}^k w_i,$$

where  $w_i$  solves the Dirichlet problem

$$\begin{cases} \Delta w_i - \lambda w_i = 0 & \text{in } A \\ w_i = \dot{\gamma}_\lambda 1_{\partial A_i} & \text{on } \partial A. \end{cases}$$

Let  $h_i$  be the same as in the proof of Lemma 4.1. By the comparison principle, we have that in  $A$

$$w_i \leq (\sup_{\partial A_i} \dot{\gamma}_\lambda) h_i. \quad (4.9)$$

Let us prove further that

$$\dot{\gamma}_\lambda - G_\lambda^{E_i} \gamma_\lambda \leq (\sup_{\partial E_i} \dot{\gamma}_\lambda) (1 - \Phi_\lambda^{E_i}) \quad \text{in } E_i. \quad (4.10)$$

Indeed, by (3.4), the function

$$\dot{\gamma}_\lambda - G_\lambda^{E_i} \gamma_\lambda = G_\lambda \gamma_\lambda - G_\lambda^{E_i} \gamma_\lambda$$

is non-negative,  $\lambda$ -harmonic, and minimal in  $E_i$ . Besides, it is bounded by  $\sup_{\partial E_i} \dot{\gamma}_\lambda$  on  $\partial E_i$ . The function  $1 - \Phi_\lambda^{E_i}$  is non-negative and  $\lambda$ -harmonic in  $E_i$ , and is equal to 1 on  $\partial E_i$ . The estimate (4.10) follows by the comparison principle in  $E_i$ .

Similarly, we have

$$\gamma_\lambda \leq (\sup_{\partial E_i} \gamma_\lambda) (1 - \Phi_\lambda^{E_i}) \quad \text{in } E_i,$$

because  $\gamma_\lambda$  is non-negative,  $\lambda$ -harmonic and minimal in  $E_i$ , and is bounded by  $\sup_{\partial E_i} \gamma_\lambda$  on  $\partial E_i$ . It follows that in  $E_i$

$$G_\lambda^{E_i} \gamma_\lambda \leq (\sup_{\partial E_i} \gamma_\lambda) G_\lambda^{E_i} (1 - \Phi_\lambda^{E_i}) = (\sup_{\partial E_i} \gamma_\lambda) \Psi_\lambda^{E_i}.$$

Combining with (4.10), we obtain that in  $E_i$

$$\dot{\gamma}_\lambda \leq (\sup_{\partial E_i} \dot{\gamma}_\lambda) (1 - \Phi_\lambda^{E_i}) + (\sup_{\partial E_i} \gamma_\lambda) \Psi_\lambda^{E_i}.$$

Substituting into (4.9), we obtain that in  $A$

$$w \leq \sum_{i=1}^k (\sup_{\partial A_i} \dot{\gamma}_\lambda) h_i \leq \sum_{i=1}^k \left( (\sup_{\partial E_i} \dot{\gamma}_\lambda) (1 - \inf_{\partial A_i} \Phi_\lambda^{E_i}) + (\sup_{\partial E_i} \gamma_\lambda) \sup_{\partial A_i} \Psi_\lambda^{E_i} \right) h_i.$$

Combining with (4.6) and (4.8), we obtain the following estimate of the function  $v = \dot{\gamma}_\lambda - \dot{\gamma}_\lambda^A$  in  $A$ :

$$\dot{\gamma}_\lambda - \dot{\gamma}_\lambda^A \leq (\sup_{\partial K} \gamma_\lambda) \Psi_\lambda^A + \sum_{i=1}^k \left( (\sup_{\partial E_i} \dot{\gamma}_\lambda) (1 - \inf_{\partial A_i} \Phi_\lambda^{E_i}) + (\sup_{\partial E_i} \gamma_\lambda) \sup_{\partial A_i} \Psi_\lambda^{E_i} \right) h_i.$$

Let  $x$  be a point of maximum of  $\dot{\gamma}_\lambda$  on  $\partial K$ . It follows that

$$\dot{\gamma}_\lambda(x) \leq \dot{\gamma}_\lambda^A(x) + (\sup_{\partial K} \gamma_\lambda) \Psi_\lambda^A(x) + \sum_{i=1}^k \left( \dot{\gamma}_\lambda(x) (1 - \inf_{\partial A_i} \Phi_\lambda^{E_i}) + (\sup_{\partial E_i} \gamma_\lambda) \sup_{\partial A_i} \Psi_\lambda^{E_i} \right) h_i(x).$$

