

Critical maximal subgroups and conjugacy of supplements in finite soluble groups

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Abstract

Let G be a finite group with an abelian normal subgroup N . When does N have a unique conjugacy class of complements in G ? We consider this question with a focus on subgroups and properties of maximal subgroups. As corollaries we obtain Theorems 1.6 and 1.7 which are closely related to a result by Parker and Rowley on supplements of a nilpotent normal subgroup (Theorem 1 of [3]). Furthermore, we consider families of maximal subgroups of G closed under conjugation whose intersection equals $\Phi(G)$. In particular, we characterize the soluble groups having a unique minimal family with this property (Theorem 2.3, Remark 2.4). In the case when $\Phi(G) = 1$, these are exactly

the soluble groups in which each abelian normal subgroup has a unique conjugacy class of complements.

1 Critical maximal subgroups and the uniqueness of complements

Throughout this paper, all groups are finite. In group theory we are often interested in cases when a fixed normal subgroup N of a group G has a unique conjugacy class of complements. Famous cases of this type are when N is a normal Hall subgroup or when N is the socle of a primitive soluble group. In this paper we study cases with this uniqueness property, in particular when N is an abelian normal subgroup of G , and we focus on the connection to properties of maximal subgroups of G .

Let G be a group. We denote by $\mathcal{M}(G)$ the set of all maximal subgroups of G and by $\mathcal{M}_0(G)$ the set of all maximal subgroups of G that do not contain the Fitting subgroup $F(G)$. A well-known concept is the Frattini subgroup $\Phi(G) := \bigcap_{M \in \mathcal{M}(G)} M$, which is a characteristic nilpotent subgroup of G and in particular contained in $F(G)$. Many important properties of $\Phi(G)$ are provided in the seminal work by Gaschütz [2]. A property that we shall constantly use is the equality $F(G/N) = F(G)/N$, which holds for every normal subgroup N of G such that $N \leq \Phi(G)$ (see [2], Satz 10). In the sequel, we call a subset of $\mathcal{M}(G)$ closed under G -conjugation a G -set. Thus the G -sets in $\mathcal{M}(G)$ are simply unions of G -conjugacy classes of maximal subgroups. We are interested in G -sets whose intersection is $\Phi(G)$.

Definition 1.1 Let G be a group and $\Lambda \subseteq \mathcal{M}(G)$ a G -subset.

- (1) We call Λ a *spanning subset (or set)* (shortly, $\Lambda \in Sp(G)$) if $\bigcap_{M \in \Lambda} M = \Phi(G)$;
- (2) We call Λ a *minimal spanning subset (or set)* (shortly, $\Lambda \in Spm(G)$) if $\Lambda \in Sp(G)$ and every G -set $\Delta \subset \Lambda$ satisfies $\Delta \notin Sp(G)$;
- (3) We call a maximal subgroup $M \in \mathcal{M}(G)$ a *critical maximal subgroup of G* if M is a member of each spanning set. The set of all critical maximal subgroups of G will be denoted by $\mathcal{M}_{cr}(G)$.

Remark 1.2 (1) Clearly $\mathcal{M}_{cr}(G)$ is a G -subset of $\mathcal{M}(G)$ (it may be empty). It is exactly the intersection of all minimal spanning subsets with respect to G .

(2) Let $H \in \mathcal{M}(G)$ and denote by $\text{Con}(H) = \text{Con}_G(H)$ the set of all G -conjugates of H . Set further $\Phi_H(G) = \bigcap_{M \in \mathcal{M}(G) - \text{Con}(H)} M$. Then $\Phi_H(G) \geq \Phi(G)$, and strict inequality holds if and only if $H \in \mathcal{M}_{cr}(G)$.

Throughout this paper we shall continue to use the notation in (2) of the above remark. It emerges (see Lemma 4.2 in Section 4) that $\mathcal{M}_{cr}(G) \subseteq \mathcal{M}_0(G)$ for every group G . One of our focuses of interest is to study the extreme case of equality $\mathcal{M}_{cr}(G) = \mathcal{M}_0(G)$ (see Section 2).

As the following proposition shows, there is a direct connection between the concept of critical maximal subgroups and the case when a minimal normal subgroup has a unique conjugacy class of complements.

Proposition 1.3 *Let G be a group with an abelian minimal normal subgroup N , and let H be a complement to N in G (so we have $H \in \mathcal{M}(G)$). Then the following conditions are equivalent:*

- (1) $H \in \mathcal{M}_{cr}(G)$;
- (2) All the complements to N in G are conjugate to H .

Furthermore, suppose that the above conditions hold and let $\overline{G} := G/\Phi(G)$. Then \overline{N} is the unique minimal normal subgroup of \overline{G} which is complemented by \overline{H} .

We note further that Lemma 4.3 (Section 4) asserts that a minimal normal subgroup which has a critical maximal subgroup as a complement *must* be abelian.

In the case of a group with a trivial Frattini subgroup we obtain a nice 1-1 correspondence between the G -conjugacy classes of critical maximal subgroups and the abelian minimal normal subgroups with a unique conjugacy class of complements. This is described in the following proposition.

Proposition 1.4 *Let G be a group with $\Phi(G) = 1$. Then, for each G -conjugacy class \mathcal{C} of subgroups in $\mathcal{M}_{cr}(G)$, there exists a **unique** minimal normal subgroup of G complementing the members of \mathcal{C} . Moreover, this minimal normal subgroup is abelian and has a unique conjugacy class of complements (namely, \mathcal{C}).*

When the group G is soluble we have a third equivalent condition that can be added to Proposition 1.3. This condition involves G -isomorphism between G -chief factors. Recall that a G -chief factor K/L is called a Frattini chief factor if $K/L \leq \Phi(G/L)$.

Proposition 1.5 *Let G be a soluble group with a minimal normal subgroup N , and let H be a complement to N in G . Then the following condition is equivalent to Conditions (1) and (2) in Proposition 1.3.*

- (3) *There does not exist a non-Frattini G -chief factor of G/N which is G -isomorphic to N .*

Propositions 1.3 and 1.5 enable us to obtain the following theorems, which are closely connected to a recent result by Parker and Rowley on supplements of a nilpotent normal subgroup (Theorem 1 of [3]). Both theorems provide equivalent conditions to the property discussed by Parker and Rowley, namely, “uniqueness (up to conjugacy) of supplements with equal intersections”. Differently from Theorem 1 of [3], in these conditions only non-Frattini G -chief factors are involved. Notice that, unlike Theorem 1 of [3], in Theorem 1.6 the normal subgroup Q is not assumed to be nilpotent. On the other hand, only supplements which are maximal subgroups of G are considered in Theorem 1.6. As Example 1.8 shows, this theorem does not hold for general supplements which are not maximal subgroups.

Theorem 1.6 *Let G be a soluble group and let $Q \trianglelefteq G$. Then the following conditions are equivalent.*

- (1) *There do not exist two G -chief factors, in Q and G/Q , which are both non-Frattini and G -isomorphic.*
- (2) *Whenever K and L are maximal subgroups of G supplementing Q and satisfying $K \cap Q = L \cap Q$, then K and L are conjugate in G .*

In Theorem 1.7 the normal subgroup Q is assumed to be nilpotent. It is even abelian because of the extra condition $Q \cap \Phi(G) = 1$ (see [4], exercise 626). Notice that the latter condition implies also that every G -chief factor in Q is non-Frattini. Example 1.8 shows that the condition $Q \cap \Phi(G) = 1$ can not be omitted.

