ON THE SMALLEST NON-ABELIAN QUOTIENT OF $\mathrm{Aut}(F_n)$

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Abstract. We show that the smallest non-abelian quotient of $\mathrm{Aut}(F_n)$ is $\mathrm{PSL}_n(\mathbb{Z}/2\mathbb{Z}) = \mathrm{L}_n(2)$, thus confirming a conjecture of Mecchia–Zimmermann. In the course of the proof we give an exponential (in n) lower bound for the cardinality of a set on which $\mathrm{SAut}(F_n)$, the unique index 2 subgroup of $\mathrm{Aut}(F_n)$, can act non-trivially. We also offer new results on the representation theory of $\mathrm{SAut}(F_n)$ in small dimensions over small, positive characteristics, and on rigidity of maps from $\mathrm{SAut}(F_n)$ to finite groups of Lie type and algebraic groups in characteristic 2.

1. Introduction

Investigating finite quotients of outer automorphism groups of free groups and mapping class groups has a long history. The first fundamental result here is that groups in both classes are residually finite – this is due to Grossman [Gro].

Once we know that the groups admit many finite quotients, we can start asking questions about the structure or size of such quotients. This is of course equivalent to studying normal subgroups of finite index in mapping class groups and $\mathrm{Out}(F_n)$.

When $n \geqslant 3$, the groups $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ have unique subgroups of index 2, denoted respectively by $\mathrm{SOut}(F_n)$ and $\mathrm{SAut}(F_n)$. Both of these subgroups are perfect, and so the abelian quotients of $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ are well understood. The situation for mapping class groups is very similar.

The simplest way of obtaining a non-abelian quotient of $\mathrm{Out}(F_n)$ or $\mathrm{Aut}(F_n)$ comes from observing that $\mathrm{Out}(F_n)$ acts on the abelianisation of F_n, that is $\mathbb{Z}^n$. In this way we obtain (surjective) maps

$$\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$$

The finite quotients of $\mathrm{GL}_n(\mathbb{Z})$ are controlled by the congruence subgroup property and are well understood. In particular, the smallest (in terms of cardinality) such quotient is $\mathrm{PSL}_n(\mathbb{Z}/2\mathbb{Z}) = \mathrm{L}_n(2)$, obtained by reducing $\mathbb{Z}$ modulo 2. According to a conjecture of Mecchia–Zimmermann [MZ], the group $\mathrm{L}_n(2)$ is the smallest non-abelian quotient of $\mathrm{Out}(F_n)$.

In [MZ] Mecchia and Zimmermann confirmed their conjecture for $n \in \{3, 4\}$. In this paper we prove it for all $n \geqslant 3$. In fact we prove more:

Theorem 9.1. *Let $n \geqslant 3$. Every non-trivial finite quotient of $\mathrm{SAut}(F_n)$ is either greater in cardinality than $\mathrm{L}_n(2)$, or isomorphic to $\mathrm{L}_n(2)$. Moreover, if the quotient is $\mathrm{L}_n(2)$, then the quotient map is the natural map postcomposed with an automorphism of $\mathrm{L}_n(2)$.*

The natural map $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$ is obtained by acting on $\mathrm{H}_1(F_n; \mathbb{Z}/2\mathbb{Z})$.

Zimmermann [Zim2] also obtained a partial solution to the corresponding conjecture for mapping class groups, but in general the question of the cardinality of the smallest non-abelian quotients of mapping class groups is still open.

Let us remark here that, even though the main result deals specifically with the smallest non-trivial quotient of $\mathrm{SAut}(F_n)$, the techniques developed in this paper

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can be used to study other finite quotients and so yield information on normal finite index subgroups of $\mathrm{SAut}(F_n)$ in general.

In order to determine the smallest non-trivial quotient of $\mathrm{SAut}(F_n)$ we can restrict our attention to the finite simple groups which, by the Classification of Finite Simple Groups (CFSG), fall into one of the following four families:

(1) the cyclic groups of prime order;
(2) the alternating groups A_n, for $n \geqslant 5$;
(3) the finite groups of Lie type, and;
(4) the 26 sporadic groups.

For the full statement of the CFSG we refer the reader to [CCN$^+$, Chapter 1] and for a more detailed exposition of the non-abelian finite simple groups to [Wil]. For the purpose of this paper, we further divide the finite groups of Lie type into the following two families:

(3C) the "classical groups": $\mathtt{A}_n$, ${}^2\mathtt{A}_n$, $\mathtt{B}_n$, $\mathtt{C}_n$, $\mathtt{D}_n$ and ${}^2\mathtt{D}_n$, and;
(3E) the "exceptional groups": ${}^2\mathtt{B}_2$, ${}^2\mathtt{G}_2$, ${}^2\mathtt{F}_4$, ${}^3\mathtt{D}_4$, ${}^2\mathtt{E}_6$, $\mathtt{G}_2$, $\mathtt{F}_4$, $\mathtt{E}_6$, $\mathtt{E}_7$ and $\mathtt{E}_8$.

We turn first to the alternating groups and prove the following.

Theorem 3.16. *Let $n \geqslant 3$. Any action of $\mathrm{SAut}(F_n)$ on a set with fewer than $k(n)$ elements is trivial, where*

$$k(n) = \begin{cases} 7 & n=3 \\ 8 & n=4 \\ 12 \quad \text{if} & n=5 \\ 14 & n=6 \end{cases}$$

and $k(n) = \max_{r \leqslant \frac{n}{2}-3} \min\{2^{n-r-p(n)}, \binom{n}{r}\}$ for $n \geqslant 7$, where $p(n)$ equals 0 when n is odd and 1 when n is even.

The bound given above for $n \geqslant 7$ is somewhat mysterious; one can however easily see that (for large n) it is bounded below by $2^{\frac{n}{2}}$.

Note that, so far, no such result was available for $\mathrm{SAut}(F_n)$ (one could extract a bound of $2n$ from the work of Bridson–Vogtmann [BV1]). Clearly, the bounds given above give precisely the same bounds for $\mathrm{SOut}(F_n)$. In this context the best bound known so far was $\frac{1}{2}\binom{n+1}{2}$ (for $n \geqslant 6$). It was obtained by the second-named author in [Kie2, Corollary 2.24] by an argument of representation theoretic flavour. The proof contained in the current paper is more direct.

The question of the smallest set on which $\mathrm{SAut}(F_n)$ or $\mathrm{SOut}(F_n)$ can act non-trivially remains open, but we do answer the question on the growth of the size of such a set with n – it is exponential. Note that the corresponding question for mapping class groups has been answered by Berrick–Gebhardt–Paris [BGP].

Let us remark here that $\mathrm{Out}(F_n)$ (and hence also $\mathrm{SAut}(F_n)$) has plenty of alternating quotients – indeed, it was shown by Gilman [Gil] that $\mathrm{Out}(F_n)$ is residually alternating.

We use the bounds above to improve on a previous result of the second-named author on rigidity of outer actions of $\mathrm{Out}(F_n)$ on free groups (see Theorem 3.19 for details).

Following the alternating groups, we rule out the sporadic groups. It was observed by Bridson–Vogtman [BV1] that any quotient of $\mathrm{SAut}(F_n)$ which does not factor through $\mathrm{SL}_n(\mathbb{Z})$ must contain a subgroup isomorphic to

$$(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes A_n = 2^{n-1} \rtimes A_n$$

(one can easily see this subgroup inside of $\mathrm{SAut}(F_n)$, as it acts on the n-rose, that is the bouquet of n circles). Thus, for large enough n, sporadic groups are never quotients of $\mathrm{SAut}(F_n)$, and therefore our proof (asymptotically) is not sensitive to whether the list of sporadic groups is really complete.

Finally we turn to the finite groups of Lie type. Our strategy differs depending on whether we are dealing with the classical or exceptional groups. The exceptional groups are handled in a similar fashion to the sporadic groups – this time we use an alternating subgroup A_{n+1} inside $\mathrm{SAut}(F_n)$, which rigidifies the group in a similar way as the subgroup $2^{n-1} \rtimes A_n$ did. The degrees of the largest alternating subgroups of exceptional groups of Lie type are known (and listed for example in [LS]); in particular this degree is bounded above by 17 across all such groups.

The most involved part of the paper deals with the classical groups. In characteristic 2 we use an inductive strategy, and prove

Theorem 6.9. *Let $n \geqslant 3$. Let K be a finite group of Lie type in characteristic 2 of twisted rank less than $n-1$, and let $\overline{K}$ be a reductive algebraic group over an algebraically closed field of characteristic 2 of rank less than $n-1$. Then any homomorphism $\mathrm{Aut}(F_n) \to K$ or $\mathrm{Aut}(F_n) \to \overline{K}$ has abelian image, and any homomorphism $\mathrm{SAut}(F_{n+1}) \to K$ or $\mathrm{SAut}(F_{n+1}) \to \overline{K}$ is trivial.*

Note that there are precisely two abelian quotients of $\mathrm{Aut}(F_n)$ (when $n \geqslant 3$), namely $\mathbb{Z}/2\mathbb{Z} = 2$ and the trivial group.

In odd characteristic we need to investigate the representation theory of $\mathrm{SAut}(F_n)$. We prove

Theorem 7.12. *Let $n \geqslant 8$. Every irreducible projective representation of $\mathrm{SAut}(F_n)$ of dimension less than $2n-4$ over a field of characteristic greater than 2 which does not factor through the natural map $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$ has dimension $n+1$.*

Note that over a field of characteristic greater than $n+1$, every linear representation of $\mathrm{Out}(F_n)$ of dimension less than $\binom{n+1}{2}$ factors through the map $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ mentioned above (see [Kie1, 3.13]). Representations of $\mathrm{SAut}(F_n)$ over characteristic other than 2 have also been studied by Varghese [Var].

The proof of the main result (Theorem 9.1) for $n \geqslant 8$ is uniform; the small values of n need special attention, and we deal with them at the end of the paper. We also need a number of computations comparing orders of various finite groups; these can be found in the appendix.

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2. Preliminaries

2.1. **Notation.** We recall a few conventions which we use frequently. When it is clear, the prime number p will denote the cyclic group of that order. Furthermore, the elementary abelian group of order p^n will be denoted as p^n. These conventions are standard in finite group theory. We also follow Artin's convention, and use $\mathrm{L}_n(q)$ to denote PSL_n over the field of cardinality q.

We conjugate on the right, and use the following commutator convention

$$[g,h] = ghg^{-1}h^{-1}$$

The abstract symmetric group of degree n is denoted by S_n. Given a set I, we define $\mathrm{Sym}(I)$ to be its symmetric group. We define A_n and $\mathrm{Alt}(I)$ in the analogous manner for the alternating groups.

We fix n and denote by N the set $\{1, \dots, n\}$.

2.2. **Some subgroups and elements of** $\mathrm{Aut}(F_n)$**.** We start by fixing n and a free generating set $a_1, \ldots, a_n$ for the free group F_n. Recall that $N = \{1, \ldots, n\}$. We will abuse notation by writing $F_n = F(N)$, and given a subset $I \subseteq N$ we will write $F(I)$ for the subgroup of $F(N)$ generated by the elements a_i with $i \in I$.

For every $i, j \in N$, $i \neq j$, set

$$\rho_{ij}(a_k) = \begin{cases} a_i a_j & \text{if} \quad k = i \\ a_k & \text{otherwise} \end{cases}$$

$$\lambda_{ij}(a_k) = \begin{cases} a_j a_i & \text{if} \quad k = i \\ a_k & \text{otherwise} \end{cases}$$

$$\sigma_{ij}(a_k) = \begin{cases} a_j & \quad k = i \\ a_i & \text{if} \quad k = j \\ a_k & \quad k \notin \{i, j\} \end{cases}$$

$$\sigma_{i(n+1)}(a_k) = \begin{cases} {a_i}^{-1} & \text{if} \quad k = i \\ a_k {a_i}^{-1} & \text{otherwise} \end{cases}$$

$$\epsilon_i(a_k) = \begin{cases} {a_i}^{-1} & \text{if} \quad k = i \\ a_k & \text{otherwise} \end{cases}$$

$$\delta(a_k) = \quad {a_k}^{-1} \quad \text{for every} \quad k$$

All of the endomorphisms of F_n defined above are in fact elements of $\mathrm{Aut}(F_n)$. The elements ρ_{ij} are the *right transvections*, the elements λ_{ij} are the *left transvections*, and the set of all transvections generates $\mathrm{SAut}(F_n)$.

The involutions ϵ_i pairwise commute, and hence generate 2^n inside $\mathrm{Aut}(F_n)$. We have $2^{n-1} = 2^n \cap \mathrm{SAut}(F_n)$. When talking about 2^n or 2^{n-1} inside $\mathrm{Aut}(F_n)$, we will always mean these subgroups.

The elements σ_{ij} with $i, j \in N$ generate a symmetric group S_n. Each of the sets

$$\{\epsilon_i \mid i \in N\}, \{\rho_{ij} \mid i, j \in N\} \text{ and } \{\lambda_{ij} \mid i, j \in N\}$$

is preserved under conjugation by elements of S_n, and the left conjugation coincides with the natural action by S_n on the indices. We have

$$A_n = S_n \cap \mathrm{SAut}(F_n)$$

The elements σ_{ij} with $i, j \in N \cup \{n+1\}$ generate a symmetric group S_{n+1}. Again, we have $A_{n+1} = S_{n+1} \cap \mathrm{SAut}(F_n)$. Again, when we talk about A_n, S_n, A_{n+1} or S_{n+1} inside $\mathrm{Aut}(F_n)$, we mean these subgroups.

Since the symmetric group S_n acts on 2^n by permuting the indices of the elements ϵ_i, we have $2^n \rtimes S_n < \mathrm{Aut}(F_n)$ (note that this is the Coxeter group of type $\mathtt{B}_n$). As usual, we will refer to this specific subgroup as $2^n \rtimes S_n$. Clearly, $\mathrm{SAut}(F_n) \cap 2^n \rtimes S_n$ contains $2^{n-1} \rtimes A_n$. We will denote this subgroup as D_n' (since it is isomorphic to the derived subgroup of the Coxeter group of type $\mathtt{D}_n$ when $n \geqslant 5$; note that we have no such isomorphism for $n = 4$). Note that 2^{n-1} inside D_n' is generated by the elements $\epsilon_i \epsilon_j$ with $i \neq j$.

Lemma 2.1. *Let $n \geqslant 3$. Then the normal closure of A_n in D_n' is the whole of D_n'.*

Proof.

$$\epsilon_1 \epsilon_2 = [\epsilon_1 \epsilon_3, \sigma_{13} \sigma_{12}] \in \langle\langle A_n \rangle\rangle$$

and so every $\epsilon_i \epsilon_j$ lies in the normal closure of A_n as well, since A_n acts transitively on unordered pairs in N. □

The Nielsen Realisation theorem for free groups (proved independently by Culler [Cul], Khramtsov [Khr] and Zimmermann [Zim1]) states that any finite subgroup of $\mathrm{Aut}(F_n)$ can be seen as a group of basepoint preserving automorphisms of a graph with fundamental group identified with F_n. From this point of view, the subgroup

$2^n \rtimes S_n$ is the automorphism group of the n-rose (the bouquet of n circles), and the subgroup S_{n+1} is the basepoint preserving automorphism group of the $n+1$-cage graph (a graph with two vertices and $n+1$ edges connecting them).

Remark 2.2. Throughout, we are going to make extensive use of the Steinberg commutator relations in $\mathrm{Aut}(F_n)$, that is

$$\rho_{ij}{}^{-1} = [\rho_{ik}{}^{-1}, \rho_{kj}{}^{-1}]$$

and

$$\lambda_{ij}{}^{-1} = [\lambda_{ik}{}^{-1}, \lambda_{kj}{}^{-1}]$$

We will also use

$$\rho_{ij}{}^{\delta} = \lambda_{ij}$$

Another two types of relations which we will frequently encounter are already present in the proof of the following lemma (based on observations of Bridson–Vogtmann [BV1]).

Lemma 2.3. *For $n \geqslant 3$, all automorphisms $\rho_{ij}{}^{\pm 1}$ and $\lambda_{ij}{}^{\pm 1}$ (with $i \neq j$) are conjugate inside* $\mathrm{SAut}(F_n)$.

Proof. Observe that

$$\rho_{ij}{}^{\epsilon_i \epsilon_j} = \lambda_{ij}$$

and that

$$\rho_{ij}{}^{\epsilon_j \epsilon_k} = \rho_{ij}{}^{-1}$$

where $k \notin \{i, j\}$.

When $n \geqslant 4$, the subgroup A_n acts transitively on ordered pairs in N, and so we are done.

Let us suppose that $n = 3$. In this case, using the 3-cycle $\sigma_{12}\sigma_{23}$, we immediately see that ρ_{12}, ρ_{23} and ρ_{31} are all conjugate. We also have

$$\rho_{12}{}^{\sigma_{12}\epsilon_3} = \rho_{21}$$

and thus all right transvections are conjugate, and we are done. □

We also note the following useful fact, following from Gersten's presentation of $\mathrm{SAut}(F_n)$ [Ger].

Proposition 2.4. *For every $n \geqslant 3$, the group $\mathrm{SAut}(F_n)$ is perfect.*

2.3. **Linear quotients.** Observe that abelianising the free group F_n gives us a map

$$\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$$

This homomorphism is in fact surjective, since each elementary matrix in $\mathrm{SL}_n(\mathbb{Z})$ has a transvection in its preimage. We will refer to this map as the natural homomorphism $\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$.

The finite quotients of $\mathrm{SL}_n(\mathbb{Z})$ (for $n \geqslant 3$) are controlled by the congruence subgroup property as proven by Mennicke [Men]). In particular, noting that $\mathrm{SAut}(F_n)$ is perfect (Proposition 2.4), we conclude that the non-trivial simple quotients of $\mathrm{SL}_n(\mathbb{Z})$ are the groups $\mathrm{L}_n(p)$ where p ranges over all primes. The smallest one is clearly $\mathrm{L}_n(2)$.

We will refer to the compositions of the natural map $\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$ and the quotient maps $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{L}_n(p)$ as natural maps as well.

We will find the following observations (due to Bridson–Vogtmann [BV1]) most useful.

Lemma 2.5. *Let $n \geqslant 3$, and let ϕ be a homomorphism with domain* $\mathrm{SAut}(F_n)$.

(1) *If n is even, and $\phi(\delta)$ is central in* $\operatorname{im}\phi$*, then ϕ factors through the natural map* $\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$.
(2) *For any n, if there exists $\xi \in 2^{n-1} \smallsetminus \{1, \delta\}$ such that $\phi(\xi)$ is central in* $\operatorname{im}\phi$*, then ϕ factors through the natural map* $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$.
(3) *For any n, if there exists $\xi \in D'_n \smallsetminus 2^{n-1}$ such that $\phi(\xi)$ is central in* $\operatorname{im}\phi$*, then ϕ is trivial.*

Proof. (1) We have

$$\delta\rho_{ij}\delta = \lambda_{ij}$$

for every i, j. Thus, ϕ factors through the group obtained by augmenting Gersten's presentation [Ger] of $\mathrm{SAut}(F_n)$ by the additional relations $\rho_{ij} = \lambda_{ij}$. But this is equivalent to Steinberg's presentation of $\mathrm{SL}_n(\mathbb{Z})$.

(2) We claim that there exists $\tau \in A_n$ such that

$$[\xi, \tau] = \epsilon_i\epsilon_j$$

We have $\xi = \prod_{i \in I} \epsilon_i$ for some $I \subset N$. Take $i \in I$ and $j \notin I$. Suppose first that there exist distinct α, β either in $I \smallsetminus \{i\}$ or in $N \smallsetminus (I \cup \{j\})$. Then $\tau = \sigma_{ij}\sigma_{\alpha\beta}$ is as claimed.

If no such α and β exist, then $n \leqslant 4$ and $\xi = \epsilon_i\epsilon_{i'}$ for some $i' \neq i$. Thus we may take $\tau = \sigma_{ii'}\sigma_{ji}$. This proves the claim

Now $\phi([\xi, \tau]) = 1$ since $\phi(\xi)$ is central. Using the action of A_n on 2^{n-1} we immediately conclude that $2^{n-1} \leqslant \ker\phi$. Thus we have

$$\phi(\rho_{ij}) = \phi(\rho_{ij})^{\phi(\epsilon_i\epsilon_j)} = \phi({\rho_{ij}}^{\epsilon_i\epsilon_j}) = \phi(\lambda_{ij})$$

Now Gersten's presentation tells us that ϕ factors through the natural map

$$\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$$

Moreover,

$$\phi(\rho_{ij}) = \phi(\rho_{ij})^{\phi(\epsilon_j\epsilon_k)} = \phi({\rho_{ij}}^{\epsilon_j\epsilon_k}) = \phi({\rho_{ij}}^{-1})$$

where $k \notin \{i, j\}$. The result of Mennicke [Men] tells us that in this case ϕ factors further through $\mathrm{SL}_n(Z) \to \mathrm{L}_n(2)$.