Since  $\sum h_i \equiv 1$ , we see that  $\dot{\gamma}_\lambda(x)$  cancels out in the both sides, and we obtain

$$\dot{\gamma}_\lambda(x) \sum_{i=1}^k (\inf_{\partial A_i} \Phi_\lambda^{E_i}) h_i(x) \leq \dot{\gamma}_\lambda^A(x) + (\sup_{\partial K} \gamma_\lambda) \Psi_\lambda^A(x) + \sum_{i=1}^k (\sup_{\partial E_i} \gamma_\lambda) (\sup_{\partial A_i} \Psi_\lambda^{E_i}) h_i(x).$$

Since  $h \leq h_i(x) \leq 1$  where  $h := \min_i \inf_K h_i > 0$ , we obtain from here (4.5). ■

## 4.2 Proof of Theorem 2.1: Upper bound

As in the statement of Theorem 2.1, let  $M$  be a connected sum of parabolic manifolds  $M_1, \dots, M_k$ , where all  $M_i$ ,  $i = 1, \dots, k$  satisfy (LY) and (RCA). Let  $V_i(r) = V_i(o_i, r)$  be the volume function on  $M_i$  at  $o_i \in K_i = M_i \setminus E_i$ . We also assume that every  $V_i(r)$  is either critical or subcritical, that is, condition (d) of Section 2.2. Let  $V(r) = V(o, r)$  be the volume function on  $M$  at a reference point  $o \in K$ .

It suffices to prove the main estimate (2.8) for large enough  $t$  because for small  $t$  we have  $p(t, o, o) \asymp t^{-N/2}$  and  $V(\sqrt{t}) \asymp t^{N/2}$ .

Fix a connected precompact open set  $A$  with smooth boundary such that  $A \supset K_\epsilon$  for large enough  $\epsilon > 0$  as in Lemmas 3.6 and 3.7 applied to all ends  $M_i$ .

Recall that the integrated resolvent  $\gamma_\lambda$  is defined by (3.3). By Lemmas 3.5 and 4.1, we have, for any  $\lambda > 0$  and any  $i = 1, \dots, k$

$$\sup_{\partial K} \gamma_\lambda \leq \frac{C}{\inf_{\partial A_i} \Phi_\lambda^{E_i}}, \quad (4.11)$$

where  $C = C(K, A)$ .

Assume first that all manifolds  $M_i$  are subcritical. Applying (3.33) on each end  $M_i$  we obtain that

$$\inf_{\partial A_i} \Phi_\lambda^{E_i} \geq c\lambda V_i\left(\frac{1}{\sqrt{\lambda}}\right)$$

provided  $\lambda \leq \lambda_0 = \lambda_0(A)$ . Substituting into (4.11), we obtain that, for  $\lambda \leq \lambda_0$ ,

$$\sup_{\partial K} \gamma_\lambda \leq \frac{C}{\lambda V_{\max}\left(\frac{1}{\sqrt{\lambda}}\right)},$$

where  $V_{\max}(r) = \max_{1 \leq i \leq k} V_i(r)$ . By Lemma 3.1(i), we conclude that, for all  $t \geq t_0 = t_0(\lambda_0)$ ,

$$p(t, o, o) \leq \frac{C}{V_{\max}(\sqrt{t})} \quad (4.12)$$

which proves the on-diagonal upper bound in (2.8) in the subcritical case.

Assume now that there exists at least one critical end. Let it be  $M_j$ . Applying (3.34) in  $M_j$ , we have

$$\inf_{\partial A} \Phi_\lambda^{E_j} \geq \frac{c}{\log \frac{1}{\lambda}}, \quad (4.13)$$

which together with (4.11) yields, for all  $\lambda \leq \lambda_0$ ,

$$\sup_{\partial K} \gamma_\lambda \leq C \log \frac{1}{\lambda}. \quad (4.14)$$

However, as we have pointed out before, in order to obtain upper bound in (2.8) in the critical case, we need some additional argument about  $\dot{\gamma}_\lambda$ .

