Complex Manifolds 2017; 4: 172–178 **DE GRUYTER** OPEN

Complex Manifolds

Research Article

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Locally conformal symplectic nilmanifolds with no locally conformal Kähler metrics

https://doi.org/10.1515/coma-2017-0011
Received July 12, 2017; accepted October 1, 2017.

Abstract: We report on a question, posed by L. Ornea and M. Verbitsky in [32], about examples of compact locally conformal symplectic manifolds without locally conformal Kähler metrics. We construct such an example on a compact 4-dimensional nilmanifold, not the product of a compact 3-manifold and a circle.

Keywords: Locally conformal symplectic manifolds, Locally conformal Kähler manifolds, Compact nilmanifolds

MSC: 53D05, 53C55

1 Introduction

In the seminal paper [17], Gray and Hervella studied almost Hermitian structures on even dimensional manifolds. Recall that an almost Hermitian structure on a manifold M of dimension $2n$ ($n \geq 2$) consists of a Riemannian metric g and a compatible almost complex structure J, that is, an endomorphism $J: TM \to TM$ with $J^2 = -\mathrm{Id}$, such that $g(JX, JY) = g(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$. Let (g, J) be an almost Hermitian structure on M; then one can consider the following tensors:

- a 2-form $\omega \in \Omega^2(M)$, the Kähler form, defined by $\omega(X, Y) = g(JX, Y)$, $X, Y \in \mathfrak{X}(M)$;
- a 1-form $\theta \in \Omega^1(M)$, the Lee form, defined by $\theta(X) = \frac{-1}{n-1}\delta\omega(JX)$, where δ is the codifferential and $X \in \mathfrak{X}(M)$.

Gray and Hervella classified almost Hermitian structures (g, J) by studying the covariant derivative of the Kähler form ω with respect to the Levi-Civita connection ∇ of g. Of particular interest for us are the following classes:

- the class of Kähler structures, for which $\nabla\omega = 0$. In this case, g is said to be a Kähler metric. Compact examples of manifolds endowed with Kähler structures are Riemann surfaces, complex tori, complex projective spaces and projective varieties; notice that the Lee form is zero on a Kähler manifold;
- the class of locally conformal Kähler (lcK from now on) structures, for which $d\omega = \omega \wedge \theta$ and $d\theta = 0$ (the second condition follows from the first one if $n \geq 3$). In this case we call g a lcK metric. Lck manifolds were introduced by Vaisman in [36]. A remarkable example of manifold endowed with a lcK structure is the Hopf surface (see [36]). The study of lcK metrics on compact complex surfaces was undertaken in [9].

In this note we will use indifferently the terminology (locally conformal) Kähler *manifold* and (locally conformal) Kähler *structure*. For a comprehensive introduction to Kähler geometry we refer to [20]; for lcK geometry, the standard reference is [13]; see also the recent survey [32].

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Kähler and locally conformal Kähler manifolds are examples of complex manifolds; this means that the almost complex structure J is integrable, hence the manifold is locally modelled on $\mathbb{C}^n$ with holomorphic transition functions. Furthermore, on Kähler and locally conformal Kähler manifolds, any two of the tensors (g, J, ω) determine the third. One can interpret Kähler manifolds as a "degenerate" case of locally conformal Kähler manifolds. The Lee form of a lcK structure is closed; suppose that it is exact; one can then easily show that there is a Kähler metric in the conformal class of g. In such a case, one says that the structure is globally conformal Kähler (gcK). In general, genuine lcK manifolds (those for which the Lee form is not exact) admit an open cover $\{U_\alpha\}$ such that the Lee form is exact on each U_α, hence the lcK metric is locally conformal to a Kähler metric.