Theorem 1.7 *Let G be a soluble group and let Q be a nilpotent normal subgroup of G with $Q \cap \Phi(G) = 1$. Then the following conditions are equivalent.*

- (1) *There do not exist two G -chief factors, in Q and G/Q , which are both non-Frattini and G -isomorphic.*
- (2) *Whenever K and L are subgroups of G supplementing Q and satisfying $K \cap Q = L \cap Q$, then K and L are conjugate in G .*
- (3) *Whenever K and L are maximal subgroups of G supplementing Q and satisfying $K \cap Q = L \cap Q$, then K and L are conjugate in G .*

Example 1.8 Let p be a prime and Q an extraspecial p -group of order p^3 which is of exponent p if p is odd and isomorphic to Q_8 if $p = 2$. Then $\Phi(Q) = Z(Q)$ (the latter denotes the center of Q) is of order p . Let further $E := \langle a, b \rangle$ be an elementary abelian group of order p^2 , and let A be the group of all automorphisms of Q which act trivially on $Z(Q)$. Then $A/\text{Inn}(Q) \simeq Sp_2(p) \simeq SL_2(p)$ (see [7, Theorem 1]). Let c be the preimage in A of an element of order $p + 1$ in $A/\text{Inn}(Q)$.

We let c act trivially on E and set

$$G := (Q \times E) : \langle c \rangle.$$

Then the only non-Frattini G -chief factor of Q is $Q/\Phi(Q)$, and the latter is not G -isomorphic to a G -chief factor of $G/Q \simeq E : \langle c \rangle \simeq E \times \langle c \rangle$.

Nevertheless, there are two non-conjugate complements to Q in G : set $K := E \times \langle c \rangle$, a complement to Q . Then every subgroup conjugate to K in G contains the normal subgroup E . Let $\langle z \rangle = Z(Q)$ and set $L := \langle az, b \rangle : \langle c \rangle$. Then L is a complement to Q in G and $E \cap L = \langle b \rangle$. The latter shows that K and L are not conjugate in G .

2 Groups with a unique minimal spanning set

We recall that the concepts of spanning sets and minimal spanning sets with respect to a group G always refers to G -sets in $\mathcal{M}(G)$. Naturally, a group may have distinct minimal spanning sets. For instance, let G be the Klein group of order 4, then $\mathcal{M}(G)$ consists of 3 members. Since each of them is a normal subgroup, every subset of $\mathcal{M}(G)$ is a G -set. Clearly each subset of $\mathcal{M}(G)$ of size 2 is a minimal spanning set. In this example $\mathcal{M}_{cr}(G) = \emptyset$.

For a less trivial example, let $G = S_3 \times C_2$ (in the sequel, C_n denotes a cyclic group of order n). Then $\Phi(G) = 1$, and $\mathcal{M}(G)$ consists of four conjugacy classes of maximal subgroups: one

class of three subgroups of order 4, and three normal maximal subgroups, from which two are isomorphic to S_3 . The class of size three together with one of the subgroups isomorphic to S_3 forms a minimal spanning set, so there are two distinct minimal spanning sets. Note further that $\mathcal{M}_{cr}(G)$ is just the class of three subgroups of order 4.

A natural question is for which groups there exists a unique minimal spanning set.

Definition 2.1 A group G is called a \mathcal{U} -group (shortly, $G \in \mathcal{U}$) if it has a unique minimal spanning set.

Since $\mathcal{M}_{cr}(G)$ is the intersection of all the minimal spanning sets of G , the following is immediate by the definitions.

Lemma 2.2 *Let G be a group. Then $G \in \mathcal{U}$ if and only if $\mathcal{M}_{cr}(G) \in Sp(G)$.*

So the \mathcal{U} -groups are exactly the groups G for which the intersection of all critical maximal subgroups is $\Phi(G)$.

It was already mentioned in Section 1 that $\mathcal{M}_{cr}(G) \subseteq \mathcal{M}_0(G)$ (see Lemma 4.2 in the sequel). It turns out that for soluble groups the equality case is equivalent to the property \mathcal{U} . Furthermore, for a soluble group G with $\Phi(G) = 1$, this is exactly the case when each nilpotent normal subgroup has a unique conjugacy class of complements. Notice that for a group G with $\Phi(G) = 1$ the nilpotent normal subgroups are all abelian, and each one of them possess at least one complement in G ([4], Exercise 626).

Theorem 2.3 *Let G be a soluble group with $\Phi(G) = 1$. Then the following conditions are equivalent.*

- (1) $G \in \mathcal{U}$;
- (2) $\mathcal{M}_{cr}(G) = \mathcal{M}_0(G)$;
- (3) For each minimal normal subgroup N of G the following holds: N is **not** G -isomorphic to any non-Frattini G -chief factor of G/N .
- (4) Every nilpotent normal subgroup of G has a **unique** conjugacy class of complements in G .

Remark 2.4 Let G be a group. Then clearly $G \in \mathcal{U}$ if and only if $G/\Phi(G) \in \mathcal{U}$. Moreover, since $F(G/\Phi(G)) = F(G)/\Phi(G)$, there is a natural 1-1 correspondence between $\mathcal{M}_0(G)$ and $\mathcal{M}_0(G/\Phi(G))$. Thus, the equivalence of (1) and (2) in Theorem 2.3 holds for every soluble group, without assuming $\Phi(G) = 1$

Remark 2.5 Theorem 2.3 is not true outside the soluble universe: let G be a simple non-abelian group. Then $\mathcal{M}_{cr}(G) = \mathcal{M}_0(G) = \emptyset$ but $G \notin \mathcal{U}$; in fact, every conjugacy class of maximal subgroups is a minimal spanning set.

Interestingly, in a direct connection with the property \mathcal{U} , the following general property of soluble groups holds:

Theorem 2.6 *Let G be a nontrivial soluble group. Then there exists a (possibly trivial) proper nilpotent normal subgroup $N \triangleleft G$ such that $G/N \in \mathcal{U}$ and $F(G/N) = F(G)/N$.*

The following theorem provides a connection between the properties $G' \in \mathcal{U}$ and $G \in \mathcal{U}$. In its proof, Theorem 2.3 is applied.

Theorem 2.7 *Let G be a soluble group satisfying $\Phi(G) = 1 = Z(G)$ and $G' \in \mathcal{U}$. Then $G \in \mathcal{U}$.*

3 A local version

By Theorem 2.3 and Remark 2.4, for G soluble the property \mathcal{U} is equivalent to the condition $\mathcal{M}_{cr}(G) = \mathcal{M}_0(G)$. It emerges that a local version of the latter condition for a specific prime is of interest, as well. First, we include the following notation for every group G and a prime number p : $\mathcal{M}^p(G)$ is the set of all maximal subgroups of G with index a p -power. Similarly, we set $\mathcal{M}_0^p(G) := \mathcal{M}_0(G) \cap \mathcal{M}^p(G)$ and $\mathcal{M}_{cr}^p(G) := \mathcal{M}_{cr}(G) \cap \mathcal{M}^p(G)$. Notice that $\mathcal{M}_0^p(G)$ is exactly the set of maximal subgroups not containing $O_p(G)$.