For that, let us use the estimate (4.5) of  $\sup_{\partial K} \dot{\gamma}_\lambda$ . Substituting into (4.5) the estimates (3.29) and (3.30), we obtain

$$\left(\sup_{\partial K} \dot{\gamma}_\lambda\right) \inf_{\partial A_j} \Phi_\lambda^{E_j} \leq C + C \sup_{\partial K} \gamma_\lambda \left(1 + \sum_{i=1}^k \sup_{\partial A_i} \Psi_\lambda^{E_i}\right).$$

Substituting here (4.13), (4.14), (3.39), we obtain, for all  $\lambda \leq \lambda_0$ ,

$$\sup_{\partial K} \dot{\gamma}_\lambda \frac{1}{\log \frac{1}{\lambda}} \leq C + C \log \frac{1}{\lambda} \left(1 + \frac{1}{\lambda \log^2 \frac{1}{\lambda}}\right) \leq \frac{C'}{\lambda \log \frac{1}{\lambda}},$$

which implies

$$\sup_{\partial K} \dot{\gamma}_\lambda \leq \frac{C}{\lambda} \text{ for all } \lambda \leq \lambda_0.$$

By Lemma 3.1 (ii), we conclude that

$$p(t, o, o) \leq \frac{C}{t} \text{ for all } t \geq t_0 \quad (4.15)$$

which finishes the proof of the upper bound in (2.8) in the critical case.

### 4.3 Proof of Theorem 2.1: Lower bound

Let  $M$  be a connected sum satisfying the assumption of Theorem 2.1. Let us observe that

$$V(r) \approx V_1(r) + V_2(r) + \cdots + V_k(r) \approx V_{\max}(r) \quad (4.16)$$

for all  $r > 0$ . By (4.12) and (4.15), we obtain that, for all  $t > 0$ ,

$$p(t, o, o) \leq \frac{C}{V(\sqrt{t})}. \quad (4.17)$$

Since each  $V_i(r)$  satisfies the doubling condition, so does  $V(r)$  by (4.16). By [3, Theorem 7.2], the upper bound (4.17) together with the doubling property of  $V(r)$  implies the matching lower bound

$$p(t, o, o) \geq \frac{c}{V(\sqrt{t})}.$$

Replacing here  $V$  by  $V_{\max}$ , we finish the proof of the lower bound in (2.8) and, hence, the proof of Theorem 2.1.

## 5 Off-diagonal estimates

In this section, we prove Theorems 2.3-2.5 by combining Theorem 2.1 with some results from [12], [13] and [15].

For any open set  $\Omega$  in any weighted manifold  $M$ , define the *exit probability function* in  $\Omega$ : for all  $x \in \Omega$  and  $t > 0$ ,

$$\psi_\Omega(y, t) = \mathbb{P}_x(\tau_\Omega \leq t).$$

Equivalently,  $\psi_\Omega(x, t)$  is the minimal non-negative solution of the heat equation  $\partial_t u = \Delta u$  in  $\Omega \times \mathbb{R}_+$  with the initial condition  $u|_{t=0} = 0$  and the boundary condition  $u|_{\partial\Omega} = 1$ .

We will use the abstract upper and lower off-diagonal estimates of [15, Theorem 3.5] for the heat kernel  $p(t, x, y)$  on an arbitrary manifold  $M$  for  $x \in A$  and  $y \in B$  where  $A, B$  are open subsets of  $M$  such either  $\bar{A}$  and  $\bar{B}$  are disjoint or  $\bar{B} \subset A$ . These estimates use the exit probabilities  $\psi_A(x, t)$  and  $\psi_B(y, t)$  and their time derivatives. Besides, they use the quantities

$$P^+(t) = \sup_{s \in [t/4, t]} \sup_{z_1 \in \partial A, z_2 \in \partial B} p(s, z_1, z_2) \text{ and } P^-(t) = \inf_{s \in [t/4, t]} \inf_{z_1 \in \partial A, z_2 \in \partial B} p(s, z_1, z_2)$$

and

$$G^+(t) = \int_0^t \sup_{z_1 \in \partial A, z_2 \in \partial B} p(s, z_1, z_2) ds \text{ and } G^-(t) = \int_0^t \inf_{z_1 \in \partial A, z_2 \in \partial B} p(s, z_1, z_2) ds.$$

