

## **$K$ -Cosymplectic manifolds**

**Giovanni Bazzoni · Oliver Goertsches**

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**Abstract** In this paper we study  $K$ -cosymplectic manifolds, i.e., smooth cosymplectic manifolds for which the Reeb field is Killing with respect to some Riemannian metric. These structures generalize coKähler structures, in the same way as  $K$ -contact structures generalize Sasakian structures. In analogy to the contact case, we distinguish between (quasi-)regular and irregular structures; in the regular case, the  $K$ -cosymplectic manifold turns out to be a flat circle bundle over an almost Kähler manifold. We investigate de Rham and basic cohomology of  $K$ -cosymplectic manifolds, as well as cosymplectic and Hamiltonian vector fields and group actions on such manifolds. The deformations of type I and II in the contact setting have natural analogues for cosymplectic manifolds; those of type I can be used to show that compact  $K$ -cosymplectic manifolds always carry quasi-regular structures. We consider Hamiltonian group actions and use the momentum map to study the equivariant cohomology of the canonical torus action on a compact  $K$ -cosymplectic manifold, resulting in relations between the basic cohomology of the characteristic foliation and the number of closed Reeb orbits on an irregular  $K$ -cosymplectic manifold.

**Keywords**  $K$ -Cosymplectic · Basic cohomology · Momentum map · Deformations · Closed Reeb orbit

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G. Bazzoni  
Fakultät für Mathematik, Universität Bielefeld, Postfach 100301, 33501 Bielefeld, Germany  
e-mail: gbazzoni@math.uni-bielefeld.de

O. Goertsches (✉)  
Mathematisches Institut der Universität München, Theresienstraße 39, 80333 Munich, Germany  
e-mail: goertsches@math.lmu.de

## 1 Introduction

In [22], Gray and Hervella organized almost Hermitian structures on even dimensional manifolds in 16 classes; this was done to generalize Kähler geometry by requiring only the presence of an almost complex structure and of a compatible Riemannian metric. The same approach was used by Chinea and González [13] to classify almost contact metric structures, the odd-dimensional analogue of almost Hermitian structures. The authors show how almost contact metric structures can be set apart in a certain number<sup>1</sup> of classes. As in the Gray–Hervella paper, the authors of [13] used the covariant derivative of the fundamental form of an almost contact metric structure. In [14], instead, almost contact metric structures are classified using the Nijenhuis tensor.

For us, an almost contact metric structure on an odd-dimensional manifold  $M$ , consisting of a 1-form  $\eta$ , the Reeb field  $\xi$ , a tensor  $\phi: TM \rightarrow TM$ , an adapted Riemannian metric  $g$ , and the associated Kähler form  $\omega$ , is called *cosymplectic* if  $\eta$  and  $\omega$  are closed. A cosymplectic structure determines a Poisson structure in a canonical way: every leaf of the foliation  $\mathcal{F} = \ker \eta$  is endowed with a symplectic structure, given by the pullback of  $\omega$ .

Cosymplectic structures were introduced by Goldberg and Yano in [20] under the name of *almost cosymplectic structures*. In fact, until very recently [7, 8, 12, 17], the word cosymplectic indicated what we call here a coKähler structure, see Sect. 2. The terminology used in this paper was introduced in [33] and seems to have been adopted since (see [4, 5, 10, 26]).

While Kähler structures play a prominent role in even dimension, it is not so clear which one, among almost contact metric structures, should be taken to be the odd dimensional analogue of Kähler structures. Of all possible candidates, Sasakian and coKähler structures seem to be the most natural. Sasakian geometry has been a very active research area for quite a long time; by now, the standard reference is [9].

CoKähler (and, more generally, cosymplectic) geometry has also drawn a great deal of interest in the past years (see for instance [3–5, 17, 26, 33]). Very recently, cosymplectic structures have appeared in a natural way in the study of  $b$ -symplectic (or *log*-symplectic) structures (see [11, 18, 27, 28]). We refer to [10] for a nice overview on cosymplectic geometry and its connection with other areas of mathematics (specially geometric mechanics) as well as with physics.

In this paper we consider cosymplectic structures for which the Reeb vector field is Killing. In analogy with the terminology used in the contact metric setting, we call such structures *K-cosymplectic*. Every coKähler structure is *K-cosymplectic*, but we will give different examples of *K-cosymplectic* non coKähler structures.

Apart from cosymplectic structures, there is another possible meaning of cosymplectic, introduced by Libermann in [34]. A manifold  $M^{2n+1}$  is *cosymplectic* if it is endowed with a 1-form  $\eta$  and a 2-form  $\omega$ , both closed, such that  $\eta \wedge \omega^n \neq 0$  at every point of  $M$ . The Reeb field is uniquely determined by the conditions  $\iota_\xi \omega = 0$  and  $\iota_\xi \eta = 1$ . Starting with this, we can define a cosymplectic manifold  $(M, \eta, \omega)$  to be *K-cosymplectic* if there exists a Riemannian metric  $g$  on  $M$  for which the Reeb field is Killing. Roughly speaking, the difference between the two definitions of *K-cosymplectic* is that in the first case we fix a metric on  $M$ , while in the second one we do not. This is akin to the difference between almost Kähler and symplectic geometry. Despite this apparent discrepancy, Proposition 2.8 shows that the two definitions are equivalent.

<sup>1</sup> 4096, to be precise.

In Sects. 3 and 6, for example, we deal with  $K$ -cosymplectic structures, whereas in Sects. 4, 5, 7 and 8, we do not choose a metric; we simply use the fact that our cosymplectic manifold admits some metric for which the Reeb field is Killing.

The overall theme of this paper is to develop a theory of  $K$ -cosymplectic manifolds, motivated by the existing theory of  $K$ -contact manifolds. We introduce several useful concepts in analogy, such as a distinction between regular, quasi-regular, and irregular structures, see Sect. 3. There, we also prove a structure theorem for compact  $K$ -cosymplectic manifolds. In Sect. 4 we show that the de Rham and the basic (with respect to the characteristic foliation) cohomology of a compact  $K$ -cosymplectic manifold contain the same information. Section 5 is devoted to cosymplectic and Hamiltonian vector fields. These have already been introduced in [1] and used to prove a reduction result *à la Marsden-Weinstein* for cosymplectic manifolds. In Sect. 6 we introduce a generalization of the deformations of type I (we briefly comment on type II deformations in Example 3.8) for general almost contact metric structures and show that these deformations respect ( $K$ -)cosymplectic and coKähler structures. On the one hand, they can be used to show that any compact manifold with a  $K$ -cosymplectic structure admits a quasi-regular  $K$ -cosymplectic structure, see Proposition 6.7; on the other hand, there are situations in which they can be used to deform  $K$ -cosymplectic structures in such a way that one has only finitely many closed Reeb orbits. This is the topic of Sect. 8, in which we use equivariant cohomology to investigate the relation between closed Reeb orbits and basic cohomology on a compact  $K$ -cosymplectic manifold, similar to what exists in the  $K$ -contact case [19]: for a compact  $K$ -cosymplectic manifold with first Betti number  $b_1 = 1$ , the number of closed Reeb orbits is either infinite or given by the total basic Betti number of  $M$ , see Corollary 8.7. The main ingredient in the proof are Morse-Bott properties of the cosymplectic momentum map, as they are shown in Sect. 7. We conclude with examples of irregular coKähler structures on the product of an odd complex quadric with  $S^1$  with minimal number of closed Reeb orbits.

## 2 $K$ -Cosymplectic manifolds and structures

An *almost contact metric structure* (see [8]) on an odd-dimensional smooth manifold  $M$  consists of

- (i) a 1-form  $\eta$  and a vector field  $\xi$ , the Reeb field, such that  $\eta(\xi) \equiv 1$ ;
- (ii) a tensor  $\phi: TM \rightarrow TM$  such that  $\phi^2 = -\text{Id} + \eta \otimes \xi$ ;
- (iii) a Riemannian metric  $g$  such that  $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$  for  $X, Y \in \mathfrak{X}(M)$ .

Conditions (i) and (ii) imply  $\eta \circ \phi = \phi \xi = 0$ . Plugging  $Y = \xi$  into (iii) gives  $\eta(X) = g(X, \xi) \forall X \in \mathfrak{X}(M)$ . Given an almost contact metric structure  $(\eta, \xi, \phi, g)$  on a  $2n + 1$ -dimensional manifold  $M$ , its *Kähler form*  $\omega \in \Omega^2(M)$  is given by  $\omega(X, Y) = g(X, \phi Y)$ . This implies that  $\eta \wedge \omega^n$  is a volume form on  $M$ . The almost contact metric structure  $(\eta, \xi, \phi, g)$  is

- *cosymplectic* if  $d\eta = d\omega = 0$ ;
- *contact metric* if  $\omega = d\eta$ ;
- *normal* if  $N_\phi + d\eta \otimes \xi = 0$ ;
- *coKähler* if it is cosymplectic and normal;
- *Sasakian* if it is contact metric and normal.

Given a tensor  $A: TM \rightarrow TM$ , its *Nijenhuis torsion*  $N_A: \wedge^2 TM \rightarrow TM$  is defined as

$$N_A(X, Y) = A^2[X, Y] - A([AX, Y] + [X, AY]) + [AX, AY].$$

**Definition 2.1** A cosymplectic structure  $(\eta, \xi, \phi, g)$  on a manifold  $M$  is  $K$ -cosymplectic if the Reeb field is Killing.

Let us consider the following four tensors on an almost contact metric manifold:

- $N^{(1)}(X, Y) = N_\phi(X, Y) + d\eta(X, Y)\xi;$
- $N^{(2)}(X, Y) = (L_{\phi X}\eta)(Y) - (L_{\phi Y}\eta)(X);$
- $N^{(3)}(X) = (L_\xi\phi)(X);$
- $N^{(4)}(X) = (L_\xi\eta)(X).$

It is well known (see [8, Theorem 6.1]) that if  $N^{(1)} = 0$ , i.e., if the almost contact metric structure is normal, then  $N^{(i)} = 0$ , for  $i = 2, 3, 4$ . We collect in the following propositions some properties of  $N^{(i)}$  for cosymplectic structures. We refer to [10, Section 3] for the proofs.

**Proposition 2.2** Assume that the structure  $(\eta, \xi, \phi, g)$  is cosymplectic. Then  $N^{(2)} = N^{(4)} = 0$  and

$$\nabla\xi = -\frac{1}{2}\phi \circ N^{(3)},$$

where  $\nabla$  is the Levi-Civita connection of  $g$ . In particular,  $N^{(3)}$  vanishes if and only if  $\xi$  is a parallel vector field.

A parallel vector field on a Riemannian manifold is always Killing; hence the vanishing of  $N^{(3)}$  is a sufficient condition for the cosymplectic structure to be  $K$ -cosymplectic. But it is also necessary, as the following Proposition shows:

**Proposition 2.3** Let  $(\eta, \xi, \phi, g)$  be a  $K$ -cosymplectic structure. Then  $N^{(3)} = 0$ .

We can, therefore, give an equivalent characterization of  $K$ -cosymplectic structures:

**Corollary 2.4** A cosymplectic structure  $(\eta, \xi, \phi, g)$  on a manifold  $M$  is  $K$ -cosymplectic if and only if  $N^{(3)} = 0$ . In particular, the Reeb field is parallel on a  $K$ -cosymplectic manifold.

*Remark 2.5* If  $(\eta, \xi, \phi, g)$  is  $K$ -cosymplectic,  $\eta(X) = g(X, \xi)$  and  $\nabla\xi = 0$  imply  $\nabla\eta = 0$ . The converse is also true: if  $(\eta, \xi, \phi, g)$  is cosymplectic and  $\nabla\eta = 0$ , then  $\xi$  is parallel, hence Killing, and  $(\eta, \xi, \phi, g)$  is  $K$ -cosymplectic. Thus we see that  $K$ -cosymplectic structures are precisely the almost contact metric structures in the class  $\mathcal{C}_2$  of [13].

*Remark 2.6* The same terminology is used in the context of contact metric structures, where the name  $K$ -contact is used when the Reeb field is Killing, which is equivalent to  $N^{(3)} = 0$ .

As mentioned in the introduction there is a second notion of cosymplectic:

**Definition 2.7** A  $2n + 1$ -dimensional manifold  $M$  is *cosymplectic* if it is endowed with a 1-form  $\eta$  and a 2-form  $\omega$ , both closed, such that  $\eta \wedge \omega^n$  is a volume form. The *Reeb field* is uniquely determined by  $\eta(\xi) = 1$  and  $t_\xi\omega = 0$ . We call  $M$   $K$ -cosymplectic if there exists a Riemannian metric on  $M$  for which  $\xi$  is a Killing vector field.

Given a cosymplectic manifold  $(M, \eta, \omega)$ , the 2-form  $\omega$  induces a symplectic structure on  $\mathcal{D} = \ker \eta$ . It is well known that in this case, there exist a Riemannian metric  $g_{\mathcal{D}}$  and an almost complex structure  $J$  on  $\mathcal{D}$  such that  $g_{\mathcal{D}}(X, Y) = \omega(JX, Y)$  and  $g_{\mathcal{D}}(JX, JY) = g_{\mathcal{D}}(X, Y)$  for all  $X, Y$  tangent to  $\mathcal{D}$  (see [2]). The pair  $(g_{\mathcal{D}}, J)$  is said to be compatible with  $\omega$ . Extend

$g_{\mathcal{D}}$  to the whole tangent bundle by requiring  $\xi$  and  $\mathcal{D}$  to be orthogonal and  $\xi$  to have length 1. We get a Riemannian metric  $g = g_{\mathcal{D}} + \eta \otimes \eta$ , which is called *adapted* to the cosymplectic structure  $(\eta, \omega)$ . Such an adapted metric is not unique, as it depends on the choice of  $g_{\mathcal{D}}$ .

We thus have two meanings of the word *K-cosymplectic*. We show next that these two definitions are equivalent.

**Proposition 2.8** *Let  $(M, \eta, \omega)$  be a cosymplectic manifold and let  $\xi$  be the Reeb field. The following are equivalent:*

1.  $(M, \eta, \omega)$  is *K-cosymplectic*;
2.  $M$  admits an adapted Riemannian metric  $g$  for which  $\xi$  is Killing;
3.  $M$  carries a *K-cosymplectic structure*  $(\eta, \xi, \phi, g)$ .

*Proof* Suppose  $(M, \eta, \omega)$  is a *K-cosymplectic manifold* and let  $\bar{g}$  be a metric for which  $\xi$  is Killing. Following [39], we show how to construct an adapted metric  $g$  for which  $\xi$  is Killing. As a byproduct, we will also get a tensor  $\phi$  such that  $(\eta, \xi, \phi, g)$  is *K-cosymplectic* with Kähler form  $\omega$ , proving basically that a *K-cosymplectic manifold*  $(M, \eta, \omega)$  with an adapted Riemannian metric  $g$  for which  $\xi$  is Killing carries a *K-cosymplectic structure*. This will give 1.  $\Rightarrow$  2.  $\Rightarrow$  3. altogether (3.  $\Rightarrow$  1. is obvious).

First of all we define a new Riemannian metric  $\tilde{g}$  by

$$\tilde{g}(X, Y) = \frac{\bar{g}(X, Y)}{\bar{g}(\xi, \xi)} - \left( \frac{\bar{g}(\xi, X)\bar{g}(\xi, Y)}{\bar{g}(\xi, \xi)^2} - \eta(X)\eta(Y) \right).$$

This metric has the property that  $\tilde{g}(\xi, \xi) = 1$  (note that it is positive definite on  $\ker \eta$  because of the Cauchy–Schwarz inequality) and that  $\xi$  is still a Killing vector field with respect to  $\tilde{g}$ . It satisfies  $\tilde{g}(\xi, X) = \eta(X)$  for all  $X$ ; in particular,  $\mathcal{D} = \ker \eta$  equals the orthogonal complement of  $\xi$  with respect to  $\tilde{g}$ .

