Weighted orbital integrals

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Abstract

This is an expanded version of a survey talk given at the Conference on Representations of Real Reductive Lie Groups, June 4 to June 8, 2006, at the Snowbird Mountain Resort, Utah. We explain the definition of weighted orbital integrals and their Fourier transforms and report their known properties, leaving aside questions of endoscopy. We also announce some new results on those Fourier transforms over the reals.

Introduction

The derivation of the trace formula requires truncation of an integral kernel to make it integrable over a non-compact quotient. On the geometric side, this produces weighted orbital integrals, which are integrals of a test function on a reductive group \( G \) with respect to a certain non-invariant measure supported on a conjugacy class. On the spectral side, truncation gives rise to weighted characters, which are traces of induced representations of \( G \) twisted by certain logarithmic derivatives of intertwining operators.

In this way one gets two families, each indexed by the Levi subgroups \( M \) of \( G \), of distributions which are non-invariant under inner automorphisms in a parallel pattern. Arthur constructed from those two families a new family of invariant distributions \( I_M \), in terms of which the trace formula can be restated. The ordinary orbital integrals reappear unchanged as the terms \( I_G \).

Recently, Arthur has proved the stabilization of the distributions \( I_M \) on real groups \( G \). An important technical device are the differential equations
satisfied by $I_M$ as a function of the orbit. In fact, $I_M$ is fully characterized by those differential equations together with the behaviour at infinity and at the singular orbits.

The differential equations form a holonomic system with a regular singularity at infinity similar to the system satisfied by the matrix coefficients of an admissible representation. Thereby the Fourier transform of $I_M$ can be explicitly calculated in terms of higher-dimensional hypergeometric series at least for groups of low real rank.

1 Definition

Let $F$ be a local field of characteristic zero and $G$ a connected reductive group defined over $F$. To simplify notation, we will use the same symbol $G$ for the set of $F$-rational points of $G$. We will denote by $A_G$ the greatest $F$-split torus in the center of $G$ and by $G^1$ the intersection of the kernels of all continuous homomorphisms $G \to \mathbb{R}$.

In the case $F = \mathbb{R}$, such homomorphism is of the form $g \mapsto \lambda(H)$ for some $\lambda \in a^*_G$, where we write $g = g^1 \exp H$ with $g^1 \in G^1$ and $H$ in the Lie algebra $a_G$ of $A_G$. For general $F$, we define an $\mathbb{R}$-vector space $a^*_G := \text{Hom}(G, \mathbb{R})$ and denote its dual space by $a_G$. For each $g \in G$, there is a unique $H \in a_G$ such that every $\lambda \in a^*_G$, considered as a homomorphism $G \to \mathbb{R}$, takes the value $\lambda(H)$ on $g$. This gives rise to a continuous homomorphism $H_G : G \to a_G$.

Every unramified character of $G$, i. e., every continuous homomorphism $G \to \mathbb{C}^*$ factoring through $\mathbb{R}_+$, is of the form $g \mapsto e^{\lambda(H_G(g))}$ for some $\lambda$ in the complexified space $a^*_{G,\mathbb{C}}$. We get an action $(\lambda, \pi) \mapsto \pi_\lambda$ of the purely imaginary subspace $i a^*_G$ on the unitary dual $\Pi(G)$, i. e., the set of equivalence classes of irreducible unitary representations, by setting $\pi_\lambda(g) = e^{\lambda(H_G(g))} \pi(g)$.

Example 1: If $G = \text{GL}(n, F)$ and we identify $A_G = F^\times$, $a_G = \mathbb{R}$, then $H_G(g) = \frac{1}{n} \log |\det g|$. Every parabolic $F$-subgroup $P$ of $G$ has a Levi decomposition $P = MN$, where $N$ is the unipotent radical of $P$ and $M$ is a connected reductive $F$-group just like $G$. We fix special maximal subgroup $K$, then $G = PK = MNK$. Writing $H_P(mnk) := H_M(m)$, we obtain a continuous map $H_P : G \to a_M$.