With these notations, the estimates of [15, Theorem 3.5] read as follows: for all  $x \in A, y \in B$  and  $t > 0$ ,

$$p(t, x, y) \approx p_A(t, x, y) + P^\pm(t) \psi_A(x, \tilde{t}) \psi_B(y, \tilde{t}) + G^\pm(\tilde{t}) [\partial_t \psi_A(x, \xi) \psi_B(y, \tilde{t}) + \partial_t \psi_B(y, \zeta) \psi_A(x, \tilde{t})], \quad (5.1)$$

where the index “+” is used for the upper bound, “-” is used for the lower bound,  $\tilde{t} = t$  for the upper bound,  $\tilde{t} = \frac{1}{4}t$  for the lower bound,  $\xi$  and  $\zeta$  are some values from  $[t/4, t]$  that may be different for upper and lower bounds.

**Proof of Theorem 2.3.** Recall that  $M$  is a connected sum of  $M_1, \dots, M_k$  with a central part  $K$ , where each  $M_i$  satisfies conditions (a)-(d) in Subsection 2.2. We apply (5.1) with  $A = E_i$  and  $B = E_j$  where  $i \neq j$ . Since  $A$  and  $B$  are disjoint, we have  $p_A(t, x, y) = 0$  for all  $x \in A$  and  $y \in B$ .

Note that, for all  $z_1 \in \partial E_i$  and  $z_2 \in \partial E_j$ , the distance  $d(z_1, z_2)$  is bounded from above and below by positive constants. Therefore, assuming  $t > 1$ , we obtain by the local Harnack inequality and Theorem 2.1 that

$$P^\pm(t) \asymp Cp(ct, o, o) \approx \frac{1}{V(\sqrt{t})}. \quad (5.2)$$

Let us estimate similarly  $G^\pm(t)$ . Assuming  $t > 1$ , we can split the integrals in the definition of  $G^\pm(t)$  into the sum of two integrals: over  $(0, 1]$  and over  $(1, t]$ . The first integral is bounded, while in the second integral we can apply the local Harnack inequality to the heat kernel and, hence, replace  $z_1, z_2$  by  $o$ . Using further the estimate (2.8) of Theorem 2.1, we obtain that, for large  $t$ ,

$$G^\pm(t) \approx \int_1^t \frac{1}{V(\sqrt{s})} ds. \quad (5.3)$$

If all ends are subcritical, then by (2.3) we have, for large  $t$ ,

$$\int_1^t \frac{ds}{V(\sqrt{s})} \leq \frac{Ct}{V(\sqrt{t})}.$$

Since also

$$\int_1^t \frac{ds}{V(\sqrt{s})} \geq \int_{t/2}^t \frac{ds}{V(\sqrt{s})} \geq \frac{t}{2V(\sqrt{t})},$$

we obtain that

$$G^\pm(\tilde{t}) \approx \frac{t}{V(\sqrt{t})}. \quad (5.4)$$

If there exists at least one critical end, then  $V(\sqrt{t}) \approx t$ , and (5.3) implies, for large  $t$ ,

$$G^\pm(\tilde{t}) \approx \log t. \quad (5.5)$$

Note that the exit probability  $\psi_i(x, t)$  depends only on the intrinsic geometry of  $E_i$ . Since each  $M_i$  satisfies (LY) and (RCA), we can use the results of [13, Theorem 4.6] that gives the following: for all  $x \in E_i$  with large enough  $|x|$ ,

$$\psi_{E_i}(x, t) \asymp \begin{cases} \frac{C|x|^2 \exp(-b|x|^2/t)}{V_i(|x|)H(|x|)} & t < 2|x|^2, \\ \frac{C}{H(\sqrt{t}) \int_{|x|}^{\sqrt{t}} \frac{s ds}{V_i(s)}}, & t \geq 2|x|^2 \end{cases} \quad (5.6)$$

and, for large enough  $|x|$  and  $t$ ,

$$\partial_t \psi_{E_i}(x, t) \asymp \frac{CH(|x|) \exp(-b|x|^2/t)}{V_i(\sqrt{t}) (H(|x|) + H(\sqrt{t})) H(\sqrt{t})}, \quad (5.7)$$

where  $H$  is the function defined in (3.32). Note that in the case of bounded  $|x|$  the estimate (5.7) matches the estimate (3.40) used in the proof of Lemma 3.7.