(3) We write $\xi = \xi'\tau$, where $\xi' \in 2^{n-1}$ and $\tau \in A_n$.

Suppose first that τ is not a product of commuting transpositions. Then, without loss of generality, we have

$${\rho_{12}}^{\xi} = {x_{23}}^{\pm 1}$$

where $x \in \{\rho, \lambda\}$. Now

$$\phi({\rho_{13}}^{-1}) = \phi([{\rho_{12}}^{-1}, {\rho_{23}}^{-1}]) = [\phi(x_{23})^{\mp 1}, \phi(\rho_{23})^{-1}] = 1$$

as ${x_{23}}^{\pm 1}$ commutes with ρ_{23}. This trivialises ϕ, since $\mathrm{SAut}(F_n)$ is generated by transvections, and every two transvections are conjugate.

Now suppose that τ is a product of commuting transpositions. Then $n \geqslant 4$, and without loss of generality

$${\rho_{12}}^{\xi} = {x_{34}}^{\pm 1}$$

where $x \in \{\rho, \lambda\}$. Now

$$\phi({\rho_{14}}^{-1}) = \phi([{\rho_{12}}^{-1}, {\rho_{24}}^{-1}]) = [\phi(x_{34})^{\mp 1}, \phi(\rho_{24})^{-1}] = 1$$

as ${x_{34}}^{\pm 1}$ commutes with ρ_{24}. This trivialises ϕ as before. □

Remark 2.6. In (2), we can draw the same conclusion if we assume that

$$\phi(\rho_{ij}) = \phi(\lambda_{ij}) = \phi(\rho_{ij})^{-1}$$

Corollary 2.7. *Let $\phi\colon \mathrm{SAut}(F_n) \to K$ be a homomorphism. If K is finite and $\phi|_{D'_n}$ is not injective, then*

(1) *ϕ is trivial; or*
(2) *$|K| > |\mathrm{L}_n(2)|$; or*
(3) *$K \cong \mathrm{L}_n(2)$ and ϕ is the natural map up to postcomposition with an automorphism of $\mathrm{L}_n(2)$.*

Many parts of the current paper are inductive in nature, and they are all based on the following observation.

Lemma 2.8. *For any $k \leqslant n+1$, the group $\mathrm{SAut}(F_n)$ contains an element ξ of order k whose centraliser contains $\mathrm{SAut}(F_{n-k})$. When k is odd then the centraliser contains $\mathrm{SAut}(F_{n-k+1})$. Moreover, when $k \geqslant 5$ then the normal closure of ξ is the whole of $\mathrm{SAut}(F_n)$.*

Proof. Let Γ be a graph with vertex set $\{u, v\}$ and edge set consisting of k distinct edges connecting u to v and $n-k+1$ distinct edges running from v to itself (see Figure 2.9). Let K denote the set of the former k edges. We identify the fundamental group of Γ with F_n. Elements of $\mathrm{SAut}(F_n)$ then correspond to based homotopy equivalences of Γ.

Let ξ denote the automorphism of Γ which cyclically permutes the edges of K. When k is odd the action of ξ on the remaining edges is trivial; when k is even, then ξ acts on $n-k$ of the remaining edges trivially, but it flips the last edge. It is clear that ξ defines an element of $\mathrm{SAut}(F_n)$ of order k. Also, ξ fixes pointwise a free factor of F_n of rank $n-k$ when k is even or $n-k+1$ when k is odd.

Observe that we have a copy of the alternating group A_k permuting the edges in K. Suppose that $k \geqslant 5$, and let τ denote some 3-cycle in A_k. It is obvious that $[\xi, \tau]$ is a non-trivial element of A_k; it is also clear that A_k is simple. These two facts imply that A_k lies in the normal closure of ξ. But this A_k contains an A_{k-1} contained in the standard $A_n < \mathrm{SAut}(F_n)$, and the result follows by Lemma 2.5(3). □

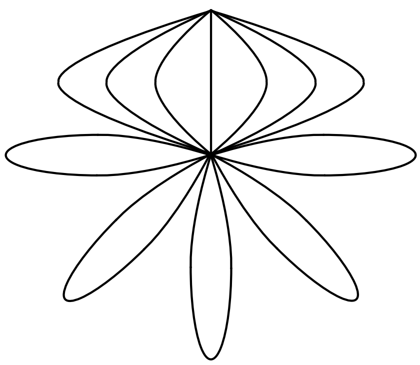

FIGURE 2.9. The graph Γ with $(n, k) = (11, 7)$

One can easily give an algebraic description of the element ξ above; in fact we will do this for $k = 3$ when we deal with classical groups in characteristic 3 in Section 7.1.

3. Alternating groups

In this section we will give lower bounds on the cardinality of a set on which the groups $\mathrm{SAut}(F_n)$ can act non-trivially. The cases $n \in \{3, \dots, 8\}$ are done in a somewhat ad-hoc manner, and we begin with these, developing the necessary tools along the way. We will conclude the section with a general result for $n \geqslant 9$.

As a corollary, we obtain that alternating groups are never the smallest quotients of $\mathrm{SAut}(F_n)$ (for $n \geqslant 3$), with a curious exception for $n = 4$, since in this case the (a fortiori smallest) quotient $\mathrm{L}_4(2)$ is isomorphic to the alternating group A_8.

Lemma 3.1 ($n = 3$)**.** *Any action of* $\mathrm{SAut}(F_3)$ *on a set* X *with fewer than* 7 *elements is trivial.*

This result can be easily verified using GAP. For this reason, we offer only a sketch proof.

Sketch of proof. The action gives us a homomorphism $\phi\colon \mathrm{SAut}(F_3) \to S_6$. Since $\mathrm{SAut}(F_3)$ is perfect, the image lies in A_6.

Consider the set of transvections

$$T = \{{\rho_{ij}}^{\pm 1}, {\lambda_{ij}}^{\pm 1}\}$$

By Lemma 2.3 all elements in T are conjugate in $\mathrm{SAut}(F_3)$, and hence also in the image of ϕ.

Consider the equivalence relation on T where elements $x, y \in T$ are equivalent if $\phi(x) = \phi(y)$. It is clear that the equivalence classes are equal in cardinality. We first show that if any (hence each) of these equivalence classes has cardinality at least 3, then ϕ is trivial. Let Φ be such a class. Without loss of generality we assume that $\rho_{12} \in \Phi$.

Suppose that

$$|\Phi \cap \{{\rho_{12}}^{\pm 1}, {\lambda_{12}}^{\pm 1}\}| \geqslant 3$$

Independently of which 3 of the 4 elements lie in Φ, their image under ϕ is centralised by $\phi(\epsilon_1\epsilon_2)$, and so in fact all four of these elements lie in Φ.

Now, we have

$$\phi(\rho_{12}) = \phi({\rho_{12}}^{-1}) = \phi(\lambda_{12}) = \phi({\lambda_{12}}^{-1})$$

In this case we conclude from Remark 2.6 that ϕ factors through $\mathrm{L}_3(2)$. But $\mathrm{L}_3(2)$ is a simple group containing an element of order 7, and so ϕ is trivial.

If $|\Phi \cap \{{\rho_{12}}^{\pm 1}, {\lambda_{12}}^{\pm 1}\}| < 3$ then there exists an element ${x_{ij}}^{\pm 1} \in \Phi$ with x being either ρ or λ, and with $(i, j) \neq (1, 2)$. If $(i, j) = (1, 3)$ then

$$\phi({\rho_{13}}^{-1}) = \phi([{\rho_{12}}^{-1}, {\rho_{23}}^{-1}]) = [\phi(x_{13})^{\mp 1}, \phi(\rho_{23})^{-1}] = 1$$

which implies that ϕ is trivial. We proceed in a similar fashion for all other values of (i, j). This way we verify our claim that if ϕ is non-trivial then $|\Phi| \leqslant 2$.

There are exactly 10 elements in T which commute with ρ_{12}, namely

$$C_T(\rho_{12}) = \{{\rho_{12}}^{\pm 1}, {\lambda_{12}}^{\pm 1}, {\lambda_{13}}^{\pm 1}, {\rho_{32}}^{\pm 1}, {\lambda_{32}}^{\pm 1}\}$$

Using what we have learned above about the cardinality of Φ, we see that ϕ is not trivial only if the conjugacy class of $\phi(\rho_{12})$ in A_6 contains at least 5 elements commuting with $\phi(\rho_{12})$. We also know that this conjugacy class has to contain $\phi(\rho_{12})^{-1}$. By inspection we see that the only such conjugacy class in A_6 is that of $\tau = (12)(34)$.

The conjugacy class of τ has exactly 5 elements, say $\{\tau, \tau_1, \tau_2, \tau_1', \tau_2'\}$, and the elements τ_i and τ_j' do not commute for any $i, j \in \{1, 2\}$. Hence the maximal subset of the conjugacy class of τ in which all elements pairwise commute is of cardinality 3. But in $C_T(\rho_{12})$ we have 8 such elements, namely

$$\{{\rho_{12}}^{\pm 1}, {\lambda_{12}}^{\pm 1}, {\rho_{32}}^{\pm 1}, {\lambda_{32}}^{\pm 1}\}$$

This implies that $|\Phi| > 2$, which forces ϕ to be trivial. □

Note that $\mathrm{L}_3(2)$ acts non-trivially on the set of non-zero vectors in 2^3 (thought of as a vector space) which has cardinality 7, and so $\mathrm{SAut}(F_3)$ has a non-trivial action on a set of 7 elements. Thus the result above is sharp.

Lemma 3.2. *Let $n \geqslant 4$. Suppose that* $\mathrm{SAut}(F_n)$ *acts non-trivially on a set X so that $\epsilon_1\epsilon_2$ acts trivially. Then $|X| \geqslant 2^{n-1}$.*

Proof. When $\epsilon_1\epsilon_2$ acts trivially then the action factors through the natural map $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$ by Lemma 2.5. Now, $\mathrm{L}_n(2)$ is simple and cannot act on a set smaller than 2^{n-1} non-trivially, by [KL, Theorem 5.2.2]. □

Lemma 3.3 ($n = 4$)**.** *Any action of* $\mathrm{SAut}(F_4)$ *on a set X with fewer than* 8 *elements is trivial.*

Proof. The group D_4' cannot act faithfully on fewer than 8 points, and the group $D_4'/\langle\delta\rangle$ cannot act faithfully on fewer than 12 points (both of these facts can be easily checked). Thus we are done by Lemma 2.5, since $\mathrm{L}_4(2) \cong A_8$ cannot act on fewer than 8 points. □

Note that in this case the result is also sharp, since we have the epimorphism

$$\mathrm{SAut}(F_4) \to \mathrm{L}_4(2) \cong A_8$$

Lemma 3.4. *Let $n \geqslant 1$. Let D_n' act on a set of cardinality less than $2^{n-1-p(n)}$ where $p(n) = 0$ for n odd, and $p(n) = 1$ for n even. Then 2^{n-1} acts trivially on every point fixed by A_n.*

Proof. Let x be a point fixed by A_n, and consider the 2^{n-1}-orbit thereof. Such an orbit corresponds to a subgroup of 2^{n-1} normalised by A_n. It cannot correspond to the trivial subgroup, as then the orbit would be too large. For the same reason it cannot correspond to the subgroup generated by δ (which is a subgroup when n is even). It is easy to see that the only remaining subgroup is the whole of 2^{n-1}, and so the action on x is trivial. □

Lemma 3.5. *Let $n \geqslant 4$. Suppose that* $\mathrm{SAut}(F_n)$ *acts on a set X of cardinality less than $2^{n-1-p(n)}$, where $p(n)$ is as above, in such a way that A_n has at most one non-trivial orbit. Then* $\mathrm{SAut}(F_n)$ *acts trivially on X.*

Proof. By Lemma 3.4, every point fixed by A_n is also fixed by 2^{n-1}. This implies that every A_n-orbit is already a D_n'-orbit, and so every point in X is fixed by 2^{n-1}, and thus the action of $\mathrm{SAut}(F_n)$ on X is trivial by Lemma 3.2. □

For higher values of n we will use the fact that actions of alternating groups on small sets are well-understood. Recall that $A_n = \mathrm{Alt}(N)$ denotes the group of even permutations of the set $N = \{1, \ldots, n\}$.

Definition 3.6. We say that a transitive action of $A_n = \mathrm{Alt}(N)$ on a set X is *associated to k* (with $k \in \mathbb{N}$) if for every (or equivalently any) $x \in X$ the stabiliser of x in A_n contains $\mathrm{Alt}(J)$ with $J \subseteq N$ and $|J| = k$, and does not contain $\mathrm{Alt}(J')$ for any $J' \subseteq N$ of larger cardinality. Notice that $k \geqslant 2$.

In the following lemma we write A_{n-1} for the subgroup $\mathrm{Alt}(N \smallsetminus \{n\})$ of $A_n = \mathrm{Alt}(N)$.

Lemma 3.7. *Let $n \geqslant 6$. Suppose that we are given a transitive action π of A_n on a set X which is associated to k. Then*

1. *the action of A_{n-1} on each of its orbits is associated to k or $k-1$;*
2. *if $k > \frac{n}{2}$ then there is exactly one orbit of A_{n-1} of the latter kind;*
3. *if $k > \frac{n}{2}$ then any other transitive action π' of A_n on a set X' isomorphic to $\pi|_{A_{n-1}}$ when restricted to A_{n-1} is isomorphic to π.*

Proof. Before we start, let us make an observation: let I and J be two subsets of N of cardinality at least 3 each, and such that $I \cap J \neq \emptyset$. Then

$$\langle \mathrm{Alt}(I), \mathrm{Alt}(J) \rangle = \mathrm{Alt}(I \cup J)$$

There are at least two quick ways of seeing it: the subgroup $\langle \mathrm{Alt}(I), \mathrm{Alt}(J) \rangle$ clearly contains all 3-cycles; the subgroup $\langle \mathrm{Alt}(I), \mathrm{Alt}(J) \rangle$ acts 2-transitively, and so primitively, on $I \cup J$, and contains a 3-cycle, which allows us to use Jordan's theorem.

(1) Let x be an element in an A_{n-1}-orbit O. If the stabiliser S of x in A_n contains $\mathrm{Alt}(J)$ with $|J| = k$ and $n \notin J$, then $\mathrm{Alt}(J) \subseteq A_{n-1}$ and so the action of A_{n-1} on O is associated to at least k. It is clear that the action cannot be associated to any integer greater than k.

If n is in J for every $J \subseteq N$ of size k with $\mathrm{Alt}(J) \subseteq S$, then $\mathrm{Alt}(J \smallsetminus \{n\})$ is contained in $S \cap A_{n-1}$ and the action of A_{n-1} on O is associated with at least $k-1$. It is clear that this action cannot be associated to any integer greater than $k-1$.

(2) Clearly, there exists $x \in X$ such that its stabiliser S in A_n contains $\mathrm{Alt}(J)$ with $|J| = k$ and $n \in J$. Note that J is unique – if there were another subset $I \subseteq N$ with $|I| = k$ and $\mathrm{Alt}(I) \leqslant S$, then $I \cap J$ would need to intersect non-trivially (as $k > \frac{n}{2}$), and so we would have $\mathrm{Alt}(I \cup J) \leqslant S$. Hence we may conclude from the proof of (1) that the action of A_{n-1} on O, its orbit of x, is associated to $k-1$.

Now suppose that there exists a point $x' \in X$ with A_{n-1}-orbit O', stabiliser S' in A_n, and subset $J' \subseteq N$ of cardinality k with $n \in J'$ and $\mathrm{Alt}(J') \leqslant S'$. There exists $\tau \in A_n$ such that

$$x = \tau.x'$$

and so $S' = S^\tau$. Thus $\tau(J') = J$, and therefore there exists $\sigma \in \mathrm{Alt}(J)$ such that $\sigma\tau(n) = n$. But then also $\sigma^{-1}.x = x$ and so $x = \sigma\tau.x'$. But $\sigma\tau \in A_{n-1}$, and therefore $O = O'$.

(3) Let us start by looking at π'. By assumption, this action is associated to at least $k-1$, since it is when restricted to A_{n-1}. It also cannot be associated to any integer larger than k, since then (2) would forbid the existence of an A_{n-1}-orbit in X' associated to $k-1$, and we know that such an orbit exists.

If $k-1 > \frac{n}{2}$ then (2) implies that $\pi'|_{A_{n-1}}$ has an orbit that is associated to $k-1$, but clearly none associated to $k-2$. Thus, by (2), π' is associated to k.

If $k-1 \leqslant \frac{n}{2}$ then in particular $k \neq n$, and so there is an A_{n-1}-orbit in X associated to k, and therefore π' cannot be associated to $k-1$. We conclude that π' is associated to k.

Pick an $x \in X$ so that the A_{n-1}-action on the A_{n-1} orbit O of x is associated to $k-1$. Let $\theta \colon X \to X'$ be a A_{n-1}-equivariant bijection (which exists by assumption). Let $x' = \theta(x)$. Let S denote the stabiliser of x in A_n, and S' the stabiliser of x' in A_n.

We have $\mathrm{Alt}(J) \times G = H \leqslant S$, where $|J| = k$, the index $|S : H|$ is at most 2, and $G \leqslant \mathrm{Alt}(N \smallsetminus J)$ – we need to observe that J is unique, as proven above. Since the A_{n-1}-orbit of x is associated to $k-1$, we see that $n \in J$.

Now the stabiliser of x' in A_{n-1} is equal to $S \cap A_{n-1}$, and S' contains $S \cap A_{n-1}$ and some $\mathrm{Alt}(J')$ with $|J'| = k$. Since $k > \frac{n}{2}$, the subsets J and J' intersect, and so $\mathrm{Alt}(J \cup J') \leqslant S'$ as before. But the A_{n-1}-action on its orbit of x' is associated to $k-1$, and so we must have $J = J'$. This implies that $S' = S$, since the index of H in S is equal to the index of $H \cap A_{n-1}$ in $S \cap A_{n-1}$. □

Theorem 3.8 (Dixon–Mortimer [DM, Theorem 5.2A])**.** *Let $n \geqslant 5$, and let $r \leqslant n/2$ be an integer. Let $H < A_n = \mathrm{Alt}(N)$ be a proper subgroup of index less than $\binom{n}{r}$. Then one of the following holds:*

(1) *The subgroup H contains a subgroup A_{n-r+1} of A_n fixing $r-1$ points in N.*
(2) *We have $n = 2m$ and $|A_n : H| = \frac{1}{2}\binom{n}{m}$. Moreover, H contains the product $A_m \times A_m$.*
(3) *The pair $(n, |A_n : H|)$ is one of the six exceptional cases:*

$$(5,6), (6,6), (6,15), (7,15), (8,15), (9,120)$$

Note that the original theorem contains more information in each of the cases; for our purposes however, the above version will suffice.

We can rephrase (1) by saying that the action $A_n \curvearrowright A_n/H$ is associated to at least $n - r + 1$.

Corollary 3.9 ($n = 5$). *Every action of* $\mathrm{SAut}(F_5)$ *on a set X of cardinality less than* 12 *is trivial.*

Proof. By Theorem 3.8, the only orbits of A_6 in X are of cardinality 1, 6 or 10. There can be at most one orbit of size greater than 1, and it contains at most two non-trivial orbits of A_5, since such orbits have cardinality at least 5. If there is at most one non-trivial A_5-orbit, we invoke Lemma 3.5. Otherwise, let x be a point on which A_6 acts non-trivially; its A_6-orbit consists of 10 points. We know from case (2) of Theorem 3.8 that it is fixed by two commuting 3-cycles. The A_5-orbit of x has cardinality 5, and so it is the natural A_5-orbit (by Theorem 3.8 again). Thus x is fixed by some standard A_4. But now any standard A_4 together with any two commuting 3-cycles generates A_6, and so x is fixed by A_6, which is a contradiction. □

Corollary 3.10 ($n = 6$). *Every action of* $\mathrm{SAut}(F_6)$ *on a set X of cardinality less than* 14 *is trivial.*

Proof. By Theorem 3.8, the only orbits of A_7 in X are either trivial or the natural orbits of size 7. There can be at most one such natural orbit, and so A_6 has at most one non-trivial orbit. We now invoke Lemma 3.5. □

We now begin the preparations towards the main tool in this section.

Lemma 3.11. *Let $n \geqslant 2$, and let $r \leqslant n/2$ be a positive integer. Let D_n' act on a set X of cardinality less than $\binom{n}{r}$, and let x be a point stabilised by*

$$\langle \{\epsilon_i \epsilon_j \mid i, j \in I\} \rangle$$

where I is a subset of N. Then I can be taken to have cardinality at least $n-r+1$, provided that

(1) *I contains more than half of the points of N; or*
(2) *I contains exactly half of the points, and x is not fixed by some $\epsilon_i\epsilon_j$ with $i, j \notin I$.*

Proof. Let S denote the stabiliser of x in 2^{n-1}. Consider a maximal (with respect to inclusion) subset J of N such that for all $i, j \in J$ we have $\epsilon_i\epsilon_j \in S$. We call such a subset a *block*. It is immediate that blocks are pairwise disjoint, and one of them, say J_0, contains I.