The set $P(M)$ of parabolic $F$-subgroups $P$ with given Levi component $M$ is in bijection with set of chambers $a^*_P$ in $a_M$. For every $x \in G$, consider the
following volume of a convex hull in $a_M^G := a_M/a_G$:

$$v_M(x) := \text{vol}_{a_M^G} \text{conv}\{-H_P(x) : P \in \mathcal{P}(M)\}. \quad (1)$$

This function is left $M$-invariant, because $H_P(mx) = H_M(m) + H_P(x)$.

**Definition 1** For $f$ in the Schwartz space $C(G)$ of rapidly decreasing $L^2$-functions on $G$ and for $m \in M$ such that the centralizer $G_m$ of $m$ in $G$ is contained in $M$, the weighted orbital integral is defined as

$$J_M(m, f) := |D(m)|^{1/2} \int_{G_m \setminus G} f(x^{-1}mx)v_M(x)\,d\hat{x},$$

where $D(m) := \det_{\mathfrak{g} \setminus \mathfrak{g}}(\text{Id} - \text{Ad}(s))$ if $s$ denotes the semisimple component of $m$.

Here we have fixed invariant measures on $G_m \setminus G$ and $a_M$, and $\mathfrak{g}$ (unlike $a_G$) denotes the Lie algebra of $G$ over $F$. It has been shown ([3], Lemma 8.1) that the integral converges and defines a tempered distribution $J_M(m)$ on $G$, i.e., a continuous linear functional on $C(G)$. Note that $v_G$ is constant equal to 1, so that $J_G(g)$ is the ordinary (unweighted) orbital integral. This is the only case in which $J_M(m)$ is invariant (under inner automorphisms).

In the above discussion, one may replace $F$ by a number field and $G$ by the group of adèlic points. The resulting global weighted orbital integrals for $F$-rational points $m$ in $M$ and a Schwartz-Bruhat function $f$ are the main terms on the geometric side of the trace formula ([2], p. 951). More general weighted orbital integrals for orbits with $G_m \not\subset M$ also occur; they have to be defined by a limiting process ([5], p. 254). Global weighted orbital integrals can be reduced to their local counterparts by splitting formulas ([6], Prop. 9.1) and will not be discussed here.

The characteristic function of the convex hull in (1) can be written as an alternating sum of characteristic functions of simplicial cones indexed by the elements of $\mathcal{P}(M)$. Their Fourier transforms as functions of $\lambda \in a_M^* \setminus G$ converge for Re$\lambda$ in a certain chamber of $a_M$, and in the limit one obtains

$$v_M(x) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} e^{-\lambda H_P(x)} \frac{\theta_P(\lambda)}{\theta_P(\lambda)}, \quad (2)$$

where $\theta_P$ is the suitably normalized product of the linear functions defining the walls of $^+a_P^*$, which is the dual cone of the chamber $a_P^*$. 

3
The definition can be made more concrete when \( G \) has \( F \)-rank one. In this case there are two parabolics \( P \) and \( \bar{P} \) containing a minimal Levi subgroup \( M \), and we may parametrise \( G_m \setminus G \) by \( N \times K \). Substituting \( n^{-1}mn = mn' \), we can express \( n \) as a function of \( n' \) and obtain

\[
J_M(m, f) = -|D^M(m)|^{1/2} \delta_P(m) \int_K \int_N f(k^{-1}mn'k)\rho_P(H_P(n)) \, dn' \, dk,
\]

where \( \delta_P(m) = e^{\rho_P(H_M(m))} = \det(\text{Ad}_n(m))^{1/2} \) is the modular character of \( P \) and \( \rho_P \) was used to normalise the measure on \( a_M \). It is standard notation to indicate by a superscript \( M \) that an object is associated to the reductive group \( M \) rather than \( G \).