The non-metric version of Kähler manifolds are symplectic manifolds (see [25]). A manifold M^{2n} is symplectic if there exists a 2-form $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and ω^n is a volume form. Clearly, the Kähler form of a Kähler structure is symplectic. It is well known that the existence of a Kähler metric on a compact manifold deeply influences its topology. In fact, suppose M is a compact Kähler manifold of dimension $2n$ and let ω be the Kähler form; then:

- the odd Betti numbers of M are even;
- the Lefschetz map $H^p(M) \to H^{2n-p}(M)$, $[\alpha] \mapsto [\alpha \wedge \omega^{n-p}]$ is an isomorphism for $0 \leq p \leq n$;
- M is formal in the sense of Sullivan.

For a long time, the only known examples of symplectic manifolds came from Kähler (or even projective) geometry. In [34], Thurston constructed the first example of a compact, symplectic 4-manifold which violates each of the three conditions given above. Since then, the problem of constructing compact symplectic manifolds without Kähler structures has inspired beautiful Mathematics (see, for instance, [14, 16, 24, 30]). The principle is that the three cohomological properties of compact Kähler manifolds listed above strongly constrain their topology; however, the topology of compact symplectic manifolds is not constrained, see for instance [12].

The non-metric version of locally conformal Kähler manifolds are locally conformal symplectic (lcs) manifolds (see for instance [3, 38]). A manifold M^{2n}, endowed with an almost symplectic structure ω (that is, ω^n is a volume form on M), is said to be lcs if there exists a closed 1-form θ such that $d\omega = \omega \wedge \theta$. The 1-form θ is uniquely determined by ω and is called the Lee form. The defining condition of a lcs structure is equivalent to the existence of an open cover $\{U_\alpha\}$ of M such that the Lee form is exact on each open set U_α; on such U_α, a suitable multiple ω' of the form ω is closed, hence ω' is a symplectic structure on U_α. Analogous to the Kähler case, the Kähler form and the Lee form of a locally conformal Kähler structure define a locally conformal symplectic structure. A lcs manifold is globally conformal symplectic (gcs) if the Lee form is exact; in this case, ω is conformal to a closed form. A lcs manifold with vanishing first Betti number is automatically gcs.

In contrast to the Kähler case, not much is known about the topology of compact lcK manifolds. In [13, Conjecture 2.1] it was conjectured that a compact lcK manifold which satisfies the topological restrictions of a Kähler manifold admits a (global) Kähler metric. A stronger conjecture (see [13, Conjecture 2.2]) is that a compact lcK manifold which is not gcK must have at least one Betti number b_{2i+1} which is odd. The latter conjecture was proven in the context of Vaisman structures, namely it was shown in [22, 37] that a compact Vaisman manifold has odd b_1 (Vaisman structures are a special class of lcK structures: a lcK metric g is of Vaisman type if, in the conformal class of g, there is a metric g' such that the Lee form is parallel with respect to the Levi-Civita connection of g'). Oeljeklaus and Toma constructed a compact lcK but non Kähler manifold whose Betti numbers satisfies the topological restrictions of a compact Kähler manifold (see [29, Proposition 2.5, Remark 2.8]).

As a consequence of the lack of topological obstructions, it is not easy to exclude that a manifold carries a locally conformal Kähler metric. This motivates the following problem, proposed by Ornea and Verbitsky in [32, Open Problem 1]:

Construct a compact lcs manifold which admits no lcK metrics.

In [2], Bande and Kotschick gave a method to construct examples of compact lcs manifolds which carry no lcK metrics; we describe it briefly. Suppose that M is an oriented compact manifold of dimension 3. Using some results on contact structures in dimension 3 [27], we have that M admits a contact structure and, then, the product manifold $M \times S^1$ is lcs (see [2, 38]). On the other hand, using some results on compact complex surfaces, one deduces that

M may be chosen in such a way that the product manifold $M \times S^1$ admits no complex structures (for instance, if M is hyperbolic) and, in particular, no lcK structures.