Define a tensor  $A$  on  $M$  by  $\omega(X, Y) = \tilde{g}(AX, Y)$ , for  $X, Y \in \mathfrak{X}(M)$ . Then  $A\xi = 0$  and  $L_{\xi}A = 0$ . By the properties of  $\tilde{g}$ , the distribution  $\mathcal{D}$  is invariant under  $A$ . By polar decomposition (see [2, 8]), we obtain  $A = \phi \circ B$ , where  $B = \sqrt{A^t A}$ . Because  $A$  is skew-symmetric with respect to  $\tilde{g}$ , it is in particular normal, and it follows that  $\phi$  and  $B$  commute. Moreover,  $B$  is given by  $B = \sqrt{A^t A} = \sqrt{-A^2}$ ; hence,  $B^2 = -A^2 = -\phi^2 B^2$ , which implies that  $\phi$  restricts to an almost complex structure on  $\mathcal{D}$ . Since  $B^2 = -A^2$ , taking the Lie derivative in the  $\xi$  direction we get

$$B \circ L_{\xi} B + L_{\xi} B \circ B = 0.$$

Let  $X \in \mathfrak{X}(M)$  be an eigenvector field of  $B$  with eigenfunction  $e^{\lambda}$ . Then

$$B(L_{\xi} B(X)) = -e^{\lambda} L_{\xi} B(X),$$

and  $L_{\xi} B(X)$  is an eigenvector field of  $B$  with negative eigenfunction  $-e^{\lambda}$ . This is impossible, hence  $L_{\xi} B = 0$ . This is equivalent to  $L_{\xi} \phi = 0$ , since  $L_{\xi} A = 0$ . Define a metric  $g$  on  $M$  by

$$\omega(X, Y) = g(\phi X, Y) \quad \text{and} \quad g(\xi, \xi) = 1. \tag{1}$$

This implies that  $g$  is adapted to  $(\eta, \omega)$ . Since  $L_{\xi} \omega = L_{\xi} \phi = 0$ , (1) also shows that  $\xi$  is Killing with respect to  $g$ . It is immediate to check that  $(\eta, \xi, \phi, g)$  is an almost contact metric structure with Kähler form  $\omega$ ; since  $d\eta = d\omega = 0$ , it is a cosymplectic structure. Finally,  $L_{\xi} \phi = N^{(3)} = 0$ ; hence the structure is *K-cosymplectic* by Corollary 2.4.  $\square$

In the compact case, we can characterize *K-cosymplectic manifolds* in terms of the existence of a certain torus action on  $M$ . More precisely, suppose that  $(M, \eta, \omega)$  is a compact

$K$ -cosymplectic manifold. Let  $\xi$  be the Reeb field and  $g$  be an adapted metric for which  $\xi$  is Killing (such a metric exists by Proposition 2.8). By the Myers-Steenrod theorem (see [36]), the isometry group of  $(M, g)$  is a compact Lie group  $G$ . The closure of the Reeb flow in  $G$  is a torus. Hence,  $M$  is endowed with a smooth torus action. In [45], Yamazaki proved, in the context of  $K$ -contact geometry, that the existence of such a torus action on a compact contact manifold  $(M, \eta)$  completely characterizes  $K$ -contactness. The same statement holds for compact  $K$ -cosymplectic manifolds. Namely, we have

**Proposition 2.9** *Let  $(M, \eta, \omega)$  be a compact cosymplectic manifold. The following are equivalent:*

1.  $(M, \eta, \omega)$  is  $K$ -cosymplectic;
2. there exist a torus  $T$ , a smooth effective  $T$ -action  $h: T \times M \rightarrow M$  and a homomorphism  $\Psi: \mathbb{R} \rightarrow T$  with dense image such that  $\psi_t = h_{\Psi(t)}$ , where  $\psi$  is the flow of the Reeb field.

*Proof* The proof is formally analogous to that of [45, Proposition 2.1]. □

According to [20],  $K$ -cosymplectic geometry coincides with coKähler geometry in dimension 3. However, in higher dimension,  $K$ -cosymplectic structures strictly generalize coKähler structures. To see this, recall that compact coKähler manifolds satisfy very strong topological properties (see [5, 12]). We collect them:

**Proposition 2.10** *Let  $M$  be a compact manifold endowed with a coKähler structure. Then*

- the first Betti number of  $M$  is odd;
- the fundamental group of  $M$  contains a finite index subgroup of the form  $\Gamma \times \mathbb{Z}$ , where  $\Gamma$  is the fundamental group of a Kähler manifold;
- $M$  is formal in the sense of Sullivan.

*Remark 2.11* A compact manifold  $M$  endowed with a cosymplectic structure must have  $b_1(M) \geq 1$ . Indeed, the 1-form  $\eta$  is closed; if it was exact,  $\eta = df$ , then it should vanish at some point  $p \in M$ , because of compactness. But this is impossible, since  $\eta(\xi) = 1$  implies that  $\eta$  is nowhere vanishing. Hence  $[\eta] \neq 0$  in  $H^1(M; \mathbb{R})$ . In particular, a compact cosymplectic manifold is never simply connected. Apart from this, the first Betti number of a compact cosymplectic manifold is not constrained; for instance in [3, Section 5], the authors study the geography of compact cosymplectic manifolds and show that for every pair  $(k, b)$  with  $k \geq 2$  and  $b \geq 1$ , there exists a compact cosymplectic non-formal manifold of dimension  $2k + 1$  with  $b_1 = b$ .

The next result allows us to construct many  $K$ -cosymplectic manifolds. We need to recall two notions. First, given a topological space  $X$  and a homeomorphism  $\varphi: X \rightarrow X$ , the *mapping torus*  $X_\varphi$  is by definition the quotient space

$$X_\varphi = \frac{X \times [0, 1]}{(x, 0) \sim (\varphi(x), 1)}$$

the projection onto the second factor endows  $X_\varphi$  with the structure of a fibre bundle with base  $S^1$  and fibre  $X$ .

An *almost Kähler* manifold is a triple  $(K, \tau, h)$  where  $(K, \tau)$  is a symplectic manifold and  $h$  is a Riemannian metric such that the tensor  $J: TK \rightarrow TK$  defined by the formula

$$h(X, JY) = \tau(X, Y)$$

satisfies  $J^2 = -\text{Id}$ .

**Proposition 2.12** *Let  $(K, \tau, h)$  be an almost Kähler manifold and let  $\varphi: K \rightarrow K$  be a diffeomorphism such that  $\varphi^*\tau = \tau$  and  $\varphi^*h = h$ . Then the mapping torus  $K_\varphi$  carries a natural  $K$ -cosymplectic structure.*

*Proof* Let  $p: K_\varphi \rightarrow S^1$  denote the natural projection. Let  $d\theta$  be the angular form on  $S^1$  and set  $\eta = p^*(d\theta)$ . Consider the projection  $\pi: K \times [0, 1] \rightarrow K$  and use it to pull back the symplectic form  $\tau$  to  $K \times [0, 1]$ . Since  $\varphi^*\tau = \tau$ , this gives a 2-form  $\omega \in \Omega^2(K_\varphi)$  which is closed and satisfies  $\omega^n \neq 0$ , where  $2n = \dim K$ . It is clear that  $\eta \wedge \omega^n \neq 0$ . Define  $\xi$  by  $i_\xi \omega = 0$  and  $\eta(\xi) = 1$ . Then  $\xi$  projects to the tangent vector to  $S^1$ . Consider an interval  $I \subset S^1$  such that  $p^{-1}(I) \cong I \times K$ . Endow  $I \times K$  with the product metric  $(dt)^2 + h$ . Also, define  $\phi: T(I \times K) \rightarrow T(I \times K)$  by  $\phi X = JX$  for  $X \in \mathfrak{X}(K)$  and  $\phi\xi = 0$ . Then  $\xi$  has length 1 and satisfies (globally)  $\phi\xi = 0$ . Since  $\varphi^*h = h$  and  $\varphi^*\tau = \tau$ , we can glue the local metric and the local tensor  $\phi$  to global objects  $g$  and  $\phi$ . By construction,  $\xi$  is Killing for the metric  $g$ . The structure  $(\eta, \xi, \phi, g)$  is  $K$ -cosymplectic. □

*Remark 2.13* Similar to Proposition 2.12 one has (see [33, Lemmata 1 and 4]):

- if  $(K, \tau)$  is a symplectic manifold and  $\varphi: K \rightarrow K$  is a symplectomorphism, then  $K_\varphi$  carries a natural cosymplectic structure;
- if  $(K, \tau, h)$  is a Kähler manifold and  $\varphi: K \rightarrow K$  is a Hermitian isometry, then  $K_\varphi$  carries a natural coKähler structure.

Recall that a Hermitian isometry of a Kähler manifold is a biholomorphism of the underlying complex manifold which is also a Riemannian isometry and hence a symplectomorphism. In all such cases, if  $p: K_\varphi \rightarrow S^1$  denotes the mapping torus projection, the 1-form  $\eta$  on  $K_\varphi$  is simply the pullback of the angular form  $d\theta$  on  $S^1$ , and hence  $[\eta] \in H^1(K_\varphi; \mathbb{Z})$ .

**Corollary 2.14** *Let  $(K, \tau, h)$  be an almost Kähler manifold. Then  $M = K \times S^1$  admits a natural  $K$ -cosymplectic structure.*

*Proof*  $K \times S^1$  is the mapping torus of the identity. □

*Example 2.15* The Kodaira–Thurston manifold  $(KT, \omega)$  is a compact symplectic manifold with  $b_1(KT) = 3$  (see [38, Example 2.1]); in particular,  $KT$  is not Kähler. One can choose a Riemannian metric  $g$  on  $KT$  in such a way that  $(KT, \omega, g)$  is an almost Kähler manifold. Set  $M = KT \times S^1$ . Then  $M$  is a compact  $K$ -cosymplectic manifold with  $b_1(M) = 4$ ; hence  $M$  is not coKähler.

*Example 2.16* By Remark 2.11, the first Betti number of a compact cosymplectic manifold is  $\geq 1$ . We give an example of a manifold  $M$  with  $b_1(M) = 1$ , endowed with a  $K$ -cosymplectic structure which is not coKähler. Let  $N$  be the compact, 8-dimensional simply connected symplectic non-formal manifold constructed in [16]. Let  $\omega$  denote the symplectic form on  $N$ . Endow  $N$  with a Riemannian metric  $\bar{g}$  adapted to  $\omega$ ; then  $(N, \omega, \bar{g})$  is almost Kähler and  $M = N \times S^1$  is a  $K$ -cosymplectic manifold with  $b_1(M) = 1$ . Notice that  $M$  is non-formal because  $N$  is, and hence  $M$  is not coKähler.

More generally, using the same argument can prove the following proposition:

**Proposition 2.17** *Let  $(K, \omega)$  be a compact symplectic non-formal manifold. Endow  $M = K \times S^1$  with the natural product  $K$ -cosymplectic structure. Then  $M$  is  $K$ -cosymplectic, but not coKähler.*

Further examples of compact  $K$ -cosymplectic non coKähler manifolds can be constructed using the fact that, by a result of Gompf (see [21]), every finitely presentable group is the fundamental group of a symplectic 4-manifold. By applying Corollary 2.14, we can obtain  $K$ -cosymplectic manifolds with fundamental group  $\Gamma \times \mathbb{Z}$ , where  $\Gamma$  is any finitely presentable group. But we have noticed above that the fundamental group of a compact coKähler manifold can not be arbitrary.

### 3 Regular $K$ -cosymplectic structures

Let  $(\eta, \xi, \phi, g)$  be an almost contact metric structure on  $M$ . Consider the distribution  $\ker \eta$ . By the Frobenius Theorem, such a distribution integrates to a foliation  $\mathcal{F}$  if and only if the integrability condition  $\eta \wedge d\eta = 0$  is satisfied. As we already observed, if  $(\eta, \xi, \phi, g)$  is cosymplectic then  $d\eta = 0$ ; hence  $\eta \wedge d\eta = 0$ . As a consequence, if  $(\eta, \xi, \phi, g)$  is cosymplectic,  $\ker \eta$  integrates to a foliation of codimension 1 on  $M$ , which we call *vertical foliation*. In the case of a cosymplectic manifold  $(M, \eta, \omega)$ , we define the vertical foliation again as  $\ker \eta$ .

For codimension 1 foliations (not necessarily given as the kernel of a 1-form), we use the following notion of regularity:

**Definition 3.1** Let  $M$  be a smooth  $n$ -dimensional manifold endowed with a codimension-1 foliation  $\mathcal{F}$ . Then  $\mathcal{F}$  is *regular* if every point  $p \in M$  has a cubic coordinate neighborhood  $U$ , with coordinates  $(x^1, \dots, x^n)$ , such that  $x^i(p) = 0$  for every  $i$  and such that the leaf  $L_p$  of  $\mathcal{F}$  passing through  $p$  is given by the equation  $x^1 = 0$ .

There are two simple situations in which a codimension 1 foliation is regular:

*Example 3.2* Suppose that  $M$  is a compact manifold of dimension  $n$  endowed with a closed and nowhere vanishing 1-form  $\sigma$ . This implies that  $[\sigma] \neq 0 \in H^1(M; \mathbb{R})$  and that  $\mathcal{F} = \ker \sigma$  is a foliation. Assume further that  $[\sigma]$  lies in  $H^1(M; \mathbb{Z}) \subset H^1(M; \mathbb{R})$ . By a result of Tischler (see [44]),  $M$  fibres over the circle; the map  $p: M \rightarrow S^1$  is given precisely by  $[\sigma]$  under the usual correspondence  $H^1(M; \mathbb{Z}) \cong [M, K(\mathbb{Z}, 1)] = [M, S^1]$ . In particular,  $\sigma = p^*(d\theta)$ , where  $d\theta$  is the angular form on  $S^1$ . One can also show that  $M$  is, in fact, the mapping torus  $N_\varphi$  of a diffeomorphism  $\varphi: N \rightarrow N$ , where  $N$  is a smooth, compact manifold of dimension  $n - 1$ . In this case, the fibres of the projection  $p: M = N_\varphi \rightarrow S^1$  coincide with the leaves of the foliation  $\mathcal{F}$ , which is therefore regular.

Tischler's argument also works when  $[\sigma] \notin H^1(M; \mathbb{Z})$ . In this case one needs first to perturb  $\sigma$  to an element in  $H^1(M; \mathbb{Q})$  and then scale it to an element of  $H^1(M; \mathbb{Z})$ . Tischler's result, then, says that every compact manifold  $M$  endowed with a closed and nowhere vanishing 1-form  $\sigma$  fibres over the circle and is a mapping torus (see [35]).

A second situation in which a codimension 1 foliation is regular is the following:

*Example 3.3* Let  $M^n$  be a compact manifold endowed with a closed and nowhere vanishing 1-form  $\sigma$ , so that  $\mathcal{F} = \ker \sigma$  is a foliation on  $M$ . Assume that  $\mathcal{F}$  has at least one compact leaf  $L$ . It is shown in [26] that  $M$  is the mapping torus of a diffeomorphism  $\varphi: L \rightarrow L$ . The fibres of the mapping torus projection  $M = L_\varphi \rightarrow S^1$  are diffeomorphic to  $L$ , and the local triviality of the fibration tells us that we are in the regular case.

We consider now the characteristic foliation, for which there is a regularity notion as well and use it to prove a structure result for compact  $K$ -cosymplectic manifolds, which mimics the contact case (see [9]).