In the special case when \( G = \GL(2, F) \) and \( M \) is the subgroup of diagonal matrices, we can explicitly calculate

\[
-\rho_P(H_P\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)) = \log \| (1, x) \|
\]
in terms of a \( K \)-invariant norm on \( F^2 \).

## 2 Invariant Fourier transform

Given a continuous representation \( \pi \) of \( G \) on a complex Hilbert space \( V \) and a continuous function \( f : G \to \mathbb{C} \) of compact support, one can define

\[
\pi(f)v = \int_G f(g)\pi(g)v \, dg
\]

for every \( v \in V \). Then

\[
\pi(f_1*f_2) = \pi(f_1)\pi(f_2).
\]

If \( \pi \) is admissible and \( f \) is smooth, which means “locally constant” in the \( p \)-adic case, the operator \( \pi(f) \) is of trace class, and the map \( f \mapsto \text{tr} \, \pi(f) \) is a distribution, called the character of \( \pi \). The set \( G_{\text{reg}} := \{ g \in G : G^0_g \text{ is a torus} \} \) of regular semisimple elements has complement of Haar measure zero. By Harish-Chandra’s regularity theorem, there exists a locally integrable smooth function \( \Theta_\pi \) on \( G_{\text{reg}} \) such that

\[
\text{tr} \, \pi(f) = \int_G \Theta_\pi(g)f(g) \, dg.
\]
Note that $\Theta_{\pi} = \bar{\Theta}_{\pi}$, where $\pi$ denotes the contragredient representation of $\pi$. An admissible representation $\pi$ is called tempered if its matrix coefficients define tempered distributions. Such a representation is unitarizable, converges for $f \in \mathcal{C}(G)$, and the character of $\pi$ is a tempered distribution as well.

We define the Fourier transform of $f \in \mathcal{C}(G)$ as the function $\hat{f}$ on the set $\Pi_{\text{temp}}(G)$ of equivalence classes of tempered representations of $G$ by

$$\hat{f}(\pi) := \text{tr } \pi(f).$$

(Taking the contragredient makes the definition consistent with the ordinary Fourier transform but is not generally agreed on.) Equation (4) implies

$$\hat{f_1} \hat{f_2} = \hat{f_2} \hat{f_1}.$$  

(6)

The trace Paley-Wiener theorem [11] claims that $f \mapsto \hat{f}$ is an open, continuous surjection of $\mathcal{C}(G)$ onto a “Schwarz space” $\mathcal{I}(G)$ of functions $\phi : \Pi_{\text{temp}}(G) \to \mathbb{C}$. This statement has substance once an independent description of the space $\mathcal{I}(G)$ is given. We will say more about that in the next section. The name of the theorem originates from its analogue [13] for the space $\mathcal{H}(G)$ of compactly supported smooth functions on $G$, also called the Hecke algebra.

**Definition 2** The tempered distribution $I : \mathcal{C}(G) \to \mathbb{C}$ has the Fourier transform $\hat{I} : \mathcal{I}(G) \to \mathbb{C}$ if $\hat{I}(f) = I(f)$ for all $f \in \mathcal{C}(G)$.

It follows from (6) that a tempered distribution $I$ possessing a Fourier transform has the property $I(f_1 \ast f_2) = I(f_2 \ast f_1)$. By letting $f_2$ converge to a delta distribution, one easily shows that the latter property is equivalent to $I$ being invariant under inner automorphisms (whence the heading). Conjecturally, every invariant tempered distribution has a Fourier transform.

**Example 2:** If $I$ is the delta distribution at the unit element, i.e., $I(f) = J_G(e, f) = f(e)$, then $\hat{I}$ is the Plancherel measure on $\Pi_{\text{temp}}(G)$.

We mention in passing that there is also a Paley-Wiener theorem for the (bijective) operator-valued Fourier transform $\pi \mapsto \pi(f)$ and a corresponding notion of non-invariant Fourier transform of distributions.
3 Tempered dual

A unitary representation of $G$ is called square-integrable (modulo the center) if the square of the absolute value of one (equivalently: all) of its matrix coefficients is integrable over $G/A_G$. We denote the set of equivalence classes of square-integrable representations by $\Pi_2(G) \subset \Pi_{\text{temp}}(G)$. In the case of trivial $A_G$, this is the set of all atoms of the Plancherel measure, also called the discrete series of $G$.