Now we define a concept that, in the soluble universe, may be viewed as a local version of the property \mathcal{U} .

Definition 3.1 Let G be a group and p a prime number. We say that $G \in \mathcal{U}^p$ if $\mathcal{M}_{cr}^p(G) = \mathcal{M}_0^p(G)$.

Remark 3.2 In view of Theorem 2.3 (and Remark 2.4) the following holds for a soluble group G : $G \in \mathcal{U}$ if and only if $G \in \mathcal{U}^p$ for every prime p .

We have the following local version of Theorem 2.3.

Theorem 3.3 *Let G be a soluble group with $\Phi(G) = 1$ and let p be a prime. Then the following conditions are equivalent.*

- (1) $G \in \mathcal{U}^p$;
- (2) For each minimal normal p -subgroup N of G the following holds: N is **not** G -isomorphic to any non-Frattini G -chief factor of G/N .
- (3) Every normal p -subgroup of G has a **unique** conjugacy class of complements in G .

Let us consider now some special cases of groups in \mathcal{U} and in \mathcal{U}^p . Since $G \in \mathcal{U}$ if and only if $G/\Phi(G) \in \mathcal{U}$ for every group G , the nilpotent \mathcal{U} -groups are exactly the cyclic groups. Let p be a prime and G a group such that $O_p(G) = 1$. In this case $\mathcal{M}_0^p(G) = \emptyset$ and hence, by Lemma 4.2, also $\mathcal{M}_{cr}^p(G) = \emptyset$. Thus every group G with $O_p(G) = 1$ is in \mathcal{U}^p (for a soluble G , this can be deduced directly by Theorem 3.3: if $O_p(G) = 1$ then Condition (2) of the theorem trivially holds).

Let G be a soluble primitive group. Then it is known that G has a unique minimal normal subgroup N , and furthermore $N = F(G)$ and $\Phi(G) = 1$. Moreover, N has a unique conjugacy class of complements (these are exactly the point stabilizers). Thus $G \in \mathcal{U}$ by Theorem 2.3. Here $\mathcal{M}_{cr}(G)$ (which is equal to $\mathcal{M}_0(G)$) is the set of point stabilizers.

Notice that the property \mathcal{U} (and \mathcal{U}^p) is not generally inherited by quotients: let G be the primitive Frobenius group of order 72 with Frobenius complement isomorphic to Q_8 . Then $G \in \mathcal{U}$ but G has a quotient isomorphic to $Q_8 \notin \mathcal{U}$. In this case $G \in \mathcal{U}^2$ but the quotient isomorphic to Q_8 is not in \mathcal{U}^2 (Q_8 is a nilpotent non-cyclic group).

A case when the property \mathcal{U}^p is preserved in quotients is as follows:

Proposition 3.4 *Let G be a group in \mathcal{U}^p , where p is a prime, and let $N \trianglelefteq G$ be a p -subgroup. Then also $G/N \in \mathcal{U}^p$.*

The next theorem describes a “local to global” situation.

Theorem 3.5 *Let G be a soluble group in \mathcal{U}^p , where p is a prime. Assume that for each minimal normal p' -subgroup N we have $G/N \notin \mathcal{U}^p$. Then $G \in \mathcal{U}$.*

The following proposition is proved by applying Theorems 2.3 and 3.3. It gives sufficient (but not necessary) conditions for the properties \mathcal{U}^p and \mathcal{U} .

Proposition 3.6 (1) *Let G be a group with a cyclic Sylow p -subgroup, where p is a prime. Then $G \in \mathcal{U}^p$.*

(2) *Let G be a group such that each Sylow subgroup of G is cyclic (then G is soluble; see [4], Theorem 10.26). Then $G \in \mathcal{U}$.*

The structure of the paper is as follows. After the presentation of our concepts and results we prove the statements of Section 1 in Section 4. Those of Sections 2 and 3 are proved in Section 5.

4 Critical maximal subgroups and proofs of theorems on conjugacy of supplements

We start with the following lemma. Recall the notation in (2) of Remark 1.2.

Lemma 4.1 *Let G be a group with a maximal subgroup K . Assume that $\Phi_K(G) > 1$. Let $N \leq \Phi_K(G)$ be a minimal normal subgroup of G . Then N is abelian. Moreover, if $N \not\leq \Phi(G)$ then the complements to N in G are exactly the members of $Con_G(K)$.*

Proof. Suppose that N is not abelian and let $P \in Syl_p(N)$, for a prime p dividing $|N|$. Then P is not normal in G . By Frattini's argument we have $G = NN_G(P)$, and $N_G(P)$ is contained in some maximal subgroup M of G . But M must be in $Con(K)$, since otherwise $M \geq \Phi_K(G) \geq N$ would lead to the contradiction $G = NM = M$. We conclude that K contains a Sylow p -subgroup of N for *all* primes p , hence $K \geq N$ and $K_G \geq N$. It follows that $N \leq \Phi(G)$ and N is abelian in contradiction to our assumption. This contradiction implies that N is abelian.

Suppose now that $N \not\leq \Phi(G)$. Then there exists $M \in \mathcal{M}$ such that $M \not\geq N$. As N is an abelian minimal normal subgroup, M is a complement to N . Moreover, since $N \leq \Phi_K(G)$ it follows that $M \in Con_G(K)$, completing the proof. \square

One of the basic properties of critical maximal subgroups, that is the connection between $\mathcal{M}_{cr}(G)$ and $\mathcal{M}_0(G)$, follows now.

Lemma 4.2 *Let G be a group with $H \in \mathcal{M}_{cr}(G)$. Set $\bar{G} := G/\Phi(G)$. Then there exists an abelian minimal normal subgroup $\bar{N} \trianglelefteq \bar{G}$ such that \bar{H} is a complement to \bar{N} . In particular, $\mathcal{M}_{cr}(G) \subseteq \mathcal{M}_0(G)$.*

Proof. Assume first $\Phi(G) = 1$. Suppose that $H \in \mathcal{M}_{cr}(G)$, then $\Phi_H(G) > 1$. Let N be a minimal normal subgroup of G such that $N \leq \Phi_H(G)$. Then N is abelian by Lemma 4.1. Now, if $N \leq H$ then $N \leq H_G$ and hence $N \leq \Phi(G) = 1$, a contradiction. Thus H is a complement to N and $H \in \mathcal{M}_0(G)$, completing the proof in the case $\Phi(G) = 1$. The lemma for an arbitrary G follows now by the natural 1-1 correspondence between $\mathcal{M}_{cr}(G)$ and $\mathcal{M}_{cr}(\bar{G})$, and by the equality $F(\bar{G}) = F(G)/\Phi(G)$. \square

The following lemma contains a large part of Proposition 1.3.