**Definition 3.4** Let  $(\eta, \xi, \phi, g)$  be an almost contact metric structure on  $M$ . The *characteristic foliation* is the 1-dimensional foliation  $\mathcal{F}_\xi$  whose leaves are given by the flow lines of  $\xi$ .

*Remark 3.5* The characteristic foliation makes sense for cosymplectic manifolds  $(M, \eta, \omega)$  as well, since the Reeb field  $\xi$  is uniquely defined by  $\iota_\xi \omega = 0$  and  $\eta(\xi) = 1$ .

**Definition 3.6** Let  $(\eta, \xi, \phi, g)$  be an almost contact metric structure and let  $\mathcal{F}_\xi$  be the characteristic foliation. The structure (or the foliation) is called *quasi-regular* if there exists a positive integer  $k$  such that each point  $p \in M$  has a neighborhood  $U$  with the following property: any integral curve of  $\xi$  intersects  $U$  at most  $k$  times. If  $k = 1$ , the structure is called *regular*. We use the term *irregular* for the non quasi-regular case.

In Proposition 6.7 below we will show that if a compact manifold admits a  $K$ -cosymplectic structure, then it also admits a quasi-regular one.

The regularity of each foliation is independent from the regularity of the other, as the following two examples show:

*Example 3.7* We consider a compact symplectic mapping torus  $N_\varphi$ , such that the symplectomorphism  $\varphi: N \rightarrow N$  has an infinite orbit. For example, take  $N = T^2$  and  $\varphi: T^2 \rightarrow T^2$  to be the symplectomorphism covered by the linear map  $\tilde{\varphi}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The mapping torus  $T_\varphi^2$  admits a cosymplectic structure  $(\eta, \xi, \phi, g)$  with  $[\eta] \in H^1(T_\varphi^2; \mathbb{Z})$  (compare with Remark 2.13). Hence the vertical foliation is regular, according to Example 3.2. The matrix  $\varphi$  has infinite order in the group of symplectomorphisms of  $T^2$ . Therefore, in  $T_\varphi^2$ , the characteristic foliation intersects each fibre of the mapping torus fibration an infinite number of times; hence it is not regular.

*Example 3.8* We can also construct examples of compact cosymplectic manifolds for which the characteristic foliation is regular, but the vertical foliation is not. To see this, consider the following analogue of a *deformation of type II* in the Sasakian setting (see [9], p. 240):

Let  $(M, \eta, \omega)$  be a cosymplectic manifold of dimension  $2n + 1$ , with Reeb field  $\xi$ , and let  $\beta \in \Omega^1(M, \mathcal{F}_\xi)$  be an arbitrary closed basic 1-form (i.e.,  $d\beta = 0$  and  $\iota_\xi \beta = 0$ , see Sect. 4 below). Then  $\eta' = \eta + \beta$  is again a closed 1-form on  $M$ . Because  $\iota_\xi \beta = 0$  and  $\iota_\xi \omega = 0$ , we have  $\beta \wedge \omega^n = 0$ , and hence  $\eta' \wedge \omega^n = \eta \wedge \omega^n$  is a volume form on  $M$ . Thus,  $(M, \eta', \omega)$  is again cosymplectic. The Reeb field of this new cosymplectic manifold is equal to the original Reeb field  $\xi$ , but the vertical foliation has changed.

For an explicit example, take  $T^3 = T^2 \times T^1$ ; let  $\langle X_1, X_2, X_3 \rangle$  be a basis of  $\mathfrak{t}$  and let  $\langle x_1, x_2, x_3 \rangle$  be the dual basis. Set  $\eta = x_3$  and  $\omega = x_1 \wedge x_2$ . Let  $g$  be the left-invariant Riemannian metric on  $T^3$  which makes  $\langle X_1, X_2, X_3 \rangle$  orthonormal. Hence  $(T^3, \eta, \omega)$  is a  $K$ -cosymplectic manifold. In this case both the vertical and the characteristic foliation are regular. Since  $\langle x_1, x_2, x_3 \rangle$  is a basis of  $H^1(T^3; \mathbb{R})$ , any closed 1-form  $\beta$  on  $T^3$  can be written uniquely as  $\beta = \sum_{i=1}^3 a_i x_i$  for some  $a_i \in \mathbb{R}$ .  $\beta$  is basic if and only if  $a_3 = 0$ . Choose  $a_1, a_2$  to be algebraically independent over  $\mathbb{Q}$ , set  $\beta = a_1 x_1 + a_2 x_2$  and  $\eta' = \eta + \beta$ . Then  $\eta' \notin H^1(T^3; \mathbb{Z})$ ; the perturbed structure  $(T^3, \eta', \omega)$  is again  $K$ -cosymplectic, by Proposition 2.9. The Reeb field has not changed; hence the characteristic foliation is still regular; however, the vertical foliation is now irregular (it is dense in  $T^3$ ). Compare also with Example 8.2 below.

There is, however, a situation in which the regularity of the vertical foliation is related to that of the characteristic foliation. Let  $K$  be a compact Kähler manifold, let  $\varphi: K \rightarrow K$  be a Hermitian isometry, and let  $K_\varphi$  be the corresponding mapping torus. Let  $(\eta, \xi, \phi, g)$  be the natural coKähler structure on  $K_\varphi$ . Then  $[\eta] \in H^1(K_\varphi; \mathbb{Z})$  (see again Remark 2.13) and the vertical foliation is regular. It is proved in [5, Theorem 6.6] that such  $\varphi$  has finite order in the group of Hermitian isometries of  $K$  (modulo the connected component of the identity). Therefore, in this specific case, the orbit of  $\varphi$  intersects each fibre only a finite number of times. Hence both the vertical and the characteristic foliation are (quasi-)regular.

We are almost ready for our structure result. In order to prove it, we need the following theorem, which is standard in almost contact metric geometry:

**Theorem 3.9** [9, Theorem 6.3.8] *Let  $(M, \xi, \eta, \phi)$  be an almost contact manifold such that the leaves of the characteristic foliation are all compact. Suppose also that  $(M, \xi, \eta, \phi)$  admits a compatible Riemannian metric  $g$  such that  $\xi$  is a Killing field which leaves  $\phi$  invariant. Then the space of leaves  $M/\mathcal{F}_\xi$  has the structure of an almost Hermitian orbifold such that the canonical projection  $\pi: M \rightarrow M/\mathcal{F}_\xi$  is an orbifold Riemannian submersion and a principal  $S^1$ -V-bundle over  $M/\mathcal{F}_\xi$  with connection 1-form  $\eta$ .*

We refer to [9, Chapter 4], and the references therein, for all the relevant definitions about orbifolds and V-bundles. However, we apply Theorem 3.9 to the case of a compact manifold  $M$  endowed with a regular  $K$ -cosymplectic structure, so that the space of leaves will be a manifold, and we can forget about orbifolds. We obtain:

**Theorem 3.10** *Let  $M$  be a compact manifold endowed with a  $K$ -cosymplectic structure  $(\xi, \eta, \phi, g)$ . Assume that the characteristic foliation  $\mathcal{F}_\xi$  is regular. Then the space of leaves  $M/\mathcal{F}_\xi$  has the structure of an almost Kähler manifold such that the canonical projection  $\pi: M \rightarrow M/\mathcal{F}_\xi$  is a Riemannian submersion and a principal, flat  $S^1$ -bundle with connection 1-form  $\eta$ .*

*Proof* First of all, regularity implies that the leaves of  $\mathcal{F}_\xi$  are all closed; hence compact. Thus they are homeomorphic to circles. Since the structure is  $K$ -cosymplectic, the Reeb field  $\xi$  is Killing with respect to  $g$  and respects the tensor  $\phi$  (these two conditions are in fact equivalent). Since  $\xi$  also respects the Kähler form  $\omega$  of the  $K$ -cosymplectic structure, the almost Hermitian manifold  $M/\mathcal{F}_\xi$  is indeed almost Kähler.  $M$  is the total space of a principal  $S^1$ -bundle  $\pi: M \rightarrow M/\mathcal{F}_\xi$ , and  $\eta$  is a connection 1-form. Since  $\eta$  is closed, the bundle is flat. □

If  $G$  is a compact Lie group, and  $P \rightarrow B$  is a principal  $G$ -bundle over a smooth manifold  $B$ , it is well-known (see for instance [41]) that  $P$  has a connection with zero curvature if and only if  $P$  is induced from the universal covering  $\tilde{B} \rightarrow B$  through a homomorphism  $\pi_1(B) \rightarrow G$ . In other words, flat  $G$ -bundles are determined by the monodromy action of the fundamental group of the base on  $G$ . In our case, when  $\mathcal{F}_\xi$  is regular, the flat  $S^1$ -bundle  $M \rightarrow M/\mathcal{F}_\xi$  is determined by a homomorphism  $\pi_1(M/\mathcal{F}_\xi) \rightarrow S^1$ . Consider the following portion of the long exact sequence of homotopy groups of the fibration  $M \rightarrow M/\mathcal{F}_\xi$ :

$$\dots \rightarrow \pi_1(S^1) \xrightarrow{\chi} \pi_1(M) \rightarrow \pi_1(M/\mathcal{F}_\xi) \rightarrow \pi_0(S^1) \rightarrow \dots \tag{2}$$

**Lemma 3.11** *Let  $M$  be a compact manifold endowed with a regular  $K$ -cosymplectic structure  $(\xi, \eta, \phi, g)$ . Then the map  $\chi: \pi_1(S^1) \rightarrow \pi_1(M)$  in (2) is injective.*

*Proof* The compactness hypothesis on  $M$  ensures that  $\pi_1(M)$  always contains a subgroup isomorphic to  $\mathbb{Z}$ . Indeed, by work of Li, any compact cosymplectic manifold  $M$  is diffeomorphic to a mapping torus (see [33]) and this displays the fundamental group of  $M$  as a semi-direct product  $\Gamma \rtimes \mathbb{Z}$  (see also [5]). This means that  $H_1(M; \mathbb{Z})$  has rank at least 1. Since the  $K$ -cosymplectic structure is regular,  $M$  is endowed with a circle action given precisely by the flow of the Reeb field. In [5] it is proved that the orbit map of this action is injective in homology. Therefore, it must also be injective in homotopy.  $\square$

When  $\pi_1(M) = \mathbb{Z}$ , (2) becomes  $0 \rightarrow \mathbb{Z} \xrightarrow{\chi} \mathbb{Z} \rightarrow \pi_1(M/\mathcal{F}_\xi) \rightarrow 0$ . Hence  $\chi$  is multiplication by some fixed integer  $k$  and  $\pi_1(M/\mathcal{F}_\xi) \cong \mathbb{Z}_k$ . Therefore, there is only a finite number of principal flat  $S^1$ -bundles over  $M/\mathcal{F}_\xi$ . All such bundles become trivial when lifted to the universal cover of  $M/\mathcal{F}_\xi$ .

### 4 Differential forms on $K$ -cosymplectic manifolds

Let  $M$  be a smooth manifold and let  $\mathcal{F}$  be a smooth foliation; we consider the following subalgebras of smooth forms  $\Omega^*(M)$ :

- *horizontal forms:*  $\Omega_{\text{hor}}^*(M) = \{\alpha \in \Omega^*(M) \mid \iota_X \alpha = 0 \ \forall X \in T\mathcal{F}\}$ ;
- *basic forms:*  $\Omega^*(M; \mathcal{F}) = \{\alpha \in \Omega^*(M) \mid \iota_X \alpha = 0 = \iota_X d\alpha \ \forall X \in T\mathcal{F}\}$ ;

The notation  $\Omega_{\text{bas}}^*(M)$  is also common to indicate basic forms. The exterior differential maps basic forms to basic forms, hence  $(\Omega^*(M; \mathcal{F}), d)$  is a differential subalgebra of  $\Omega^*(M)$ . The corresponding cohomology  $H^*(M; \mathcal{F})$  is called the *basic cohomology* of  $\mathcal{F}$ .

**Lemma 4.1** *Let  $(M^{2n+1}, \eta, \omega)$  be a cosymplectic manifold; let  $\xi$  be the Reeb field and let  $\mathcal{F}_\xi$  be the characteristic foliation. Then*

$$\Omega^p(M) = \Omega_{\text{hor}}^p(M) \oplus \eta \wedge \Omega_{\text{hor}}^{p-1}(M)$$

as  $C^\infty(M)$ -modules, for  $1 \leq p \leq 2n + 1$ . Also,  $C^\infty(M) = \Omega^0(M) = \Omega_{\text{hor}}^0(M)$ .

*Proof* Since the tangent space to  $\mathcal{F}_\xi$  at  $x \in M$  is spanned by  $\xi_x$ ,  $\Omega_{\text{hor}}^p(M) = \{\alpha \in \Omega^p(M) \mid \iota_{\xi} \alpha = 0\}$ . Given any  $\alpha \in \Omega^p(M)$ , we can write

$$\alpha = (\alpha - \eta \wedge \iota_{\xi} \alpha) + \eta \wedge \iota_{\xi} \alpha =: \alpha_1 + \eta \wedge \alpha_2.$$

Since  $\eta(\xi) = 1$ , we see that  $\iota_{\xi} \alpha_1 = 0$ ; hence  $\alpha_1 \in \Omega_{\text{hor}}^p(M)$ . Furthermore  $\alpha_2 \in \Omega_{\text{hor}}^{p-1}(M)$ ; hence

$$\Omega^p(M) = \Omega_{\text{hor}}^p(M) + \eta \wedge \Omega_{\text{hor}}^{p-1}(M)$$

Suppose  $\beta \in \Omega_{\text{hor}}^p(M) \cap \eta \wedge \Omega_{\text{hor}}^{p-1}(M)$ . Hence  $\eta \wedge \beta = 0$ ; by contracting the latter with  $\xi$ , we get  $0 = \beta - \eta \wedge \iota_{\xi} \beta = \beta$ , which gives  $\beta = 0$ .  $\square$

If  $(M, \eta, \omega)$  is a cosymplectic manifold, the map  $\mathfrak{X}(M) \rightarrow \Omega^1(M)$  defined by

$$X \mapsto \iota_X \omega + \eta(X)\eta \tag{3}$$

is an isomorphism (see [1, Proposition 1]). By Lemma 4.1,

$$\Omega^1(M) = \Omega_{\text{hor}}^1(M) \oplus \langle \eta \rangle,$$

where  $\langle \eta \rangle$  denotes the  $C^\infty(M)$ -module generated by  $\eta$ . We have  $\iota_{\xi}(\iota_X \omega) = -\iota_X(\iota_{\xi} \omega) = 0$ ; hence  $\iota_X \omega \in \Omega_{\text{hor}}^1(M)$ . We can rephrase (3) in the following way:

**Proposition 4.2** *Let  $(M, \eta, \omega)$  be a cosymplectic manifold. Then the map*

$$\begin{aligned} \Psi: \mathfrak{X}(M) &\rightarrow \Omega_{\text{hor}}^1(M) \oplus C^\infty(M) \\ X &\mapsto (\iota_X \omega, \eta(X)) \end{aligned}$$

*is an isomorphism.*

#### 4.1 Basic cohomology of $K$ -cosymplectic structures

We prove a splitting result for the de Rham cohomology of a compact  $K$ -cosymplectic manifold, generalizing the corresponding result for compact coKähler manifolds (see [4]).