If $L$ is a Levi component of a parabolic subgroup $P$ of $G$ (for short, a Levi subgroup of $G$) and $\sigma$ a tempered representation of $L$, considered as a representation of $P$, then the induced representation $\text{Ind}_P^G \sigma$ is tempered and completely reducible of finite length. Every tempered representation is a constituent of such a representation parabolically induced from some square-integrable representation $\sigma$ of some Levi subgroup $L$.

If $f \in \mathcal{C}(G)$, then $\text{tr} \text{Ind}_P^G \sigma(\lambda)(f)$ is a Schwartz function of $\lambda \in i\mathfrak{a}_L^* / \text{Stab} \sigma$. Thus, for $\phi \in \mathcal{I}(G)$, $\phi(\text{Ind}_P^G \sigma(\lambda))$ has to extend across the points of reducibility to a Schwartz function. Here, the stabilizer of $\sigma$ is trivial for archimedean $F$, while it is a lattice in $i\mathfrak{a}_L^*$ for $p$-adic $F$, in which case any smooth function on the quotient torus is called a Schwartz function. Incidentally, if $f$ belongs to the Hecke algebra $\mathcal{H}(G)$, then $\text{tr} \text{Ind}_P^G \sigma(\lambda)(f)$ is a Paley-Wiener function of $\lambda \in \mathfrak{a}_L^* / \text{Stab} \sigma$ in the archimedean case and a finite Fourier series in the $p$-adic case.

If $P' = LN'$ is another parabolic with the same Levi component $L$, there is an operator $J_{P'|P}(\sigma_\lambda)$ intertwining $\text{Ind}_P^G \sigma_\lambda$ with $\text{Ind}_{P'}^G \sigma_\lambda$. It maps $\psi$ to the function $\psi'$ defined by

$$
\psi'(g) = \int_{N \cap N' \backslash N'} \psi(n'g) \, dn',
$$

where the integral converges for $\text{Re} \, \lambda$ in some cone in $\mathfrak{a}_L^*$. By restriction $\psi|_K$, the space of $\text{Ind}_P^G \sigma_\lambda$ is realized independently of $\lambda$ as a space of functions on $K$ with values in the space of $\sigma$, called the compact picture. For $K$-finite $\psi$, the vector $J_{P'|P}(\sigma_\lambda)\psi$ has a meromorphic continuation to $\mathfrak{a}_L^*$. One can choose nonzero meromorphic functions $r_{P'|P}$ as in (8, Theorem 2.1) so that

$$
J_{P'|P}(\sigma_\lambda) = r_{P'|P}(\sigma_\lambda)R_{P'|P}(\sigma_\lambda),
$$

(7)

where the normalized intertwining operator $R_{P'|P}(\sigma_\lambda)$ is holomorphic at $\lambda =$
0 and satisfies
\[ R_{P'\mid P}(\sigma) R_{P'\mid P}(\sigma) = R_{P''\mid P}(\sigma), \quad R_{P'\mid P}(\sigma)^* = R_{P'\mid P}(\sigma). \quad (8) \]
In particular, \( R_{P'\mid P}(\sigma) \) is unitary, hence invertible. Thus, for the generic \( \lambda \in i\mathfrak{a}_L^* \) for which the representation
\[ \sigma^G := \text{Ind}_{P'}^G \sigma \]
is irreducible, it does not depend on \( P \in \mathcal{P}(L) \).

The trace of a representation induced from a subgroup is supported on the conjugacy classes that meet the subgroup. It follows that \( \Theta_{\sigma^G} \) for a representation \( \sigma \) of a proper Levi subgroup vanishes on the set \( G_{\text{ell}} \) of all elliptic semisimple elements, i.e., elements that are not contained in any proper parabolic subgroup.