Lemma 4.3 *Let G be a group with a minimal normal subgroup N , and let $H \in \mathcal{M}_{cr}(G)$ be a complement to N in G . Then N is abelian, $N \leq \Phi_H(G)$ and all the complements to N in G are conjugate to H .*

Proof. Let $\bar{G} := G/\Phi(G)$. Since $\bar{H} \in \mathcal{M}_{cr}(\bar{G})$ we have $\Phi_{\bar{H}}(\bar{G}) > 1$. Let \bar{L} be a minimal normal subgroup of \bar{G} with $\bar{L} \leq \Phi_{\bar{H}}(\bar{G})$. Then, by Lemma 4.1, \bar{L} is abelian and it is complemented by \bar{H} . Thus $|N| = [G : H] = [\bar{G} : \bar{H}] = |\bar{L}|$ is a prime power and hence N is abelian. Suppose to the contrary that N has a complement K which is not conjugate to H . Then $K \in \mathcal{M}(G)$ and we have $\bar{G} = \bar{H}\bar{N} = \bar{K}\bar{N}$. Furthermore, $\bar{G} = \bar{H}\bar{L}$ and $\bar{L} \leq \Phi_{\bar{H}}(\bar{G})$. Hence $\bar{K} \geq \bar{L}$ and $\bar{K} = (\bar{K} \cap \bar{H})\bar{L}$.

We obtain $\bar{G} = \bar{K}\bar{N} = (\bar{K} \cap \bar{H})\bar{N}\bar{L}$, and $(\bar{K} \cap \bar{H})\bar{N}$ is a complement to \bar{L} in \bar{G} . Since $\bar{L} \leq \Phi_{\bar{H}}(\bar{G})$, this forces by Lemma 4.1 that $(\bar{K} \cap \bar{H})\bar{N}$ is \bar{G} -conjugate to \bar{H} . This implies $\bar{H} \geq \bar{N}$, contradicting $\bar{G} = \bar{H}\bar{N}$. It follows that all the complements to N in G are conjugate to H , hence each maximal subgroup not conjugate to H must contain N . Therefore $N \leq \Phi_H(G)$ and the proof is completed. \square

Remark 4.4 An almost immediate corollary of Lemmas 4.2 and 4.3 is that if $H \in \mathcal{M}_{cr}(G)$ then H contains each non-abelian minimal normal subgroup of G . Indeed, let U be a

non-abelian minimal normal subgroup of G . We have $H \in \mathcal{M}_0(G)$ by Lemma 4.2, hence $G = HF(G)$. Since $F(G)$ centralizes U we obtain $H \cap U \triangleleft G$, forcing $H \cap U = 1$ or $H \geq U$. Since the first option is impossible by Lemma 4.3, we are done.

Our next lemma describes a general property of abelian minimal normal subgroups which is of independent interest.

Lemma 4.5 *Let G be a group with $\Phi(G) = 1$. Let N and L be distinct abelian minimal normal subgroups of G . Then there is a complement to L containing N , and there is a complement to N containing L .*

Proof. It suffices to prove the existence of a complement to N containing L . Suppose to the contrary that each complement to N is a complement to L . Set $\overline{G} := G/L$. Suppose $M \in \mathcal{M}(G)$, $M \geq L$ and \overline{M} is a complement to \overline{N} in \overline{G} . Then $G = MN$ and M is a complement to N in G , contradicting our assumption. Thus \overline{N} has no complements in \overline{G} , implying (since \overline{N} is an abelian minimal normal subgroup) that every maximal subgroup of \overline{G} contains \overline{N} . Therefore $\overline{N} \leq \Phi(\overline{G})$. Now let H be a complement to L in G and consider the natural isomorphism $\overline{G} \simeq H$. Under this isomorphism, there is a 1-1 correspondence $\overline{A} \leftrightarrow AL \cap H$ between the subgroups of \overline{G} and the subgroups of H (where $A \leq G$). Thus, since $\overline{N} \leq \Phi(\overline{G})$ we obtain $NL \cap H \leq \Phi(H)$. Furthermore $1 < NL \cap H \triangleleft G$, implying by [4], Lemma 11.7, $1 < NL \cap H \leq \Phi(G) = 1$, a contradiction. This completes the proof. \square

We are ready for

Proof of Proposition 1.3. Suppose (2) holds. Since N is abelian, every maximal subgroup not in $Con_G(H)$ must contain N , hence $N \leq \Phi_H(G)$. Therefore, $\Phi_H(G) > \Phi(G)$, implying (1) (recall (2) of Remark 1.2). The other direction follows immediately by Lemma 4.3. Now suppose \overline{L} is a minimal normal subgroup of \overline{G} , $\overline{L} \neq \overline{N}$, and \overline{H} is a complement to \overline{L} . Notice that $\overline{H} \in \mathcal{M}_{cr}(\overline{G})$, implying that \overline{L} is abelian by Lemma 4.3. By the same lemma, the complements to \overline{N} and to \overline{L} are exactly the same, namely the members of $Con_{\overline{G}}(\overline{H})$. This forces $\overline{L} = \overline{N}$ by Lemma 4.5 (note $\Phi(\overline{G}) = 1$), a contradiction which completes the proof. \square

Now we can prove Proposition 1.4.

Proof of Proposition 1.4. Let \mathcal{C} be a G -conjugacy class of subgroups in $\mathcal{M}_{cr}(G)$ and let $H \in \mathcal{C}$. We have $H \not\leq F(G)$ since $\mathcal{M}_{cr}(G) \subseteq \mathcal{M}_0(G)$ (Lemma 4.2). Since $\Phi(G) = 1$ it follows

by [4], Exercise 626, that $F(G)$ is a product of (abelian) minimal normal subgroups of G . Thus there exists an abelian minimal normal subgroup N such that $H \not\leq N$. This forces that H and all the members of \mathcal{C} are complements to N . Moreover, $H \in \mathcal{M}_{cr}(G)$, so by Proposition 1.3 N has a unique conjugacy class of complements, which must be \mathcal{C} . Finally, as $\Phi(G) = 1$, Proposition 1.3 implies also that N is the unique minimal normal subgroup of G complementing the members of \mathcal{C} . \square

The following remark on critical maximal subgroups of G and quotients of G is very useful.

Remark 4.6 Let G be a group and let $H \in \mathcal{M}_{cr}(G)$. Suppose $N \triangleleft G$ satisfies $N \leq H$ and set $\overline{G} := G/N$. Then $\overline{H} \in \mathcal{M}_{cr}(\overline{G})$. Indeed, we have $\Phi_H(G) > \Phi(G)$ by Remark 1.2 (2), implying $\Phi_H(G) \not\leq H$. It follows that $\Phi_{\overline{H}}(\overline{G}) \not\leq \overline{H}$, so $\Phi_{\overline{H}}(\overline{G}) > \Phi(\overline{G})$, implying our claim.

The following lemma includes one direction of Proposition 1.5, which holds even without solubility assumption.

Lemma 4.7 *Let G be a group with an abelian minimal normal subgroup N , and let H be a complement to N in G (so we have $H \in \mathcal{M}(G)$). Suppose there exists a non-Frattini G -chief factor of G/N which is G -isomorphic to N . Then $H \notin \mathcal{M}_{cr}(G)$.*

Proof. Denote the given G -chief factor of G/N by K/L . We have $C_G(K/L) = C_G(N) = N \times H_G$, where the first equality holds by the G -isomorphism between K/L and N . Thus $L = L \cap H_G N = (L \cap H_G)N$. Set $\overline{G} := G/L \cap H_G$. If $L \cap H_G > 1$ we may apply induction and deduce $\overline{H} \notin \mathcal{M}_{cr}(\overline{G})$. Then $H \notin \mathcal{M}_{cr}(G)$ by Remark 4.6 and the proof is completed.