**Theorem 4.3** *Let  $(M, \eta, \omega)$  be a compact  $K$ -cosymplectic manifold. Then the cohomology  $H^*(M; \mathbb{R})$  splits as  $H^*(M; \mathcal{F}_\xi) \otimes \Lambda([\eta])$ . In particular, for each  $0 \leq p \leq 2n + 1$ ,*

$$H^p(M; \mathbb{R}) = H^p(M; \mathcal{F}_\xi) \oplus [\eta] \wedge H^{p-1}(M; \mathcal{F}_\xi).$$

*Proof* Let  $(M, \eta, \omega)$  be a compact  $K$ -cosymplectic manifold. Set

$$\Omega_\xi^p(M) = \{\alpha \in \Omega^p(M) \mid L_\xi \alpha = 0\}.$$

Since the Lie derivative commutes with the exterior derivative,  $(\Omega_\xi^*(M), d)$  is a differential subalgebra of  $(\Omega^*(M), d)$ . In [4, Corollary 4.3] it is proven that if  $M$  is coKähler, the inclusion  $\iota: (\Omega_\xi^*(M), d) \hookrightarrow (\Omega^*(M), d)$  is a quasi-isomorphism, i.e., it induces an isomorphism in cohomology. The proof relies on the fact that  $\eta$  is a parallel form in the coKähler setting, and this remains true when the structure is  $K$ -cosymplectic (see Remark 2.5). Alternatively, one can argue as follows: since  $M$  is compact, the closure of the Reeb flow generates a torus action on  $M$  (see the discussion before Proposition 2.9); the fact that  $\iota$  is a quasi-isomorphism is then a special case of a general result on the cohomology of invariant forms, see for instance [37, §9, Theorem 1]. Because  $\Omega_\xi^*(M) \cap \Omega_{\text{hor}}^*(M) = \Omega^*(M; \mathcal{F}_\xi)$ , Lemma 4.1 implies that

$$\Omega_\xi^p(M) = \Omega^p(M; \mathcal{F}_\xi) \oplus \eta \wedge \Omega^{p-1}(M; \mathcal{F}_\xi)$$

for all  $p$ . Each summand of the right-hand side is a differential subalgebra; taking cohomology, the right-hand side gives  $H^p(M; \mathcal{F}_\xi) \oplus [\eta] \wedge H^{p-1}(M; \mathcal{F}_\xi)$ . By the above discussion, the cohomology of the left-hand side is isomorphic to  $H^p(M; \mathbb{R})$ .  $\square$

As a consequence, we deduce some properties of the basic cohomology of the characteristic foliation on a compact  $K$ -cosymplectic manifold.

**Proposition 4.4** *Let  $(M, \eta, \omega)$  be a compact, connected  $K$ -cosymplectic manifold of dimension  $2n + 1$ , let  $\mathcal{F}_\xi$  denote the characteristic foliation, and let  $H^*(M; \mathcal{F}_\xi)$  be the basic cohomology. Then*

1. *the groups  $H^p(M; \mathcal{F}_\xi)$  are finite dimensional;*
2.  *$H^{2n}(M; \mathcal{F}_\xi) \cong \mathbb{R}$ ,  $H^0(M; \mathcal{F}_\xi) \cong \mathbb{R}$  and  $H^p(M; \mathcal{F}_\xi) = 0$  for  $p > 2n$ ;*
3. *the class  $[\omega]^p \in H^{2p}(M; \mathcal{F}_\xi)$  is non-trivial for  $1 \leq p \leq n$ ;*
4.  *$H^1(M; \mathbb{R}) \cong H^1(M; \mathcal{F}_\xi) \oplus \mathbb{R}[\eta]$ ;*
5. *there is a non-degenerate pairing*

$$\Psi: H^p(M; \mathcal{F}_\xi) \otimes H^{2n-p}(M; \mathcal{F}_\xi) \longrightarrow \mathbb{R}.$$

*Proof* A theorem of El Kacimi-Alaoui et al. [15] states that the basic cohomology of a Riemannian foliation on a compact manifold is always finite-dimensional, which directly

implies 1. However, in our simple situation, one does not need to invoke this result: since  $M$  is a compact manifold,  $b_p(M) = \dim H^p(M; \mathbb{R}) < \infty$  for every  $0 \leq p \leq 2n + 1$ , and by Theorem 4.3,  $b_p(M) = \dim H^p(M; \mathcal{F}_\xi) + \dim H^{p-1}(M; \mathcal{F}_\xi)$  which gives finiteness recursively.

As  $\mathcal{F}_\xi$  is a foliation of codimension  $2n$ , the basic cohomology  $H^*(M; \mathcal{F}_\xi)$  vanishes in degrees larger than  $2n$ . Also,  $b_0(M) = 1$  implies that  $H^0(M; \mathcal{F}_\xi) \cong \mathbb{R}$ . The Kähler form  $\omega$  is closed and non-degenerate, meaning that  $\omega^p \neq 0$  for  $1 \leq p \leq n$ . It is also basic, and cannot be exact by compactness of  $M$ , according to Stokes’ theorem. Therefore,  $[\omega]^p$  is non-trivial in  $H^{2p}(M; \mathcal{F}_\xi)$  for  $1 \leq p \leq n$ . Since  $M$  is compact and  $\eta \wedge \omega^n$  is a volume form, we have  $b_{2n+1}(M) = 1$ , which implies that  $H^{2n}(M; \mathcal{F}_\xi)$  is generated by  $[\omega]^n$ . This completes the proof of 2. and 3.

Number 4. follows immediately from Theorem 4.3 and number 2. To construct the pairing  $\Psi$  we proceed as follows: given a cohomology class  $[\alpha] \in H^p(M; \mathcal{F}_\xi)$ , by Poincaré duality on  $M$  there exists  $[\beta] \in H^{2n+1-p}(M; \mathbb{R})$  such that  $([\alpha], [\beta]) \neq 0$ , where  $(\cdot, \cdot)$  is the usual pairing on  $M$ . Now write  $[\beta] = [\sigma] + [\eta \wedge \tau]$  according to the splitting of Theorem 4.3, with  $[\sigma] \in H^{2n+1-p}(M; \mathcal{F}_\xi)$  and  $[\tau] \in H^{2n-p}(M; \mathcal{F}_\xi)$ . Then  $[\alpha \wedge \sigma] \in H^{2n+1}(M; \mathcal{F}_\xi)$ , which vanishes by number 2. Therefore, we see that the Poincaré pairing on  $M$  is given by

$$\int_M \alpha \wedge \eta \wedge \tau.$$

For such  $\alpha$ , therefore, the pairing  $\Psi : H^p(M; \mathcal{F}_\xi) \otimes H^{2n-p}(M; \mathcal{F}_\xi) \longrightarrow \mathbb{R}$

$$\Psi([\alpha], [\tau]) = \int_M \alpha \wedge \eta \wedge \tau$$

is non-degenerate, and we have number 5. □

**Definition 4.5** Given a compact  $K$ -cosymplectic manifold  $(M, \eta, \omega)$  of dimension  $2n + 1$ , we define the *basic Betti numbers* as

$$b_p(M; \mathcal{F}_\xi) = \dim H^p(M; \mathcal{F}_\xi), \quad 0 \leq p \leq 2n + 1.$$

**Corollary 4.6** Let  $(M, \eta, \omega)$  be a compact  $K$ -cosymplectic manifold of dimension  $2n + 1$ . Then

- $b_{2n-p}(M; \mathcal{F}_\xi) = b_p(M; \mathcal{F}_\xi)$ ;
- For any  $1 \leq p \leq 2n$ , the basic Betti numbers  $b_p(M; \mathcal{F}_\xi)$  are determined by

$$b_p(M; \mathcal{F}_\xi) = \sum_{i=0}^p (-1)^i b_{p-i}(M).$$

In particular, they are topological invariants of  $M$ .

#### 4.2 The Lefschetz map on $K$ -cosymplectic manifolds

Let  $(\eta, \xi, \phi, g)$  be a coKähler structure on a manifold  $M$  of dimension  $2n + 1$  and let  $\omega$  be the Kähler form. Let  $\mathcal{L} : \Omega^p(M) \rightarrow \Omega^{2n+1-p}(M)$  be the map

$$\mathcal{L}(\alpha) = \omega^{n-p} \wedge (\omega \wedge \iota_\xi \alpha + \eta \wedge \alpha). \tag{4}$$

$\mathcal{L}$  is called the *Lefschetz map*. Unlike the symplectic Lefschetz map,  $\mathcal{L}$  does not commute with the exterior differential; hence it does not descend to cohomology. It is proven in [12, Theorem 12] that, when  $M$  is compact,  $\mathcal{L} : \mathcal{H}^p(M) \rightarrow \mathcal{H}^{2n+1-p}(M)$  is an isomorphism,

where  $\mathcal{H}^*(M)$  denotes harmonic forms. If the structure  $(\eta, \xi, \phi, g)$  is not coKähler, then  $\mathcal{L}$  does not necessarily send harmonic forms to harmonic forms. Thus the Lefschetz map is not well defined for arbitrary cosymplectic structures. It was observed in [4] that the restriction of the Lefschetz map to the differential subalgebra  $(\Omega_\xi^*(M), d)$  supercommutes with the exterior differential. Furthermore, since the inclusion  $\iota: (\Omega_\xi^*(M), d) \rightarrow (\Omega^*(M), d)$  is a quasi-isomorphism when the structure is coKähler and  $M$  is compact (this is again [4, Corollary 4.3]),  $\mathcal{L}$  descends to the cohomology  $H^*(M; \mathbb{R})$ , and is shown to be an isomorphism, recovering the result of [12].

We observed before that  $\iota: (\Omega_\xi^*(M), d) \rightarrow (\Omega^*(M), d)$  is a quasi-isomorphism also in the compact  $K$ -cosymplectic case. Hence we get the following:

**Lemma 4.7** *Let  $(\eta, \xi, \phi, g)$  be a  $K$ -cosymplectic structure on a compact manifold  $M$  of dimension  $2n + 1$ . Then the Lefschetz map  $\mathcal{L}: \Omega_\xi^p(M) \rightarrow \Omega_\xi^{2n+1-p}(M)$  is well defined and descends to cohomology.*

We are not claiming here that  $\mathcal{L}$  is an isomorphism in the  $K$ -cosymplectic case. This is in fact false; to see this, notice that there exist symplectic manifolds which do not satisfy the usual Lefschetz property (any symplectic non-toral nilmanifold does the job, see [6]). Choose one such  $(K, \tau)$  and fix an adapted metric  $h$ , so that  $(K, \tau, h)$  is almost Kähler. Then  $M = K \times S^1$  admits a  $K$ -cosymplectic structure by Corollary 2.14. One can see that the Lefschetz map (4) on  $M$  is not an isomorphism.

### 5 Cosymplectic and Hamiltonian vector fields

In this section we recall the notions of cosymplectic and Hamiltonian vector fields, which were introduced by Albert [1]; however, he used a slightly different terminology. Furthermore, in his paper Albert deals with the cosymplectic and the contact case simultaneously. So far, only the contact set-up has caught attention. In order to make our exposition self-contained, we allow ourselves to provide a proof of some of the results we quote.

#### 5.1 Cosymplectic vector fields

**Definition 5.1** Let  $(M, \eta, \omega)$  be a cosymplectic manifold. Let  $\psi: M \rightarrow M$  be a diffeomorphism.  $\psi$  is a *weak cosymplectomorphism* if  $\psi^*\eta = \eta$  and there exists a function  $h_\psi \in C^\infty(M)$  such that  $\psi^*\omega = \omega - dh_\psi \wedge \eta$ . Such  $\psi$  is a *cosymplectomorphism* if one can choose  $h_\psi$  to vanish.

*Remark 5.2* A cosymplectomorphism  $\psi$  satisfies  $\psi^*\eta = \eta$  and  $\psi^*\omega = \omega$ . Hence it respects the Reeb field and the characteristic foliation.

**Definition 5.3** Let  $(M, \eta, \omega)$  be a cosymplectic manifold and let  $X \in \mathfrak{X}(M)$  be a vector field.  $X$  is *weakly cosymplectic* if  $L_X\eta = 0$  and there exists a function  $h_X \in C^\infty(M)$  such that  $L_X\omega = -dh_X \wedge \eta$ .  $X$  is *cosymplectic* if one can choose  $h_X$  to vanish.

**Lemma 5.4** *Let  $X$  and  $Y$  be two weakly cosymplectic vector fields such that  $L_X\omega = -dh_X \wedge \eta$  and  $L_Y\omega = -dh_Y \wedge \eta$  for  $h_X, h_Y \in C^\infty(M)$ . Then  $[X, Y]$  belongs to  $\ker \eta$  and is weakly cosymplectic, with  $h_{[X, Y]} = X(h_Y) - Y(h_X)$ . In particular, if both  $X$  and  $Y$  are cosymplectic, then so is  $[X, Y]$ .*

*Proof* Suppose  $X$  and  $Y$  are weakly cosymplectic. Then  $\eta(X)$  and  $\eta(Y)$  are constant functions on  $M$ , and  $d\eta = 0$  implies  $\iota_{[X,Y]}\eta = X\eta(Y) - Y\eta(X) = 0$ . We must prove that  $L_{[X,Y]}\omega = d(\iota_{[X,Y]}\omega) = -d(X(h_X) - Y(h_X)) \wedge \eta$ :

$$\begin{aligned} d(\iota_{[X,Y]}\omega) &= dL_X\iota_Y\omega - d\iota_YL_X\omega = d\iota_Xd\iota_Y\omega + d\iota_Y(dh_X \wedge \eta) \\ &= d\iota_XL_Y\omega + (d\iota_Ydh_X) \wedge \eta = -d\iota_X(dh_Y \wedge \eta) + (d(Y(h_X))) \wedge \eta \\ &= -d(X(h_Y) - Y(h_X)) \wedge \eta. \end{aligned}$$

□

**Corollary 5.5** *Let  $(M, \eta, \omega)$  be a cosymplectic manifold. Then (weakly) cosymplectic vector fields form a Lie subalgebra in  $\mathfrak{X}(M)$ .*

The Reeb field  $\xi$  is cosymplectic. We denote by  $\mathfrak{X}^{\text{cosymp}}(M) \subset \mathfrak{X}(M)$  the Lie subalgebra of cosymplectic vector fields. For a cosymplectic vector field  $X$ , the function  $\eta(X)$  is constant; we also consider the following subset of  $\mathfrak{X}^{\text{cosymp}}(M)$ :

$$\mathfrak{X}_0^{\text{cosymp}}(M) = \{X \in \mathfrak{X}^{\text{cosymp}}(M) \mid \eta(X) = 0\}.$$

### 5.2 Hamiltonian vector fields

**Definition 5.6** Let  $(M, \eta, \omega)$  be a cosymplectic manifold, let  $\xi$  denote the Reeb field, and let  $X \in \mathfrak{X}(M)$  be a vector field.  $X$  is *weakly Hamiltonian* if  $\eta(X) = 0$  and there exists  $f \in C^\infty(M)$  such that  $\iota_X\omega = df - \xi(f)\eta$ .  $X$  is *Hamiltonian* if, in addition,  $f$  can be chosen to be invariant along the flow of  $\xi$ , i.e.,  $\xi(f) = 0$ . In both cases,  $f$  is the *Hamiltonian function* of  $X$ .

If  $X \in \mathfrak{X}(M)$  is a weakly Hamiltonian vector field then  $X$  is not, in general, cosymplectic, but only weakly cosymplectic, and the functions  $h_X$  and  $\xi(f)$  can be chosen to coincide. We denote by  $X_f$  the weakly Hamiltonian vector field such that  $\iota_X\omega = df - \xi(f)\eta$  and by  $\mathfrak{X}_w^{\text{ham}}(M)$  the set of weakly Hamiltonian fields. When  $X \in \mathfrak{X}(M)$  is Hamiltonian, then it is cosymplectic, and the 1-form  $\iota_X\omega$  is exact. We denote by  $\mathfrak{X}_0^{\text{ham}}(M) \subset \mathfrak{X}_w^{\text{ham}}(M)$  the subset of Hamiltonian vector fields.