If  $M_i$  is subcritical then  $H(r) \approx r^2/V_i(r)$ . Substituting this into then (5.6) and (5.7), we obtain, for all large enough  $t$  and  $|x|$ ,

$$\psi_{E_i}(x, t) \asymp C e^{-b \frac{|x|^2}{t}}, \quad (5.8)$$

$$\partial_t \psi_{E_i}(x, t) \asymp \frac{C}{t} D(x, t) e^{-b \frac{|x|^2}{t}}, \quad (5.9)$$

where  $D$  is defined in (2.14).

If  $M_i$  is critical then  $H(r) \approx \log r$  which yields

$$\psi_{E_i}(x, t) \asymp C U(x, t) e^{-b \frac{|x|^2}{t}}, \quad (5.10)$$

$$\partial_t \psi_{E_i}(x, t) \asymp \frac{C}{t \log t} W(x, t) e^{-b \frac{|x|^2}{t}}, \quad (5.11)$$

where  $U$  is defined in (2.15) and  $W$  is defined in (2.16).

Now we are in position to verify all the heat kernel estimates claimed in Theorem 2.3 for  $x \in E_i, y \in E_j$  with  $i \neq j$ . It suffices to prove all the estimates for large enough  $|x|, |y|$  and  $t$ . Then the estimates for all  $x \in E_i$  and  $y \in E_j$  (while  $t$  is still large enough) follow by application of the local Harnack inequality.

(i) If all ends are subcritical, then (5.1), (5.2), (5.4), (5.8), (5.9) yield:

$$p(t, x, y) \asymp \frac{C}{V(\sqrt{t})} [1 + D(x, t) + D(y, t)] e^{-b \frac{|x|^2 + |y|^2}{t}}.$$

Observing that that by (2.14)  $D(x, t)$  is bounded and that

$$|x|^2 + |y|^2 \approx d^2(x, y)$$

we obtain (2.18).

(ii) Now let at least one of the ends be critical, so that  $V(r) \approx r^2$ .

(ii)<sub>1</sub> Let  $M_i, M_j$  are subcritical, then (5.1), (5.2), (5.5), (5.8), (5.9) yield:

$$p(t, x, y) \asymp \frac{C}{t} (1 + (D(x, t) + D(y, t)) \log t) e^{-b \frac{|x|^2 + |y|^2}{t}},$$

which proves (2.19).

(ii)<sub>2</sub> Let both  $M_i$  and  $M_j$  be critical. Then we obtain from (5.1), (5.2), (5.5), (5.10), (5.11) that

$$p(t, x, y) \asymp \frac{C}{t} [U(x, t)U(y, t) + W(x, t)U(y, t) + U(x, t)W(y, t)] e^{-b \frac{|x|^2 + |y|^2}{t}},$$

that is, (2.20).

(ii)<sub>3</sub> Let  $M_i$  be subcritical and  $M_j$  be critical. Then we obtain similarly

$$p(t, x, y) \asymp \frac{C}{t} [U(x, t) + D(x, t)U(y, t) \log t + W(x, t)] e^{-b \frac{|x|^2 + |y|^2}{t}}.$$

By (2.17) we can replace here  $U + W$  by 1, which yields (2.21). ■

For the proof of Theorems 2.4 and 2.5, we will use again the estimate (5.1) but this time we take  $A = E_i$  and  $B = E'_i$  where  $E'_i = E_i \setminus K'$  and  $K'$  is a closed  $\epsilon$ -neighborhood of  $K$  for large enough  $\epsilon$ . In this case we have  $\bar{B} \subset A$ .

Note that, for all  $z_1 \in \partial E_i$  and  $z_2 \in \partial E'_i$ , the distance  $d(z_1, z_2)$  is bounded from above and below by positive constants. Hence, arguing as above, we obtain the same estimates of  $P^\pm(t), G^\pm(t)$  as stated in the proof of Theorem 2.3. The estimates of  $\psi_{E_i}$  and  $\partial_t \psi_{E_i}$  also remain the same. Clearly,  $\psi_{E'_i}$  and  $\partial_t \psi_{E'_i}$  satisfy similar estimates.