Suppose then that $L \cap H_G = 1$, which implies $L = N$. We have $K = K \cap H_G N = (K \cap H_G)N$, therefore $K/L = (K \cap H_G)N/N \simeq K \cap H_G$, a G -isomorphism of G -chief factors. There exists $M \in \mathcal{M}(G)$ such that $M \geq L = N$ and $M \not\leq K$, forcing $M \not\leq K \cap H_G$. Thus $K \cap H_G$, like N , is a non-Frattini minimal normal subgroup of G . Now set $\overline{G} := G/\Phi(G)$, and note that $\overline{K \cap H_G}$ and \overline{N} are two distinct G -isomorphic minimal normal subgroups of \overline{G} (one of them is contained in \overline{H} , the other not). Let $\alpha : \overline{N} \rightarrow \overline{K \cap H_G}$ be a G -isomorphism. Then the diagonal group $D := \{(\overline{u}, \overline{u}^\alpha) \mid u \in N\}$ is a minimal normal subgroup of \overline{G} and $D \not\leq \overline{H}$. Thus \overline{H} is a complement both to \overline{N} and to D , implying $H \notin \mathcal{M}_{cr}(G)$ by Proposition 1.3. The proof is completed. \square

Now Proposition 1.5 can be proved.

Proof of Proposition 1.5. (1) implies (3) is exactly the content of Lemma 4.7, and it holds even without assuming that G is soluble. We complete the proof by proving that (3) implies (2). Suppose that (2) does not hold and there exists $M \notin \text{Con}_G(H)$ such that M is a complement to N . Then $C_G(N) = H_G \times N = M_G \times N$, and by Ore's theorem on soluble groups ([1], Theorem (16.1)) we have $M_G \neq H_G$. Thus $N \simeq NM_G/M_G = H_G M_G/M_G \simeq H_G/H_G \cap M_G \simeq H_G N/(H_G \cap M_G)N$, where all the isomorphisms are G -isomorphisms of G -chief factors.

In order to complete the proof and deduce that (3) does not hold, we show now that $H_G N/(H_G \cap M_G)N$ is a non-Frattini chief factor. We have $M_G \not\leq H_G$ and so ([1], Theorem (16.6)) $H \cap M$ is a maximal subgroup of M . Thus $N(H \cap M)$ is maximal in G . Clearly $N(H \cap M) \geq (H_G \cap M_G)N$, whereas $N(H \cap M) \not\leq H_G N$ since $(H \cap M)H_G N = HN = G$. Thus $H_G N/(H_G \cap M_G)N$ is a non-Frattini chief factor, as required. \square

In the remaining part of this section we prove Theorems 1.6 and 1.7.

Proof of Theorem 1.6. We show first that the negation of (2) implies the negation of (1). So let $K, L \in \mathcal{M}(G)$ satisfy $KQ = G = LQ$, $K \cap Q = L \cap Q$ and K and L are not conjugate in G .

Set $U := K \cap Q = L \cap Q$, then $U \trianglelefteq K$ and $U \trianglelefteq L$, implying $U \trianglelefteq G$ since $G = \langle K, l \rangle$. Let $\overline{G} := G/U$. Then $\overline{K}, \overline{L}$ are non-conjugate maximal subgroups of \overline{G} and \overline{Q} is a normal complement to both of them. Moreover, \overline{Q} is a minimal normal subgroup of \overline{G} since it is a complement to maximal subgroups of \overline{G} . By Proposition 1.5 there exists a non-Frattini \overline{G} -chief factor in $\overline{G}/\overline{Q}$ which is \overline{G} -isomorphic to \overline{Q} . Thus there exists a non-Frattini G -chief factor of G/Q which is G -isomorphic to Q/U . Since Q/U is a non-Frattini G -chief factor (we have $K \geq U$ and $K \not\leq Q$), we have proved the negation of (1).

Next we show that the negation of (1) implies the negation of (2). Let G be a minimal counterexample. So we have a non-Frattini G -chief factor S_1/S_2 in G/Q and a non-Frattini G -chief factor Q_1/Q_2 in Q , and these G -chief factors are G -isomorphic. Suppose first that $Q_2 > 1$. Then, by the minimality assumption, $\overline{G} := G/Q_2$ satisfies the negation of (2), that is \overline{G} has two non-conjugate maximal subgroups which are supplements to \overline{Q} and their intersections with \overline{Q} are equal. This clearly implies that also G satisfies the negation of (2),

so it is not a counterexample. Therefore $Q_2 = 1$, that is, Q_1 is a minimal normal subgroup of G .

Since $Q_1 \not\leq \Phi(G)$, the minimal normal (and abelian) subgroup Q_1 has a supplement which is automatically also a complement to Q_1 in G . By Proposition 1.5 a complement to Q_1 in G is not a critical maximal subgroup, and there exist two non-conjugate maximal subgroups K and L of G such that $KQ_1 = G = LQ_1$.

Notice that Q centralizes S_1/S_2 and hence it centralizes also Q_1 . Since $KQ_1 = G = LQ_1$, it follows that $K \cap Q$ and $L \cap Q$ are normal in G . Suppose $K \cap Q > 1$ and let $\overline{G} := G/K \cap Q$. The \overline{G} -chief factor $\overline{Q_1}$ is non-Frattini since $\overline{K} \not\leq \overline{Q_1}$. Thus \overline{G} satisfies the negation of (1) and, by minimality, also the negation of (2). Thus \overline{G} has two non-conjugate maximal subgroups which are supplements to \overline{Q} and their intersections with \overline{Q} are equal. This implies that also G satisfies the negation of (2), so it is not a counterexample. Thus $K \cap Q = 1$ and similarly $L \cap Q = 1$.

This implies $Q = Q_1$ as well as $K \cap Q = L \cap Q (= 1)$. Thus G again satisfies the negation of (2). This shows that there does not exist a counterexample, the final contradiction. \square

Proof of Theorem 1.7. By [4], Exercise 626, Q is abelian and $Q = \Pi_{1 \leq i \leq t} N_i$ a direct product, where each N_i is minimal normal in G . We prove (1) implies (2). Assume that (1) holds and apply induction on $|G|$. Assume first that $K \cap Q = P = L \cap Q > 1$. Notice that $P \trianglelefteq G$ since it is normal in K and centralized by Q . Consider the quotient $\overline{G} := G/P$. Then $\overline{K}, \overline{L}$ are complements to \overline{Q} . By [4], Exercise 626, P has a complement U in G and we have a natural isomorphism $G/P \simeq U$. Set $\overline{G} := G/P$ and notice that $\Phi(\overline{G}) = \Phi(U)P/P$. Thus $\overline{Q} \cap \Phi(\overline{G}) = (Q/P) \cap (\Phi(U)P/P) = (Q \cap \Phi(U))P/P$.

We have that $Q \cap \Phi(U)$ is normal in U and centralized by Q , hence it is normal in G . Thus, by [4], Lemma 11.7, $Q \cap \Phi(U) \leq Q \cap \Phi(G)$, which is trivial by assumption. Thus we obtain $Q \cap \Phi(U) = 1$, implying by the above $\overline{Q} \cap \Phi(\overline{G}) = 1$. It follows that we may apply the induction to the group \overline{G} with its normal subgroup \overline{Q} and its complements \overline{K} and \overline{L} . Hence \overline{K} and \overline{L} are conjugate in \overline{G} , implying that K and L are conjugate in G as claimed.