If the block J_0 contains more than half of the points in N (which is guaranteed to happen in the case of assumption (1)), then it is the unique largest block of S. In the case of assumption (2), the block J_0 may contain exactly half of the points, but all of the other blocks contain strictly fewer elements. Thus, again, J_0 is the unique largest block.

Since J_0 is unique, it is clear that any element $\tau \in A_n$ which does not preserve J_0 gives $S^\tau \neq S$, and so in particular $\tau.x \neq x$. Using this argument we see that X

has to contain at least $\binom{n}{n-|J_0|}$ elements. But $|X| < \binom{n}{r}$, and so $|J_0| > n - r$, and we are done. □

Definition 3.12. Let $\mathrm{SAut}(F_n)$ act on a set X. For each point $x \in X$ we define

(1) I_x to be a subset of N such that
$$\langle \{\epsilon_i \epsilon_j \mid i, j \in I_x\} \rangle$$
fixes x, and I_x has maximal cardinality among such subsets.

(2) J_x to be a subset of N such that x is fixed by
$$\mathrm{Alt}(J_x) \leqslant \mathrm{Alt}(N) = A_n$$
and J_x is of maximal cardinality among such subsets.

The following is the main technical tool of this part of the paper.

Lemma 3.13. *Let $n \geqslant 5$. Suppose that $\mathrm{SAut}(F_n)$ acts transitively on a set X in such a way that*

(1) *there exists a point $x_0 \in X$ with I_{x_0} containing more than half of the points in N; and*

(2) *for every $x \in X$ we have $|J_x| \geqslant \frac{n+3}{2}$.*

Then every point x is fixed by $\mathrm{SAut}(F(J_x))$, provided that $|X| < \min\{2^{n-r}, \binom{n}{r}\}$ for some positive integer $r < \frac{n}{2} - 1$.

Proof. Lemma 3.11 tells us immediately that I_{x_0} contains at least $\nu = n - r + 1$ points. We claim that in fact every I_x contains at least ν points.

Since the action of $\mathrm{SAut}(F_n)$ on X is transitive, and $\mathrm{SAut}(F_n)$ is generated by transvections, it is enough to prove that for every point x with I_x of size at least ν, and every transvection, the image y of x under the transvection has $|I_y| \geqslant \nu$. For concreteness, let us assume that the transvection in question is ρ_{ij} (the situation is analogous for the left transvections). Since ρ_{ij} commutes with every involution $\epsilon_\alpha \epsilon_\beta$ with $\alpha, \beta \in I_x \smallsetminus \{i, j\}$, we see that y is fixed by $\epsilon_\alpha \epsilon_\beta$. But
$$|I_x| - 2 \geqslant \nu - 2 = n - r - 1 > \frac{n}{2}$$
and so $I_y \supseteq I_x \smallsetminus \{i, j\}$ (here we use the fact that I_y is defined to be the largest block). Therefore $|I_y| \geqslant \nu$ by Lemma 3.11. We have thus established that I_x contains at least ν points for every $x \in X$.

Note that the sets I_x form a poset under inclusion. Pick an element $z \in X$ so that I_z is minimal in this poset. Let Z denote the subset of X consisting of points w with $I_w = I_z$. Now for every $i, j \in I_z$ and every $w \in Z$ we have
$$I_z \smallsetminus \{i, j\} \subseteq I_{\rho_{ij}.w}$$
since ρ_{ij} commutes with involutions $\epsilon_\alpha \epsilon_\beta$ with $\alpha, \beta \in I_z \smallsetminus \{i, j\}$ as before.

Assume there exists $k \in I_{\rho_{ij}.w} \smallsetminus I_z$. By definition of $I_{\rho_{ij}.w}$ and using the above inclusion, there exists $l \in I_z \smallsetminus \{i, j\}$ such that $\epsilon_l \epsilon_k$ fixes $\rho_{ij}.w$. But $\epsilon_l \epsilon_k$ commutes with ρ_{ij}, and hence fixes w, which forces $k \in I_z$, a contradiction. Thus
$$I_{\rho_{ij}.w} \subseteq I_z$$
Since I_z is minimal, we conclude that $\rho_{ij}.w \in Z$. An analogous argument applies to left transvections, and so Z is preserved by
$$\mathrm{SAut}(F(I_z)) = \langle \{\rho_{ij}, \lambda_{ij} \mid i, j \in I_z\} \rangle \leqslant \mathrm{SAut}(F_n)$$
But in the action of $\mathrm{SAut}(F(I_z))$ on Z the involutions $\epsilon_\alpha \epsilon_\beta$ with $\alpha, \beta \in I_z$ act trivially. Therefore this action is trivial by Lemma 3.2, since X has fewer than 2^{n-r} points and $n - r + 1 > \frac{n}{2} + 2 \geqslant 3\frac{1}{2}$. In particular, we have $I_w \subseteq J_w$ for every $w \in Z$.

Every $w \in Z$ is fixed by $\mathrm{SAut}(F(I_w))$, but also by $\mathrm{Alt}(J_w)$ by assumption. Thus, it is fixed by the subgroup of $\mathrm{SAut}(F_n)$ generated by the two subgroups. It is clear that this is $\mathrm{SAut}(F(J_w \cup I_w))$ and so we have finished the proof for points in Z. Now we also see that in fact $I_w = J_w$, since the subgroup of 2^{n-1} corresponding to J_w lies in $\mathrm{SAut}(F(J_w))$ and hence fixes w.

Let $x \in X$ be any point. Since the action of $\mathrm{SAut}(F_n)$ is transitive, there exists a finite sequence $z = x_0, x_1, \dots, x_{m-1}, x_m = x$ such that for every i there exists a transvection τ_i with $\tau_i.x_i = x_{i+1}$ (we assume as well that the elements of the sequence are pairwise disjoint). We claim that every x_i is fixed by $\mathrm{SAut}(F(J_{x_i}))$. Let i be the smallest index so that our claim is not true for x_i. As usual, for concreteness, let us assume that $\tau_{i-1} = \rho_{\alpha\beta}$. Note that we cannot have both α and β in $J_{x_{i-1}}$, since then the action of $\rho_{\alpha\beta}$ on x_{i-1} would be trivial.

Consider the intersection $(J_{x_{i-1}} \cap J_{x_i}) \smallsetminus \{\alpha, \beta\}$. By assumption, the intersection $J_{x_{i-1}} \cap J_{x_i}$ contains at least 3 points, and at most one of these points lies in $\{\alpha, \beta\}$. Thus there exist $\alpha', \beta' \in J_{x_{i-1}} \cap J_{x_i}$ such that $\rho_{\alpha'\beta'}$ commutes with $\rho_{\alpha\beta}$. The action of $\rho_{\alpha'\beta'}$ on x_{i-1} is trivial, and thus it must also be trivial on $x_i = \rho_{\alpha\beta}.x_{i-1}$. We also know that $\mathrm{Alt}(J_{x_i})$ acts trivially on x_i, and so every right transvection with indices in J_{x_i} acts trivially on x_i. This implies that x_i is fixed by $\mathrm{SAut}(F(J_{x_i}))$, which contradicts the minimality of x_i, and so proves the claim, and therefore the result. □

Proposition 3.14 ($n \geqslant 7$)**.** *Let $n \geqslant 7$. Every action of $\mathrm{SAut}(F_n)$ on a set of cardinality less than*

$$\max_{r \leqslant \frac{n}{2}-3} \min\left\{2^{n-r-p(n)}, \binom{n}{r}\right\}$$

is trivial, where $p(n)$ equals 0 when n is odd and 1 when n is even.

Proof. Let X denote the set on which we are acting. Without loss of generality we will assume that $\mathrm{SAut}(F_n)$ acts on X transitively, and that X is non-empty.

Let R denote a value of r for which

$$\max_{r \leqslant \frac{n}{2}-3} \min\left\{2^{n-r-p(n)}, \binom{n}{r}\right\}$$

is attained. Note that $R > 1$ by Lemma A.1 for $n \geqslant 8$; a direct computation shows that $R = 2$ for $n \in \{7, 8\}$.

Let us first look at the action of A_{n+1}. Since $|X| < \binom{n}{R} < \binom{n+1}{R}$, Theorem 3.8 tells us that each orbit of A_{n+1} is

(1) associated to at least $n - R + 1$; or
(2) as described in case (2) of the theorem – this is immediately ruled out, since X would have to be too large by Lemma A.2 for $n \geqslant 12$, and by direct computation for $n \in \{8, 10\}$; or
(3) one of the two exceptional actions $(8, 15)$ or $(9, 120)$ as in case (3) of the theorem.

For now let us assume that we are in case (1). Thus $|J_x| \geqslant n - R$ for each $x \in X$, and so the action of $\mathrm{SAut}(F_n)$ on X satisfies assumption (2) of Lemma 3.13. Also, by Lemma 3.7, there is at least one point $y \in X$ with $|J_y| \geqslant n - R + 1$.

Let J_0 denote a largest (with respect to cardinality) subset of N such that $\mathrm{Alt}(J_0)$ has a fixed point in X. Let x_0 be such a fixed point. Note that J_0 has at least $n - R + 1$ elements, and $|X| < 2^{n-R-p(n)}$, which implies by Lemma 3.4 that I_{x_0} contains at least $n - R + 1$ elements. As $R < \frac{n}{2}$, we conclude that the action of $\mathrm{SAut}(F_n)$ on X satisfies assumption (1) of Lemma 3.13.

We are now in position to apply Lemma 3.13. We conclude that x_0 is fixed by $\mathrm{SAut}(F(J_{x_0}))$. Let us consider the graph Γ from Figure 2.9 with $k = |J_{x_0}| + 1$

and fundamental group isomorphic to F_n. We can choose such an isomorphism so that $\mathrm{Alt}(J_{x_0})$ acts on Γ by permuting (in a natural way) all but one of the edges which are not loops. But it is clear that we also have a supergroup G of $\mathrm{Alt}(J_{x_0})$, which is isomorphic to an alternating group of rank $|J_{x_0}|+1$, and acts by permuting all such edges. By construction, $G < \mathrm{SAut}(F(J_{x_0}))$ and so $G.x_0 = x_0$. But now consider the action of G on X – by Lemma 3.7, it has to agree with the action of $\mathrm{Alt}(J')$, where J' is a superset of J_{x_0} with a single new element. By assumption, $\mathrm{Alt}(J')$ does not fix any point in X. However G does, and this is a contradiction. This implies that there is no superset J', but then we must have $J_{x_0} = N$, and so $\mathrm{SAut}(F_n)$ fixes a point in X. But the action is transitive, and so X is a single point. This proves the result.

Now let us investigate the exceptional cases. The first one occurs when $n = 7$, and the A_8-orbit of x has cardinality 15. We have $R = 2$ in this case, and so X has fewer than $\binom{7}{2} = 21$ elements. In this case, there are at most 5 points in $X \smallsetminus A_8.x$, and so the action of A_8 on each of these is trivial. Thus A_7 also fixes these points.

Now consider the action of A_7 on $A_8.x$. Since A_7 cannot fix any point here, and the smallest orbit of A_7 of size other than 1 and 7 has to be of size 15 by Theorem 3.8, we conclude, noting that $15 = 2 \cdot 7 + 1$, that A_7 acts transitively on $A_8.x$. Therefore the action of A_7 on X has exactly 1 non-trivial orbit, and so we may apply Lemma 3.5 – note that X has fewer than 2^6 points.

The remaining case occurs for $n = 8$; we have $R = 2$ and so X has fewer than $\binom{8}{2} = 28$ elements. But then we cannot have an orbit of size 120, and thus this exceptional case does not occur. □

Remark 3.15. In particular, we can put $r = \lfloor \frac{n}{2} \rfloor - 3$ in the above result; we see that (asymptotically) $\binom{n}{r}$ grows much faster than 2^{n-r} (in fact it grows like $n^{-1/2}2^n$), and so we obtain an exponential bound on the size of a set on which we can act non-trivially. The smallest set with a non-trivial action of $\mathrm{SAut}(F_n)$ known is also exponential in size – coming from the action of $\mathrm{L}_n(2)$ on the cosets of its largest maximal subgroup (see [KL, Table 5.2.A]). Hence the result above answers the question about the asymptotic size of such a set.

Theorem 3.16. *Let $n \geqslant 3$. Any action of $\mathrm{SAut}(F_n)$ on a set with fewer than $k(n)$ elements is trivial, where*

$$k(n) = \begin{cases} 7 & & n = 3 \\ 8 & & n = 4 \\ 12 & \text{if} & n = 5 \\ 14 & & n = 6 \end{cases}$$

and $k(n) = \max\limits_{r \leqslant \frac{n}{2}-3} \min\{2^{n-r-p(n)}, \binom{n}{r}\}$ for $n \geqslant 7$, where $p(n)$ equals 0 when n is odd and 1 when n is even.

Proof. This follows from Lemmata 3.1 and 3.3, Corollaries 3.9 and 3.10, and Proposition 3.14. □

As commented after the proofs of Lemmata 3.1 and 3.3, the bounds are sharp when $n \in \{3, 4\}$.

Corollary 3.18. *Let $n \geqslant 3$ and K be a quotient of $\mathrm{SAut}(F_n)$ with $|K| \leqslant \mathrm{L}_n(2)$. If K is isomorphic to an alternating group, then $n = 4$ and $K \cong A_8 \cong L_4(2)$.*

Proof. The proof consists of two parts. Firstly, Lemma A.3 tells us that

$$|A_{\binom{n}{2}}| > |\mathrm{L}_n(2)|$$

	Order
A_5	60
$\mathrm{L}_3(2)$	168
A_6	360
A_7	2520
$\mathrm{L}_4(2) \cong A_8$	21060
A_9	181440
A_{10}	1814400
$\mathrm{L}_5(2)$	9999360
A_{11}	19958400
A_{12}	239500800
A_{13}	3113510400
$\mathrm{L}_6(2)$	20158709760

TABLE 3.17. Small alternating groups

for $n \geqslant 7$. In view of the bounds in Theorem 3.16, this proves the result for $n \geqslant 7$ – for $n \in \{7, 8\}$ we have computed above that $r = 2$; for larger values of n we have $2^{n-3} > \binom{n}{2}$ by Lemma A.1.

Secondly, for $3 \leqslant n \leqslant 6$, Table 3.17 lists all alternating groups of degree at least 5 smaller or equal (in cardinality) than $\mathrm{L}_6(2)$. The table also lists the groups $\mathrm{L}_n(2)$ in the relevant range. All these groups are listed in increasing order. The result follows from inspecting the table and comparing it to the bounds in Theorem 3.16. □

3.1. **An application.** We record here a further application of the bounds established in Theorem 3.16.

Theorem 3.19. *Let $n \geqslant 12$ be an even integer, and let $m \neq n$ satisfy $m < \binom{n+1}{2}$. Then every homomorphism*

$$\phi\colon \mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

has image of cardinality at most 2.

Proof. [Kie1, Theorems 6.8 and 6.10] tell us that ϕ has a finite image. Every finite subgroup of $\mathrm{Out}(F_m)$ can be realised by a faithful action on a finite connected graph Γ with $\delta(\Gamma) \geqslant 3$ and Euler characteristic $1 - m$. These two facts immediately imply that Γ has fewer than $2m$ vertices. But now

$$2 \cdot \binom{n+1}{2} < \binom{n}{2} + 2 \cdot \binom{n+1}{2} + \binom{n+2}{2} = \binom{n}{4}$$

and

$$2 \cdot \binom{n+1}{2} < 2^{n-5}$$

for $n \geqslant 14$ by an argument analogous to Lemma A.1. Therefore, for $n \geqslant 14$ we have

$$2 \cdot \binom{n+1}{2} < \min\left\{\binom{n}{r}, 2^{n-r-1}\right\}$$

with $r = 4$ (and such an r satisfies $r \leqslant \frac{n}{2} - 3$).

For $n = 12$ we take $r = 3$ and compute directly that the inequality also holds.

In any case, the action of $\mathrm{SOut}(F_n)$ (via ϕ) on the vertices of Γ is trivial. Now each vertex has at most $2m - 2$ edges emanating from it, and so again the action of $\mathrm{SOut}(F_n)$ on these edges is trivial. Thus $\phi(\mathrm{SOut}(F_n))$ is trivial, which proves the result. □

4. Sporadic groups

In this section we show that sporadic groups are never the smallest quotients of $\mathrm{SAut}(F_n)$. The proof relies on determining for each sporadic group its *D'-rank*, that is the largest n such that the group contains D'_n. This information can be extracted from the lists of maximal subgroups contained in [CCN+] or in [Wil]; the lists are complete with the exception of the Monster group, in which case the list of possible maximal subgroups is known. The upper bound for the D'-rank of each sporadic group is recorded in Table 4.1 (which also lists the groups $\mathrm{L}_n(2)$ for comparison).

If K is a sporadic group of D'-rank smaller than n, then K is not the smallest quotient of $\mathrm{SAut}(F_n)$ by Lemma 2.5 (observing that K is not the smallest quotient of $\mathrm{SL}_n(\mathbb{Z})$). This observation allows us to rule out all but one sporadic group; the Deucalion is Fi_{22}, and we deal with it by other means.

Lemma 4.2. *Every homomorphism* $\phi\colon \mathrm{SAut}(F_7) \to \mathrm{Fi}_{22}$ *is trivial.*

Proof. In the ATLAS [CCN+] we see that there is a single conjugacy class of elements of order 5 in Fi_{22} denoted $5A$; moreover, the centraliser of an element $x \in 5A$ is of cardinality 600. We also see that Fi_{22} contains a copy of S_{10}, and we may without loss of generality assume that x is a 5-cycle in S_{10}. But then the centraliser of x inside S_{10} is $5 \times S_5$, which is already of order 600, and thus coincides with the centraliser of x in Fi_{22}.

Let τ be the element of order 5 given by Lemma 2.8; since its normal closure is $\mathrm{SAut}(F_7)$, its image in Fi_{22} is not trivial. Looking at the centraliser of τ, we obtain a homomorphism

$$\psi\colon \mathrm{SAut}(F_3) \to 5 \times S_5$$

Since $\mathrm{SAut}(F_3)$ is perfect (Proposition 2.4), the image of ψ must lie within A_5. Lemma 3.1 tells us that then ψ is trivial. But ψ is a restriction of ϕ, and so ϕ trivialises a transvection, say ρ_{67}, and thus ϕ is trivial. □

Proposition 4.3. *Let* $n \geqslant 3$ *and* K *be a sporadic simple group. Then* K *is not the smallest finite quotient of* $\mathrm{SAut}(F_n)$.

Proof. Let K be a sporadic group, and suppose that it is a smallest finite quotient of $\mathrm{SAut}(F_n)$. We must have

$$|K| \leqslant |\,\mathrm{L}_n(2)|$$

In fact, the inequality is strict, since for each n the group $\mathrm{L}_n(2)$ is not isomorphic to a sporadic group (this is visible in Table 4.1). Thus, by Lemma 2.5, we see that the epimorphism $\phi\colon \mathrm{SAut}(F_n) \to K$ has to be injective on D'_n. Inspection of Table 4.1 shows that this is only possible for $K = \mathrm{Fi}_{22}$, in which case $n \geqslant 7$. But this is ruled out by Lemma 4.2. □

5. Algebraic groups and groups of Lie type

In this section we review the necessary information about algebraic groups over fields of positive characteristic, and the (closely related) finite groups of Lie type.

5.1. Algebraic groups.

We begin by discussing connected algebraic groups. Following [GLS], we will denote such groups by $\overline{K}$. We review here only the facts that will be useful to us, focusing on simple, semi-simple, and reductive algebraic groups.

Let r be a prime, and let $\overline{\mathbb{F}}$ be an algebraically closed field of characteristic r. The simple algebraic groups over $\overline{\mathbb{F}}$ are classified by the Dynkin diagrams $\mathtt{A}_n$ (for each n), $\mathtt{B}_n$ (for $n \geqslant 3$), $\mathtt{C}_n$ (for $n \geqslant 2$), $\mathtt{D}_n$ (for $n \geqslant 4$), $\mathtt{E}_n$ (for $n \in \{6,7,8\}$), $\mathtt{F}_4$, and $\mathtt{G}_2$. The index of the diagram is defined to be the *rank* of the associated group.

K	Bound for D'-rank	Order of K
M_{11}	3	7920
$\mathrm{L}_4(2)$		21060
M_{12}	3	95040
J_1	4	175560
M_{22}	3	443520
J_2	4	604800
$\mathrm{L}_5(2)$		9999360
M_{23}	3	10200960
HS	4	44352000
J_3	4	50232960
M_{24}	3	244823040
$\mathrm{M^cL}$	4	898128000
He	4	4030387200
$\mathrm{L}_6(2)$		20158709760
Ru	5	145926144000
Suz	5	448345497600
O′N	4	460815505920
Co_3	5	495766656000
Co_2	6	42305421312000
Fi_{22}	7	64561751654400
$\mathrm{L}_7(2)$		163849992929280
HN	6	273030912000000
Ly	5	51765179004000000
Th	6	90745943887872000
Fi_{23}	7	4089470473293004800
Co_1	6	4157776806543360000
$\mathrm{L}_8(2)$		5348063769211699200
J_4	7	86775571046077562880
$\mathrm{L}_9(2)$		699612310033197642547200
Fi'_{24}	9	1255205709190661721292800
$\mathrm{L}_{10}(2)$		366440137299948128422802227200
B	10	4154781481226426191177580544000000
$\mathrm{L}_{11}(2)$		768105432118265670534631586896281600
$\mathrm{L}_{12}(2)$		6441762292785762141878919881400879415296000
$\mathrm{L}_{13}(2)$		216123289355092695876117433338079655078664339456000
M	12	808017424794512875886459904961710757005754368000000000

TABLE 4.1. Upper bounds for the D'-ranks of the sporadic groups.