To handle the term  $p_A(t, x, y) = p_{E_i}(t, x, y)$  in (5.1), we use the result of [12, Theorem 4.9] that says the following: for all  $t > 0$  and all  $x, y \in E_i$  with large  $|x|, |y|$ ,

$$p_{E_i}(t, x, y) \asymp \frac{C}{V_i(x, \sqrt{t})} \left( \frac{H(|x|)}{H(|x|) + H(\sqrt{t})} \right) \left( \frac{H(|y|)}{H(|y|) + H(\sqrt{t})} \right) e^{-b \frac{d^2}{t}},$$

where  $d = d(x, y)$ . If  $M_i$  is subcritical, then  $H(r) \approx r^2/V(r)$ , which gives

$$p_{E_i}(t, x, y) \asymp C \frac{D(x, t)D(y, t)}{V_i(x, \sqrt{t})} e^{-b \frac{d^2}{t}}. \quad (5.12)$$

If  $M_i$  is critical, then  $H(r) \approx \log r$ , which gives

$$p_{E_i}(t, x, y) \asymp C \frac{W(x, t)W(y, t)}{V_i(x, \sqrt{t})} e^{-b \frac{d^2}{t}}. \quad (5.13)$$

For the proof of Theorems 2.4 and 2.5 we need the following lemma.

**Lemma 5.1** *For all  $x, y \in E_i$  and  $\sqrt{t} \geq \min(|x|, |y|)$  we have*

$$C e^{-b \frac{|x|^2 + |y|^2}{t}} \asymp C' e^{-b' \frac{d^2(x, y)}{t}}. \quad (5.14)$$

Moreover, if  $\sqrt{t} \geq |x|$  then

$$\frac{C}{V_i(x, \sqrt{t})} e^{-b \frac{d^2(x, y)}{t}} \asymp \frac{C'}{V_i(\sqrt{t})} e^{-b' \frac{d^2(x, y)}{t}} \quad (5.15)$$

**Proof.** Set  $\delta = \text{diam}K$ . The triangle inequality  $|x| + |y| + \delta \geq d(x, y)$  implies

$$e^{-b \frac{|x|^2 + |y|^2}{t}} \leq e^{-b' \frac{d^2(x, y) - \delta^2}{t}} \leq C' e^{-b' \frac{d^2(x, y)}{t}}. \quad (5.16)$$

To prove the opposite inequality, assume that  $|x| \leq \sqrt{t}$  (the case  $|y| \leq \sqrt{t}$  is similar). The triangle inequality

$$|y| \leq |x| + \delta + d(x, y)$$

implies

$$|x| + |y| \leq 2|x| + \delta + d(x, y) \leq 2\sqrt{t} + \delta + d(x, y),$$

whence it follows that

$$\frac{|x|^2 + |y|^2}{t} \leq b' \frac{d^2(x, y)}{t} + \text{const},$$

which completes the proof of (5.14).

To prove (5.15) observe first that by (5.14), the term  $d^2(x, y)$  in the both sides of (5.15) can be replaced by  $|x|^2 + |y|^2$ . The doubling property of  $V_i(x, r)$  yields

$$\frac{V_i(o_i, \sqrt{t})}{V_i(x, \sqrt{t})} \leq C \left( 1 + \frac{|x|}{\sqrt{t}} \right)^\beta \leq C e^{\varepsilon \frac{|x|^2}{t}},$$

for arbitrarily small  $\varepsilon > 0$ , which implies that

$$\begin{aligned} \frac{C}{V_i(x, \sqrt{t})} e^{-b\frac{|x|^2+|y|^2}{t}} &\leq \frac{C'}{V_i(o, \sqrt{t})} e^{\varepsilon\frac{|x|^2}{t}} e^{-b\frac{|x|^2+|y|^2}{t}} \\ &\leq \frac{C'}{V_i(o, \sqrt{t})} e^{-b'\frac{|x|^2+|y|^2}{t}}. \end{aligned} \quad (5.17)$$

The opposite inequality is proved similarly. ■

**Proof of Theorem 2.4(a).** We consider the same cases as in Theorem 2.3 and use the same estimates of all the terms in (5.1), except for the Dirichlet heat kernel. Note that the case  $(ii)_3$  cannot occur because  $x, y$  are at the same end  $E_i$ .