So assume from now on $K \cap Q = 1 = L \cap Q$, that is K and L are complements to Q in G . Set $R := \Pi_{2 \leq i \leq t} N_i$ (in the case $t = 1$ take $R = 1$), then $Q = N_1 \times R$. Note that KR and LR are complements to N_1 , hence they are maximal subgroups of G . Consider the quotient $\overline{G} := G/R$. Then $\overline{N_1} = \overline{Q}$ is a minimal normal subgroup of \overline{G} with complements \overline{K} and \overline{L} . Moreover, by (1), $\overline{N_1}$ is not \overline{G} -isomorphic to a non-Frattini \overline{G} -chief factor of $\overline{G}/\overline{N_1}$.

It follows by Proposition 1.5 that \overline{K} and \overline{L} are conjugate in \overline{G} . Thus KR and LR are conjugate in G . Let $x \in G$ satisfy $LR = K^xR$ and set $C := LR$, a complement to N_1 . We have $R \cap \Phi(C) \trianglelefteq G$ since it is normalized by C and centralized by N_1 . Thus, by [4], Lemma 11.7, $R \cap \Phi(C) \leq R \cap \Phi(G) \leq Q \cap \Phi(G) = 1$.

Suppose that we have two C -isomorphic non-Frattini C -chief factors, T/S in R , A/B in C/R . We have a natural isomorphism $C \simeq G/N_1$. Under this isomorphism, we have a 1-1 correspondence $D \leftrightarrow \overline{D} := DN_1/N_1$ (for every $D \leq C$) between the subgroups of C and the subgroups of G/N_1 . Thus $\overline{T}/\overline{S}$, $\overline{A}/\overline{B}$ are two G/N_1 -isomorphic non-Frattini G/N_1 -chief factors. Notice that $\overline{T} = TN_1/N_1 \leq RN_1/N_1 = Q/N_1$ and $\overline{B} = BN_1/N_1 \geq RN_1/N_1 = Q/N_1$. Therefore, TN_1/SN_1 , AN_1/BN_1 are two G -isomorphic non-Frattini G -chief factors, the first of them is in Q and the second in G/Q . This contradicts the assumption that (1) holds. Thus there do not exist two C -isomorphic non-Frattini C -chief factors as described in the beginning of the paragraph. Now we see that the induction may be applied to the group C with its normal subgroup R and its complements L , K^x . Hence L , K^x are conjugate in C and so K , L are conjugate in G , completing the proof that (1) implies (2). Clearly (2) implies (3). Finally, that (3) implies (1) follows directly by (2) implies (1) of Theorem 1.6. This completes the proof. \square

5 \mathcal{U} -groups and characterizations of soluble \mathcal{U} -groups

The following lemmas will be needed for proving Theorem 2.3.

Lemma 5.1 *Let G be a group with $\Phi(G) = 1$. Let $\Lambda \subseteq \mathcal{M}_0(G)$ be a G -set which is $\text{Aut}(G)$ -invariant. Then $\bigcap_{M \in \Lambda} M = \bigcap_{M \in \mathcal{M}_0(G)} M$ if and only if $\Lambda = \mathcal{M}_0(G)$.*

Proof. The sufficient part is clear, so we prove only the other direction. Let $\Lambda \subset \mathcal{M}_0(G)$ and let $H \in \mathcal{M}_0(G) - \Lambda$. Since $\Phi(G) = 1$, the Fitting subgroup $F(G)$ is generated by the abelian minimal normal subgroups of G . By the definition of $\mathcal{M}_0(G)$ it follows that there is an abelian minimal normal subgroup N of G such that $H \not\geq N$. Then $G = HN$. By [5, Chapter 2, corollary to (8.7)] each complement to N in G has the form H^α , $\alpha \in \text{Aut}(G)$. Since Λ is $\text{Aut}(G)$ -invariant, no complement to N in G is in Λ . Thus $\bigcap_{M \in \Lambda} M \geq N$. As N is not contained in $\Phi(G) = 1$, that is, there exists $K \in \mathcal{M}_0(G)$ with $K \not\geq N$. Therefore, $\bigcap_{M \in \mathcal{M}_0(G)} M \not\geq N$, implying $\bigcap_{M \in \Lambda} M \neq \bigcap_{M \in \mathcal{M}_0(G)} M$. The proof is completed. \square

Lemma 5.2 *Let G be a group. Then $\bigcap_{M \in \mathcal{M}_{cr}(G)} M = \bigcap_{M \in \mathcal{M}_0(G)} M$ if and only if $\mathcal{M}_{cr}(G) = \mathcal{M}_0(G)$.*

Proof. Again the sufficient part is clear, so we prove only the other direction. Assume to the contrary that $\bigcap_{M \in \mathcal{M}_{cr}(G)} M = \bigcap_{M \in \mathcal{M}_0(G)} M$ but $\mathcal{M}_{cr}(G) \subset \mathcal{M}_0(G)$. Set $\bar{G} = G/\Phi(G)$ and notice that $\mathcal{M}_{cr}(\bar{G}) = \overline{\mathcal{M}_{cr}(G)}$, $\mathcal{M}_0(\bar{G}) = \overline{\mathcal{M}_0(G)}$ (the latter follows since $F(\bar{G}) = \overline{F(G)}$). Thus we obtain $\mathcal{M}_{cr}(\bar{G}) \subset \mathcal{M}_0(\bar{G})$. As $\mathcal{M}_{cr}(\bar{G})$ is $Aut(\bar{G})$ -invariant, it follows by Lemma 5.1 that $\bigcap_{\bar{M} \in \mathcal{M}_{cr}(\bar{G})} \bar{M} \supset \bigcap_{\bar{M} \in \mathcal{M}_0(\bar{G})} \bar{M}$. Hence $\bigcap_{M \in \mathcal{M}_{cr}(G)} M \supset \bigcap_{M \in \mathcal{M}_0(G)} M$, the desired contradiction. \square

Lemma 5.3 *Let G be a soluble group. Then $\bigcap_{M \in \mathcal{M}_0(G)} M = \Phi(G)$ (that is, $\mathcal{M}_0(G) \in Sp(G)$).*

Proof. Since $F(G/\Phi(G)) = F(G)/\Phi(G)$ we may assume $\Phi(G) = 1$. Suppose to the contrary that $\bigcap_{M \in \mathcal{M}_0(G)} M > 1$ and let $N \leq \bigcap_{M \in \mathcal{M}_0(G)} M$, where N is minimal normal in G . Since G is soluble, N is abelian, hence $N \leq F(G) \leq M$ for each maximal subgroup M not in $\mathcal{M}_0(G)$. Thus $N \leq \bigcap_{M \in \mathcal{M}(G)} M = \Phi(G) = 1$, the desired contradiction. \square

We are ready for

Proof of Theorem 2.3. We prove first the equivalence of (1) and (2). By Lemma 2.2, (1) is equivalent to the condition $\bigcap_{M \in \mathcal{M}_{cr}(G)} M = \Phi(G) = 1$. This, in turn, is equivalent to $\bigcap_{M \in \mathcal{M}_{cr}(G)} M = \bigcap_{M \in \mathcal{M}_0(G)} M$ by Lemma 5.3. Now, Lemma 5.2 implies the equivalence of (1) and (2).