To each Dynkin diagram we associate a finite number of simple algebraic groups; two such groups associated to the same diagram are called *versions*; they become isomorphic upon dividing them by their respective finite centres. Two versions are particularly important: the *universal* (or *simply-connected*) one, which maps onto every other version with a finite central kernel, and the *adjoint* version, which is a quotient of every other version with a finite central kernel.

Every semi-simple algebraic group over $\overline{\mathbb{F}}$ is a central product of finitely many simple algebraic groups over $\overline{\mathbb{F}}$. The rank of such a group is defined to be the sum of the ranks of the simple factors, and is well-defined.

Every reductive algebraic group is a product of an abelian group and a semi-simple group. Its rank is defined to be the rank of the semi-simple factor, and again it is well-defined.

Given an algebraic group $\overline{K}$, a maximal with respect to inclusion closed connected solvable subgroup of $\overline{K}$ will be referred to as a *Borel subgroup*. The Borel subgroups always exist, and are conjugate, and hence one can talk about the Borel subgroup (up to conjugation). When $\overline{K}$ is reductive, any closed subgroup thereof containing a Borel subgroup is called *parabolic*. Let us now state the main tool in our approach towards algebraic groups and groups of Lie type.

Theorem 5.1 (Borel–Tits [GLS, Theorem 3.1.1(a)]). *Let $\overline{K}$ be a reductive algebraic group over an algebraically closed field, let $\overline{X}$ be a closed unipotent subgroup, and let $\overline{N}$ denote the normaliser of $\overline{X}$ in $\overline{K}$. Then there exists a parabolic subgroup $\overline{P} \leqslant \overline{K}$ such that $\overline{X}$ lies in the unipotent radical of $\overline{P}$, and $\overline{N} \leqslant \overline{P}$.*

We will not discuss the other various terms appearing above beyond what is strictly necessary. For our purpose we only need to observe the following.

Remark 5.2.
(1) The unipotent radical of a reductive group is trivial.
(2) If $\overline{K}$ is defined in characteristic r, then every finite r-group in $\overline{K}$ is a closed unipotent subgroup.

Theorem 5.3 (Levi decomposition [GLS, Theorem 1.13.2, Proposition 1.13.3]). *Let $\overline{P}$ be a proper parabolic subgroup in a reductive algebraic group $\overline{K}$.*
(1) *Let $\overline{U}$ denote the unipotent radical of $\overline{P}$ (note that $\overline{U}$ is nilpotent). There exists a subgroup $\overline{L} \leqslant \overline{P}$, such that $P = \overline{U} \rtimes \overline{L}$.*
(2) *The subgroup $\overline{L}$ (the* Levi factor*) is a reductive algebraic group of rank smaller than $\overline{K}$.*

5.2. **Finite groups of Lie type.** Let r be a prime, and q a power thereof.

Any finite group of Lie type K is obtained as a fixed point set of a Steinberg endomorphism of a connected simple algebraic group $\overline{K}$ defined over an algebraically closed field of characteristic r. Such groups have a *type*, which is related to the Dynkin diagram of $\overline{K}$, and an associated *twisted rank*. As stated in Section 1, the finite groups of Lie type fall into two families: the types $\mathtt{A}_n, {}^2\mathtt{A}_n, \mathtt{B}_n, \mathtt{C}_n, \mathtt{D}_n$ and ${}^2\mathtt{D}_n$ are called *classical*, and the types ${}^2\mathtt{B}_2$, ${}^3\mathtt{D}_4$, $\mathtt{E}_6$, ${}^2\mathtt{E}_6$, $\mathtt{E}_7$, $\mathtt{E}_8$, $\mathtt{F}_4$, ${}^2\mathtt{F}_4$, $\mathtt{G}_2$ and ${}^2\mathtt{G}_2$ are called *exceptional*.

The types of the classical groups and their twisted ranks are listed in Table 5.4. Note that $\lceil . \rceil$ denotes the ceiling function. In the case of the exceptional groups, for the groups of types $\mathtt{G}_2$, $\mathtt{F}_4$, $\mathtt{E}_6$, $\mathtt{E}_7$ and $\mathtt{E}_8$ the twisted rank is equal to the rank. Groups of type ${}^2\mathtt{B}_2$ or ${}^2\mathtt{G}_2$ have twisted rank 1, those of type ${}^3\mathtt{D}_4$ or ${}^2\mathtt{F}_4$ have twisted rank 2 and groups of type ${}^2\mathtt{E}_6$ have twisted rank 4. Groups of types ${}^2\mathtt{B}_2$ and ${}^2\mathtt{F}_4$ are defined only over fields of order 2^{2m+1} while groups of type ${}^2\mathtt{G}_2$ are defined only over fields of order 3^{2m+1}. All groups of all other types are defined in all characteristics.

As was the case with algebraic groups, each type corresponds to a finite number of finite groups (the *versions*), and two such are related by dividing by the centre as before. The smallest version (in cardinality, say) is called *adjoint* as before; the adjoint version is a simple group with the following exceptions [CCN+, Chapter 3.5]

$$\mathtt{A}_1(2) \cong S_3, \quad \mathtt{A}_1(3) \cong A_4, \quad \mathtt{C}_2(2) \cong S_6, \quad {}^2\mathtt{A}_2(2) \cong 3^2 \rtimes Q_8,$$
$$\mathtt{G}_2(2) \cong {}^2\mathtt{A}_2(3) \rtimes 2, \quad {}^2\mathtt{B}_2(2) \cong 5 \rtimes 4, \quad {}^2\mathtt{G}_2(3) \cong \mathtt{A}_1(8) \rtimes 3, \quad {}^2\mathtt{F}_4(2)$$

where Q_8 denotes the quaternion group of order 8, and the index 2 derived subgroup of ${}^2\mathtt{F}_4(2)$ is simple, known as the Tits group. For the purpose of this paper, we treat T as a finite group of Lie type.

Type	Conditions	Twisted rank	Dimension	Classical isomorphism
$\mathtt{A}_n(q)$	$n \geqslant 1$	n	$n+1$	$\mathrm{L}_{n+1}(q)$
${}^2\mathtt{A}_n(q)$	$n \geqslant 2$	$\lceil \frac{n}{2} \rceil$	$n+1$	$\mathrm{U}_{n+1}(q)$
$\mathtt{B}_n(q)$	$n \geqslant 2$	n	$2n+1$	$\mathrm{O}_{2n+1}(q)$
$\mathtt{C}_n(q)$	$n \geqslant 3$	n	$2n$	$\mathrm{S}_{2n}(q)$
$\mathtt{D}_n(q)$	$n \geqslant 4$	n	$2n$	$\mathrm{O}^+_{2n}(q)$
${}^2\mathtt{D}_n(q)$	$n \geqslant 4$	$n-1$	$2n$	$\mathrm{O}^-_{2n}(q)$

TABLE 5.4. The classical groups of Lie type

For reference, we also recall the following additional exceptional isomorphisms

$$\mathtt{A}_1(4) \cong \mathtt{A}_1(5) \cong A_5, \quad \mathtt{A}_1(9) \cong A_6, \quad \mathtt{A}_1(7) \cong \mathtt{A}_2(2), \quad \mathtt{A}_3(2) \cong A_8, \quad {}^2\mathtt{A}_3(2) \cong \mathtt{C}_2(3)$$

In addition, $\mathtt{B}_n(2^e) \cong \mathtt{C}_n(2^e)$ for all $n \geqslant 3$ and $e \geqslant 1$. We will sometimes abuse the notation, and denote the adjoint version of some type by the type itself.

The adjoint version of a classical group over q comes with a natural projective module over an algebraically closed field in characteristic r; the dimensions of these modules are taken from [KL, Table 5.4.C] and listed in Table 5.4. Note that these projective modules are irreducible.

A *parabolic* subgroup of K is any subgroup containing the Borel subgroup of K, which is obtained by taking an α-invariant Borel subgroup in $\overline{K}$ (where α denotes a Steinberg endomorphism), and intersecting it with K. Note that such a Borel subgroup always exists – in fact, its intersection with K is equal to the normaliser of some Sylow r-subgroup of K.

Theorem 5.5 (Borel–Tits [GLS, Theorem 3.1.3(a)]). *Let K be a finite group of Lie type in characteristic r, and let R be a non-trivial r-subgroup of K. Then there exists a proper parabolic subgroup $P \leqslant K$ such that R lies in the normal r-core of P, and $N_K(R) \leqslant P$.*

Note that the normal r-core of K is trivial.

Theorem 5.6 (Levi decomposition [GLS, Theorem 2.6.5(e,f,g), Proposition 2.6.2(a,b)]). *Let P be a proper parabolic in a finite group K of Lie type in characteristic r.*

(1) *Let U denote the normal r-core of P (note that U is nilpotent). There exists a subgroup $L \leqslant P$, such that $L \cap U = \{1\}$ and $LU = P$.*
(2) *The subgroup L (the* Levi factor*) contains a normal subgroup M such that L/M is abelian of order coprime to r.*
(3) *The subgroup M is isomorphic to a central product of finite groups of Lie type (the* simple factors of L*) in characteristic r such that the sum of the twisted ranks of these groups is lower than the twisted rank of K.*
(4) *When K is of classical type other than ${}^2\mathtt{D}$ or $\mathtt{B}$, then each simple factor of L is either of the same type as K, or of type $\mathtt{A}$. For type ${}^2\mathtt{D}$ we also get factors of type ${}^2\mathtt{A}_3$; for type $\mathtt{B}$ we also get factors of type $\mathtt{C}_2$.*
(5) *When K is of classical type other than ${}^2\mathtt{A}$ or ${}^2\mathtt{D}$, then the simple factors of L are defined over the same field. The groups ${}^2\mathtt{A}(q)$ admit simple factors of L of type $\mathtt{A}(q^2)$, and the groups ${}^2\mathtt{D}(q)$ admit a simple factor of L of type $\mathtt{A}_1(q^2)$.*

6. GROUPS OF LIE TYPE IN CHARACTERISTIC 2

Because of the special role the involutions $\epsilon_1, \ldots, \epsilon_n$ play in the structure of $\mathrm{Aut}(F_n)$, groups of Lie type in characteristic 2 require a different approach than groups in odd characteristic. The strategy is to look at the centraliser of ϵ_n in

$\mathrm{Aut}(F_n)$, note that it contains $\mathrm{Aut}(F_{n-1})$, and then use the Borel-Tits theorem (Theorem 5.5) for its image. The same strategy works for reductive algebraic groups in characteristic 2.

Before we proceed to the main part of this section, we will investigate maps $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$, with the aim of showing that there is essentially only one such non-trivial map.

Lemma 6.1 ([KL, Proposition 5.3.7]). *Let A_n be the alternating group of degree n where $3 \leqslant n \leqslant 8$. The degree $R_p(A_n)$ of the smallest nontrivial irreducible projective representation of A_n over a field of characteristic p is as given in Table 6.2. If $n \geqslant 9$, then the degree of the smallest nontrivial projective representation of A_n is $n-2$.*

n	$R_2(A_n)$	$R_3(A_n)$	$R_5(A_n)$	$R_7(A_n)$
5	2	2	2	2
6	3	2	3	3
7	4	4	3	4
8	4	7	7	7

TABLE 6.2.

Remark 6.3. Moreover, the result of Wagner [Wag] tells us the following: Assume that the characteristic is 2 and that $n \geqslant 9$. Then the smallest non-trivial A_n-module appears in dimension $n-1$ when n is odd and $n-2$ when n is even, and is unique. This module appears as an irreducible module in our group $D'_n = 2^{n-1} \rtimes A_n$, where for n even we take a quotient by $\langle \delta \rangle$.

Lemma 6.4. *Let $n \geqslant 4$. Let $\phi\colon D'_n \to \mathrm{L}_m(2) = \mathrm{GL}(V)$ be a homomorphism.*

1. *When $m < n-1$ then $\phi(2^{n-1})$ is trivial.*
2. *Suppose that $m = n-1$ and $\phi(2^{n-1})$ is non-trivial. Then n is even, $\phi(\delta) = 1$ and we can choose a basis of V in such a way that either for $i < n-1$ the element $\phi(\epsilon_i \epsilon_{i+1})$ is given by the elementary matrix E_{1i}, that is the matrix equal to the identity except at the position $(1i)$, or each element $\phi(\epsilon_i\epsilon_{i+1})$ is given by the elementary matrix E_{i1}.*
3. *When $m = n$ and we additionally assume that ϕ is injective and that when $n = 8$ the representation $\phi|_{A_8}$ is the 8-dimensional permutation representation, then we can choose a basis of V in such a way that either for each $i < n-2$ the element $\phi(\epsilon_i\epsilon_{i+1})$ is given by the elementary matrix E_{1i}, or each element $\phi(\epsilon_i\epsilon_{i+1})$ is given by the elementary matrix E_{i1}.*

Proof. Fix n, and proceed by induction on m. Clearly $m > 1$.

Consider the subgroup $V \rtimes \phi(2^{n-1}) < V \rtimes \mathrm{GL}(V)$. It is a 2-group, hence it is nilpotent, and therefore it has a non-trivial centre Z. Since $\phi(2^{n-1})$ acts faithfully, we have $Z \leqslant V$ as a subgroup, and hence also as a 2-vector subspace. Clearly,

$$Z = \{v \in V \mid \phi(\xi)(v) = v \text{ for all } \xi \in 2^{n-1}\}$$

and therefore Z is preserved setwise by $\phi(D'_n)$, as 2^{n-1} is a normal subgroup of D'_n.

Suppose that $\dim Z \leqslant \dim V/Z = m - \dim Z$. If $n \geqslant 5$, this implies that Z is a trivial A_n-module: for $n \geqslant 9$ and $n = 7$ this follows from Lemma 6.1, for $n \in \{5, 6\}$ we observe that A_n is simple and larger in cardinality than $\mathrm{L}_{n-3}(2)$; and $n = 8$ we see that $A_8 \cong \mathrm{L}_4(2)$ is larger than $\mathrm{L}_3(2)$, which is enough for (1) and (2); for (3) we us the additional hypothesis on the A_8-representation.

When $n = 4$ we could have $m = n$ and $\dim Z = 2$, in which case Z does not have to be a trivial A_4-module. But in this case we have

$$\mathrm{GL}(Z) \cong \mathrm{GL}(V/Z) \cong \mathrm{L}_2(2) \cong S_3$$

and every homomorphism $D'_4 \to S_3$ has $2^3 \rtimes V_4$ in its kernel, where V_4 denotes the Klein four-group. But then ϕ takes $2^3 \rtimes V_4$ to an abelian group of matrices which differ from the identity only in the top-right 2×2 corner. Thus $\phi(\delta) = \phi([\epsilon_1\epsilon_2, \sigma_{13}\sigma_{24}]) = 1$, contradicting the injectivity of ϕ.

We may therefore assume that Z is a trivial A_n module even when $n = 4$.

Suppose that $m < n - 1$. Then, by the inductive hypothesis, we know that the action of 2^{n-1} on V/Z is trivial; it is also trivial on Z by construction. Hence $\phi(D'_n)$ is a subgroup of

$$2^{\dim Z(m-\dim Z)} \rtimes \mathrm{GL}(V/Z)$$

and ϕ takes 2^{n-1} into the $2^{\dim Z(m-\dim Z)}$ part. But this subgroup cannot contain 2^{n-1} as an A_n-module, since as a $\mathrm{GL}(V/Z)$-module it is a direct sum of $(m-\dim Z)$-dimensional modules, and $\dim Z \geqslant 1$. This shows that ϕ is not injective on 2^{n-1}. But the only subgroup of 2^{n-1} which can lie in $\ker \phi$ is $\langle \delta \rangle$, and therefore if $\phi(2^{n-1})$ is not trivial, then we need to be able to fit a $(n-2)$-dimensional module into $2^{\dim Z(m-\dim Z)}$. This is impossible when $m < n - 1$, and so (1) follows.

All of the above was conducted under the assumption that $\dim Z \leqslant m - \dim Z$. If this is not true, then we take the transpose inverse of ϕ; for this representation the inequality is true, and the kernel of this representation coincides with the kernel of ϕ.

When $m = n - 1$ then we have just proven (2) – it is clear that we can change the basis of V if necessary to have each $\phi(\epsilon_i\epsilon_{i+1})$ as required.

In case (3) we immediately see that $\dim Z = 1$. If 2^{n-1} does not act trivially on V/Z then we apply (2). Since now ϕ is injective, it must take 2^{n-1} to the subgroup of $\mathrm{L}_n(2)$ generated by E_{ji} with $j \in \{1, 2\}$ and $i > j$. Suppose that for some i we have

$$\phi(\epsilon_i\epsilon_{i+1}) = E_{12} + E_{2(i+1)} + M$$

where $M \in \langle\{E_{1i} \mid i > 2\}\rangle$. Then $\phi(\epsilon_i\epsilon_{i+1})$ is of order 4, which is impossible. So

$$\phi(2^{n-1}) \leqslant \langle\{E_{ji} \mid j \in \{1, 2\}, i > 2\}\rangle \cong 2^{n-2} \oplus 2^{n-2}$$

as an A_n-module, which contradicts injectivity of ϕ. Therefore 2^{n-1} acts trivially on V/Z, and the result follows as before. □

Remark 6.5. In fact, for $n = 3$ we can obtain identical conclusions, with the exception that in (3) we may need to postcompose ϕ with an outer automorphism of $\mathrm{L}_3(2)$. To see this note that in (1) we have $\mathrm{L}_m(2) = \mathrm{L}_1(2) = \{1\}$; in (2) we have $\mathrm{L}_m(2) = \mathrm{L}_2(2) \cong S_3$, and every map from $\mathtt{D}'_3 \cong A_4$ to S_3 has $2^2 \cong V_4$ in the kernel. For (3) we see that $\mathrm{L}_m(2) = \mathrm{L}_3(2)$ contains exactly two conjugacy classes of $A_4 \cong D'_3$, and these are related by an outer automorphism of $\mathrm{L}_3(2)$ (see [CCN+]). Thus, up to postcomposing ϕ with an outer automorphism of $\mathrm{L}_3(2)$, we may assume that ϕ maps the involutions $\epsilon_i\epsilon_j$ in the desired manner.

Proposition 6.6. *Let $n \geqslant 3$ and $m \leqslant n$ be integers. If*

$$\phi\colon\ \mathrm{SAut}(F_n) \to \mathrm{L}_m(2)$$

is a non-trivial homomorphism, then $m = n$ and ϕ is equal to the natural map $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$ postcomposed with an automorphism of $\mathrm{L}_n(2)$.

Proof. Assume that $n \geqslant 3$. Observe that if ϕ is not injective on D'_n then we are done by Lemma 2.5 – one has to note that when $\phi(\delta) = 1$ then we know that ϕ factors through $\mathrm{SL}_n(\mathbb{Z})$, and so we know using the congruence subgroup property that every non-trivial map $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{L}_n(2)$ factors through the natural such map. We will assume that ϕ is injective on D_n'.

We apply Lemma 6.4 (and Remark 6.5 when $n = 3$); for $n = 8$ we consider $\phi(A_9)$ – by Remark 6.3, ϕ must be the unique 8-dimensional representation, and so as an A_8-module V is the natural permutation representation. Up to possibly taking the transpose inverse of ϕ, we see that $m = n$, and

$$\phi(\epsilon_i \epsilon_{i+1}) = E_{1i}$$

Let Z denote the subspace of V generated by the first basis vector, note that Z is precisely the centraliser in V of $\phi(2^{n-1})$ and coincides with the commutator $[V, \phi(\xi)]$ for every $\xi \in 2^{n-1} \smallsetminus \{1\}$. A direct computation shows that if a matrix in $\mathrm{L}_n(2)$ commutes with some E_{1i}, then it preserves Z. In fact this remains true for any non-zero sum of matrices E_{1i}.

The group $\mathrm{SAut}(F_n)$ is generated by transvections, and each of them commutes with some $\epsilon_i \epsilon_j$ as $n \geqslant 4$, and so $\mathrm{SAut}(F_n)$ preserves Z. Thus we have a representation

$$\mathrm{SAut}(F_n) \to \mathrm{GL}(V/Z) \cong \mathrm{L}_{n-1}(2)$$

and such a representation is trivial, or n is even and the representation has δ in its kernel by Lemma 6.4. But then it factors through $\mathrm{SL}_n(\mathbb{Z})$, and therefore must be trivial, since the smallest quotient of $\mathrm{SL}_n(\mathbb{Z})$ is $\mathrm{L}_n(2)$.