(i) Assume that all ends are subcritical. Substituting (5.12), (5.2), (5.4), (5.8) and (5.9) into (5.1), we obtain

$$\begin{aligned} p(t, x, y) &\asymp C \frac{D(x, t)D(y, t)}{V_i(x, \sqrt{t})} e^{-b\frac{d^2}{t}} \\ &\quad + \frac{C}{V(\sqrt{t})} (1 + D(x, t) + D(y, t)) e^{-b\frac{|x|^2+|y|^2}{t}}. \end{aligned} \quad (5.18)$$

By (2.14) and the assumption  $\sqrt{t} \leq \min(|x|, |y|)$  we have

$$D(x, t) = D(y, t) = 1$$

and, hence,

$$p(t, x, y) \asymp \frac{C}{V_i(x, \sqrt{t})} e^{-b\frac{d^2(x, y)}{t}} + \frac{C}{V(\sqrt{t})} e^{-b\frac{|x|^2+|y|^2}{t}}. \quad (5.19)$$

Using the volume doubling property of  $V_i$ , we obtain

$$\begin{aligned} \frac{1}{V(\sqrt{t})} e^{-b\frac{|x|^2+|y|^2}{t}} &= \frac{V_i(o_i, \sqrt{t})}{V_{\max}(\sqrt{t})} \frac{V_i(x, \sqrt{t})}{V_i(o_i, \sqrt{t})} \frac{1}{V_i(x, \sqrt{t})} e^{-b\frac{|x|^2+|y|^2}{t}} \\ &\leq C \left(1 + \frac{|x|}{\sqrt{t}}\right)^\beta \frac{1}{V_i(x, \sqrt{t})} e^{-b\frac{|x|^2+|y|^2}{t}} \\ &\leq \frac{C'}{V_i(x, \sqrt{t})} e^{-b'\frac{d^2(x, y)}{t}}, \end{aligned} \quad (5.20)$$

which shows that the first term in (5.19) is dominant, hence yielding (2.22).

(ii) Let at least one of the ends be critical.

(ii)<sub>1</sub> Let  $M_i$  be subcritical. In this case we have as above

$$p(t, x, y) \asymp \frac{C}{V_i(x, \sqrt{t})} e^{-b\frac{d^2}{t}} + C \frac{\log t}{t} e^{-b\frac{|x|^2+|y|^2}{t}}. \quad (5.21)$$

By (2.7) and the volume doubling property of  $M_i$ , we obtain

$$\begin{aligned} \frac{\log t}{t} e^{-b\frac{|x|^2+|y|^2}{t}} &= \frac{\log t}{t} V_i(o_i, \sqrt{t}) \frac{1}{V_i(x, \sqrt{t})} \frac{V_i(x, \sqrt{t})}{V_i(o_i, \sqrt{t})} e^{-b\frac{|x|^2+|y|^2}{t}} \\ &\leq \frac{C}{V_i(x, \sqrt{t})} \left(1 + \frac{|x|}{\sqrt{t}}\right)^\beta e^{-b\frac{|x|^2+|y|^2}{t}} \\ &\leq \frac{C'}{V_i(x, \sqrt{t})} e^{-b'\frac{d^2(x, y)}{t}}. \end{aligned} \quad (5.22)$$

Substituting (5.22) into (5.21), we obtain (2.22).

(ii)<sub>2</sub> Let  $M_i$  be critical. Substituting (5.13), (5.2), (5.5), (5.10) and (5.11) into (5.1), we obtain

$$p(t, x, y) \asymp C \frac{W(x, t)W(y, t)}{V_i(x, \sqrt{t})} e^{-b \frac{d^2}{t}} + \frac{C}{t} [U(x, t)U(y, t) + W(x, t)U(y, t) + W(y, t)U(x, t)] e^{-b \frac{|x|^2 + |y|^2}{t}}. \quad (5.23)$$

By (2.16) and  $\sqrt{t} \leq \min(|x|, |y|)$ , we have

$$W(x, t) = W(y, t) = 1.$$

Substituting into (5.23) we obtain

$$p(t, x, y) \asymp \frac{C}{V_i(x, \sqrt{t})} e^{-b \frac{d^2}{t}} + \frac{C}{t} [U(x, t)U(y, t) + U(y, t) + U(x, t)] e^{-b \frac{|x|^2 + |y|^2}{t}}.$$