The main result of this note is to present another kind of examples of compact lcs manifolds which admit no lcK metrics. More precisely, we prove:

Theorem 1.1. *There exists an example of a 4-dimensional compact nilmanifold which carries a locally conformal symplectic structure but no locally conformal Kähler metrics. This manifold is not a product of a 3-dimensional compact manifold and a circle.*

The example of Thurston that we mentioned above (which admits a Vaisman structure) is also a nilmanifold. This makes the parallel between the symplectic and the locally conformal symplectic case particularly transparent.

This note is organized as follows. In Section 2 we review some basics about nilmanifolds and describe briefly Thurston's example. In Section 3 we prove the main result.

2 Basics on nilmanifolds

Let G be a simply connected nilpotent Lie group. A *nilmanifold* is a homogeneous space $N = \Gamma\backslash G$ where $\Gamma \subset G$ is a lattice. Then $\mathfrak{g}$, the Lie algebra of G, is nilpotent and the exponential map $\exp\colon \mathfrak{g} \to G$ is a global diffeomorphism, hence G is diffeomorphic to $\mathbb{R}^n$ for some n. This entails that $p\colon G \to N$ is a covering map and that N is the Eilenberg-MacLane space $K(\Gamma, 1)$. Tori are simple examples of nilmanifolds. A slightly less trivial example is the following: let H be the Heisenberg group, defined as

$$H = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Then H is a nilpotent Lie group. A lattice $H_{\mathbb{Z}}$ in H is provided by matrices in H such that $x, y, z \in \mathbb{Z}$. Hence $H_{\mathbb{Z}}\backslash H$ is a compact nilmanifold.

There is a simple criterion for the existence of a lattice Γ in a nilpotent Lie group G:

Theorem 2.1 (Mal'cev, [26])**.** *Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then G admits a lattice if and only if there exists a basis of $\mathfrak{g}$ with respect to which the structure constants are rational numbers.*

Suppose $N = \Gamma\backslash G$ is a compact nilmanifold and let us identify $\mathfrak{g}$ with left-invariant vector fields on G. These vector fields are, in particular, invariant under the (left) action of Γ on G. We can also identify $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, with left-invariant 1-forms on G. In order to describe a geometric structure on a nilmanifold N (given, for example, in terms of forms and vector fields), we can specify its "value" at the (class of) the identity element $[e] \in N$, that is, by giving a corresponding geometric structure in $\mathfrak{g}$.

Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ and let $\{x_1 \ldots, x_n\}$ denote the dual basis of $\mathfrak{g}^*$, defined by $x_i(X_j) = \delta_{ij}$. Dualizing the bracket, we obtain a map $d\colon \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$; for $x_i \in \mathfrak{g}^*$, $dx_i(X_j, X_k) = -x_i([X_j, X_k])$. d is then extended to the whole exterior algebra $\wedge^*\mathfrak{g}^*$ by imposing the graded Leibniz rule. We thus obtain a differential complex $(\wedge^*\mathfrak{g}^*, d)$, called Chevalley-Eilenberg complex. The property $d^2 = 0$ is equivalent to the Jacobi identity in $\mathfrak{g}$. The Chevalley-Eilenberg complex $(\wedge^*\mathfrak{g}^*, d)$ computes the Lie algebra cohomology of $\mathfrak{g}$.

Example 2.2. *Let $N = \Gamma\backslash G$ be a nilmanifold of dimension $2n$. An invariant symplectic structure on N is described by means of a 2-form $\omega \in \wedge^2\mathfrak{g}^*$ such that $d\omega = 0$ and $\omega^n \neq 0$.*