Therefore ϕ take $\mathrm{SAut}(F_n)$ to 2^{n-1}, and hence must be trivial. □

We now proceed to the main discussion. We start by looking at small values of n. These considerations will form the base of our induction.

Lemma 6.7. *Let $\overline{\mathbb{F}}$ be an algebraically closed field of characteristic 2. Every homomorphism $\phi\colon \mathrm{SAut}(F_3) \to \mathrm{L}_2(\overline{\mathbb{F}}) = \mathrm{PSL}_2(\overline{\mathbb{F}})$ is trivial.*

Proof. We start by observing that $\mathrm{PSL}_2(\overline{\mathbb{F}}) = \mathrm{SL}_2(\overline{\mathbb{F}})$, since the only element in $\overline{\mathbb{F}}$ which squares to 1 is 1 itself.

Suppose first that $\epsilon_1\epsilon_2$ lies in the kernel of ϕ. Then ϕ descends to a map

$$\mathrm{L}_3(2) \to \mathrm{L}_2(\overline{\mathbb{F}})$$

by Lemma 2.5. Since $\mathrm{L}_3(2)$ is simple, this map is either faithful or trivial. But it cannot be faithful, since the upper triangular matrices in $\mathrm{L}_3(2)$ form a 2-group (the dihedral group of order 8) which is nilpotent of class 2, whereas every non-trivial 2-subgroup of $\mathrm{L}_2(\overline{\mathbb{F}})$ is abelian. (Alternatively, one can use the fact that $\mathrm{L}_3(2)$ has no non-trivial projective representations in dimension 2 in characteristic 2, as can be seen from the 2-modular Brauer table which exists in GAP.)

Hence we may assume that $\phi(\epsilon_1\epsilon_2) \neq 1$. Consider the 2-subgroup of $\mathrm{L}_2(\overline{\mathbb{F}})$ generated by $\phi(\epsilon_i\epsilon_j)$ with $1 \leqslant i, j \leqslant 3$ (it is isomorphic to 2^2). As before, up to conjugation, this subgroup lies within the unipotent subgroup of upper triangular matrices with ones on the diagonal. Now a direct computation shows that the matrices which commute with

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $x \neq 0$ are precisely the matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

In particular, this implies that if an element of $\mathrm{L}_2(\overline{\mathbb{F}})$ commutes with $\phi(\epsilon_1\epsilon_2)$, then it also commutes with $\phi(\epsilon_2\epsilon_3)$. This applies to $\phi(\sigma_{12}\epsilon_3)$, and so

$$\phi(\epsilon_2\epsilon_3) = \phi(\epsilon_2\epsilon_3)^{\phi(\sigma_{12}\epsilon_3)} = \phi(\epsilon_1\epsilon_3)$$

and so $\phi(\epsilon_1\epsilon_2) = 1$, contradicting our assumption. □

Lemma 6.8. *Every homomorphism ϕ from $\mathrm{Aut}(F_3)$ to a finite group of Lie type in characteristic 2 of twisted rank 1 has abelian image.*

Proof. The groups we have to consider as targets here are the versions of $\mathtt{A}_1(q)$, ${}^2\mathtt{A}_2(q)$, and ${}^2\mathtt{B}_2(q)$, where q is a power of 2 (where the exponent is odd in type ${}^2\mathtt{B}_2$). Since $\mathrm{SAut}(F_3)$ is perfect, and we claim that it has to be contained in the kernel of our homomorphism, we need only look at the adjoint versions.

For type $\mathtt{A}_1$ the result follows from Lemma 6.7, since $\mathrm{L}_2(q) \leqslant \mathrm{L}_2(\overline{\mathbb{F}})$ with $\overline{\mathbb{F}}$ algebraically closed and of characteristic 2. The simple group of type ${}^2\mathtt{B}_2(q)$ has no elements of order 3 (this can easily be seen from the order of the group), and so $\mathrm{SAut}(F_3)$ lies in the kernel of the homomorphism by Lemma 2.5.

We are left with the type ${}^2\mathtt{A}_2(q)$. In this case we observe that, up to conjugation, there are only two parabolic subgroups of $K = {}^2\mathtt{A}_2(q)$, namely K itself and a Borel subgroup B.

Suppose first that ϵ_3 has a non-trivial image in K. By Theorem 5.5, the image of the centraliser of ϵ_3 in $\mathrm{Aut}(F_n)$ lies in B. The Borel subgroup B is a semi-direct product of the unipotent subgroup by the torus. The torus contains no elements of order 2. Moreover, the only elements of order 2 in the unipotent subgroup lie in its centre – this can be verified by a direct computation with matrices.

The centraliser of ϵ_3 in $\mathrm{Aut}(F_3)$ contains $\mathrm{Aut}(F_2)$, which is generated by involutions $\epsilon_1, \epsilon_2, \rho_{12}\epsilon_2$ and $\rho_{21}\epsilon_1$. Thus the image of $\mathrm{Aut}(F_2)$ lies in the centre of the unipotent subgroup of B, which is abelian. Therefore, we have

$$\phi(\rho_{12}) = \phi({\rho_{12}}^{\epsilon_1\epsilon_2\sigma_{12}}) = \phi(\lambda_{21})$$

and therefore

$$\phi(\rho_{13})^{-1} = [\phi(\rho_{12})^{-1}, \phi(\rho_{23})^{-1}] = [\phi(\lambda_{21})^{-1}, \phi(\rho_{23})^{-1}] = 1$$

This trivialises the subgroup $\mathrm{SAut}(F_3)$ as claimed.

Recall that we have assumed that ϵ_3 is not in the kernel of ϕ; when it is, then the homomorphism factors through $\mathrm{L}_3(2)$, which is simple and not a subgroup of ${}^2\mathtt{A}_2(q)$ whenever q is a power of 2 – this can be seen by inspecting the maximal subgroups of ${}^2\mathtt{A}_2(q)$ [BHRD]. □

Theorem 6.9. *Let $n \geqslant 3$. Let K be a finite group of Lie type in characteristic 2 of twisted rank less than $n-1$, and let $\overline{K}$ be a reductive algebraic group over an algebraically closed field of characteristic 2 of rank less than $n-1$. Then any homomorphism $\mathrm{Aut}(F_n) \to K$ or $\mathrm{Aut}(F_n) \to \overline{K}$ has abelian image, and any homomorphism $\mathrm{SAut}(F_{n+1}) \to K$ or $\mathrm{SAut}(F_{n+1}) \to \overline{K}$ is trivial.*

Proof. We start by looking at the finite group K, and a homomorphism

$$\phi\colon\ \mathrm{Aut}(F_n) \to K$$

Since $\mathrm{SAut}(F_n)$ is perfect and of index 2 in $\mathrm{Aut}(F_n)$, we may without loss of generality divide K by its centre; we may also assume that K is not solvable.

Our proof is an induction on n. The base case ($n = 3$) is covered by Lemma 6.8. In what follows let us assume that $n > 3$.

We claim that $\phi(\mathrm{SAut}(F_{n-1}))$ lies in a proper parabolic subgroup P of K. If $\phi(\epsilon_n)$ is central then $\phi(\epsilon_{n-1}\epsilon_n)$ is trivial, since ϵ_n and ϵ_{n-1} are conjugate. Thus ϕ factors through

$$\mathrm{Aut}(F_n) \to \mathrm{L}_n(2)$$

by Lemma 2.5. Let $\eta\colon\ \mathrm{L}_n(2) \to K$ denote the induced homomorphism.

The group $\mathrm{L}_n(2)$ contains $\mathrm{L}_{n-1}(2)$ inside a proper parabolic subgroup which normalises a non-trivial 2-group G. This 2-group contains an elementary matrix, and so if $\eta(G)$ is trivial, then so is every elementary matrix in $\mathrm{L}_n(2)$, and therefore η is trivial (as $\mathrm{L}_n(2)$ is generated by elementary matrices). This proves the claim.

Now let us assume that G has a non-trivial image in K. Thus $\eta(G)$ does not lie in the normal 2-core of K, and therefore, by Theorem 5.5, the normaliser of G in $\mathrm{L}_n(2)$ is mapped by η into a proper parabolic subgroup P. Clearly, we may choose G so that it is normalised by the image of $\mathrm{Aut}(F_{n-1})$ in $\mathrm{L}_n(2)$. This way we have shown that $\phi(\mathrm{Aut}(F_{n-1}))$ lies in P.

Now assume that $\phi(\epsilon_n)$ is not central, and so in particular not trivial. We conclude, using Theorem 5.5, that $\phi(\mathrm{Aut}(F_{n-1}))$ lies in a parabolic P inside K such that $P \neq K$. Hence we have

$$\phi(\mathrm{Aut}(F_{n-1})) \leqslant P < K$$

irrespectively of what happens to ϵ_n, which proves the claim.

Consider the induced map $\psi\colon \mathrm{Aut}(F_{n-1}) \to P/U \cong L$ (using the notation of Theorem 5.6). Note that in fact the image of ψ lies in M, since L/M is abelian and contains no element of order 2. Now M is a central product of finite groups of Lie type in characteristic 2, where the sum of the twisted ranks is lower than that of K. Thus, using the projections, we get maps from $\mathrm{Aut}(F_{n-1})$ to finite groups of Lie type of twisted rank less than $n-2$. By the inductive assumption all such maps have abelian image, and so the image of $\mathrm{Aut}(F_{n-1})$ in M is abelian. This forces ϕ to contain $\mathrm{SAut}(F_{n-1})$ in its kernel, and the result follows, since U is nilpotent and $\mathrm{SAut}(F_{n-1})$ is perfect as $n \geqslant 4$.

Now let us look at a homomorphism $\phi\colon \mathrm{Aut}(F_n) \to \overline{K}$. We proceed as above; the base case ($n = 3$) is covered by Lemma 6.7.

We claim that, as before, $\phi(\mathrm{Aut}(F_{n-1}))$ is contained in a proper parabolic subgroup $\overline{P}$ of $\overline{K}$. This is proved exactly as before using Theorem 5.1, except that now we use the fact that every finite 2-group in $\overline{K}$ is a closed unipotent subgroup. Note that $\overline{P}$ is a proper subgroup, since $\overline{K}$ is reductive, and thus its unipotent radical is trivial.

Again as before we look at the induced map $\psi\colon \mathrm{Aut}(F_{n-1}) \to \overline{P}/\overline{U} \cong \overline{L}$. By Theorem 5.3, the group L is reductive of lower rank, and so ψ has abelian image by induction. But then $\phi|_{\mathrm{Aut}(F_{n-1})}$ has solvable image, and so $\phi(\mathrm{SAut}(F_{n-1})) = \{1\}$. Therefore

$$\phi(\mathrm{SAut}(F_n)) = \{1\}$$

as well, and the image of ϕ is abelian.

The statements for $\mathrm{SAut}(F_{n+1})$ follow from observing that the natural embedding $\mathrm{SAut}(F_n) \hookrightarrow \mathrm{SAut}(F_{n+1})$ extends to an embedding $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{SAut}(F_{n+1})$, where we map an element $x \in \mathrm{Aut}(F_n)$ of determinant -1 to $x\epsilon_{n+1}$. When this copy of $\mathrm{Aut}(F_n)$ has an abelian image under a homomorphism, then the homomorphism is trivial on $\mathrm{SAut}(F_n)$, and hence on the whole of $\mathrm{SAut}(F_{n+1})$. □

Theorem 6.10. *Let $n \geqslant 8$. Let K be a finite simple group of Lie type in characteristic 2 which is a quotient of $\mathrm{SAut}(F_n)$. Then either $|K| > |\,\mathrm{L}_n(2)|$, or $K = \mathrm{L}_n(2)$ and ϕ is obtained by postcomposing the natural map $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$ by an automorphism of $\mathrm{L}_n(2)$.*

Proof. By Theorem 6.9, K is of twisted rank at least $n-2$.

Since $n \geqslant 8$, by Lemma A.4 we see that all the finite simple groups of Lie type in characteristic 2 and twisted rank at least $n-2$ are larger than $\mathrm{L}_n(2)$, with the exception of $\mathtt{A}_{n-2}(2)$ and $\mathtt{A}_{n-1}(2)$. Proposition 6.6 immediately tells us that $K = \mathrm{L}_n(2)$ and ϕ is as claimed. □

7. Classical groups in odd characteristic

7.1. **Field of 3 elements.** In this subsection we use Borel–Tits in characteristic 3. To do this, we need to find suitable elements of order 3 in $\mathrm{SAut}(F_n)$.

Table 7.2. Some conjugacy classes in $\mathtt{C}_2(3)$

Class	$2A$	$2B$	$3C$	$3D$	$4A$	$4B$	$5A$	$6E$	$6F$
$\lvert x^G \cap C_G(x)\rvert$	13	22	6	12	8	4	4	2	2

Let $\gamma = \epsilon_{n-1}\epsilon_n \lambda_{(n-1)n}{}^{-1}\rho_{n(n-1)}$. A direct computation immediately shows that γ is of order 3. Also, the centraliser of γ in $\mathrm{SAut}(F_n)$ contains $\mathrm{SAut}(F_{n-2})$. In fact, γ is the element constructed in Lemma 2.8. We define it here algebraically, since it allows us to easily show the following.

Lemma 7.1. *Let $n \geqslant 4$. The normal closure of γ inside $\mathrm{SAut}(F_n)$ is the whole of $\mathrm{SAut}(F_n)$.*

Proof. Let C denote the normal closure. Then

$$\begin{aligned}\rho_{n1}{}^{-1}C &= [\rho_{n(n-1)}{}^{-1}, \rho_{(n-1)1}{}^{-1}]C \\ &= [\epsilon_{n-1}\epsilon_n\lambda_{(n-1)n}{}^{-1}, \rho_{(n-1)1}{}^{-1}]C \\ &= \lambda_{(n-1)1}\rho_{(n-1)1}C\end{aligned}$$

where the last equality follows by expanding the commutator. Now

$$\begin{aligned}\rho_{21}{}^{-1}C &= [\rho_{2n}{}^{-1}, \rho_{n1}{}^{-1}]C \\ &= [\rho_{2n}{}^{-1}, \lambda_{(n-1)1}\rho_{(n-1)1}]C \\ &= C\end{aligned}$$

and we are done. □

Lemma 7.3. *Every homomorphism $\phi\colon \mathrm{SAut}(F_4) \to K$, where K is a finite group of Lie type of type $\mathtt{A}_2(3)$ or $\mathtt{C}_2(3)$, is trivial.*

Proof. Since $\mathrm{SAut}(F_4)$ is perfect, we may assume that K is simple. If K is of type $\mathtt{A}$, then $K \cong \mathrm{L}_3(3)$ which has no element of order 5 – this follows immediately from the order of the group. But then ϕ trivialises the five cycle in A_5, and so ϕ is trivial by Lemma 2.5.

Suppose that K is of type $\mathtt{B}$. We are now going to argue as in the proof of Lemma 3.1. Consider the set of transvections

$$T = \{\rho_{ij}{}^{\pm 1}, \lambda_{ij}{}^{\pm 1}\}$$

Recall that any two elements in T are conjugate in $\mathrm{SAut}(F_4)$ (by Lemma 2.3). Let $C_T(\rho_{12})$ denote the set of elements in T which commute with ρ_{12}. There are exactly 24 elements in $C_T(\rho_{12})$, namely

$$\{\rho_{12}{}^{\pm1}, \lambda_{12}{}^{\pm1}, \lambda_{13}{}^{\pm1}, \lambda_{14}{}^{\pm1}, \rho_{32}{}^{\pm1}, \lambda_{32}{}^{\pm1}, \rho_{42}{}^{\pm1}, \lambda_{42}{}^{\pm1}, \rho_{34}{}^{\pm1}, \lambda_{34}{}^{\pm1}, \rho_{43}{}^{\pm1}, \lambda_{43}{}^{\pm1}\}$$

Table 7.2 lists every conjugacy class in K conjugate to its own inverse, as can be computed in GAP; it also lists the number of elements in the conjugacy class which commute with a fixed representative of the class.

Note that if $\phi(\rho_{12})$ is an involution, then a direct computation with GAP reveals that ϕ factors through

$$\mathrm{SAut}(F_4)/\langle\!\langle \rho_{12}{}^2 \rangle\!\rangle \cong 2^4 \rtimes \mathrm{L}_4(2)$$

(Note that an analogous statement is true for $n = 3$, but for large enough n the quotient is infinite, as shown in [BV1].) The group K is simple and non abelian, and so ϕ factors through $\mathrm{L}_4(2)$. Hence ϕ is trivial, as $\mathrm{L}_4(2)$ is simple and not isomorphic to K.

We may thus assume that $\phi(\rho_{12})$ is not an involution. Inspecting Table 7.2 we see that there are at most 12 elements in the conjugacy class of $\phi(\rho_{12})$ which commute with $\phi(\rho_{12})$. Thus there exist two elements in $C_T(\rho_{12})$ which get identified under ϕ. Without loss of generality we may assume that we have

$$\phi(\rho_{12}) = \phi({x_{ij}}^{\pm 1})$$

where x is either ρ or λ, and ${x_{ij}}^{\pm 1} \notin \{\rho_{12}, {\rho_{12}}^{-1}\}$.

If $j > 2$, take k such that $k \in \{2,3,4\} \smallsetminus \{i,j\}$. Now

$$\phi({x_{ik}}^{-1}) = \phi([{x_{ij}}^{-1}, {x_{jk}}^{-1}]) = [\phi(\rho_{12})^{\mp 1}, \phi(x_{jk})^{-1}] = 1$$

and so ϕ is trivial. Let us assume that $j \leqslant 2$.

Similarly, if $i > 1$, take $k \in \{3,4\} \smallsetminus \{i\}$. Now

$$\phi({x_{kj}}^{-1}) = \phi([{x_{ki}}^{-1}, {x_{ij}}^{-1}]) = [\phi(x_{ki})^{-1}, \phi(\rho_{12})^{\mp 1}] = 1$$

and so ϕ is trivial.

We are left with the case $(i,j) = (1,2)$ and $x = \lambda$. If ${x_{ij}}^{\pm 1} = \lambda_{12}$ then ϕ factors through $\mathrm{SL}_4(\mathbb{Z})$, since adding the relation $\rho_{12}{\lambda_{12}}^{-1}$ takes the Gersten's presentation of $\mathrm{SAut}(F_n)$ to the Steinberg's presentation of $\mathrm{SL}_n(\mathbb{Z})$. But we know all the finite simple quotients of $\mathrm{SL}_4(\mathbb{Z})$, and K is not one of them. Hence ϕ is trivial.

We are left with the case ${x_{ij}}^{\pm 1} = {\lambda_{12}}^{-1}$. Gersten's presentation contains the relation

$$(\rho_{12}{\rho_{21}}^{-1}\lambda_{12})^4$$

Using the relation $\rho_{12}\lambda_{12}$ gives

$$(\rho_{12}{\rho_{21}}^{-1}{\rho_{12}}^{-1})^4$$

which is equivalent to ${\rho_{21}}^4$. Thus $\phi(\rho_{21})$, and hence also $\phi(\rho_{12})$, has order 4. Inspecting Table 7.2 again we see that in fact we have at least three elements in $C_T(\rho_{12})$ which coincide under ϕ, and so, without loss of generality, there exists ${x_{ij}}^{\pm 1} \notin \{\rho_{12}, {\rho_{12}}^{-1}, {\lambda_{12}}^{-1}\}$ such that

$$\phi(\rho_{12}) = \phi({x_{ij}}^{\pm 1})$$

Thus we are in one of the cases already considered. □

Lemma 7.4. *Every homomorphism $\phi\colon \mathrm{SAut}(F_5) \to K$ where K is a finite group of Lie type of type $\mathtt{A}_3(3), {}^2\mathtt{A}_3(3)$ or $\mathtt{C}_3(3)$ is trivial.*

Proof. As always, we assume that K is simple. The simple group $\mathtt{A}_3(3) \cong \mathrm{L}_4(3)$ contains two conjugacy classes of involutions [CCN+] where they are denoted $2A$ and $2B$. The ATLAS also gives the order of their centralisers. The centraliser of an involution in class $2B$ has order $1152 = 2^7 3^2$ and hence is solvable by Burnside's $p^a q^b$-Theorem. The structure of the centraliser of an involution in class $2A$ is given in [CCN+] and is isomorphic to

$$(4 \times A_6) \rtimes 2$$

Consider $\epsilon_4\epsilon_5 \in \mathrm{SAut}(F_5)$. If $\phi(\epsilon_4\epsilon_5) = 1$ then ϕ factors through $\mathrm{L}_5(2)$ (by Lemma 2.5), which is simple and non-isomorphic to $\mathtt{A}_2(3)$. This trivialises ϕ.