Since  $U$  is bounded, (5.20) implies that the second term is dominated by the first one, which yields (2.22). ■

**Proof of Theorem 2.4(b).** Let  $V_i(r) \approx V_{\max}(r)$ . In the view of part (a), we can assume that  $\sqrt{t} > \min(|x|, |y|)$ . Since by the doubling property of  $V_i$

$$\frac{C}{V_i(x, \sqrt{t})} e^{-b \frac{d^2(x, y)}{t}} \asymp \frac{C'}{V_i(y, \sqrt{t})} e^{-b' \frac{d^2(x, y)}{t}}$$

(cf. (5.17)), the estimate (2.22) is symmetric in  $x, y$ . Hence, we can assume that  $\sqrt{t} > |x|$ . As in Theorem 2.3, we can also assume that  $|x|, |y|$  are large enough.

(i) Let all the ends be subcritical. Then we have again (5.18). Using  $\sqrt{t} > |x|$  and (5.14), we can replace  $e^{-b \frac{|x|^2 + |y|^2}{t}}$  in the right hand side of (5.18) by  $e^{-b \frac{d^2(x, y)}{t}}$ . Using further (5.15), we can replace  $V_i(x, \sqrt{t})$  by  $V_i(\sqrt{t})$  and, hence, by  $V(\sqrt{t})$ , which yields

$$p(t, x, y) \asymp \frac{C}{V(\sqrt{t})} (D(x, t)D(y, t) + 1 + D(x, t) + D(y, t)) e^{-b \frac{d^2(x, y)}{t}},$$

and which implies (2.22) since  $D(x, t), D(y, t)$  are bounded.

(ii) Let at least one of the ends be critical. Then by  $V_i(r) \approx V(r)$ , the end  $M_i$  has to be critical, too. As in the case (ii)<sub>2</sub> of the proof of Theorem 2.4(a), we obtain again (5.23), where by (5.14) we can replace  $e^{-b \frac{|x|^2 + |y|^2}{t}}$  in the right hand side of (5.23) by  $e^{-b \frac{d^2}{t}}$ . Using further (5.15), we replace  $V_i(x, \sqrt{t})$  by  $V_i(\sqrt{t}) \approx V(\sqrt{t}) \approx t$ , which yields

$$\begin{aligned} p(t, x, y) &\asymp \frac{C}{t} [W(x, t)W(y, t) + U(x, t)U(y, t) + W(x, t)U(y, t) + W(y, t)U(x, t)] e^{-b \frac{d^2}{t}} \\ &= \frac{C}{t} \{W(x, t) + U(x, t)\} \{W(y, t) + U(y, t)\} e^{-b \frac{d^2}{t}}. \end{aligned}$$

Using (2.17), we conclude (2.22). ■

**Proof of Theorem 2.5.** As in Theorem 2.3, we can assume that  $|x|, |y|$  are large enough. Since  $\sqrt{t} \geq \min(|x|, |y|)$  and the both estimates (2.23) and (2.24) are symmetric in  $x, y$ , so we can assume without loss of generality that  $\sqrt{t} \geq |x|$ . Then, by Lemma 5.1, the function  $V_i(x, \sqrt{t})$  in the estimates (5.12) and (5.13) can be replaced by  $V_i(\sqrt{t})$ .

(i) Assume that all ends are subcritical. Applying (5.14) to (5.18) and observing that the function  $D$  is bounded, we obtain (2.23).

(ii) Let at least one of the ends be critical. Since  $M_i$  is subcritical, substituting (5.12), (5.2), (5.5), (5.8) and (5.9) into (5.1), we obtain

$$p(t, x, y) \asymp C \frac{D(x, t)D(y, t)}{V_i(\sqrt{t})} e^{-b\frac{a^2}{t}} + \frac{C}{t} e^{-b\frac{|x|^2+|y|^2}{t}} \\ + C \frac{\log t}{t} (D(x, t) + D(y, t)) e^{-b\frac{|x|^2+|y|^2}{t}},$$

which together with (5.14) implies (2.24). ■

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