Example 2.3. *Let G denote the product $H \times \mathbb{R}$, where H is the Heisenberg group described above. It is easy to see that the Lie algebra $\mathfrak{g}$ of G admits a basis $\{X_1, \ldots, X_4\}$ with only non-zero bracket $[X_1, X_2] = -[X_2, X_1] = -X_4$. Accordingly, $\mathfrak{g}^*$ has a basis $\{x_1, \ldots, x_4\}$ with $dx_1 = dx_2 = dx_3 = 0$ and $dx_4 = x_1 \wedge x_2$. Then $\omega = x_1 \wedge x_4 + x_2 \wedge x_3 \in \wedge^2\mathfrak{g}^*$ is closed and satisfies $\omega^2 \neq 0$. Let $\Gamma \subset G$ denote the lattice $H_{\mathbb{Z}} \times \mathbb{Z}$ and let $N = \Gamma\backslash G$ be the* *corresponding nilmanifold. By abuse of notation, we denote also by ω the 2-form on N obtained from $\omega \in \wedge^2\mathfrak{g}^*$. Then $\omega \in \Omega^2(N)$ is closed and nondegenerate, hence (N,ω) is a compact symplectic nilmanifold.*

One feature of nilmanifolds, which makes them particularly tractable from a cohomological point of view, is that their de Rham cohomology is computed by the Chevalley-Eilenberg complex. More precisely,

Theorem 2.4 (Nomizu, [28]). *Let $N = \Gamma\backslash G$ be a compact nilmanifold. Then the natural inclusion $(\wedge^*\mathfrak{g}^*, d) \hookrightarrow (\Omega^*(N), d)$ induces an isomorphism in cohomology.*

Corollary 2.5. *Let $N = \Gamma\backslash G$ be a compact nilmanifold of dimension n. Then, for every $0 \le i \le n$, $b_i(N) = b_i(\mathfrak{g})$.*

If we apply Corollary 2.5 to the nilmanifold N of Example 2.3, we see that $b_1(N) = 3$, since 3 of the 4 generators of $\mathfrak{g}^*$ are closed. Comparing this with the topological restrictions imposed by the existence of a Kähler metric on a compact manifold, we see immediately that N can not carry any Kähler metric. One can also easily show that N is neither Hard Lefschetz nor formal. The nilmanifold N of Example 2.3 is known as the Kodaira-Thurston manifold. On the one hand, it appeared in the classification of compact complex surfaces carried out by Kodaira (see [23]); on the other hand, it was the first example of a compact symplectic manifold without any Kähler structure in the celebrated paper [34] by Thurston.

As a matter of fact, compact Kähler nilmanifolds are very rare. More precisely, we have the following theorem:

Theorem 2.6 (Benson-Gordon, [10] and Hasegawa, [19]). *Let N be a $2n$-dimensional compact nilmanifold endowed with a Kähler structure. Then N is diffeomorphic to the torus T^{2n}.*

In particular, Benson and Gordon proved that a compact symplectic nilmanifold which satisfies the Hard Lefeschetz property must be diffeomorphic to a torus. Hasegawa showed that the same conclusion holds for a formal compact nilmanifold.

As a consequence, compact symplectic non-toral nilmanifolds are never Kähler. The problem of determining which even-dimensional nilmanifolds carry a symplectic structure is still open (see for instance [18]), but is completely solved in dimension 2, 4 and 6 (see [7]). Notice that, by Nomizu Theorem, a compact nilmanifold has a symplectic structure if and only if it has a left-invariant one.

In what follows, we shall adopt the standard notation for nilpotent Lie algebras/nilmanifolds. It is best explained by means of an example: $(0,0,0,0,12,34)$ denotes a 6-dimensional nilmanifold $N = \Gamma\backslash G$ such that $\mathfrak{g}$ has a basis $\{X_1,\ldots,X_6\}$ with dual basis $\{x_1,\ldots,x_6\}$ such that $dx_1 = \ldots = dx_4 = 0$, $dx_5 = x_1 \wedge x_2$ and $dx_6 = x_3 \wedge x_4$, where d is the Chevalley-Eilenberg differential. Different choices of lattices in G give rise to coverings between the corresponding nilmanifolds.