If $\phi(\epsilon_4\epsilon_5) \neq 1$ then ϕ maps $\mathrm{SAut}(F_3)$ (which centralises $\epsilon_4\epsilon_5$) to either a solvable group, or to $(4 \times A_6) \rtimes 2$. In both cases we have $\mathrm{SAut}(F_3) \leqslant \ker\phi$, as $\mathrm{SAut}(F_3)$ is perfect and has no non-trivial homomorphisms to A_6 by Lemma 3.1. This trivialises ϕ.

The simple group ${}^2\mathtt{A}_3(3)$ has a single conjugacy class of involutions, denoted $2A$ in [CCN+], and the centraliser of an involution in this class again has order 1152, hence it is solvable, and so we argue as before.

The conjugacy classes of maximal subgroups of the simple group $\mathtt{C}_3(3)$ are known [CCN+, pg.113]. By inspection we see that it does not contain D_5' and so ϕ factors through $\mathrm{L}_5(2)$ by Lemma 2.5. But $\mathrm{L}_5(2)$ contains an element of order 31, whereas $\mathtt{C}_3(3)$ does not. Thus ϕ is trivial. □

Lemma 7.5. *Let $n \geqslant 4$. Every homomorphism $\phi\colon \mathrm{SAut}(F_n) \to K$ is trivial, where*

(1) *n is even, and K is the of type $\mathtt{A}_k(3)$ or $\mathtt{B}_k(3)$ or $\mathtt{C}_2(3)$ with $k \leqslant \frac{n}{2}$; or*

(2) *n is odd, and K is of type $\mathtt{A}_k(3), {}^2\mathtt{A}_3(3), \mathtt{C}_k(3), \mathtt{D}_k(3)$ or ${}^2\mathtt{D}_k(3)$ with $k \leqslant \frac{n+1}{2}$.*

Proof. As usual, since $\mathrm{SAut}(F_n)$ is perfect, we may assume that we are dealing with adjoint versions; therefore we will use type to denote its adjoint version.

The proof is an induction; the base case when n is even is covered by Lemma 7.3, upon noting that for $k = 1$ we only have to consider $\mathtt{A}_1(3)$, which is solvable.

When n is odd, the base case consists of the groups $\mathtt{A}_1(3)$, $\mathtt{A}_2(3)$, $\mathtt{A}_3(3)$, $\mathtt{C}_2(3)$, $\mathtt{C}_3(3)$, and ${}^2\mathtt{A}_3(3)$. The first two are subgroups of the third, which is covered by Lemma 7.4, and so is $\mathtt{C}_3(3)$. The group $\mathtt{C}_2(3)$ is covered by Lemma 7.3. The remaining group ${}^2\mathtt{A}_3(3)$ is again covered by Lemma 7.4.

Now suppose that $n > 4$. Consider $\phi(\gamma)$. If this is trivial, then we are done by Lemma 7.1. Otherwise, Theorem 5.5 tells us that ϕ maps $\mathrm{SAut}(F_{n-2})$ to a parabolic subgroup P of K. We will now use the notation of Theorem 5.6.

Let

$$\psi\colon \mathrm{SAut}(F_{n-2}) \to L$$

be the map induced by taking the quotient $P \to P/U \cong L$. Since L/M is abelian, and $\mathrm{SAut}(F_{n-2})$ is perfect, we immediately see that $\mathrm{im}\,\psi \leqslant M$.

Suppose that n is even. Then M admits projections onto groups of type $\mathtt{A}_l(3)$ or $\mathtt{B}_l(3)$ with $l < k$ or $\mathtt{C}_2(3)$. The inductive hypothesis shows that $\psi(\mathrm{SAut}(F_{n-2}))$ lies in the intersection of the kernels of such projections. But M is a central product of the images of these projections, and so ψ is trivial. But then ϕ trivialises $\mathrm{SAut}(F_{n-2})$, and the result follows.

When n is odd the situation is similar: the group M admits projections to groups of type $\mathtt{A}_l(3)$, ${}^2\mathtt{A}_3(3)$, $\mathtt{C}_l(3), \mathtt{D}_l(3), {}^2\mathtt{D}_l(3)$ or $\mathtt{A}_1(9)$. The last group is isomorphic to A_6, and every homomorphism from $\mathrm{SAut}(F_3)$ to A_6 is trivial by Lemma 3.1. The other groups are covered by the inductive hypothesis, and we conclude as before. □

Remark 7.6. In fact the groups of type $\mathtt{A}_k(3)$ are not quotients of $\mathrm{SAut}(F_n)$ when $k \leqslant n-2$ which will become clear in the following section.

7.2. Representations of D'_n.

Our aim now is to control projective representations of $\mathrm{SAut}(F_n)$ in small dimensions over fields of odd characteristic. To do this we will first develop some representation theory of the subgroup D'_n.

Definition 7.7. The action of D'_n on $\mathbb{Z}^n$ obtained by abelianising F_n is the *standard action*. Tensoring $\mathbb{Z}^n$ with a field $\mathbb{F}$ gives us the *standard D'_n-module* $\mathbb{F}^n$, and the image of the generators $a_1, \ldots, a_n$ in $\mathbb{F}^n$ is the *standard basis*.

Definition 7.8. Let π be a representation of 2^{n-1}. We set

$$E_I = \{v \in V \mid \pi(\epsilon_i\epsilon_j)(v) = (-1)^{\chi_I(i)+\chi_I(j)}v\}$$

with χ_I standing for the characteristic function of $I \subseteq N$.

Note that $E_I = E_{N \smallsetminus I}$, but otherwise these subspaces intersect trivially.

Lemma 7.9. *Let $n \geqslant 7$. Let $\pi\colon D'_n \to \mathrm{GL}(V)$ be a linear representation of D'_n over a field of characteristic other than 2 in dimension $k < 2n$, such that there is no vector fixed by all elements $\pi(\epsilon_i\epsilon_j)$. Then $k = n$ and π is the standard representation.*

Proof. The elements $\pi(\epsilon_i\epsilon_j) \in \mathrm{GL}(V)$ are all commuting involutions, and so we can simultaneously diagonalise them (since the characteristic of the ground field is not 2). This implies that

$$V = \bigoplus_{|I| \leqslant \frac{n}{2}} E_I$$

Note that for each $m \leqslant \frac{n}{2}$, the subgroup A_n acts on $\bigoplus_{|I|=m} E_I$; such a subspace is also preserved by the subgroup 2^{n-1}, and so by the whole of D_n'. Since there are no vectors fixed by each $\pi(\epsilon_i \epsilon_j)$, we have

$$V = \bigoplus_{|I|>0} E_I$$

The action of A_n permutes the subspaces E_I according to the natural action of A_n on the subsets of N. Hence for any $k < \frac{n}{2}$ we have

$$\dim \bigoplus_{|I|=k} E_I = \binom{n}{k} \dim E_I$$

for any $I \subseteq N$ with $|I| = k$, and for $k = \frac{n}{2}$ (assuming that n is even) we have

$$\dim \bigoplus_{|I|=k} E_I = \frac{1}{2} \binom{n}{k} \dim E_I$$

Since $\dim V < 2n$, we conclude that

$$V = \bigoplus_{i \in N} E_{\{i\}}$$

and each $E_{\{i\}}$ is 1-dimensional. Let us pick a non-zero vector in $E_{\{i\}}$ for each i; these vectors form a basis of V.

It is immediate that with respect to this basis, the action of 2^{n-1} agrees with that of the restriction of the standard representation of D_n' to 2^{n-1}; moreover, it also shows that for each $\tau \in A_n$ the matrix $\pi(\tau)$ is a monomial matrix obtainable from the matrix given by the standard representation of D_n' by multiplication by a diagonal matrix.

Since $n \geqslant 6$, the setwise stabiliser in A_n of any $E_{\{i\}}$ is simple (as it is isomorphic to A_{n-1}), and so we can rescale each vector in our basis so that $\pi(\tau)$ becomes a permutation matrix for each $\tau \in A_n$, and this concludes our proof. □

Recall that $R_p(A_n)$ (occurring in the statement of the following result) denotes the minimal dimension of a faithful projective representation of A_n as in [KL]

Proposition 7.10. *Let $n \geqslant 8$ be even. Let $\pi\colon D_n' \to \mathrm{PGL}(V)$ be a faithful projective representation of dimension less than $n + R_p(A_n)$ over an algebraically closed field $\overline{\mathbb{F}}$ of characteristic $p > 2$. Then the projective representation lifts to a representation $\overline{\pi}\colon D_n' \to \mathrm{GL}(V)$, and the module V splits as $W \oplus U$ where W is a sum of trivial modules, and U is the standard module of D_n'.*

Proof. Let $d \in \mathrm{GL}(V)$ be a lift of $\pi(\delta)$. Since δ is an involution, d^2 is central, and so the characteristic polynomial of d is $x^2 - \lambda$ for some $\lambda \in \mathbb{F}^\times$. Since the field $\mathbb{F}$ is algebraically closed and not of characteristic 2, this characteristic polynomial has two distinct roots, and so d is diagonalisable. Upon multiplying d by a central matrix we may assume that at least one of the entries in the diagonal matrix of d is 1. Thus all the entries are ± 1, and in particular d is also an involution.

For any $\xi \in D_n'$, let $\overline{\xi} \in \mathrm{GL}(V)$ denote a lift of $\pi(\xi)$. Since δ is central in D_n', every $\overline{\xi}$ either preserves the eigenspaces of d, or permutes them. This way we obtain a homomorphism $D_n' \to \mathbb{Z}/2\mathbb{Z}$, which has to be trivial by Lemma 2.1. Thus every $\overline{\xi}$ preserves the eigenspaces of d.

Since $\pi(\delta)$ is not trivial (as π is faithful), the involution d has a non-trivial eigenspace for each eigenvalue, and the same is true for any other involution lifting $\pi(\delta)$.

Take an eigenspace of d of dimension less than n. By [KL, Corollary 5.5.4], the projective module obtained by restricting to this eigenspace is not faithful –

in fact, the action of D'_n on W has the whole of 2^{n-1} in its kernel. Therefore, if both eigenspaces of d are of dimension less than n, then the kernel of π contains an index two subgroup of 2^{n-1}, and therefore is not trivial. This contradicts the assumption on faithfulness of π.

We conclude that one of the eigenspaces of d, say U, has dimension at least n. But then the other eigenspace W, has dimension less than $R_p(A_n)$, and so the restricted projective A_n-module W is trivial. Hence it is also a trivial projective D'_n-module. The abelianisation of D'_n is trivial, and so for each ξ we may choose $\overline{\xi}$ so that its restriction to W is the identity matrix. In this way we obtain a homomorphism $\overline{\pi}\colon D'_n \to \mathrm{GL}(V)$ by declaring $\overline{\pi}(\xi) = \overline{\xi}$. Note that W is a sum of trivial submodules of this representation, and so in particular it is the $(+1)$-eigenspace of $\overline{\delta}$.

It is easy to see that in fact $V = U \oplus W$ as a D'_n-module, since $\overline{\xi}$ preserves the eigenspaces of $\overline{\delta}$ for every $\xi \in D'_n$, as remarked above.

Suppose that there is a non-zero vector in U fixed by each $\overline{\epsilon_i\epsilon_j}$. Then it is also fixed by $\overline{\delta}$, as $\delta = \epsilon_1 \cdots \epsilon_n$ and n is even. But $\overline{\delta}$ acts as minus the identity on U, which is a contradiction. Hence we may apply Lemma 7.9 to U and finish the proof. ☐

Corollary 7.11. *Let $n \geqslant 8$. Let $\pi\colon D'_n \to \mathrm{PGL}(V)$ be a faithful projective representation over an algebraically closed field $\overline{\mathbb{F}}$ of characteristic $p > 2$ of dimension less than $2 \cdot R_p(A_n)$ when n is even or less than $n + R_p(A_{n-1}) - 1$ when n is odd. Then the representation lifts to a representation $\overline{\pi}\colon D'_n \to \mathrm{GL}(V)$, and the module V splits as $W \oplus U$ where W is a sum of trivial and U is the standard module of D'_n.*

Proof. When n is even the result is covered by Proposition 7.10; let us assume that n is odd.

We apply Proposition 7.10 to two subgroups P_1 and P_2 of D'_n isomorphic to D'_{n-1}, where P_i is the stabiliser of a_i in D'_n.

If $\dim V < n-1$ then we immediately learn that V is sum of trivial P_1 modules, and thus it is also a sum of trivial D'_n-modules, as D'_n is the closure of A_n which is simple and has a non-trivial intersection with P_1. Let as assume that

$$\dim V \geqslant n - 1 \geqslant 5$$

We obtain a lift of the projective representations of P_1 and P_2 into $\mathrm{GL}(V)$; it is immediate that the two lifts agree on each $\epsilon_i\epsilon_j$ with $i, j > 2$, since each of the lifts of such an element is an involution with (-1)-eigenspace of dimension 2 and $(+1)$-eigenspace of dimension $\dim V - 2 > 2$, lifting $\pi(\epsilon_i\epsilon_j)$. Similarly, the lifts of the elements $\sigma_{ij}\sigma_{kl}$ (with $i, j, k, l > 2$ all distinct) also agree. It follows that the lifts agree on $P_1 \cap P_2 \cong D'_{n-2}$.

We now repeat the argument for any two stabilisers P_i and P_j. This way we have defined a map from generators of D'_n to $\mathrm{GL}(V)$, which respects all relations supported by some P_i. But it is easy to see that such relations are sufficient for defining the group, and so the map induces a homomorphism $\overline{\pi}\colon D'_n \to \mathrm{GL}(V)$ as required.

Let U_i denote the standard P_i-module, and W_i its complement which is a sum of trivial P_i-modules. Let $U = \sum U_i$. We claim that U is D'_n-invariant: take a generator ξ of D'_n lying in, say, $P_1 \smallsetminus P_2$. Let $x \in U_2$. Then $x = y + z$ with $y \in U_1$ and $z \in W_1$, and so

$$\overline{\pi}(\xi)(x) = y' + z = x - (y - y') \in U_2 + U_1$$

Similar computations for arbitrary indices prove the claim. Now Lemma 7.9 implies that U is the standard representation of D'_n.

Consider V as a 2^{n-1}-representation. Since V is a vector space over a field of characteristic $p > 2$, this representation is semi-simple, and so U has a complement W. It is clear that W is is a sum of trivial 2^{n-1}-representations, since all the non-trivial modules of elements $\epsilon_i\epsilon_j$ are contained in some U_l, and thus in U. For the same reason it is clear that W is a sum of trivial A_n-modules – for this we look at elements $\sigma_{ij}\sigma_{kl}$. We conclude that W is a sum of trivial D_n'-modules. □

7.3. **Projective representations of** $\mathrm{SAut}(F_n)$**.** Now we use the rigidity of D_n'-representations developed above in the context of projective representations of $\mathrm{SAut}(F_n)$.

Theorem 7.12. *Let $n \geqslant 8$. Let $\pi\colon \mathrm{SAut}(F_n) \to \mathrm{PGL}(V)$ be a projective representation of dimension k with $k < 2n - 4$ over an algebraically closed field $\overline{\mathbb{F}}$ of characteristic other than 2. If π does not factor through the natural map $\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$, then $k \geqslant n+1$ and the projective module V contains a trivial projective module of dimension $k - n - 1$.*

Proof. Since $\overline{\mathbb{F}}$ is algebraically closed, $\mathrm{PGL}(V) = \mathrm{PSL}(V)$. As π does not factor through the natural map $\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$, Lemma 2.5 tells us that π restricted to D_n' is injective.

By Corollary 7.11 we see that there is a lifting $\overline{\pi}$ of the projective representation of D_n' to a linear representation on V such that $V = W \oplus U$ as a D_n'-module, where U is standard and W is a sum of trivial modules. Let $u_1, \dots, u_n$ denote the standard basis for U. For notational convenience we will write

$$\overline{\xi} = \overline{\pi}(\xi)$$

for $\xi \in D_n'$.

Note that Corollary 7.11 implies that

$$U = \bigoplus_{i \in N} E_{\{i\}}$$

where $E_{\{i\}}$ is spanned by u_i.

Let us pick a lift of $\pi(\rho_{12})$ acting linearly on V; we will call it $\overline{\rho_{12}}$. Since ρ_{12} commutes with $\epsilon_i\epsilon_j$ with $i, j > 2$, the element $\overline{\rho_{12}}$ permutes the eigenspaces of $\overline{\epsilon_i\epsilon_j}$. But for a given pair (i, j), the eigenspaces of $\overline{\epsilon_i\epsilon_j}$ have dimensions 2 and $\dim V - 2 \geqslant n - 2 > 2$. Thus $\overline{\rho_{12}}$ preserves each eigenspace of $\overline{\epsilon_i\epsilon_j}$. It follows that

$$\overline{\rho_{12}}(u_i) \in \langle u_i \rangle = E_{\{i\}}$$

for each $i > 2$.

Let us choose lifts $\overline{\rho_{ij}}$ of $\pi(\rho_{ij})$ for each pair (i, j). By a discussion identical to the one above we see that $\overline{\rho_{ij}}$ preserves $E_{\{l\}}$ for $l \notin \{i, j\}$.

We may choose $\overline{\rho_{12}}$ so that it fixes u_3. We have

$$[\overline{\rho_{14}}^{-1}, \overline{\rho_{42}}^{-1}] = \lambda \cdot \overline{\rho_{12}}^{-1}$$

for some $\lambda \in \overline{\mathbb{F}} \smallsetminus \{0\}$. But clearly $[\overline{\rho_{14}}^{-1}, \overline{\rho_{42}}^{-1}](u_3) = u_3$, since both $\overline{\rho_{14}}^{-1}$ and $\overline{\rho_{42}}^{-1}$ preserve u_3 up to homothety. Therefore $\lambda = 1$, and thus

$$\overline{\rho_{12}}^{-1}(u_i) = u_i$$

for all $i > 4$. Replacing 4 by another number greater than 3 in the calculation above yields the same result for any $i > 3$. Using analogous argument we may choose each $\overline{\rho_{ij}}$ so that it fixes u_l for all $l \notin \{i, j\}$. It follows that conjugating $\overline{\rho_{12}}$ by an element $\overline{\xi}$ (with $\xi \in A_n$) yields an appropriate element $\overline{\rho_{ij}}$, and not just $\overline{\rho_{ij}}$ up to homothety.

We also see that $\overline{\rho_{12}}$ preserves $Z = W \oplus \langle u_1, u_2 \rangle$, as this is the centraliser of

$$\langle \{\overline{\epsilon_i\epsilon_j}\} \mid i, j > 2\} \rangle$$

Note that W is a subspace of Z of codimension 2; therefore $W' = \overline{\rho_{12}}^{-1}(W) \cap W$ is a subspace of W of codimension at most 2, and so of dimension at least $k-n-2$. Let $x \in W'$ be any vector. Now $\overline{\rho_{12}}(x)$ lies in W, and so

$$\overline{\rho_{12}}(x) = \overline{\sigma_{12}\sigma_{13}\epsilon_3\epsilon_2}\,\overline{\rho_{12}}(x)$$

Thus

$$\overline{\rho_{31}}\,\overline{\rho_{12}}(x) = \overline{\rho_{31}}\,\overline{\sigma_{12}\sigma_{13}\epsilon_3\epsilon_2}\,\overline{\rho_{12}}(x) = \overline{\sigma_{12}\sigma_{13}\epsilon_3\epsilon_2}\,\overline{\rho_{12}}^{-1}\overline{\rho_{12}}(x) = \overline{\sigma_{12}\sigma_{13}\epsilon_3\epsilon_2}(x) = x$$

where the last equality follows from the fact that $x \in W$. Observe that

$$\overline{\rho_{12}}.x = \overline{\epsilon_2\epsilon_3}\,\overline{\rho_{12}}.x = \overline{\rho_{12}}^{-1}\,\overline{\epsilon_2\epsilon_3}.x = \overline{\rho_{12}}^{-1}.x$$

Using a similar argument we show that

$$\overline{\rho_{31}}^{-1}\overline{\rho_{12}}^{-1}(x) = x$$

and so

$$[\overline{\rho_{31}}^{-1}, \overline{\rho_{12}}^{-1}](x) = x$$

But $[\overline{\rho_{31}}^{-1}, \overline{\rho_{12}}^{-1}] = \overline{\rho_{32}}^{-1}$, and so $\overline{\rho_{32}}(x) = x$. Conjugating by elements $\overline{\xi}$ with $\xi \in D'_n$ we conclude that

$$\overline{\rho_{ij}}(x) = x = \overline{\lambda_{ij}}(x)$$

for every i and j. This implies that W' is preserved by $\mathrm{SAut}(F_n)$, and the restricted projective module is trivial.

If $\dim W' = k-n-1$ then we are done. Let us assume that this is not the case, that is that W' is of codimension 2 in W. Consider the involution $\rho_{12}\epsilon_2\epsilon_3$. We set

$$\overline{\rho_{12}\epsilon_2\epsilon_3} = \overline{\rho_{12}}\,\overline{\epsilon_2\epsilon_3}$$

Note that this element satisfies

$$\overline{\rho_{12}\epsilon_2\epsilon_3}^2 = \nu I$$

for some $\nu \in \mathbb{F}^\times$. But $\overline{\rho_{12}\epsilon_2\epsilon_3}(u_3) = -u_3$ and so $\nu = 1$. Therefore $\overline{\rho_{12}\epsilon_2\epsilon_3}$ is an involution.