We continue the proof by showing the equivalence of (2) and (3). Suppose that (3) does not hold and let N be a minimal normal subgroup of G which is G -isomorphic to a non-Frattini G -chief factor of G/N . Since $\Phi(G) = 1$, the minimal normal subgroup N has a complement H in $\mathcal{M}_0(G)$. Then Proposition 1.5 gives $H \notin \mathcal{M}_{cr}(G)$. Thus (2) does not hold.

In the other direction, suppose that (3) holds and let $H \in \mathcal{M}_0(G)$. Since $\Phi(G) = 1$, we have that $F(G)$ is generated by the minimal normal subgroups of G . Thus there exists a minimal normal subgroup N such that H is a complement to N . Since N is not G -isomorphic to a non-Frattini G -chief factor of G/N , we obtain by Proposition 1.5 that $H \in \mathcal{M}_{cr}(G)$. We have proved $\mathcal{M}_0(G) \subseteq \mathcal{M}_{cr}(G)$, implying equality by Lemma 4.2, thus (2) holds.

Till now we have proved the equivalence of (1), (2) and (3). Suppose now that these conditions hold. We show that (4) follows. Let N be a nilpotent normal subgroup of G . Since $\Phi(G) = 1$, we have that N is abelian, that N has at least one complement in G and that $N = \prod_{1 \leq i \leq t} N_i$, a direct product of N_i in G ([4], Exercise 626).

We prove (4) by induction on t . If $t = 1$ then N is minimal normal in G . As just noticed, there exists a complement, say L , to N in G , and L is maximal in G . Then $L \in \mathcal{M}_0(G)$, and Condition (2) yields $L \in \mathcal{M}_{cr}(G)$. Therefore, the complements to N in G are exactly the members of $Con_G(H)$, see Proposition 1.3, completing the case $t = 1$. Now let $t > 1$ and set $R := \prod_{1 \leq i \leq t-1} N_i$. Then $N = R \times N_t$. We already know that there exists a complement L to N . Let K be another complement to N . We show that K is conjugate to L . Notice that LR and KR are complements to N_t in G . So they are conjugate in G by the case $t = 1$.

So let $x \in N_t$ satisfy $KR = (LR)^x$. Furthermore, LN_t and KN_t are complements to R in G and so, by induction, they are conjugate, as well. Hence there exists $y \in R$, such that $KN_t = (LN_t)^y$. Since x, y commute we obtain $KR = (LR)^{xy}$ and $KN_t = (LN_t)^{xy}$. Furthermore, $|KR \cap KN_t| = \frac{|KR||KN_t|}{|KRKN_t|} = \frac{|K|^2 \cdot |R||N_t|}{|G|} = |K|$, thus $KR \cap KN_t = K$ and similarly $LR \cap LN_t = L$. Therefore, $K = KR \cap KN_t = (LR \cap LN_t)^{xy} = L^{xy}$ as required.

This completes the induction, and we proved that Conditions (1), (2), (3) imply (4).

Finally, assume that (4) holds. In particular, each minimal normal subgroup of G has a unique conjugacy class of complements. Let $H \in \mathcal{M}_0(G)$, then (as $F(G) = Soc(G)$, a product of minimal normal subgroups) there exists a minimal normal subgroup N complemented by H . It follows by Proposition 1.3 that $H \in \mathcal{M}_{cr}(G)$. Thus Condition (2) follows. This completes the proof of the theorem. \square

Remark 5.4 The proof of Theorem 3.3 is very similar to that of Theorem 2.3. Notice that the Conditions (1), (2), (3) in Theorem 3.3 are parallel to the Conditions (1), (3), (4) of Theorem 2.3 (Theorem 3.3 has one item less than Theorem 2.3 since the “local parallel” to Condition (2) of the latter theorem is just the definition of the property \mathcal{U}^p). The proof of the local theorem goes on the same lines, by replacing $\mathcal{M}_0(G)$ and $\mathcal{M}_{cr}(G)$ by their local versions $\mathcal{M}_0^p(G)$ and $\mathcal{M}_{cr}^p(G)$. Because of this similarity we omit the proof.

We include now the following generalization of Proposition 3.4, describing a case when the property \mathcal{U}^p is preserved by quotients, which then also proves the statement of Proposition 3.4.

Proposition 5.5 *Let G be a group in \mathcal{U}^p , where p is a prime, and let $N \trianglelefteq G$ satisfy $O_p(G/N) = O_p(G)N/N$ (this holds in particular when N is a p -group). Then also $G/N \in \mathcal{U}^p$.*

Proof: We have to show $\mathcal{M}_{cr}^p(G/N) = \mathcal{M}_0^p(G/N)$. By Lemma 4.2, it suffices to prove $\mathcal{M}_0^p(G/N) \subseteq \mathcal{M}_{cr}^p(G/N)$. Let $H/N \in \mathcal{M}_0^p(G/N)$. Then $H/N \not\leq O_p(G/N) = O_p(G)N/N$. Thus $H \not\leq O_p(G)$, that is $H \in \mathcal{M}_0^p(G)$. As $G \in \mathcal{U}^p$, we deduce $H \in \mathcal{M}_{cr}^p(G)$. This, in turn, implies $H/N \in \mathcal{M}_{cr}^p(G/N)$ by Remark 4.6, completing the proof. \square

The following claim provides a sufficient condition for a minimal normal subgroup to have a unique conjugacy class of complements. We shall need it for proving Theorems 2.6 and 3.5.

Proposition 5.6 *Let G be a group and p a prime. Let N be an abelian minimal normal subgroup satisfying $O_p(G/N) > O_p(G)N/N$ (so N must be a p' -group). Then N has a complement H , and all the complements to N in G are conjugate to H . Furthermore, $H \in \mathcal{M}_{cr}(G)$.*

Proof: If $N \leq \Phi(G)$ then $F(G/N) = F(G)/N$, contradicting $O_p(G/N) > O_p(G)N/N$. Thus $N \not\leq \Phi(G)$ and therefore N is not contained in some $H \in \mathcal{M}(G)$. Since N is an abelian minimal normal subgroup we obtain that H is a complement to N . Let M be another complement to N in G . We have natural isomorphisms $H \simeq G/N \simeq M$.

Let $A/N = O_p(G/N)$, then $A = (A \cap H)N = (A \cap M)N$, hence $A \cap H$ and $A \cap M$ are Sylow p -subgroups of A . Thus there exists $a \in A$ such that $A \cap M = (A \cap H)^a$. We have $A/N = O_p(G/N) = O_p(H)N/N = O_p(M)N/N$. Thus we obtain $A \cap H = O_p(H)$, $A \cap M = O_p(M)$. Furthermore, the latter two subgroups are not normal in G since they are not contained in $O_p(G)$ (recall the assumption $O_p(G/N) > O_p(G)N/N$). Since H and M are maximal in G we obtain $H = N_G(A \cap H)$ and $M = N_G(A \cap M)$. Thus $M = N_G(A \cap M) = N_G((A \cap H)^a) = (N_G(A \cap H))^a = H^a$. This shows that all the complements to N in G are conjugate to H . It follows now by Proposition 1.3 that $H \in \mathcal{M}_{cr}(G)$, completing the proof. \square

Next we prove Theorems 2.6 and 3.5.