We have the following result:

Proposition 2.7 (see for instance [7, 30]). *Let $N = \Gamma\backslash G$ be a compact 4-dimensional nilmanifold. Then $\mathfrak{g}^*$ is one of the following:*

1. $(0,0,0,0)$ *and N is diffeomorphic to a torus T^4*
2. $(0,0,0,12)$ *and N is diffeomorphic to (a finite quotient of) the Kodaira-Thurston manifold*
3. $(0,0,12,13)$ *and N is diffeomorphic to (a finite quotient of) an iterated circle bundle over T^2.*

Correspondingly, in each cases N carries a (left-invariant) symplectic structure, which we describe in terms of a basis $\{x_1,\ldots,x_4\}$ of $\mathfrak{g}^*$;

1. $\omega = x_1 \wedge x_2 + x_3 \wedge x_4$
2. $\omega = x_1 \wedge x_4 + x_2 \wedge x_3$
3. $\omega = x_1 \wedge x_4 + x_2 \wedge x_3$.

3 Proof of the main theorem

We prove next two propositions, of which Theorem 1.1 is a corollary.

Proposition 3.1. *Let G be the nilpotent Lie group* $(0,0,12,13)$*, let $\Gamma \subset G$ be any lattice and let $N = \Gamma \backslash G$ be the corresponding nilmanifold. Then N is symplectic and locally conformal symplectic but does not admit any complex structure.*

Proof. A symplectic structure on N has been described explicitly in Proposition 2.7. In terms of the basis of $\mathfrak{g}^*$ considered above, the locally conformal symplectic structure is described by taking the almost symplectic 2-form to be $\omega = x_1 \wedge x_3 + x_4 \wedge x_2$ and the Lee form to be $\theta = x_2$. Then $d\omega = \theta \wedge \omega$. Notice that θ is not exact, hence the structure is genuinely lcs. Nomizu Theorem implies that $b_1(N) = 2$. Assume that N admits a complex structure. By standard results in the theory of compact complex surfaces (see for instance [4, Theorem 3.1, page 144]), a complex surface with even first Betti number admits a Kähler metric. Since N is clearly not diffeomorphic to T^4, this would give a contradiction with Theorem 2.6. Hence N does not carry any complex structure. □

Proposition 3.2. *The nilmanifold N considered above is not the product of a compact 3-dimensional manifold and a circle.*

Proof. By Nomizu theorem, we see that $b_1(N) = 2$. Assume N is a product $M \times S^1$, where M is a compact 3-dimensional manifold. Since N is a compact nilmanifold, the fundamental group Γ of N is nilpotent and torsion-free. If $\Lambda = \pi_1(M)$, then $\Lambda \subset \Gamma$ is also nilpotent and torsion-free. In [26], Mal'cev showed that for such a group Λ there exists a real nilpotent Lie group $\Lambda_{\mathbb{R}}$ such that $P = \Lambda \backslash \Lambda_{\mathbb{R}}$ is a nilmanifold. It is well known that a compact nilmanifold has first Betti number at least 2, hence $b_1(P) \geq 2$. Now P is an aspherical manifold; in this case, $H^*(P;\mathbb{Z})$ is isomorphic to the group cohomology $H^*(\Lambda;\mathbb{Z})$ of Λ with coefficients in the trivial Λ-module $\mathbb{Z}$ (see for instance [11, page 40]). Hence $b_1(\Lambda) \geq 2$. Now M is also an aspherical space with fundamental group Λ, hence, by the same token, $b_1(M) \geq 2$. However, by the Künnet formula applied to $N = M \times S^1$, we get $b_1(M) = 1$. Alternatively, we could apply [15, Theorem 3] directly to our manifold M. □

Proof of Theorem 1.1. Assume N carries a lcK metric. Then N must be in particular a complex manifold, and this contradicts Proposition 3.1. By Proposition 3.2, N is not a product of a compact 3-manifold and a circle. □

The lcs structure of the compact nilmanifold $(0,0,12,13)$, which was considered in the proof of Proposition 3.1, is of the first kind in the terminology of Vaisman [38]. A lcs manifold of the first kind is locally the product of a contact manifold with the real line [38] and the typical example of a compact lcs manifold of the first kind is the product of a compact contact manifold with the circle S^1. However, our compact example is not globally a product manifold. This is a good motivation in order to discuss the global structure of lcs manifolds of the first kind. This topic is the subject of a second paper by the authors, see [6].