**Proposition 5.7** *Let  $X, Y$  be cosymplectic vector fields. Then  $[X, Y]$  is Hamiltonian with Hamiltonian function  $-\omega(X, Y)$ .*

*Proof* By Lemma 5.4,  $\eta([X, Y]) = 0$ . Since  $\omega$  is closed, we have

$$\begin{aligned} 0 &= d\omega(X, Y, Z) \\ &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X). \end{aligned} \tag{5}$$

Since  $Y$  is cosymplectic,

$$0 = d(\iota_Y\omega)(X, Z) = -(\iota_Y\omega)([X, Z]) + X(\iota_Y\omega)(Z) - Z(\iota_Y\omega)(X); \tag{6}$$

Plugging (6) into (5) we get

$$(\iota_{[X,Y]}\omega)(Z) = -Y(\iota_X\omega)(Z) + (\iota_X\omega)([Y, Z]) = -(L_Y(\iota_X\omega))(Z) = -(d(\iota_Y\iota_X\omega))(Z),$$

where we used that  $X$  is cosymplectic in the last equality. Hence  $\iota_{[X,Y]}\omega = -d(\omega(X, Y))$ . We need to check that  $\omega(X, Y)$  satisfies  $\xi\omega(X, Y) = 0$ . By definition

$$\xi\omega(X, Y) = d(\omega(X, Y))(\xi) = \iota_\xi d(\omega(X, Y)) = -\iota_\xi \iota_{[X, Y]} \omega = \iota_{[X, Y]} \iota_\xi \omega = 0.$$

□

**Proposition 5.8** [1, Proposition 3] *Let  $Z$  be a weakly cosymplectic vector field and let  $X_f$  be a weakly Hamiltonian vector field. Then the following two formulae hold true:*

- $[\xi, Z] = X_{h_Z}$ ;
- $[Z, X_f] = X_{Z(f)}$ .

*Proof* Recall that the weakly Hamiltonian vector field  $X_{h_Z}$  is defined by the conditions  $\eta(X_{h_Z}) = 0$  and  $\iota_{X_{h_Z}} \omega = dh_Z - \xi(h_Z)\eta$ . We prove that the same is true for  $[\xi, Z]$ . Now  $\eta(Z)$  is constant; hence  $\eta([\xi, Z]) = 0$ . Furthermore,

$$\begin{aligned} \iota_{[\xi, Z]} \omega(Y) &= \omega([\xi, Z], Y) \stackrel{(\diamond)}{=} \omega(Z, [Y, \xi]) + \xi\omega(Z, Y) = (\iota_Z \omega)([Y, \xi]) + \xi(\iota_Z \omega)(Y) \\ &= -d(\iota_Z \omega)(Y, \xi) = (dh_Z \wedge \eta)(Y, \xi) = dh_Z(Y) - \xi(h_Z)\eta(Y), \end{aligned}$$

where  $(\diamond)$  holds because  $\omega$  is closed. For the second formula, it will be enough to prove that  $\eta([Z, X_f]) = 0$  and  $\iota_{[Z, X_f]} \omega = d(Z(f)) - \xi(Z(f))\eta$ . For the first equality, we have

$$\eta([Z, X_f]) = Z\eta(X_f) - X_f\eta(Z) = 0,$$

since  $\eta(X_f) = 0$  and  $\eta(Z)$  is a constant function. Notice that

$$[\xi, Z](f) = X_{h_Z}(f) = df(X_{h_Z}) = \omega(X_f, X_{h_Z}) = -dh_Z(X_f) = -X_f(h_Z). \tag{7}$$

Next, recalling that  $d\omega = 0$ , we have

$$\begin{aligned} (\iota_{[Z, X_f]} \omega)(Y) &= -\omega([X_f, Y], Z) - \omega([Y, Z], X_f) + Z\omega(X_f, Y) + X_f\omega(Y, Z) + Y\omega(Z, X_f) \\ &= -d(\iota_Z \omega)(X_f, Y) + df([Y, Z]) - \xi(f)\eta([Y, Z]) + Z(df(Y) - \xi(f)\eta(Y)) \\ &= (dh_Z \wedge \eta)(X_f, Y) + Y(Z(f)) - Z(\xi(f))\eta(Y) \\ &= X_f(h_Z)\eta(Y) + d(Z(f))(Y) - Z(\xi(f))\eta(Y) \\ &\stackrel{(\dagger)}{=} [Z, \xi](f)\eta(Y) + d(Z(f))(Y) - Z(\xi(f))\eta(Y) \\ &= (d(Z(f)) - \xi(Z(f))\eta)(Y), \end{aligned}$$

where  $(\dagger)$  holds in view of (7). □

**Corollary 5.9** *Let  $(M, \eta, \omega)$  be a cosymplectic manifold. Then*

- $\xi$  is a central in  $\mathfrak{X}^{\text{cosymp}}(M)$ ;
- (weakly) Hamiltonian vector fields form an ideal in (weakly) cosymplectic vector fields.

Using this we obtain the following result:

**Lemma 5.10**  $\mathfrak{X}_0^{\text{cosymp}}(M) \subset \mathfrak{X}^{\text{cosymp}}(M)$  is an ideal and there is a Lie algebra isomorphism

$$\mathfrak{X}^{\text{cosymp}}(M) \cong \mathfrak{X}_0^{\text{cosymp}}(M) \oplus \langle \xi \rangle \tag{8}$$

*Proof* The first statement is clear. For the second one, we consider the map  $\mathfrak{X}^{\text{cosymp}}(M) \rightarrow \mathfrak{X}_0^{\text{cosymp}}(M) \oplus \langle \xi \rangle$  defined by  $X \mapsto (X - \eta(X)\xi, \eta(X)\xi)$ . Its inverse is simply  $(X, \xi) \mapsto X + \xi$ . Since  $\xi$  is central in  $\mathfrak{X}^{\text{cosymp}}(M)$  and  $\mathfrak{X}_0^{\text{cosymp}}(M)$  is in ideal, the splitting (8) is also of Lie algebras. □

We denote by  $C_\xi^\infty(M)$  the subalgebra of  $C^\infty(M)$  consisting of functions that are constant along the flow lines of  $\xi$ , namely

$$C_\xi^\infty(M) = \{f \in C^\infty(M) \mid \xi(f) = 0\}.$$

*Remark 5.11* Let  $(M, \eta, \omega)$  be a compact  $K$ -cosymplectic manifold. We can characterize cosymplectic and Hamiltonian vector fields in terms of the isomorphism  $\Psi: \mathfrak{X}(M) \rightarrow \Omega_{\text{hor}}^1(M) \oplus C^\infty(M)$  of Proposition 4.2 as follows:

- $\mathfrak{X}^{\text{cosymp}}(M) = \Psi^{-1}(\Omega_{\text{hor}}^1(M) \cap \ker d, \mathbb{R})$ ;
- $\mathfrak{X}_0^{\text{cosymp}}(M) = \Psi^{-1}(\Omega_{\text{hor}}^1(M) \cap \ker d, 0)$ ;
- $\mathfrak{X}^{\text{ham}}(M) = \Psi^{-1}(d(C_\xi^\infty(M)), 0)$ .

*Remark 5.12* The subalgebra  $C_\xi^\infty(M)$  is invariant under the action of the group of cosymplectomorphisms of  $M$ . This holds because  $\psi$  respects the characteristic foliation by Remark 5.2.

Let  $(M, \eta, \omega)$  be a cosymplectic manifold. We want to find sufficient conditions under which a cosymplectic vector field is Hamiltonian. If  $M$  is compact then  $H^1(M; \mathbb{R}) \neq 0$  (see Remark 2.11); hence we will not be able, in general, to solve the equation  $d(\iota_X \omega) = 0$ , i.e., to find a function  $f \in C_\xi^\infty(M)$  such that  $X_f = X$ . However, we can handle some special cases.

**Proposition 5.13** *Let  $(M, \eta, \omega)$  be a compact cosymplectic manifold with  $b_1(M) = 1$ . Then a vector field in  $\mathfrak{X}_0^{\text{cosymp}}(M)$  is Hamiltonian.*

*Proof* Since  $b_1(M) = 1$ ,  $H^1(M; \mathbb{R}) = \langle [\eta] \rangle$  by Proposition 4.4. As  $X \in \mathfrak{X}_0^{\text{cosymp}}(M)$ ,  $\iota_X \omega$  is a closed 1-form, hence it defines a cohomology class in  $H^1(M; \mathbb{R})$ . We have then  $\iota_X \omega = a\eta - df$  for some  $f \in C^\infty(M)$  and  $a \in \mathbb{R}$ . We would like to show that  $a = 0$ . Plugging the Reeb field in the last equation, we obtain  $a = \xi(f)$ . Fix a point  $p \in M$  and let  $\gamma = \gamma(t)$  be the integral curve of  $\xi$  passing through  $p$  at time  $t = 0$ . Then

$$\left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma)(t) = df(\xi) = \xi(f) = a.$$

This shows that  $(f \circ \gamma)(t) = at + b$  for  $b \in \mathbb{R}$ . Now, if  $a \neq 0$ , the image of  $f \circ \gamma$  is an unbounded set in  $\mathbb{R}$ , hence not compact; but this is absurd, since  $M$  is compact. Therefore  $a = 0$ . □

**Proposition 5.14** *Let  $(M, \eta, \omega)$  be a compact  $K$ -cosymplectic manifold. Then there is an exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{X}^{\text{ham}}(M) \xrightarrow{i} \mathfrak{X}_0^{\text{cosymp}}(M) \xrightarrow{\pi} H^1(M; \mathcal{F}_\xi) \longrightarrow 0,$$

where  $i$  is inclusion and  $\pi(X) = [\iota_X \omega]$ . Here  $H^1(M; \mathcal{F}_\xi)$  is intended as an abelian Lie algebra.

*Proof* It is clear that  $i$  is injective. We show that  $\pi$  is well defined. If  $X \in \mathfrak{X}_0^{\text{cosymp}}(M)$ , then  $\eta(X) = 0$  and  $\iota_X \omega$  is closed. Therefore,  $\iota_X \omega \in H^1(M; \mathbb{R})$ . In view of Proposition 4.4,  $H^1(M; \mathbb{R}) = H^1(M; \mathcal{F}_\xi) \oplus \mathbb{R}[\eta]$ . To show that  $\pi(X) \in H^1(M; \mathcal{F}_\xi)$  we must show that  $\iota_\xi(\iota_X \omega) = 0$ . We have  $\iota_\xi(\iota_X \omega) = -\iota_X(\iota_\xi \omega) = 0$ . Take  $X \in \mathfrak{X}^{\text{ham}}(M)$ ; then  $\iota_X \omega = df$  for some  $f \in C_\xi^\infty(M)$ ; hence  $\pi(X) = [df] = 0$ . Take  $X \in \mathfrak{X}_0^{\text{cosymp}}(M)$  with  $\pi(X) = 0$ . Then  $\iota_X \omega = df$  with  $\iota_\xi(\iota_X \omega) = 0$ ; hence  $\iota_\xi df = \xi(f) = 0$  and  $f \in C_\xi^\infty(M)$ ; thus  $X \in \mathfrak{X}^{\text{ham}}(M)$

and  $\ker(\pi) = \text{im}(i)$ . We prove surjectivity of  $\pi$ . Take  $[\alpha] \in H^1(M; \mathcal{F}_\xi)$ ; then  $d\alpha = 0$  and  $\alpha(\xi) = 0$ . Since  $\alpha \in \Omega^1(M; \mathbb{R})$ , there exists  $X \in \mathfrak{X}(M)$  such that  $\alpha = \iota_X \omega + \eta(X)\eta$ ; now  $\alpha(\xi) = 0$ ; therefore, contracting with  $\xi$ , we see that  $\eta(X) = 0$  and  $\alpha = \iota_X \omega$ . Since  $\alpha$  is closed,  $d(\iota_X \omega) = 0$ ; hence  $X \in \mathfrak{X}_0^{\text{cosymp}}(M)$  and  $\pi$  is surjective. We are left with showing that  $\pi$  is a Lie algebra homomorphism. By definition we have  $[\pi(X), \pi(Y)] = 0$ . On the other hand, by Proposition 5.7,

$$\pi([X, Y]) = [\iota_{[X, Y]}\omega] = [-d(\iota_Y \iota_X \omega)] = 0.$$

□

*Remark 5.15* Proposition 5.14 follows from Proposition 5.13 when the  $K$ -cosymplectic manifold  $M$  has  $b_1(M) = 1$ .

### 5.3 The Poisson structure

Here we use the Lie algebra structure of (weakly) Hamiltonian vector fields to induce a skew-symmetric bilinear form on  $C^\infty(M)$ .

**Definition 5.16** Let  $(M, \eta, \omega)$  be a cosymplectic manifold. The *Poisson bracket* is the bilinear map  $\{\cdot, \cdot\}: C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$  defined by

$$\{f, g\} = X_f(g).$$

**Proposition 5.17** *The Poisson bracket is skew-symmetric and satisfies the following properties:*

1.  $\{f + g, h\} = \{f, h\} + \{g, h\}$ ;
2.  $\{f, g\} = -\omega(X_f, X_g)$ ; in particular,  $\{\cdot, \cdot\}$  is skew-symmetric;
3.  $[X_f, X_g] = X_{\{f, g\}}$ ;
4.  $\{f, gh\} = \{f, g\}h + g\{f, h\}$ ;
5.  $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ .

As a consequence,  $(C^\infty(M), \{\cdot, \cdot\})$  is a Lie algebra and the map

$$\Theta: \begin{matrix} (C^\infty(M), \{\cdot, \cdot\}) \\ f \end{matrix} \rightarrow \begin{matrix} (\mathfrak{X}_w^{\text{ham}}(M), [\cdot, \cdot]) \\ X_f \end{matrix}$$

is a morphism of Lie algebras.

*Remark 5.18* A proof of this Proposition can be found in [1]; notice that we are using a different sign convention.

**Corollary 5.19**  $C_\xi^\infty(M) \subset C^\infty(M)$  is a subalgebra. The map  $\Theta$  induces a Lie algebra morphism

$$\bar{\Theta}: \begin{matrix} (C_\xi^\infty(M), \{\cdot, \cdot\}) \\ f \end{matrix} \rightarrow \begin{matrix} (\mathfrak{X}^{\text{ham}}(M), [\cdot, \cdot]) \\ X_f \end{matrix}$$

*Proof* It is enough to prove that if  $f, g \in C_\xi^\infty(M)$ , so does  $\{f, g\}$ . This is checked as follows:

$$\xi(\{f, g\}) = \xi(X_f(g)) = [\xi, X_f](g) = 0,$$

where we used Corollary 5.9 in the last equality. □

We obtain a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow C_\xi^\infty(M) \longrightarrow \mathfrak{X}^{\text{ham}}(M) \longrightarrow 0,$$

where  $\mathbb{R}$  represents constant functions.

### 6 Deformations of cosymplectic structures

In this section we describe a type of deformation of ( $K$ -)cosymplectic manifolds, respectively, structures, which are a generalization of the so-called *deformations of type I*, see [9, Section 8.2.3]. Deformations of this kind were introduced by Takahashi (see [42]), in the particular context of Sasakian structures. Also, for the special case in which the vector field  $\theta$  (see below) is a scalar multiple of the Reeb field, they had been studied previously by Tanno (see [43]). Such deformations modify the characteristic foliation  $\mathcal{F}_\xi$  but preserve the distribution  $\mathcal{D} = \ker \eta$ .

#### 6.1 Deformations of almost contact metric structures

We will first describe the type of deformations for an arbitrary almost contact metric structure. Below we will apply it to ( $K$ -)cosymplectic and coKähler structures; afterwards we also mention that they can be formulated in the setting of a ( $K$ -)cosymplectic manifold, without any fixed metric.