Proof of Theorem 2.6. Suppose that G is a minimal counterexample with respect to the order of $|G|$. The theorem clearly holds when G is nilpotent (take $N \triangleleft G$ such that G/N is cyclic). So we may assume $G > F(G) > 1$. Let L be a minimal normal subgroup of G .

Assume $F(G/L) = F(G)/L$. As G/L is not a counterexample, there exists $N/L \triangleleft G/L$ that is contained in $F(G/L)$, such that $(G/L)/(N/L) \in \mathcal{U}$ and $F((G/L)/(N/L)) = F(G/L)/(N/L)$. The latter equals by assumption $(F(G)/L)/(N/L)$. This implies by natural isomorphisms that $G/N \in \mathcal{U}$ and $F(G/N) \simeq F(G)/N$, hence the equality $F(G/N) = F(G)/N$. Thus G is not a counterexample in this case.

Therefore, we obtain that $F(G/L) > F(G)/L$ for each minimal normal subgroup L of G . This implies that $\Phi(G) = 1$, since if $L \leq \Phi(G)$ is normal in G then $F(G/L) = F(G)/L$. Hence $F(G)$ is a product of minimal normal subgroups ([4], Exercise 626).

Notice that $F(G) > \Phi(G) = 1$, hence $\mathcal{M}_0(G) \neq \emptyset$. Let $H \in \mathcal{M}_0(G)$. Then $H \not\cong F(G)$ and there exists a minimal normal subgroup L such that $H \not\cong L$. It follows that H is a complement to L . We have $F(G/L) > F(G)/L$, forcing the existence of a prime p satisfying $O_p(G/L) > O_p(G)L/L$. It follows by Proposition 5.6 that $H \in \mathcal{M}_{cr}(G)$. We have proved $\mathcal{M}_0(G) \subseteq \mathcal{M}_{cr}(G)$, and equality follows by Lemma 4.2. Thus $G \in \mathcal{U}$ and G is not a counterexample. This contradiction completes the proof. \square

Proof of Theorem 3.5. We already know that $G \in \mathcal{U}^p$. Thus, in order to prove that $G \in \mathcal{U}$ it suffices to show $G \in \mathcal{U}^q$ for every prime q except p .

First we show that $\Phi(G)$ is a (possibly trivial) p -subgroup. Indeed, suppose to the contrary that there exists a minimal normal p' -subgroup, say N , such that $N \leq \Phi(G)$. Then by assumption $G/N \notin \mathcal{U}^p$. On the other hand, we obtain from $N \leq \Phi(G)$ and $G \in \mathcal{U}^p$ that $G/N \in \mathcal{U}^p$. This contradiction implies that $\Phi(G)$ is indeed a (possibly trivial) p -subgroup.

If $O_q(G) = 1$ for every prime $q \neq p$, then G is in \mathcal{U}^q for every q . Thus we may assume $O_q(G) = 1$ for some $q \neq p$. Then there is $H \in \mathcal{M}_0^q(G)$. Set $\overline{G} := G/\Phi(G)$, then $\overline{H} \not\cong \overline{O_q(G)}$. Hence there exists a minimal normal q -subgroup \overline{N} of \overline{G} such that $\overline{H} \not\cong \overline{N}$. Notice that the (full) preimage N of \overline{N} is nilpotent, since $N/\Phi(G)$ is nilpotent. Let Q be the Sylow q -subgroup of N , then $Q \trianglelefteq G$. Moreover, Q is minimal normal in G since $\overline{N} = \overline{Q}$ is minimal normal in \overline{G} . By assumption $G/Q \notin \mathcal{U}^p$. Thus, by Proposition 5.5, $O_p(G/Q) > O_p(G)Q/Q$. As H is a complement to Q , this implies $H \in \mathcal{M}_{cr}^q(G)$ by Proposition 5.6. We have proved $\mathcal{M}_0^q(G) \subseteq \mathcal{M}_{cr}^q(G)$, and equality follows by Lemma 4.2. Thus $G \in \mathcal{U}^q$, completing the proof. \square

We continue with the proofs of Theorem 2.7 and Proposition 3.6.

Proof of Theorem 2.7. Suppose to the contrary that $G \notin \mathcal{U}$. Then, by Theorem 2.3, there exists a minimal normal subgroup N of G which is G -isomorphic to a non-Frattini G -chief factor of G/N . Let K/L be this G -chief factor. If $KG' > LG'$ then K/L is G -isomorphic to the chief factor KG'/LG' , which is central. Thus $N \leq Z(G) = 1$, a contradiction. Therefore, the equality $KG' = LG'$ holds, implying $K = L(K \cap G')$. We obtain that $K \cap G'/L \cap G'$ is a G -chief factor which is G -isomorphic to K/L , hence also to N . We claim that $K \cap G'/L \cap G'$ is non-Frattini. Indeed, since K/L is non-Frattini there exists $M \in \mathcal{M}(G)$ such that $M \geq L$, $M \not\leq K$. By the equality $K = L(K \cap G')$ we deduce that $M \geq L \cap G'$ and $M \not\leq K \cap G'$. Thus $K \cap G'/L \cap G'$ is a non-Frattini G -chief factor as claimed. It follows further by [4], Lemma 11.7, that $K \cap G'/L \cap G' \not\leq \Phi(G'/L \cap G')$. This means that $K \cap G'/L \cap G'$ is a non-Frattini G' -factor.

We have $N \leq G'$, since $Z(G) = 1$, and N is G' -isomorphic to $K/L \cong K \cap G'/L \cap G'$. By Clifford's theorem (see [6, Chapter 6, Paragraph 1 (1.5)]) N and $K \cap G'/L \cap G'$ are semi-simple G' -modules, that is each one of them is a direct product of simple G' -modules. Thus there exist a minimal normal subgroup M of G' in N and a non-Frattini G' -chief factor $A/L \cap G'$ of G' (where $L \cap G' < A \leq K \cap G'$), which are G' -isomorphic. It follows by Theorem 2.3 that $G' \notin \mathcal{U}$ (notice $\Phi(G') = 1$), a contradiction which completes the proof. \square

Proof of Proposition 3.6. We prove (1) first. The equivalence of $G \in \mathcal{U}^p$ and $G/\Phi(G) \in \mathcal{U}^p$ holds for every group G . Thus we may consider $G/\Phi(G)$ instead of G . So we may assume that $\Phi(G) = 1$. We show that Condition (2) of Theorem 3.3 holds. If $O_p(G) = 1$ then this condition holds in a trivial way. Assume $O_p(G) > 1$ and let N be a minimal normal p -subgroup. Let P be a Sylow p -subgroup of G , then $N \leq P$. If $N < P$ then, as P is cyclic, we have $N \leq \Phi(P)$, implying $N \leq \Phi(G)$ by [4], Lemma 11.7. This contradicts $\Phi(G) = 1$, so $N = P$, hence Condition (2) of Theorem 3.3 holds indeed. Part (2) follows by (1) and Remark 3.2, since G is soluble ([4], Theorem 10.26). \square

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