Let $Y = Z/W'$. Note that $\overline{\epsilon_2\epsilon_3}$ acts on Y, and its (-1)-eigenspace of dimension exactly 1, and the $(+1)$-eigenspace of dimension 3. We also have an action of the involution $\overline{\rho_{12}\epsilon_2\epsilon_3}$ on Y.

Since $\overline{\rho_{12}\epsilon_2\epsilon_3}$ acts trivially on W' and its (-1)-eigenspace in the complement of Z in V is of dimension 1, the dimension of its (-1)-eigenspace in Y must be odd (here we use the fact that π is a map to $\mathrm{PSL}(\mathbb{F})$). Thus there are at least two linearly independent vectors v_1 and v_2 lying in the intersection of the $(+1)$-eigenspace of $\overline{\epsilon_2\epsilon_3}$ and some eigenspace of $\overline{\rho_{12}\epsilon_2\epsilon_3}$.

Since $\overline{\rho_{12}\epsilon_2\epsilon_3}$ is an involution and the characteristic of $\mathbb{F}$ is odd, there exists a complement of W' in Z on which $\overline{\rho_{12}\epsilon_2\epsilon_3}$ acts as on Y. Thus, we have the two vectors corresponding to v_1 and v_2; we will abuse the notation by calling them v_1 and v_2 as well.

Since W' lies in the $(+1)$-eigenspace of $\overline{\epsilon_2\epsilon_3}$, so do v_1 and v_2. Thus there is a non-zero linear combination v_3 of v_1 and v_2 which lies in W, since the codimension of W in the $(+1)$-eigenspace of $\overline{\epsilon_2\epsilon_3}$ is 1. Also, $\overline{\epsilon_2\epsilon_3}$ act trivially on this vector, and so we have found another vector in W which is mapped to W by $\overline{\rho_{12}}$. Arguing exactly as before we show that $\langle v_3 \rangle$ is $\mathrm{SAut}(F_n)$ invariant and trivial as a projective module. Hence $W' \oplus \langle v_3 \rangle$ is also $\mathrm{SAut}(F_n)$ invariant, and is trivial as a projective module since $\mathrm{SAut}(F_n)$ is perfect (Proposition 2.4). □

Let us remark here that there do exist representations of $\mathrm{SAut}(F_n)$ in dimension $n+1$ over any field (over $\mathbb{Z}$ in fact) which do not factor through the natural map $\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$ – see [BV2, Proposition 3.2].

Theorem 7.13. *Let $n \geqslant 10$. Then every finite simple classical group of Lie type in odd characteristic which is a quotient of* $\mathrm{SAut}(F_n)$ *is larger in order than* $\mathrm{L}_n(2)$.

Proof. Let K be such a quotient, and suppose that $|K| \leqslant |\,\mathrm{L}_n(2)|$. Let k denote the rank of K.

If K is of type $\mathtt{A}$ or ${}^2\mathtt{A}$, then Lemma A.5 tells us that $k \leqslant 2n-8$. If K is of any other classical type, then Lemma A.6 tells us that $k \leqslant n-4$.

Let V be the natural projective module of K, and let m denote its dimension. Note that V is an irreducible projective K-module, and $m < 2n-6$. Thus Theorem 7.12 implies that either the representation

$$\mathrm{SAut}(F_n) \to K \to \mathrm{PGL}(V)$$

factors through the natural map $\mathrm{SAut}(F_n) \to \mathrm{SL}_n(\mathbb{Z})$, or $m = n+1$. In the former case we must have $K \cong \mathrm{L}_n(p)$ for some prime p, as K is simple. But $\mathrm{L}_n(p)$ is larger than $\mathrm{L}_n(2)$ for $p \geqslant 3$.

We may thus assume that $m = n+1$. If K is of type $\mathtt{A}$ or ${}^2\mathtt{A}$ then it is immediate that it is too big.

When n is even this means that K is the simple group $\mathtt{B}_{\frac{n}{2}}(q)$. This is larger (in cardinality) than $\mathrm{L}_n(2)$ for every $q > 3$ by Lemma A.7, and so we may assume that $q = 3$. But this is impossible by Lemma 7.5.

When n is odd, K is one of the simple groups $\mathtt{C}_{\frac{n+1}{2}}(q)$, $\mathtt{D}_{\frac{n+1}{2}}(q)$ or ${}^2\mathtt{D}_{\frac{n+1}{2}}(q)$. Lemma A.7 immediately rules out all values of q except for $q = 3$, and again we are done by applying Lemma 7.5. ☐

8. The exceptional groups of Lie type

In this section we focus on exceptional groups of Lie type. These are

(1) the Suzuki-Ree groups ${}^2\mathtt{B}_2(2^{2m+1})$, ${}^2\mathtt{G}_2(3^{2m+1})$, ${}^2\mathtt{F}_4(2^{2m+1})$ and ${}^2F_4(2)'$,
(2) the Steinberg groups ${}^3\mathtt{D}_4(q)$, ${}^2\mathtt{E}_6(q)$ and
(3) the exceptional Chevalley groups $\mathtt{G}_2(q)$, $\mathtt{F}_4(q)$, $\mathtt{E}_6(q)$, $\mathtt{E}_7(q)$ and $\mathtt{E}_8(q)$.

They are defined for all $q \geqslant 2$, $m \geqslant 0$ and are all simple with the following exceptions: the group $\mathrm{Sz}(2) \cong 5 \rtimes 4$ which is visibly solvable; the group ${}^2\mathtt{G}_2(3)$ whose index 3 derived subgroup is isomorphic to $\mathtt{A}_1(8)$; the group $\mathtt{G}_2(2)$ whose index 2 derived subgroup is isomorphic to ${}^2\mathtt{A}_2(3)$; and the group ${}^2\mathtt{F}_4(2)$ whose derived subgroup ${}^2\mathtt{F}_4(2)'$ is simple.

For simplicity, in this section we always use the type symbols to denote the adjoint versions.

We now introduce the following notation: for a group K the *A-rank* of K is the largest n such that K contains a copy of the alternating group A_n. In particular, we will make use of the bounds on the A-rank of the exceptional groups given in [LS, Table 10.1].

Generally, to show that a group K is not the smallest quotient of some $\mathrm{SAut}(F_n)$ we argue as follows: let $n(K)$ be the smallest integer such that

$$|\,\mathrm{L}_{n(K)-1}(2)| < |K| \leqslant |\,\mathrm{L}_{n(K)}(2)|$$

and assume that we have an epimorphism $\phi\colon\ \mathrm{SAut}(F_n) \to K$ with $n \geqslant n(K)$. Now we compare n to the 2-rank, the D'-rank, and A-rank of K. If the 2-rank is smaller than $n-1$ the we use Lemma 2.5 applied to the subgroup 2^{n-1} and conclude that K is in fact a quotient of $\mathrm{SL}_n(\mathbb{Z})$. But the smallest such quotient is $\mathrm{L}_n(2)$. If the A-rank of K is smaller than $n+1$, then we use Lemma 2.5 applied to the subgroup A_n (we observe that if A_{n+1} is not mapped injectively, then neither is A_n for any $n \geqslant 3$). Similarly for the D'-rank.

If the 2-rank and A-rank arguments fail, we look at centralisers. If $n \geqslant 5$ and the simple non-abelian factors of every involution in K have already been shown not to be quotients of $\mathrm{SAut}(F_{n-2})$, then we look at $\phi(\epsilon_1\epsilon_2)$. If this is trivial then we are done by Lemma 2.5; otherwise we obtain a map from $\mathrm{SAut}(F_{n-2})$ (which centralises $\epsilon_1\epsilon_2$) to a group whose simple composition factors are not quotients of $\mathrm{SAut}(F_{n-2})$ (note that the abelian factors are ruled out by the fact that $\mathrm{SAut}(F_{n-2})$ is perfect). Thus $\mathrm{SAut}(F_{n-2})$ lies in the kernel of ϕ, and so in particular ϕ trivialises some transvection. But then it trivialises every transvection since they are all conjugate, and thus ϕ is trivial.

If $n \geqslant 5$ we may argue analogously using the element γ of order 3 from Lemma 7.1; if $n \geqslant 2+k$ for $k \geqslant 5$ odd we may argue in an analogous manner using Lemma 2.8.

Lemma 8.1. *Let K be a finite simple group belonging to one of the following families:*

(1) *the Suzuki groups* ${}^2\mathtt{B}_2(2^{2m+1})$,
(2) *the small Ree groups* ${}^2\mathtt{G}_2(3^{2m+1})$,
(3) *the large Ree groups* ${}^2\mathtt{F}_4(2^{2m+1})$, *or,*
(4) *the Tits group* ${}^2\mathtt{F}_4(2)'$,

where $m \geqslant 1$ is an integer. Then K is not the smallest finite non-trivial quotient of $\mathrm{SAut}(F_n)$.

Proof. The smallest K among the families considered is isomorphic to $\mathrm{Sz}(8) = {}^2\mathtt{B}_2(8)$, and has order greater than $|\,\mathrm{L}_4(2)|$; thus $n \geqslant 5$. The order of ${}^2\mathtt{B}_2(2^{2m+1})$ is coprime to 3, and the order of ${}^2\mathtt{G}_2(3^{2m+1})$ is coprime to 5. Hence it is clear that the simple Suzuki and small Ree groups cannot be quotients of $\mathrm{SAut}(F_n)$ for $n \geqslant 4$, since the alternating group A_5 cannot be mapped injectively, and so we may use Lemma 2.5.

Now assume that K is ${}^2\mathtt{F}_4(2^{2m+1})$ or the Tits group; observe that the smallest member of this family, the Tits group ${}^2\mathtt{F}_4(2)'$, has order greater than $\mathrm{L}_5(2)$ and so $n \geqslant 6$. But, by [Mal, Proposition 2.2] we see that the A-rank of K is 6. □

Lemma 8.2. *Let K be a finite simple group belonging to one of the following families:*

(1) *the exceptional groups of type* $\mathtt{G}_2(q)$, *where $q \geqslant 3$, or*
(2) *the exceptional groups of type* ${}^3\mathtt{D}_4(q)$, *where $q \geqslant 2$.*

Then, K is not the smallest finite non-trivial quotient of $\mathrm{SAut}(F_n)$.

Proof. First, let $K \cong \mathtt{G}_2(q)$. We divide the proof into the case that q is either odd or even. When q is odd the 2-rank of K is 3 by [Kle1, Lemma 2.4], but

$$|K| \geqslant |G_2(3)| > |\,\mathrm{L}_4(2)|$$

and so $n \geqslant 5$.

When $q \geqslant 4$ is even, $|K| > |\,\mathrm{L}_5(2)|$ but from inspection of the list of maximal subgroups of K (see [Coo]) we see that the A-rank of K is at most 5.

For $K \cong {}^3\mathtt{D}_4(q)$ note that the smallest member of this family is ${}^3\mathtt{D}_4(2)$ and has order greater than $|\,\mathrm{L}_5(2)|$. The maximal subgroups of K are known (see [Kle2]) and we see that the A-rank of K is 5. □

For the remaining groups we again split into the odd and even characteristic case.

Lemma 8.3. *Let K be a finite simple exceptional group of type* $\mathtt{F}_4$, $\mathtt{E}_6$, ${}^2\mathtt{E}_6$, $\mathtt{E}_7$ *or* $\mathtt{E}_8$ *in odd characteristic. Then K is not the smallest non-trivial finite quotient of* $\mathrm{SAut}(F_n)$.

Proof. It is easy to see that if K belongs to any of these families, then the order of $|K|$ is bounded below when $q = 3$. If $K \cong \mathtt{F}_4(q)$, $\mathtt{E}_6(q)$ or ${}^2\mathtt{E}_6(q)$, then the A-rank of K is at most 7 [LS, Table 10.1] but the order of K is bounded below by the order of $\mathtt{F}_4(3)$ which has order greater than $\mathrm{L}_9(2)$.

If $K \cong \mathtt{E}_7(q)$, then then the A-rank of K is at most 10, but the smallest member of this family $\mathtt{E}_7(3)$ has order greater than $|\,\mathrm{L}_{14}(2)|$.

Finally, if $K \cong \mathtt{E}_8(q)$, then the A-rank of K is at most 11, but the smallest member of this family $\mathtt{E}_8(3)$ has order greater than $|\,\mathrm{L}_{19}(2)|$. □

Lemma 8.4. *Let K be a finite simple exceptional group of type $\mathtt{F}_4$, $\mathtt{E}_6$, ${}^2\mathtt{E}_6$, $\mathtt{E}_7$ or $\mathtt{E}_8$ defined over a finite field of order $q = 2^m \geqslant 4$. Then K is not the smallest non-trivial finite quotient of* $\mathrm{SAut}(F_n)$.

Proof. It is easy to see that if K belongs to any of these families, then the order of $|K|$ is bounded below when $q = 4$. The degree of the largest alternating group in each of these groups can be found in [LS, Table 10.1]. If $K \cong \mathtt{F}_4(q)$, then the A-rank of K is 10, but K has order greater than $\mathrm{L}_{10}(2)$. If $K \cong \mathtt{E}_6(q)$ or ${}^2\mathtt{E}_6(q)$, then the A-rank is bounded above by 12, but the smallest such group $\mathtt{E}_6(4)$ has order greater than $\mathrm{L}_{12}(2)$. Finally, if $K \cong \mathtt{E}_7(q)$ or $\mathtt{E}_8(q)$, then the A-rank of K is at most 17, but the smallest member of this family $\mathtt{E}_7(4)$ has order greater than $\mathrm{L}_{16}(2)$. □

In order to dispose of the remaining five cases, we state the following result whose proof can be found in [AS, Sections 15-17].

Lemma 8.5.
(1) *Any non-abelian composition factor of an involution centraliser in $\mathtt{E}_6(2)$ is isomorphic to one of $\mathtt{A}_2(2)$, $\mathtt{A}_5(2)$ or $\mathtt{B}_3(2)$.*
(2) *Any non-abelian composition factor of an involution centraliser in $\mathtt{E}_7(2)$ is isomorphic to one of $\mathtt{B}_3(2)$, $\mathtt{B}_4(2)$, $\mathtt{D}_6(2)$ or $\mathtt{F}_4(2)$.*
(3) *Any non-abelian composition factor of an involution centraliser in $\mathtt{E}_8(2)$ is isomorphic to one of $\mathtt{B}_4(2)$, $\mathtt{B}_6(2)$, $\mathtt{F}_4(2)$ or $\mathtt{E}_7(2)$.*

We are now in a position to prove the following.

Lemma 8.6. *Let K be a finite simple exceptional group of type $\mathtt{F}_4(2)$, $\mathtt{E}_6(2)$, ${}^2\mathtt{E}_6(2)$, $\mathtt{E}_7(2)$ or $\mathtt{E}_8(2)$. Then K is not the smallest non-trivial finite quotient of* $\mathrm{SAut}(F_n)$.

Proof. If $K \cong \mathtt{F}_4(2)$, then $n \geqslant 8$, but from the comparison of the character tables of K [CCN$^+$] and of D_8' which can be performed in GAP, we see that D_8' is not a subgroup of K, and we use Lemma 2.5. We eliminate the case $K \cong {}^2\mathtt{E}_6(2)$ in the same way, except that here $n \geqslant 9$.

If $K \cong \mathtt{E}_6(2)$, then $|K| > |\,\mathrm{L}_8(2)|$. By the preceding lemma, it remains to show that any homomorphism from $\mathrm{SAut}(F_n)$ with $n \geqslant 7$ to $\mathtt{A}_2(2)$, $\mathtt{A}_5(2)$ or $\mathtt{B}_3(2)$ is trivial. It can easily be checked in GAP that none of these groups contains a subgroup isomorphic to D_7', hence the result follows from Lemma 2.5. (Also, we will revisit $\mathrm{SAut}(F_7)$ in the next section.)

If $K \cong \mathtt{E}_7(2)$, then $|K| > |\,\mathrm{L}_{11}(2)|$. By the preceding lemma, it remains to show that any homomorphism from $\mathrm{SAut}(F_n)$ with $n \geqslant 10$ to $\mathtt{B}_3(2)$, $\mathtt{B}_4(2)$, $\mathtt{D}_6(2)$ or $\mathtt{F}_4(2)$ is trivial. The groups $\mathtt{B}_3(2)$, $\mathtt{B}_4(2)$ and $\mathtt{D}_6(2)$ are of classical type in even characteristic and smaller in cardinality than $\mathrm{L}_{10}(2)$, hence we can apply Theorem 6.10. The maximal subgroups of $\mathtt{F}_4(2)$ are known and can be found in [CCN$^+$]; it is clear by inspection that the A-rank of $\mathtt{F}_4(2)$ is 10.

Finally, if $K \cong \mathtt{E}_8(2)$, then $|K| > |\,\mathrm{L}_{15}(2)|$. By the preceding lemma, it remains to show that any homomorphism from $\mathrm{SAut}(F_n)$ with $n \geqslant 14$ to $\mathtt{B}_4(2)$, $\mathtt{B}_6(2)$, $\mathtt{F}_4(2)$ or $\mathtt{E}_7(2)$ is trivial. As before, Theorem 6.10 takes care of $\mathtt{B}_4(2)$ and $\mathtt{B}_6(2)$ since they are smaller in cardinality than $\mathrm{L}_{14}(2)$, whereas the A-rank of $\mathtt{F}_4(2)$ and $\mathtt{E}_7(2)$ is at most 13. This completes the proof. □

We can now summarise the preceding lemmata.

Theorem 8.7. *Let K be a finite simple group of exceptional type. If K is a quotient of* $\mathrm{SAut}(F_n)$*, then* $|K| > |\,\mathrm{L}_n(2)|$.

In fact, using the A-rank we can say more: when $n > 16$ then the exceptional groups of Lie type are never quotients of $\mathrm{SAut}(F_n)$, see [LS].

9. Small values of n and the conclusion

We can now conclude the paper.

Theorem 9.1. *Let $n \geqslant 3$. Every non-trivial finite quotient of* $\mathrm{SAut}(F_n)$ *is either greater in cardinality than* $\mathrm{L}_n(2)$*, or isomorphic to* $\mathrm{L}_n(2)$*. Moreover, if the quotient is* $\mathrm{L}_n(2)$*, then the quotient map is the natural map postcomposed with an automorphism of* $\mathrm{L}_n(2)$.

Proof. Suppose that $n \geqslant 8$, and let K be a smallest non-abelian quotient of $\mathrm{SAut}(F_n)$. Since $\mathrm{SAut}(F_n)$ is perfect, K is simple. By Corollary 3.18, K is not an alternating group; by Proposition 4.3, K is not a sporadic group; by Theorem 7.13, K is not a classical group of Lie type in odd characteristic; by Theorem 8.7, K is not an exceptional group of Lie type. Finally, by Theorem 6.10, K is isomorphic to $\mathrm{L}_n(2)$, and the quotient map is obtained by postcomposing the natural map $\mathrm{SAut}(F_n) \to \mathrm{L}_n(2)$ by an automorphism of $\mathrm{L}_n(2)$.

For $3 \leqslant n < 8$, the result follows from Lemmata 9.3 to 9.7 below. □

As indicated above, we now verify Theorem 9.1 for $n \in \{3, \ldots, 7\}$. Note that in view of Proposition 6.6, it is enough to show that a smallest quotient of $\mathrm{SAut}(F_n)$ is isomorphic to $\mathrm{L}_n(2)$.

By Corollary 3.18, Proposition 4.3, and Theorem 8.7 we can assume that K is of classical type. We make use of the list of simple groups in order of size appearing in [CCN+, pgs. 239–242]. Note that this list does not contain all members of the families of types $\mathtt{A}_1(q)$, $\mathtt{A}_2(q)$, ${}^2\mathtt{A}_2(q)$, $\mathtt{A}_3(q)$, $\mathtt{C}_2(q)$ or $\mathtt{G}_2(q)$; we can exclude $\mathtt{G}_2(q)$ by Lemma 8.2.

Lemma 9.2 ([GLS, Theorem 4.10.5])**.** *Let $K \leqslant \mathrm{L}_n(q)$ where q is odd. If $n \leqslant 4$, then the 2-rank of K is bounded above by n.*

The general strategy is exactly as described in the previous section. As before, we use types to denote the adjoint versions.

We now look at each value of n separately.