**Proposition 6.1** *Let  $(\eta, \xi, \phi, g)$  be an almost contact metric structure on a manifold  $M$ . Assume  $\theta$  is a vector field on  $M$  satisfying  $1 + \eta(\theta) > 0$ . Set*

$$\begin{aligned} \xi' &= \xi + \theta \\ \eta' &= \frac{\eta}{\eta(\xi')} \\ \phi' &= \phi \circ (\text{Id} - \eta' \otimes \xi') \\ g' &= \frac{1}{\eta(\xi')} g \circ (\text{Id} - \eta' \otimes \xi', \text{Id} - \eta' \otimes \xi') + \eta' \otimes \eta'. \end{aligned}$$

*Then  $(\eta', \xi', \phi', g')$  is again an almost contact metric structure on  $M$ . Its Kähler form is given by*

$$\omega' = \frac{1}{\eta(\xi')} (\omega + (\iota_\theta \omega) \wedge \eta'). \tag{9}$$

*Proof* First of all it is clear that  $\eta'(\xi') = 1$ . Next we need to verify that  $(\phi')^2 = -\text{Id} + \eta' \otimes \xi'$ . Notice that  $\eta' \circ \phi = 0$ , because  $\ker \eta = \ker \eta'$ . Thus we have

$$\begin{aligned} (\phi')^2(X) &= \phi'(\phi(X) - \eta'(X)\phi(\xi')) = \phi^2(X) - \eta'(X)\phi^2(\xi') \\ &= -X + \eta(X)\xi + \eta'(X)\xi' - \eta'(X)\eta(\xi')\xi \\ &= (-\text{Id} + \eta' \otimes \xi')(X). \end{aligned}$$

This alone already implies that  $\phi'(\xi') = 0$ . Next,  $g'$  is a Riemannian metric on  $M$ . Indeed,

$$g'(X, X) = \frac{1}{\eta(\xi')} g(X - \eta'(X)\xi', X - \eta'(X)\xi') + \eta'(X)^2$$

which is zero if and only if  $X - \eta'(X)\xi' = 0$  and  $\eta'(X) = 0$ , since  $g$  is Riemannian. But this holds only if  $X = 0$ . Last, we need to check the compatibility between  $g'$  and  $\phi'$ . We compute

$$\begin{aligned} g'(\phi'(X), \phi'(Y)) &= g'(\phi(X - \eta'(X)\xi'), \phi(Y - \eta'(Y)\xi')) \\ &= \frac{1}{\eta(\xi')} g(\phi(X - \eta'(X)\xi'), \phi(Y - \eta'(Y)\xi')) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\eta(\xi')} g(X - \eta'(X)\xi', Y - \eta'(Y)\xi') \\
 &= g'(X, Y) - \eta'(X)\eta'(Y).
 \end{aligned}$$

Finally, we compute the Kähler form of the deformed almost contact metric structure. We have  $\eta' \circ \phi' = 0$ ; also, note that  $\phi(\xi') = \phi(\theta)$  and that  $i_\xi \omega = 0$ . We obtain

$$\begin{aligned}
 \omega'(X, Y) &= g'(X, \phi'(Y)) = \frac{1}{\eta(\xi')} g(X - \eta'(X)\xi', \phi(Y) - \eta'(Y)\phi(\xi')) \\
 &= \frac{1}{\eta(\xi')} (\omega(X, Y) - \eta'(Y)g(X, \phi(\theta)) - \eta'(X)g(\xi', \phi(Y)) + \eta'(X)\eta'(Y)g(\xi', \phi(\xi'))) \\
 &= \frac{1}{\eta(\xi')} (\omega(X, Y) + \eta'(Y)(i_\theta \omega)(X) - \eta'(X)(i_\theta \omega)(Y) + \eta'(X)\eta'(Y)\omega(\xi', \xi')) \\
 &= \frac{1}{\eta(\xi')} (\omega + (i_\theta \omega) \wedge \eta')(X, Y).
 \end{aligned}$$

This finishes the proof. □

**Proposition 6.2** *Let  $(\eta, \xi, \phi, g)$  be a cosymplectic (resp. K-cosymplectic resp. coKähler) structure on a manifold  $M$ . Assume  $\theta$  is a vector field satisfying*

- i)  $1 + \eta(\theta) > 0$ ;
- ii)  $[\xi, \theta] = 0$ ;
- iii)  $L_\theta g = 0 = L_\theta \omega$ .

*Then the deformed structure  $(\eta', \xi', \phi', g')$  is again cosymplectic (resp. K-cosymplectic resp. coKähler).*

*Proof* Let us first assume that  $(\eta, \xi, \phi, g)$  is cosymplectic and show that the same holds for  $(\eta', \xi', \phi', g')$ . It is enough to prove that  $d\eta' = 0 = d\omega'$ , where  $\omega'$  is the Kähler form of  $(\eta', \xi', \phi', g')$ . The function  $\eta(\theta)$  is constant on a cosymplectic manifold; indeed, since  $\eta$  is closed,  $d(i_\theta \eta) = L_\theta \eta$  and

$$(L_\theta \eta)(X) = \theta(\eta(X)) - \eta([\theta, X]) = \theta(g(\xi, X)) - g(\xi, [\theta, X]) \stackrel{(\dagger)}{=} g([\theta, \xi], X) \stackrel{(*)}{=} 0,$$

where  $(\dagger)$  holds because  $\theta$  is Killing and  $(*)$  holds because  $\theta$  and  $\xi$  commute. Since  $\eta(\theta)$  is constant, so is  $\eta(\xi')$  and hence  $d\eta' = \frac{1}{\eta(\xi')} d\eta = 0$ . We show next that  $\omega'$  is closed. By (9),

$$d\omega' = \frac{1}{\eta(\xi')} d(i_\theta \omega) \wedge \eta' = 0$$

since  $d\omega = 0$  and  $L_\theta \omega = 0$  by hypothesis.

Now assume that  $(\eta, \xi, \phi, g)$  is K-cosymplectic. In order to show that  $(\eta', \xi', \phi', g')$  is K-cosymplectic, we use Corollary 2.4 and prove that  $L_{\xi'} \phi' = 0$ . Notice that since  $(\eta, \xi, \phi, g)$  is K-cosymplectic, we have

$$L_\xi \phi' = L_\xi \phi - L_\xi(\eta' \otimes \phi \xi') = -(L_\xi \eta') \otimes \phi \xi' + \eta' \otimes L_\xi(\phi \xi') = \eta' \otimes \phi(L_\xi \xi') = 0.$$

Moreover,  $L_\theta \phi = 0$  because  $L_\theta g = L_\theta \omega = 0$ . Finally,

$$L_\theta \phi' = L_\theta \phi - L_\theta(\eta \otimes \phi \xi') = \eta \otimes L_\theta(\phi \xi') = \eta \otimes \phi(L_\theta \xi') = 0.$$

To finish we need to consider the case in which  $(\eta, \xi, \phi, g)$  is coKähler and show that  $N_{\phi'} = 0$ , where  $N_{\phi'}$  is the Nijenhuis torsion of  $\phi'$ . Since  $N_{\phi'}$  is a tensor, it is enough to compute it on pairs  $(X, Y)$  and  $(X, \xi')$ , with  $X, Y$  sections of  $\ker \eta$ . Take such  $X$  and  $Y$ ; then, because  $\ker \eta$  is an integrable distribution,

$$\begin{aligned}
 N_{\phi'}(X, Y) &= (\phi')^2[X, Y] - \phi'[\phi'X, Y] - \phi'[X, \phi'Y] + [\phi'X, \phi'Y] \\
 &= -[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + [\phi X, \phi Y] = N_{\phi}(X, Y) = 0.
 \end{aligned}$$

Further, because the Lie bracket of  $\xi'$  with a section of  $\ker \eta$  is again in  $\ker \eta$ , we have

$$N_{\phi'}(X, \xi') = (\phi')^2[X, \xi'] - \phi'[\phi'X, \xi'] = -[X, \xi'] - \phi[\phi X, \xi'] = 0$$

using  $L_{\xi'}\phi = 0$  in the last equality. □

Of course, we can interpret this proposition also in terms of cosymplectic manifolds, without having fixed a Riemannian metric. Indeed, if  $(M, \eta, \omega)$  is a cosymplectic manifold with Reeb field  $\xi$ , and  $\theta \in \mathfrak{X}(M)$  is a cosymplectic vector field which commutes with  $\xi$  and is such that  $1 + \eta(\theta) > 0$ , then

$$\eta' = \frac{\eta}{1 + \eta(\theta)} \quad \text{and} \quad \omega' = \frac{\omega + (i_{\theta}\omega) \wedge \eta'}{1 + \eta(\theta)}$$

define a new cosymplectic structure on  $M$  with Reeb field  $\xi' = \xi + \theta$ .

### 6.2 Cosymplectic group actions

Cosymplectic group actions were introduced by Albert in [1]. We recall the definition and prove a proposition which is needed in Sect. 8.1.

**Definition 6.3** Let  $(M, \eta, \omega)$  be a cosymplectic manifold and let  $G$  be a Lie group acting smoothly on  $M$ ; we will denote the diffeomorphism of  $M$  given by  $g \in G$  with  $g: M \rightarrow M; x \mapsto g \cdot x$ . Such an action is *cosymplectic* if, for every  $g \in G$ ,

$$g^*\eta = \eta \quad \text{and} \quad g^*\omega = \omega,$$

In this case, we say that  $G$  acts by cosymplectomorphisms on  $M$ .

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ ; for every element  $A \in \mathfrak{g}$ , let  $\bar{A} \in \mathfrak{X}(M)$  denote the fundamental vector field of the action, defined by

$$\bar{A}(x) = \left. \frac{d}{dt} \right|_{t=0} \exp(tA) \cdot x.$$

Then, if  $G$  acts on  $M$  by cosymplectomorphisms, one has

$$L_{\bar{A}}\eta = 0 \quad \text{and} \quad L_{\bar{A}}\omega = 0 \quad \forall A \in \mathfrak{g}.$$

In particular, every fundamental vector field is cosymplectic.

**Corollary 6.4** *If  $G$  acts on  $M$  by cosymplectomorphisms, then the 1-form  $\eta$  defines a linear form on  $\mathfrak{g}$ , and  $\mathfrak{g}' = \ker \eta$  is an ideal in  $\mathfrak{g}$  which is either equal to  $\mathfrak{g}$  or of codimension one.*

*Proof* Because the  $G$ -action is cosymplectic, we have  $d(i_{\bar{A}}\eta) = 0$ ; hence  $\eta(\bar{A})$  is a constant function. Thus  $\eta$  is well defined on  $\mathfrak{g}$ , and so is its kernel  $\mathfrak{g}'$ , which has codimension 0 or 1. By Lemma 5.10, this kernel is in fact an ideal in  $\mathfrak{g}$ . □

Let us assume now that the acting Lie group  $G$  is compact; this has the important consequence that each component of the fixed point set is a submanifold of  $M$ . With this fact in mind, we can prove the following:

**Proposition 6.5** *Suppose  $G$  is a compact Lie group acting cosymplectically on  $M$ . If  $\eta(\bar{A}) = 0$  for all  $A \in \mathfrak{g}$ , then every component of the fixed point set  $M^G$  is a cosymplectic submanifold of  $M$ .*

*Proof* The proof is analogous to the contact case, see [19, Lemma 9.15]. Let  $N$  be a component of  $M^G$ , and  $p \in N$ . Then the tangent space  $T_p N$  is exactly the subspace of  $T_p M$  consisting of elements fixed by the isotropy representation of  $G$ . In particular, this shows  $\xi_p \in T_p N$ .

We want to show that  $(N, \eta|_N, \omega|_N)$  is a cosymplectic manifold. In other words,  $\omega$  has to be nondegenerate on  $\ker(\eta|_N) = TN \cap \ker \eta$ . To see this, we decompose, for  $p \in N$ , the tangent space

$$T_p M = T_p N \oplus \bigoplus_{\mu} V_{\mu}$$

into the weight spaces of the isotropy representation. More precisely, we fix a maximal torus  $T \subset G$ ; then, each  $V_{\mu}$  is the complex one-dimensional vector subspace such that for  $A \in \mathfrak{t}$  and  $v \in V_{\mu}$  we have  $\bar{A} \cdot v = \mu(A)Jv$ . Thus, for  $v \in T_p N$  and  $w \in V_{\mu}$  we have

$$0 = (L_{\bar{A}}\omega)(v, w) = -\omega(\bar{A} \cdot v, w) - \omega(v, \bar{A} \cdot w) = -\mu(A)\omega(v, Jw).$$

Hence,  $T_p N$  is  $\omega$ -orthogonal to every  $V_{\mu}$ . This implies that  $\omega$  is nondegenerate on  $TN \cap \ker \eta$ . □

### 6.3 Deformations induced by cosymplectic actions

Consider a cosymplectic action of a Lie group  $G$  on a compact cosymplectic manifold  $(M, \eta, \omega)$ . The fundamental vector fields of the action are cosymplectic; it follows from Corollary 5.9 that they commute with  $\xi$ . Thus we can apply the general deformation described in the previous section to any fundamental vector field  $\bar{A}$ , where  $A \in \mathfrak{g}$ , as long as  $1 + \eta(\bar{A}) > 0$ , to obtain a new cosymplectic structure on  $M$  with Reeb field  $\xi + \bar{A}$ .

*Example 6.6* Consider the standard symplectic structure on  $\mathbb{C}P^n$ , together with the induced  $K$ -cosymplectic manifold  $(M = \mathbb{C}P^n \times S^1, \eta, \omega)$ . On  $\mathbb{C}P^n$  we have a natural  $T^n$ -action defined by

$$(t_1, \dots, t_n) \cdot [z_1 : \dots : z_{n+1}] = [t_1 z_1 : \dots : t_n z_n : z_{n+1}]$$

and we consider the induced product action of  $T^{n+1} = T^n \times S^1$  on  $M$ . This  $K$ -cosymplectic structure is regular, with the Reeb field being a fundamental field of the  $S^1$ -factor. The action is cosymplectic, so by choosing  $\bar{A}$  close to the original Reeb field (i.e., such that  $\eta(\bar{A}) > 0$ ) and such that the one-parameter subgroup defined by  $A$  is dense in  $T^{n+1}$ , we can find a new  $K$ -cosymplectic structure on  $\mathbb{C}P^n \times S^1$  with Reeb field  $\bar{A}$ ; hence it has the property that each Reeb orbit is dense in the corresponding  $T^{n+1}$ -orbit.

We observe that the  $T^{n+1}$ -action has precisely  $n + 1$  one-dimensional orbits, given by  $\{[1 : 0 : \dots : 0]\} \times S^1, \dots, \{[0 : \dots : 0 : 1]\} \times S^1$ . These coincide with the closed Reeb orbits of the deformed structure. Compare also Corollaries 8.8 and 8.9 below.

We thus see that we can use the type of deformation introduced above to obtain examples of irregular cosymplectic manifolds. But we can also use it in the converse direction: Assume that we are given an irregular  $K$ -cosymplectic structure on a compact manifold  $M$ . By [36],

the isometry group of  $(M, g)$  is a compact Lie group. The Reeb field  $\xi$  generates a 1-parameter subgroup. Set

$$T = \overline{\{\exp(t\xi) \mid t \in \mathbb{R}\}}.$$

Then  $T$  is a torus acting on  $M$ , and the irregularity assumption means that its dimension is strictly larger than 1; notice that, since  $\xi$  preserves the  $K$ -cosymplectic structure, and its flow is by construction dense on  $T$ , by continuity every element  $t \in T$  acts on  $M$  preserving the  $K$ -cosymplectic structure. Infinitesimally, we thus obtain a Lie algebra homomorphism  $\mathfrak{t} \rightarrow \mathfrak{X}^{\text{cosymp}}(M)$ .