An explicit realization of the compact nilmanifold $(0,0,12,13)$ is the following one. Let G be the connected simply connected Lie group $G = \mathbb{R}^4$ endowed with the multiplication

$$(x,y,z,t)\cdot(x',y',z',t') = (x+x', y+y', z+z'+yx', t+t'+zx'+y\frac{(x')^2}{2}).$$

Then, $(\mathbb{R}^4,\cdot)$ is a nilpotent Lie group and

$$\{x_1 = dx,\ x_2 = dy,\ x_3 = dz - y\,dx,\ x_4 = dt - z\,dx\}$$

is a basis of left-invariant 1-forms on G. Moreover,

$$\Gamma = 2\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$$

is a co-compact discrete subgroup of G.

The proof of Theorem 1.1 relies heavily on the classification of compact complex surfaces. It is therefore unlikely that our argument can be extended to the higher dimensional case.

It was conjectured in [2, Conjecture 5.3] that any compact complex surface admits a lcK structure (not necessarily compatible with the given complex structure). By what we argued so far, this conjecture is true in the context of 4-dimensional compact nilmanifolds. The torus T^4 admits a (globally) conformally Kähler structure and the Kodaira-Thurston carries a Vaisman structure (see [33]).

4-dimensional nilmanifolds display interesting behaviors with respect to the geometric structure we are dealing with:

– the torus T^4 is Kähler;
– the Kodaira-Thurston manifold is complex, symplectic, Vaisman but not Kähler;
– the nilmanifold $(0, 0, 12, 13)$ is symplectic, locally conformal symplectic, but not complex (hence neither Kähler nor lcK).

Remark 3.3. *An example of a manifold which is genuinely lcs and complex but neither symplectic nor lcK was constructed in [1]. The question is easier if one puts the more restrictive condition of not being of Vaisman type – see [6].*

It was conjectured in [35] that if a $2n$-dimensional compact nilmanifold $N = \Gamma\backslash G$ carries a lcK metric, then $\mathfrak{g} = \mathfrak{h}_{2n-1} \oplus \mathbb{R}$, where $\mathfrak{h}_{2n-1}$ is the generalized Heisenberg algebra; this is the Lie algebra with basis $\{X_1, Y_1, \ldots, X_{n-1}, Y_{n-1}, Z\}$ and non-zero brackets $[X_i, Y_i] = -[Y_i, X_i] = -Z$ for $i = 1, \ldots, n-1$ (notice that $\mathfrak{h}_3$ is the Lie algebra of the Heisenberg group). This conjecture has been proved by Sawai in [33], assuming that the complex structure of the lcK metric is left-invariant. A proof of this conjecture, under the stronger hypothesis that the lcK structure is of Vaisman type, but withouth any assumption on the complex structure, was given by the first author in [5]. A positive answer to this conjecture would produce immediately many examples of lcs non lcK manifolds. Vaisman structures on solvmanifolds have been considered in [21].

Remark 3.4. *Since the Kodaira-Thurston manifold carries a Vaisman metric, it is clear that compact locally conformal Kähler manifolds need not be formal. Can one, in some sense to be made precise, "bound" the non-formality of a compact lcK manifold (for instance, in terms of non-zero Massey products)? The study of geometric formality in the context of Vaisman metrics was tackled in [31].*

Acknowledgement: G. Bazzoni is partially supported by a Juan de la Cierva grant. J. C. Marrero has been partially supported by Ministerio de Economía y Competitividad (MINECO, Spain) under grant MTM 2015-64166-C2-2P. The authors would like to thank M. Fernández, L. Ornea and M. Verbitsky for useful comments and remarks.

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