Lemma 9.3 ($n = 3$)**.** *Let K be a non-abelian finite simple group with $|K| \leqslant |\,\mathrm{L}_3(2)|$. If K is a quotient of* $\mathrm{SAut}(F_3)$*, then $K \cong \mathrm{L}_3(2)$.*

Proof. If K is a non-abelian simple group not isomorphic to $\mathrm{L}_3(2)$ and order at most $|\,\mathrm{L}_3(2)|$, then $K \cong A_5$. But A_5 is not a quotient of $\mathrm{SAut}(F_3)$ by Lemma 3.1. □

Lemma 9.4 ($n = 4$)**.** *Let K be a non-abelian finite simple group with $|K| \leqslant |\,\mathrm{L}_4(2)|$. If K is a quotient of* $\mathrm{SAut}(F_4)$*, then $K \cong \mathrm{L}_4(2)$.*

Proof. Let K be a simple group of order at most $|\,\mathrm{L}_4(2)|$ and a quotient of $\mathrm{SAut}(F_4)$. Assume that K is not isomorphic to $\mathrm{L}_4(2)$. By Lemma 9.2, K is not a subgroup of $\mathtt{A}_2(q)$ where q is odd. Hence, K is isomorphic to one of the following.

$$\mathtt{A}_1(8),\ \mathtt{A}_1(16),\ \mathtt{A}_2(4)$$

With the exception of $\mathrm{L}_3(4)$ (which has the same order as $\mathrm{L}_4(2)$), it is clear from the inspection of their maximal subgroups [CCN+] that they do not contain subgroups isomorphic to D_4'. In the case of $\mathrm{L}_3(4)$, there is a subgroup isomorphic to $2^4 \rtimes A_5$, however this is not isomorphic to the group D_5'. It can be computed in GAP that $\mathrm{L}_3(4)$ does not contain subgroups isomorphic to D_4'. This completes the proof. □

Lemma 9.5 ($n = 5$)**.** *Let K be a non-abelian finite simple group with $|K| \leqslant |\,\mathrm{L}_5(2)|$. If K is a quotient of* $\mathrm{SAut}(F_5)$*, then $K \cong \mathrm{L}_5(2)$.*

Proof. Assume that K is not isomorphic to $\mathrm{L}_5(2)$. By Lemmata 6.1 and 9.2 we can exclude all but the following groups.

$$\mathtt{A}_2(4),\ {}^2\mathtt{A}_2(4),\ {}^2\mathtt{A}_2(8),\ \mathtt{A}_3(3),\ {}^2\mathtt{A}_3(2) \cong \mathtt{C}_2(3),\ \mathtt{C}_2(4),\ \mathtt{C}_2(5),\ {}^2\mathtt{A}_3(3),\ \mathtt{C}_3(2)$$

The groups $\mathtt{A}_3(3)$ and ${}^2\mathtt{A}_3(3)$ are dealt with in Lemma 7.4. Excluding those which also do not contain D_5' as subgroups we are left with the possibilities $\mathtt{C}_3(2)$ and $\mathtt{C}_2(5)$. If $K \cong \mathtt{C}_2(5)$ or $\mathtt{C}_3(2)$, then any non-abelian composition factor of an involution centraliser is isomorphic to A_5 or A_6, neither of which is a quotient of $\mathrm{SAut}(F_3)$ by Lemma 3.1, a contradiction. □

Lemma 9.6 ($n = 6$)**.** *Let K be a non-abelian finite simple group with $|K| \leqslant |\,\mathrm{L}_6(2)|$. If K is a quotient of* $\mathrm{SAut}(F_6)$*, then $K \cong \mathrm{L}_6(2)$.*

Proof. Assume that K is not isomorphic to $\mathrm{L}_6(2)$. By Lemmata 6.1 and 9.2 we can assume that K has dimension at least 4 in even characteristic, in order to contain a subgroup isomorphic to A_7, and dimension at least 5 in odd characteristic in order for the 2-rank to be at least 5. Hence K is isomorphic to one of the following:

$$\mathtt{C}_2(8),\ \mathtt{A}_3(4),\ {}^2\mathtt{A}_3(4),\ \mathtt{C}_3(3),\ \mathtt{C}_3(3),\ {}^2\mathtt{A}_4(2),\ \mathtt{D}_4(2),\ {}^2\mathtt{D}_4(2),\ {}^2\mathtt{A}_5(2)$$

Those groups which contain subgroups isomorphic to D_6' are isomorphic to ${}^2\mathtt{A}_5(2)$, $\mathtt{B}_3(3)$, $\mathtt{D}_4(2)$ and ${}^2\mathtt{D}_4(2)$. The simple factors of the centralisers of elements of order 3 in ${}^2\mathtt{A}_5(2)$ and $\mathtt{B}_3(3)$ can be computed in GAP and are isomorphic to $\mathtt{A}_1(9)$ or $\mathtt{C}_2(3)$, neither of which is a quotient of $\mathrm{SAut}(F_4)$ – this follows from Lemma 7.3 for $\mathtt{C}_2(3)$ and from Lemma 9.4 for $\mathtt{A}_1(9)$, since they are smaller in cardinality than $\mathrm{L}_4(2)$.

The simple factors of the involution centralisers of $\mathtt{D}_4(2)$ and ${}^2\mathtt{D}_4(2)$ are isomorphic to $\mathtt{A}_1(4)$ or $\mathtt{A}_1(9)$, neither of which is a quotient of $\mathrm{SAut}(F_4)$, by Lemma 9.4, since they are both smaller in cardinality than $\mathrm{L}_4(2)$. □

Lemma 9.7 ($n = 7$)**.** *Let K be a non-abelian finite simple group with $|K| \leqslant |\,\mathrm{L}_7(2)|$. If K is a quotient of* $\mathrm{SAut}(F_7)$*, then $K \cong \mathrm{L}_7(2)$.*

Proof. Assume that K is not isomorphic to $\mathrm{L}_7(2)$. Again we make use of the values listed in Lemma 6.1 for $R_p(A_8)$, hence we need to consider groups of dimension at least 7 in odd characteristic and dimension at least 4 in even characteristic. In even characteristic we are left with the following

$$\mathtt{A}_3(8),\ {}^2\mathtt{A}_3(8),\ {}^2\mathtt{A}_4(4),\ \mathtt{B}_2(16),\ \mathtt{C}_3(4),\ \mathtt{C}_4(2),\ \mathtt{D}_5(2),\ {}^2\mathtt{D}_5(2)$$

none of which is a quotient of $\mathrm{SAut}(F_7)$ by Theorem 6.9, with the exception of ${}^2\mathtt{D}_5(2)$.

Now let $K \cong \mathtt{D}_5(2)$. The non-abelian simple quotients of the involution centralisers in K are isomorphic to A_6, A_8 or $\mathtt{C}_3(2)$. Since all of these are smaller than $\mathrm{L}_5(2)$ we apply Lemma 9.5 which completes the proof.

In odd characteristic we have the groups

$$\mathtt{D}_4(3),\ {}^2\mathtt{D}_4(3)$$

Any non-abelian simple factor of a centraliser of an element of order 3 in either of these groups is isomorphic to $\mathtt{A}_1(9)$ or to $\mathtt{C}_2(3)$. By Lemma 9.5, neither of these groups is a quotient of $\mathrm{SAut}(F_5)$ since they are smaller in cardinality than $\mathrm{L}_5(2)$. □

Appendix A. Computations

This appendix contains all the necessary computations. Note that we use type symbols to denote the adjoint versions of the groups of Lie type.

Lemma A.1. *For $n \geqslant 8$ we have*

$$2^{n-3} > \binom{n}{2}$$

Proof. It is enough to observe that the result is true for $n = 8$, and

$$\frac{\binom{n+1}{2}}{\binom{n}{2}} = \frac{n+1}{n-1} \leqslant 2$$

for all $n \geqslant 3$. □

Lemma A.2. *For an even $n \geqslant 12$ we have*

$$\frac{1}{2}\binom{n}{\frac{n}{2}} \geqslant \min\left\{\binom{n}{\lfloor\frac{n}{4}\rfloor}, 2^{n-\lfloor\frac{n}{4}\rfloor-1}\right\}$$

Proof. Let $n = 2m$. We have

$$\begin{aligned}
\frac{\frac{1}{2}\binom{n}{\frac{n}{2}}}{\binom{n}{\lfloor\frac{n}{4}\rfloor}} &= \frac{(\lfloor\frac{m}{2}\rfloor)!(2m-\lfloor\frac{m}{2}\rfloor)!}{2\cdot m!m!} \\
&= \frac{1}{2}\cdot\prod_{i=1}^{\lceil\frac{m}{2}\rceil}\frac{m+i}{\lfloor\frac{m}{2}\rfloor+i} \\
&= \frac{(m+1)(m+2)}{2(\lfloor\frac{m}{2}\rfloor+1)(\lfloor\frac{m}{2}\rfloor+2)}\cdot\prod_{i=3}^{\lceil\frac{m}{2}\rceil}\frac{m+i}{\lfloor\frac{m}{2}\rfloor+i} \\
&\geqslant \frac{(m+1)(m+2)}{(m+2)(\frac{m}{2}+2)} \\
&\geqslant \frac{2m+2}{m+4} \\
&\geqslant 1
\end{aligned}$$

for any $m \geqslant 2$.

We also have

$$2^{n-\lfloor\frac{n}{4}\rfloor-1} \leqslant 2^{n-\frac{n}{4}-\frac{1}{2}} = 2^{\frac{3m-1}{2}}$$

and

$$\frac{2^{\frac{3(m+1)-1}{2}}}{2^{\frac{3m-1}{2}}} = 2^{\frac{3}{2}} < 3$$

Now

$$\frac{\frac{1}{2}\binom{n+2}{\frac{n+2}{2}}}{\frac{1}{2}\binom{n}{\frac{n}{2}}} = \frac{(2m+1)(2m+2)}{(m+1)^2} \geqslant 3$$

We conclude by remarking that $\frac{1}{2}\binom{n}{\frac{n}{2}} \geqslant 2^{\frac{3n-2}{4}}$ for $n = 12$. □

Lemma A.3. *For $n \geqslant 7$ we have $\binom{n}{2}! \cdot \frac{1}{2} > |\,\mathrm{L}_n(2)|$.*

Proof. We have

$$2^{n^2} > |\,\mathrm{L}_n(2)|$$

since the left-hand side is the number of $n \times n$ matrices over the field of 2 elements. We also have

$$m! \geqslant (2\pi m)^{\frac{1}{2}}(\frac{m}{e})^m$$

by Stirling's approximation. Putting $m = \binom{n}{2} = \frac{n(n-1)}{2}$ we obtain

$$\begin{aligned}\binom{n}{2}! \cdot \frac{1}{2} &\geqslant \frac{1}{2}(\pi n(n-1))^{\frac{1}{2}}(\frac{n(n-1)}{2e})^{\frac{n(n-1)}{2}} \\ &= \frac{\pi^{\frac{1}{2}}}{2} \cdot \frac{(n(n-1))^{\frac{n^2-n+1}{2}}}{(2e)^{\frac{n(n-1)}{2}}} \\ &\geqslant \frac{1}{2} \cdot \frac{2^{\frac{5(n^2-n+1)}{2}}}{2^{\frac{5n(n-1)}{4}}} \\ &= 2^{\frac{10(n^2-n+1)-5n(n-1)-4}{4}} \\ &= 2^{\frac{5n^2-5n+6}{4}}\end{aligned}$$

where we have used the fact that $2^{\frac{5}{2}} > 2e$ and that $n(n-1) \geqslant 2^5$, as $n \geqslant 7$.

Now

$$5n^2 - 5n + 6 > 4n^2$$

holds for every $n \geqslant 4$ and we are done. □

We will now proceed to compute certain inequalities between orders of adjoint versions of finite groups of Lie type – these orders be found in [CCN$^+$, pg. xvi]. Let us start by some general remarks.

Firstly, if we fix the type, rank and characteristic, then enlarging the field always results in enlarging the group: this is obvious for the universal versions, and for adjoint versions requires comparing the sizes of centres of the universal versions; such a comparison can easily be performed. Since we will be looking at the smallest groups of a given type, rank and characteristic, we may therefore assume that the field is of prime cardinality.

In fact, arguing as above, we see that for odd characteristics we may assume that the field is of size 3, and for odd characteristics greater than 3 we may assume the field to be of size 5.

Secondly, if we fix the type and field, then increasing the rank always results in enlarging the group. The argument is precisely as above. The same holds for twisted rank, since to increase the twisted rank we have to increase the rank.

Lemma A.4. *Let $n \geqslant 8$. Then every finite group of Lie type in characteristic 2 of twisted rank at least $n-2$ is larger than $\mathrm{L}_n(2)$, with the exception of $\mathtt{A}_{n-2}(2)$ and $\mathtt{A}_{n-1}(2)$.*

Proof. By the discussion above, it is enough to prove the result for

$$\mathtt{A}_{2n-2}(4), {}^2\mathtt{A}_{2n-3}(2), \mathtt{B}_{n-2}(2), \mathtt{C}_{n-2}(2), \mathtt{D}_{n-2}(2), {}^2\mathtt{D}_{n-1}(2)$$

and $\mathtt{E}_6(2), \mathtt{E}_7(2)$ and $\mathtt{E}_8(2)$ for small values of n.

For the groups of type $\mathtt{E}$ we confirm the result by a direct computation.

The orders of $\mathtt{B}_{n-2}(2)$ and $\mathtt{C}_{n-2}(2)$ are equal, and for all $n \geqslant 1$ we have the following identities

$$\frac{|\,\mathtt{B}_n(2)|}{2^n(2^n+1)} = |\,\mathtt{D}_n(2)| = \frac{|\,{}^2\mathtt{D}_{n+1}(2)|}{2^{2n}(2^{n+1}+1)(2^n+1)}$$

Furthermore, $\mathtt{D}_{n-2}(2)$ is a subgroup of $\mathtt{A}_{2n-5}(2)$, and $|\,\mathtt{A}_{2n-5}(2)| < |\,{}^2\mathtt{A}_{2n-4}(2)|$ when $n \geqslant 4$. We also have $|\,\mathtt{A}_{2n-5}(2)| < |\,{}^2\mathtt{A}_{2n-2}(4)|$ Therefore, $\mathtt{D}_{n-2}(2)$ is the smallest group we are considering, and so it remains to prove that $|\,\mathtt{D}_{n-2}(2)| > |\,\mathtt{A}_{n-1}(2)|$.

$$\begin{aligned}\frac{|\,\mathtt{D}_{n-2}(2)|}{|\,\mathtt{A}_{n-1}(2)|} &= \frac{2^{(n-2)(n-3)}(2^{n-2}-1)\prod_{i=1}^{n-3}(2^{2i}-1)}{2^{n(n-1)/2}\prod_{i=1}^{n-1}(2^{i+1}-1)} \\ &= \frac{2^{(n-2)(n-3)}(2^{n-2}-1)\prod_{i=1}^{n-3}(2^{i}-1)\prod_{i=1}^{n-3}(2^{i}+1)}{2^{n(n-1)/2}\prod_{i=1}^{n-1}(2^{i+1}-1)} \\ &= 2^{\frac{n^2-9n+12}{2}}\frac{\prod_{i=1}^{n-3}(2^i+1)}{(2^n-1)(2^{n-1}-1)} \\ &> 2^{\frac{n^2-9n+12}{2}}\frac{2^{(n-2)(n-3)/2}}{2^{2n-1}} \\ &= 2^{\frac{n^2-9n+12}{2}}2^{\frac{(n^2-9n+8)}{2}} \\ &= 2^{n^2-9n+10}\end{aligned}$$

which is at least 1 for all $n \geqslant 8$. □

Lemma A.5. *Let K be any version of a finite classical group of type $\mathtt{A}_k$ or ${}^2\mathtt{A}_k$ in odd characteristic. For every $n \geqslant 6$, if $k \geqslant 2n-7$ then $|K| > |\,\mathrm{L}_n(2)|$.*

Proof. By the previous discussion, it is clear that it is enough to consider the smallest rank, that is $k = 2n-7$, and the simple group K. Also, it is enough to consider $q = 3$, as the orders increase with the field − this is obvious for the universal versions, and for the simple groups follows from inspecting the sizes of the centres of the universal versions.

We have $|\,{}^2\mathtt{A}_k(3)| \geqslant \frac{1}{2}|\,\mathtt{A}_k(3)|$, and

$$\begin{aligned}\frac{1}{2}|\,\mathtt{A}_{2n-7}(3)| &\geqslant \frac{1}{4}\cdot 3^{\binom{2n-6}{2}}\cdot\prod_{i=1}^{2n-7}(3^{i+1}-1) \\ &\geqslant 2^{-2}\cdot 2^{\frac{3(2n-6)(2n-7)}{4}}\cdot\prod_{i=1}^{2n-7}2^{\frac{3i}{2}} \\ &= 2^{\frac{-8+3(2n-6)(2n-7)+3(2n-7)(2n-6)}{4}} \\ &= 2^{6n^2-39n+61} \\ &= 2^{\binom{n}{2}}\cdot 2^{\frac{11n^2-77n+122}{2}} \\ &= 2^{\binom{n}{2}}\cdot\prod_{i=1}^{n}2^{i+1}\cdot 2^{5n^2-40n+60} \\ &> 2^{\binom{n}{2}}\cdot\prod_{i=1}^{n}(2^{i+1}-1) \\ &= |\,\mathtt{A}_{n-1}(2)|\end{aligned}$$

where the last inequality holds for $n \geqslant 6$. □

Lemma A.6. *Let $n \geqslant 8$, and let K be any version of a finite classical group of of type $\mathtt{B}_k$, $\mathtt{C}_k$, $\mathtt{D}_k$ or ${}^2\mathtt{D}_k$ in odd characteristic. IF $k \geqslant n-3$ then $|K| > |\,\mathrm{L}_n(2)|$.*

Proof. By the previous discussion we take $k = n-4$, $q = 3$, and the adjoint version K.

Note that $|\mathtt{B}_k(3)| = |\mathtt{C}_k(3)|$; also $|\mathtt{B}_k(3)| > |\mathtt{D}_k(3)|$. We also have $|{}^2\mathtt{D}_k(3)| \geqslant \frac{1}{2} \cdot |\mathtt{D}_k(3)|$. We then have

$$\begin{aligned}
\frac{1}{2} \cdot |\mathtt{D}_{n-3}(3)| &\geqslant \frac{1}{8} \cdot 3^{(n-3)(n-4)} \cdot (3^{n-3} - 1) \cdot \prod_{i=1}^{n-4} (3^{2i} - 1) \\
&> 2^{-3} \cdot 2^{\frac{3(n-3)(n-4)}{2}} \cdot 2^{\frac{3(n-3)}{2}} \cdot \prod_{i=1}^{n-4} 2^{3i} \\
&= 2^{\frac{3}{2}(-2+(n-3)(n-4)+n-3+(n-3)(n-4))} \\
&= 2^{\frac{3}{2}(2n^2-13n+19)} \\
&= 2^{\binom{n}{2}} 2^{\frac{1}{2}(5n^2-38n+19)} \\
&= 2^{\binom{n}{2}} \cdot \prod_{i=1}^{n} 2^{i+1} \cdot 2^{\frac{1}{2}(4n^2-41n+17)} \\
&> 2^{\binom{n}{2}} \cdot \prod_{i=1}^{n} (2^{i+1} - 1) \\
&= |\mathtt{A}_{n-1}(2)|
\end{aligned}$$

where the last inequality holds for $n \geqslant 10$. In the cases $n = 8$ or 9, it can be verified directly that our claim holds. □

Lemma A.7. *For $n \geqslant 4$ and $q > 3$ odd, the simple groups $\mathtt{B}_{\frac{n}{2}}(q)$ (when n is even), $\mathtt{C}_{\frac{n+1}{2}}(q)$, $\mathtt{D}_{\frac{n+1}{2}}(q)$ and ${}^2\mathtt{D}_{\frac{n+1}{2}}(q)$ (when n is odd) are larger in cardinality than $\mathrm{L}_n(2)$.*

Proof. When n is odd, all of the orders are bounded below by the order of $\mathtt{D}_{\frac{n+1}{2}}(5)$, which is

$$\begin{aligned}
5^{\frac{n^2-1}{4}} \cdot \prod_{i=1}^{\frac{n-1}{2}} (5^{2i} - 1) \cdot (5^{\frac{n+1}{2}} - 1) \cdot \frac{1}{4} &\geqslant 2^{\frac{n^2-1}{2}} \cdot \prod_{i=1}^{\frac{n-1}{2}} 2^{4i} \cdot 2^{n+1} \cdot \frac{1}{4} \\
&= 2^{\frac{n^2-1}{2} + \frac{n^2-n}{2} + n - 1} \\
&= 2^{n^2 + \frac{n}{2} - \frac{1}{2}} \\
&> 2^{n^2} \\
&= 2^{\binom{n}{2}} \cdot 2^{\binom{n+1}{2}} \\
&> 2^{\binom{n}{2}} \cdot \prod_{i=1}^{n} (2^i - 1) \\
&= |\mathrm{L}_n(2)|
\end{aligned}$$

When n is even, we have

$$\begin{aligned}
|\mathtt{B}_{\frac{n}{2}}(5)| &= 5^{\frac{n^2}{4}} \prod_{i=1}^{\frac{n}{2}} (5^{2i} - 1) \\
&> 2^{\frac{n^2}{2}} \prod_{i=1}^{\frac{n}{2}} 2^{4i} \\
&= 2^{\frac{1}{2}(n^2+n(n+2))} \\
&= 2^{n^2+1} \\
&> |L_n(2)|
\end{aligned}$$

□

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