For any element  $A \in \mathfrak{t}$  such that  $\eta(\bar{A}) > 0$  we find a  $K$ -cosymplectic structure with  $\bar{A}$  as Reeb field. Choosing  $A$  such that it generates a circle in  $T$  we have shown

**Proposition 6.7** *On any compact  $K$ -cosymplectic manifold there exists a  $K$ -cosymplectic structure which is either regular or quasi-regular.*

### 7 Hamiltonian group actions

In this section we recall the notion of Hamiltonian action with the corresponding momentum map, as defined by Albert [1]. We provide some existence and uniqueness statements for the momentum map in the spirit of Hamiltonian group actions on symplectic manifolds.

**Definition 7.1** Let  $(M, \eta, \omega)$  be a cosymplectic manifold and let  $G$  be a Lie group acting on  $M$  by cosymplectomorphisms. The  $G$ -action is *Hamiltonian* if there exists a smooth map  $\mu: M \rightarrow \mathfrak{g}^*$  such that

$$\bar{A} = X_{\mu^A} \quad \forall A \in \mathfrak{g},$$

where  $\mu^A: M \rightarrow \mathbb{R}$  is the function defined by the rule  $\mu^A(x) = \mu(x)(A)$  and  $X_{\mu^A}$  is the Hamiltonian vector field of this function. Furthermore, we require  $\mu$  to be equivariant with respect to the natural coadjoint action of  $G$  on  $\mathfrak{g}^*$ :

$$\mu(g \cdot x) = g \cdot \mu(x) = \mu(x) \circ \text{Ad}_{g^{-1}}.$$

The map  $\mu$  is called *momentum map* of the Hamiltonian  $G$ -action.

By definition of a Hamiltonian vector field, we have  $\eta(X_{\mu^A}) = 0$ . One sees easily that the conditions of the definition imply that each component of the momentum map satisfies

$$\iota_{\bar{A}}\omega = d\mu^A.$$

*Remark 7.2* Not every group action on a cosymplectic manifold is Hamiltonian. Indeed, if  $M$  is compact (or if the flow of the Reeb field  $\xi$  is complete), then  $M$  is endowed with a natural  $\mathbb{R}$ -action, given by the flow of  $\xi$ . The fundamental field of this action is  $\xi$  and since  $\eta(\xi) = 1$ , such an action will never be Hamiltonian. This is different from what happens in the contact case.

It is possible to give another, equivalent, definition of Hamiltonian action, which uses the so-called comomentum map.

**Definition 7.3** Let  $(M, \eta, \omega)$  be a cosymplectic manifold and let  $G$  be a Lie group acting on  $M$  by cosymplectomorphisms. The  $G$ -action is *weakly Hamiltonian* if there exists a map  $\nu: \mathfrak{g} \rightarrow C_\xi^\infty(M)$  which makes the following diagram commute:

$$\begin{array}{ccc}
 \mathfrak{g} & \xrightarrow{\nu} & C_\xi^\infty(M) \\
 \downarrow & & \downarrow \bar{\Theta} \\
 \mathfrak{X}^{\text{cosymp}}(M) & \longleftarrow & \mathfrak{X}^{\text{ham}}(M)
 \end{array} \tag{10}$$

The map  $\bar{\Theta}$  comes from Corollary 5.19. The action is *Hamiltonian* if  $\nu$  is an anti-morphism of Lie algebras. Such map  $\nu$  is called *comomentum map* of the Hamiltonian  $G$ -action.

**Lemma 7.4** Let  $(M, \eta, \omega)$  be a connected cosymplectic manifold and let  $G$  be a Lie group acting on  $M$  by cosymplectomorphisms. Assume that the action is weakly Hamiltonian. Then, for  $A, B \in \mathfrak{g}$ , the function  $\nu([A, B]) + \{\nu(A), \nu(B)\}$  is constant.

*Proof* Since the action is weakly Hamiltonian, given  $A \in \mathfrak{g}$ ,  $\nu(A) \in C_\xi^\infty(M)$  is a Hamiltonian function for  $\bar{A}$ . Now we have  $[\bar{A}, \bar{B}] = -[\bar{A}, \bar{B}]$ , therefore,  $-\{\nu(A), \nu(B)\}$  is a Hamiltonian function for  $[\bar{A}, \bar{B}]$ . Since the same is true for  $\nu([A, B])$ , their difference  $\nu([A, B]) + \{\nu(A), \nu(B)\}$  is constant.  $\square$

**Proposition 7.5** Let  $(M, \eta, \omega)$  be a connected cosymplectic manifold and let  $G$  be a connected Lie group. Let  $\Phi: G \times M \rightarrow M$  be a cosymplectic action of  $G$  on  $M$ . The two notions of Hamiltonian  $G$ -action are equivalent.

*Proof* Assume we have a momentum map  $\mu: M \rightarrow \mathfrak{g}^*$  which is equivariant and such that, for every  $A \in \mathfrak{g}$ ,  $\xi(\mu^A) = 0$ . Define  $\nu: \mathfrak{g} \rightarrow C_\xi^\infty(M)$  by  $\nu(A)(x) = \mu^A(x)$ , where  $x \in M$ . Then  $\nu$  is well defined. For  $A, B \in \mathfrak{g}$  and  $x \in M$ , we compute

$$\begin{aligned}
 \nu([A, B])(x) &= \mu^{[A, B]}(x) = \mu(x)([A, B]) = \left. \frac{d}{dt} \right|_{t=0} \mu(x)(\text{Ad}_{\exp(tA)} B) \\
 &= \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(tA)}^* (\mu(x))(B) = \left. \frac{d}{dt} \right|_{t=0} (\mu(\Phi_{\exp(-tA)}(x)))(B) \\
 &= -d\mu^B(X_{\mu^A})(x) = -X_{\mu^A}(\mu^B)(x) = -\{\mu^A, \mu^B\}(x) \\
 &= -\{\nu(A), \nu(B)\}(x).
 \end{aligned}$$

For the converse, suppose that we have a comomentum map  $\nu: \mathfrak{g} \rightarrow C_\xi^\infty(M)$  which is an anti-morphism of Lie algebras. Define the map  $\mu: M \rightarrow \mathfrak{g}^*$  in the following way:  $\mu(x)(A) = \nu(A)(x)$ , for any  $A \in \mathfrak{g}$ . Also, for a fixed  $A \in \mathfrak{g}$ , set  $\mu^A(x) = \mu(x)(A)$ . We prove first that  $\xi(\mu^A) = 0$  for every  $A \in \mathfrak{g}$ . Indeed,

$$\xi(\mu^A) = d\mu^A(\xi) = d(\nu(A))(\xi) = \xi(\nu(A)) = 0,$$

since  $\nu(A) \in C_\xi^\infty(M)$  for any  $A \in \mathfrak{g}$ . By diagram (10), the vector field  $X_{\nu(A)}$  is Hamiltonian. Now  $\nu(A) = \mu^A$ ; hence  $\bar{A} = X_{\nu(A)} = X_{\mu^A}$ . Finally, we prove the equivariance of  $\mu$ . Since  $G$  is connected, it is enough to prove this infinitesimally. We must, therefore, show that the map  $\mu: M \rightarrow \mathfrak{g}^*$  we just defined satisfies, for every  $x \in M$ ,

$$d_x \mu(\bar{A}_x) = \hat{A}_{\mu(x)},$$

where  $\hat{A}$  denotes the infinitesimal vector field of the coadjoint action of  $G$  on  $\mathfrak{g}^*$ , defined by  $\hat{A}_\alpha(B) = \alpha([B, A])$  for  $\alpha \in \mathfrak{g}^*$  and  $A, B \in \mathfrak{g}$ . We compute

$$\begin{aligned} \hat{A}_{\mu(x)}(B) &= \mu(x)([B, A]) = \nu([B, A])(x) = \{\nu(A), \nu(B)\}(x) = X_{\nu(A)}(\nu(B))(x) \\ &= \bar{A}_x(\nu(B)) = d_x \nu(B)(\bar{A}_x) = d_x \mu^B(\bar{A}_x) \\ &= d_x \mu(\bar{A}_x)(B). \end{aligned}$$

□

### 7.1 Existence and uniqueness of the momentum map

We are interested in sufficient conditions under which a  $G$ -action on a cosymplectic manifold  $M$  is Hamiltonian. We can prove the existence of the momentum map in some simple cases which are of our interest.

**Theorem 7.6** *Let  $(M, \eta, \omega)$  be a compact cosymplectic manifold with  $b_1(M) = 1$  and let  $T$  be a torus which acts on  $M$ . Suppose that the  $T$ -action is cosymplectic and that  $\eta(\bar{A}) = 0$  for every  $A \in \mathfrak{t}$ . Then the  $T$ -action is Hamiltonian.*

*Proof* Set  $\mathfrak{t} = \text{Lie}(T)$ , take  $A \in \mathfrak{t}$ , and let  $\bar{A} \in \mathfrak{X}_0^{\text{cosymp}}(M)$  be the fundamental vector field of the action. By Proposition 5.13,  $\bar{A}$  is Hamiltonian; hence,  $\iota_{\bar{A}}\omega = df^A$  for some function  $f^A \in C^\infty(M)$ ; in fact,  $f^A \in C_\xi^\infty(M)$ . Next, we collect all these functions in a map  $\nu: \mathfrak{t} \rightarrow C_\xi^\infty(M)$  as follows: choose a basis  $\{A_1, \dots, A_n\}$  of  $\mathfrak{t}$ , define  $\nu(A_i) = f^{A_i}$  for  $1 \leq i \leq n$  and extend  $\nu$  to  $\mathfrak{t}$  by linearity. We proceed to show that  $\nu$  is an anti-morphism of Lie algebras. The function  $\{\nu(A), \nu(B)\} + \nu([A, B])$  is constant on  $M$  (see Lemma 7.4). Since the Lie algebra  $\mathfrak{t}$  is abelian, this implies that  $\{\nu(A), \nu(B)\}$  is constant. But  $M$  is a compact manifold, so  $\nu(A)$  must have at least one critical point. Hence  $\{\nu(A), \nu(B)\}$  vanishes at some point, so it is identically zero. Then  $\nu([A, B]) = 0 = -\{\nu(A), \nu(B)\}$ , i.e.,  $\nu$  is an anti-morphism of Lie algebras and the  $T$ -action is Hamiltonian. □

The above theorem gives conditions on the cosymplectic manifold  $M$  for a cosymplectic torus action to be Hamiltonian. Next, we give conditions on a compact Lie group  $G$  that ensure that a cosymplectic  $G$ -action is Hamiltonian. The kind of conditions we give mirrors what happens in the symplectic case (see [2, 29]).

**Theorem 7.7** *Let  $(M, \eta, \omega)$  be a compact cosymplectic manifold and let  $G$  be a connected, compact semisimple Lie group acting on  $M$  by cosymplectomorphisms. Assume that  $\eta(\bar{A}) = 0$  for every  $A \in \mathfrak{g}$ . Then the action is Hamiltonian and the momentum map  $\mu: M \rightarrow \mathfrak{g}^*$  is unique.*

*Proof* The proof is formally equal to the symplectic case, where every symplectic action of a compact semisimple Lie group is Hamiltonian. One shows that the obstruction to lifting the map  $\mathfrak{g} \rightarrow \mathfrak{X}_0^{\text{cosymp}}(M)$  to a map  $\mathfrak{g} \rightarrow \mathfrak{X}^{\text{ham}}(M)$  lies in  $H^2(\mathfrak{g}; \mathbb{R})$ , which vanishes if  $G$  is semisimple. We obtain, therefore, a comomentum map  $\nu: \mathfrak{g} \rightarrow C_\xi^\infty(M)$ , and the action is Hamiltonian, since we assume  $G$  connected. The lack of uniqueness is measured by  $H^1(\mathfrak{g}; \mathbb{R})$ , which again vanishes since  $G$  is semisimple. □

## 8 Closed Reeb orbits and basic cohomology

In this section we derive a relation between the topology of the union  $C$  of closed Reeb orbits on a compact  $K$ -cosymplectic manifold  $M$  and the (basic) cohomology of  $M$ , similar to the

$K$ -contact case treated in [19]. As in [19], the proof follows from considering the equivariant cohomology of the torus action defined by the closure of the Reeb field, using the statement that  $C$  arises as the critical locus of a generic component of the momentum map, which was proven in the  $K$ -contact case by Rukimbira [40]. The analogous statement for fixed points of Hamiltonian group actions on symplectic manifolds is well known, see, e.g. [2, Proposition III.2.2].

### 8.1 The canonical torus action on an irregular $K$ -cosymplectic manifold

Consider a Hamiltonian action of a torus  $T$  on a cosymplectic manifold  $(M, \eta, \omega)$ , with momentum map  $\mu: M \rightarrow \mathfrak{g}^*$ .

**Proposition 8.1** *For any  $A \in \mathfrak{t}$  we have  $\text{crit}(\mu^A) = \{p \in M \mid \bar{A}_p = 0\}$ .*

*Proof* Because  $\bar{A} = X_{\mu^A}$ , we have

$$\iota_{\bar{A}}\omega = \iota_{X_{\mu^A}}\omega = d\mu^A,$$

and hence the critical points of  $\mu^A$  are precisely those points at which  $\bar{A}$  is a multiple of the Reeb field  $\xi$ . But by definition of a momentum map, we have  $\eta(\bar{A}) = \eta(X_{\mu^A}) = 0$ ; hence at such a point  $\bar{A}$  necessarily vanishes. □

Assume now that  $M$  is a compact  $K$ -cosymplectic manifold. Let  $T$  denote the torus given by the closure of the flow of the Reeb field; more precisely, we choose a metric with respect to which the Reeb field is Killing and define  $T$  as the closure of the 1-parameter subgroup of the corresponding isometry group defined by  $\xi$ . We additionally assume that our  $K$ -cosymplectic manifold is irregular, which in terms of  $T$  just means that  $\dim T \geq 2$ . By Corollary 6.4, the element  $\xi \in \mathfrak{t}$  has a canonical complement in  $\mathfrak{t}$  given by  $\mathfrak{s} = \ker \eta$ . Let  $S$  denote the connected Lie subgroup of  $T$  with Lie algebra  $\mathfrak{s}$ .

Because  $T$  is defined as the closure of the 1-parameter subgroup defined by  $\xi$ , and  $\xi$  is cosymplectic, the torus  $T$  acts on  $M$  by cosymplectomorphisms. Hence,  $S$  also acts by cosymplectomorphisms and satisfies  $\eta(A) = 0$  for all  $A \in \mathfrak{s}$ . We now assume additionally that the  $S$ -action on  $M$  is Hamiltonian (which by Theorem 7.6 is automatic if  $b_1 = 1$ ) and denote a momentum map by  $\mu: M \rightarrow \mathfrak{s}^*$ . We will argue below that, under this assumption,  $S$  is closed in  $T$ . Notice that in general the corresponding statement for cosymplectic non-Hamiltonian actions is false:

*Example 8.2* We consider  $(T^3, \eta, \omega)$  as a  $K$ -cosymplectic manifold, as we did in Example 3.8; let  $\xi$  denote the Reeb field. Pick a vector  $\xi' \in \mathfrak{t}$  with the property that the closure of its flow is dense in  $T^3 \subset \text{Isom}(T^3)$ ; for instance, one whose coefficients are pairwise algebraically independent over  $\mathbb{Q}$  does the job. Using  $\xi'$  we can perform a deformation of type I, as explained in Sect. 6, and obtain another  $K$ -cosymplectic manifold  $(T^3, \eta', \omega')$ . We use next a deformation of type II to produce a third  $K$ -cosymplectic manifold  $(T^3, \eta'', \omega'')$  with Reeb field  $\xi'' = \xi'$  such that the foliation  $\ker \eta''$  is dense in  $T^3$ ; more precisely, if  $S$  is the unique connected Lie group with Lie algebra  $\ker \eta''$ , then  $S$  is dense in  $T$ ; in particular,  $S$  is not a closed subtorus. The action of  $T^3$  on itself is cosymplectic but not Hamiltonian.

We denote by  $C \subset M$  the union of all closed orbits of  $\xi$ , i.e., all flow lines which are homeomorphic to a circle. Note that by definition,  $C$  is equal to the union of all one-dimensional  $T$ -orbits, and then also equal to the fixed point set of the  $S$ -action. We then obtain the following statement about the functions  $\mu^A: M \rightarrow \mathbb{R}$ , where  $A \in \mathfrak{s}$ :

**Proposition 8.3** *For generic  $A \in \mathfrak{s}$ , i.e., such that the 1-parameter subgroup defined by  $A$  is dense in  $S$ , the critical set of  $\mu^A$  is precisely  $C$ .*

*Proof* By Proposition 8.1 the critical set of  $\mu^A$  is the set of points where  $\bar{A}$  vanishes. But  $\bar{A}$  vanishes at a point  $p$  if and only if  $p$  is a fixed point of the 1-parameter subgroup defined by  $A$ , and these fixed points coincide by assumption on  $A$  with the fixed points of  $S$ .  $\square$

Note that by compactness of  $M$ , the functions  $\mu^A$  always have critical points. Thus, the proposition directly implies the existence of closed Reeb orbits on compact  $K$ -cosymplectic manifolds. Note that below, in Corollary 8.8, we will prove a more precise existence statement for closed Reeb orbits.

The fact that  $C \neq \emptyset$  also implies that  $S$  is closed in  $T$ , i.e., a subtorus: if the codimension one subgroup  $S$  was not closed in  $T$ , then its closure would be equal to  $T$ . This would imply that the  $S$ -fixed point set equals the  $T$ -fixed point set, but this contradicts the fact that the Reeb field has no zeros.

**Proposition 8.4** *For generic  $A \in \mathfrak{s}$  the function  $\mu^A$  is a Morse-Bott function.*

*Proof* The critical set of  $\mu^A$  was shown to be the fixed point set of the  $S$ -action; hence each of its components is a submanifold of  $M$ . We have to show that the Hessian of  $\mu^A$  is non-degenerate in the normal directions. The computation is structurally similar to the  $K$ -contact case, see [40].

Let  $p \in C$ , and  $v, w \in T_pM$  nonzero vectors normal to  $C$ . Extend  $v$  and  $w$  to vector fields  $V$  and  $W$  in a neighborhood of  $p$ . Then we compute using  $\bar{A}_p = 0$ :

$$\begin{aligned} \text{Hess}_{\mu^A}(p)(v, w) &= V(W(\mu^A))(p) = V(d\mu^A(W))(p) \\ &= V(\omega(\bar{A}, W))(p) = L_V(\omega(\bar{A}, W))(p) \\ &= (L_V\omega)(\bar{A}, W)(p) + \omega([V, \bar{A}], W)(p) + \omega(\bar{A}, [V, W])(p) \\ &= (\iota_V\omega)([\bar{A}, W])(p) + \omega(\nabla_V\bar{A}, W)(p) \\ &= -\omega(v, \nabla_w\bar{A}) + \omega(\nabla_v\bar{A}, w) \\ &= 2\omega(\nabla_v\bar{A}, w). \end{aligned}$$

We now claim that  $\nabla_v\bar{A}$  is always nonzero and perpendicular to  $C$ . For that we fix a Riemannian metric on  $M$  for which  $\xi$  is Killing. Then the  $T$ -action is isometric, and all its fundamental vector fields are also Killing vector fields. The restriction of the Killing field  $\bar{A}$  to the geodesic  $\gamma$  through  $p$  in direction  $v$  is a Jacobi field with initial conditions  $\bar{A}_p = 0$  and  $\nabla_v\bar{A}$ ; hence, if  $\nabla_v\bar{A}$  was zero, then  $\bar{A}$  would vanish along  $\gamma$ . But this would mean that  $\gamma$  consists entirely of  $S$ -fixed points, contradicting the fact that  $v$  points in a direction perpendicular to  $C$ . Hence,  $\nabla_v\bar{A} \neq 0$ . Moreover, we observe that

$$T_pC = \{u \in T_pM \mid [\bar{A}, u] = 0\};$$

note that the expression  $[\bar{A}, u] \in T_pM$  is well defined because  $\bar{A}$  vanishes at  $p$ . We then compute

$$g(\nabla_v\bar{A}, u) = g([v, \bar{A}], u) = g(v, [\bar{A}, u]) = 0$$

for all  $u \in T_pC$ , using the facts that  $\bar{A}$  is Killing and vanishes at  $p$ . Hence,  $\nabla_v\bar{A}$  is perpendicular to  $C$ .

Because  $C$  is the fixed point set of the  $S$ -action all of whose fundamental vector fields are in  $\mathfrak{X}_0^{\text{cosymp}}(M)$  (and because we have argued above that  $S$  is indeed compact) Proposition

6.5 shows that every component of  $C$  is a cosymplectic submanifold. Hence,  $\phi \nabla_v \bar{A}$  is also a nonzero vector perpendicular to  $C$ . Then we obtain

$$\text{Hess}_{\mu^A}(p)(v, \phi \nabla_v \bar{A}) = 2\omega(\nabla_v \bar{A}, \phi \nabla_v \bar{A}) \neq 0.$$

□

### 8.2 Equivariant cohomology of the canonical action

Recall that the (Cartan model of) *equivariant cohomology* of an action of a compact Lie group  $G$  on a manifold  $M$  is defined as the cohomology  $H_G^*(M)$  of the complex  $(C_G(M), d_G)$ , where

$$C_G(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G$$

is the space of  $G$ -equivariant differential forms, i.e.,  $G$ -equivariant polynomials  $\omega: \mathfrak{g} \rightarrow \Omega(M)$ , and the equivariant differential  $d_G$  is defined by

$$(d_G \omega)(X) = d(\omega(X)) - \iota_{\bar{X}}(\omega(X)).$$

The grading on  $H_G^*(M)$  is defined by imposing that a linear form on  $\mathfrak{g}$  has degree two; more precisely, the equivariant differential forms of degree  $k$  are

$$C_G^k(M) = \bigoplus_{2p+q=k} (S^p(\mathfrak{g}^*) \otimes \Omega^q(M))^G,$$

and with this grading  $d_G$  increases the degree by one. The ring homomorphism

$$S(\mathfrak{g}^*)^G \longrightarrow C_G(M); f \longmapsto f \otimes 1$$

induces on  $H_G^*(M)$  the structure of an  $S(\mathfrak{g}^*)^G$ -algebra.

**Definition 8.5** The  $G$ -action is called *equivariantly formal* if  $H_G^*(M) \cong H^*(M; \mathbb{R}) \otimes S(\mathfrak{g}^*)^G$  as  $S(\mathfrak{g}^*)^G$ -modules.

Using the notation and assumptions of the previous subsection, we now consider the equivariant cohomology  $H_S^*(M)$  of the  $S$ -action on the compact  $K$ -cosymplectic manifold  $M$ . We have

**Theorem 8.6** *If the  $S$ -action on a compact  $K$ -cosymplectic manifold is Hamiltonian, then it is equivariantly formal.*

*Proof* The existence of a Morse-Bott function whose critical set is the fixed point set of the action implies that the action is equivariantly formal; this was proven by Kirwan for the momentum map of a Hamiltonian action on a compact symplectic manifold [31, Proposition 5.8] but is true for arbitrary Morse-Bott functions, see, e.g. [25, Theorem G.9]. In our situation Propositions 8.1 and 8.4 show the existence of a Morse-Bott function with the desired property. □

For the action of a torus  $T$  on a manifold  $M$  we have that equivariant formality is equivalent to the equality of total Betti numbers  $\dim H^*(M; \mathbb{R}) = \dim H^*(M^T)$ , where  $M^T$  is the  $T$ -fixed point set, as follows from Borel’s localization theorem, see, e.g. [25, Theorems C.20 and C.24]. Applied to our canonical  $S$ -action this gives the following corollary:

**Corollary 8.7** *For a compact  $K$ -cosymplectic manifold such that the canonical  $S$ -action is Hamiltonian we have  $2 \dim H^*(M; \mathcal{F}_\xi) = \dim H^*(M; \mathbb{R}) = \dim H^*(C; \mathbb{R})$ .*

*Proof* The first equality follows directly from Theorem 4.3. The second follows from the equivariant formality of the  $S$ -action, because the  $S$ -fixed points are exactly  $C$ .  $\square$

In particular, if there are only finitely many closed Reeb orbits, then their number is given by  $\dim H^*(M; \mathcal{F}_\xi)$ .

**Corollary 8.8** *A compact  $2n + 1$ -dimensional  $K$ -cosymplectic manifold  $M$  such that the canonical  $S$ -action is Hamiltonian has at least  $n + 1$  closed Reeb orbits.*

*Proof* By item 3. of Proposition 4.4 the  $p$ -th power of  $\omega$  defines a nontrivial class in  $H^{2p}(M; \mathcal{F}_\xi)$  for  $1 \leq p \leq n$ . Thus,  $\dim H^*(M; \mathcal{F}_\xi) \geq n + 1$ .  $\square$

**Corollary 8.9** *Assume that the compact  $2n + 1$ -dimensional  $K$ -cosymplectic manifold  $M$  has only finitely many closed Reeb orbits and that the canonical  $S$ -action is Hamiltonian. Then the following conditions are equivalent:*

1.  $M$  has exactly  $n + 1$  closed Reeb orbits.
2.  $H^*(M; \mathcal{F}_\xi)$  is generated by  $[\omega] \in H^2(M; \mathcal{F}_\xi)$ .
3.  $M$  has the real cohomology ring of  $\mathbb{C}P^n \times S^1$ .

*Proof* The equivalence of 1. and 2. is just the fact that the number of closed Reeb orbits is equal to the dimension of  $H^*(M; \mathcal{F}_\xi)$ . Conditions 2. and 3. are equivalent because by Theorem 4.3 we have  $H^*(M; \mathbb{R}) = H^*(M; \mathcal{F}_\xi) \otimes \Lambda\langle[\eta]\rangle$  for any  $K$ -cosymplectic manifold.  $\square$

*Remark 8.10* Instead of considering ordinary  $S$ -equivariant cohomology, we could have also used, in the same way as in [19], the equivariant basic cohomology of the associated transverse action on the foliated manifold  $(M, \mathcal{F}_\xi)$ . In this way we would directly obtain information on the basic cohomology  $H^*(M; \mathcal{F}_\xi)$ . Our situation is, however, simpler than the  $K$ -contact case considered in [19], because in the  $K$ -cosymplectic case the union  $C$  of closed Reeb orbits appears as the fixed point set of the subtorus  $S$  of  $T$ .

It is an interesting question to ask which compact manifolds have the same real cohomology of  $\mathbb{C}P^n \times S^1$ . By the Künneth formula, it is sufficient to find a manifold with the same real cohomology as  $\mathbb{C}P^n$ . We are, therefore, looking for a smooth manifold  $M$  of dimension  $2n$  whose real cohomology is

$$H^*(M; \mathbb{R}) \cong \mathbb{R}[x]/x^{n+1},$$

where  $x$  has degree 2. Such a manifold is called a rational or real cohomology  $\mathbb{C}P^n$ .

### 8.3 An example

In this section we will construct, for any  $m \geq 2$ , an example of a real cohomology  $\mathbb{C}P^{2m-1} \times S^1$  with a minimal number of closed Reeb orbits which is not diffeomorphic to  $\mathbb{C}P^{2m-1} \times S^1$ . These examples can be considered as the coKähler analogues of the Sasaki structures with minimal number of closed Reeb orbits on the Stiefel manifold considered in [19, Section 8].

Let  $\mathbb{Q}^{2m-1}$  denote the odd complex quadric, the zero locus of a single quadratic equation in  $\mathbb{C}P^{2m}$ . As a homogeneous space, the quadric  $\mathbb{Q}^{2m-1}$  can be described as

$$\mathbb{Q}^{2m-1} = \frac{\text{SO}(2m + 1)}{\text{SO}(2m - 1) \times \text{SO}(2)}.$$

A good reference for this is [32, Page 278]. As this is a homogeneous space of two compact Lie groups of equal rank, its real cohomology vanishes in odd degrees [24, Volume III,

Sec. 11.7, Theorem 7]. Moreover, its Euler characteristic is given by the quotient of the orders of the corresponding Weyl groups, see [23, Volume II, Sec. 4.21], and hence equal to  $2m$ . As  $\mathcal{Q}^{2m-1}$  is a Kähler manifold, it follows that the real cohomology ring of  $\mathcal{Q}^{2m-1}$  is precisely

$$H^*(\mathcal{Q}^{2m-1}; \mathbb{R}) \cong \mathbb{R}[x]/x^{2m},$$

where  $x$  is of degree 2. Hence  $\mathcal{Q}^{2m-1}$  is a real cohomology  $\mathbb{C}P^{2m-1}$ . However,  $\mathcal{Q}^{2m-1}$  and  $\mathbb{C}P^{2m-1}$  are not homeomorphic. Indeed, the odd quadric sits in the principal circle bundle  $S^1 \rightarrow V_2(\mathbb{R}^{2m+1}) \rightarrow \mathcal{Q}^{2m-1}$ , where  $V_2(\mathbb{R}^{2m+1})$  is the Stiefel manifold of orthonormal 2-frames in  $\mathbb{R}^{2m+1}$ . The long exact sequence of homotopy groups of this principal bundle gives

$$\pi_k(\mathcal{Q}^{2m-1}) = \pi_k(V_2(\mathbb{R}^{2m+1})) \quad \text{for } k \geq 3.$$

One can see that  $\pi_{\ell-j}(V_j(\mathbb{R}^\ell)) = \mathbb{Z}_2$  for  $j \geq 2$  and  $\ell - j$  an odd number ([30, Proposition 11.2]); hence  $\pi_{2m-1}(V_2(\mathbb{R}^{2m+1})) = \mathbb{Z}_2$  and also  $\pi_{2m-1}(\mathcal{Q}^{2m-1}) = \mathbb{Z}_2$ . However, the long exact homotopy sequence of the Hopf fibration  $S^1 \rightarrow S^{4m-1} \rightarrow \mathbb{C}P^{2m-1}$  shows that  $\pi_{2m-1}(\mathbb{C}P^{2m-1}) = 0$ .

As  $\mathcal{Q}^{2m-1}$  is a Kähler manifold,  $M = \mathcal{Q}^{2m-1} \times S^1$  is a coKähler manifold with the same real cohomology as  $\mathbb{C}P^{2m-1} \times S^1$ . Since the structure is coKähler, the flow of the Reeb field consists of isometries.  $M$  admits an isometric action of a torus  $T = T^m \times S^1$  of dimension  $m + 1$ . Using our deformation theory, we can perturb the coKähler structure to one for which the flow of the Reeb field is dense in this torus  $T$ . The unique connected subgroup of  $T$  with Lie algebra  $\ker \eta$  is  $S := T^m$ . Since  $b_1(M) = 1$ , the  $S$ -action is automatically Hamiltonian by Theorem 7.6. The closed Reeb orbits of the deformed structure are now given by the orbits through  $S$ -fixed points in  $\mathcal{Q}^{2m-1}$ . A general fact about homogeneous spaces  $G/H$  of two compact Lie groups of equal rank is now that the fixed point set of the action of a maximal torus in  $H$  on  $G/H$  by left multiplication is finite and given explicitly by the quotient of Weyl groups  $W(G)/W(H)$ . In our case, there are hence precisely  $2m$  closed Reeb orbits, as predicted by Corollary 8.9.

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