**Example 3:** Let \( I(f) = J_G(l, f) \), where \( l \in L \cap G_{\text{reg}} \), and let \( \phi \in \mathcal{I}(G) \) such that \( \text{supp} \phi \subset \{ \sigma^G : \sigma \in \Pi_2(L) \} \). Then the Fourier transform \( \hat{I}(\phi) = \hat{J}_G(l, \phi) \) vanishes unless \( l \in L_{\text{ell}} \), in which case
\[ \hat{J}_G(l, \phi) = |D^L(l)|^{1/2} \sum_{\sigma \in \Pi_2(L)/i\mathfrak{a}_L^*} \int_{i\mathfrak{a}_L^*/\text{Stab} \sigma} \Theta_{\sigma^G}(l)(\sigma^G) d\lambda. \quad (9) \]

Note that for \( l \in L_{\text{ell}} \cap G_{\text{reg}} \), the torus \( G_{l}/A_L \) is compact and has a Haar measure of total mass 1. Thus the measure on \( G_l \) used in the definition of \( J_G(l, f) \) corresponds to a measure on \( A_L \), and the measure in \( (9) \) is the corresponding Plancherel measure. We can combine sum and integral into one integral over \( \Pi_2(L) \) with respect to a measure \( d\sigma \) (different from the Plancherel measure of \( L \)). We would like to rewrite it as an integral over a subset of \( \Pi_{\text{temp}}(G) \).

An element \( w \in W_L := \text{Norm}_G L/\text{Cent}_G L \), whose representative \( \tilde{w} \) can be chosen in \( K \), induces a unitary intertwining operator
\[ \text{Ind}_{P'}^G \sigma \sim \text{Ind}_{P'}^G (w \sigma) \]
simply by left translation, where \( P^w = \tilde{w}^{-1} P \tilde{w} \) and \( w \sigma(l) = \sigma(\tilde{w}^{-1} l \tilde{w}) \). In fact, two representations \( \sigma^G \) and \( \sigma'^G \) are equivalent iff the pairs \( (L, \sigma) \) and \( (L', \sigma') \) are \( G \)-conjugate. For \( \phi \in \mathcal{I}(G) \), we see that \( \phi(\sigma^G) \) is \( W_L \)-invariant as a function of \( \lambda \), and we can rewrite the right-hand side of \( (9) \) as an integral over a subset of \( W_L \setminus \Pi_2(L) \) of full measure, which is embedded into \( \Pi_{\text{temp}}(G) \).
If \( \phi \in \mathcal{I}(G) \), then the limit of \( \phi(\sigma^G_\lambda) \) as \( \lambda \) tends to a point of reducibility has to coincide with the sum of \( \phi(\pi) \) over the constituents \( \pi \). In the \( \mathbb{C} \)-vector space with basis \( \Pi_{\text{temp}}(G) \), Arthur has defined (\cite{4}, p. 93) a new basis \( T(G) \) that includes all the representations \( \text{Ind}^G_P \sigma, \sigma \in \Pi_2(L) \), in order to avoid the aforementioned compatibility conditions in the description of \( \mathcal{I}(G) \). The subset

\[ T_{\text{ell}}(G) := \{ \tau \in T(G) : \Theta_\tau|_{G_{\text{ell}}} \neq 0 \} \]

is a union of \( \mathfrak{a}^*_G \)-orbits, and the map \( \tau \mapsto \tau^G \) determines an embedding \( \mathbb{W}_L \setminus T_{\text{ell}}(L) \rightarrow T(G) \), the latter being the disjoint union of the images over all conjugacy classes of Levi subgroups \( L \). In this way, one gets a description

\[ \mathcal{I}(G) \cong \bigoplus_{[L]} C(T_{\text{ell}}(L))^{\mathbb{W}_L} \]

as a direct sum of Weyl group invariants in Schwartz spaces completed with respect to a natural topology.

**Example 4:** For \( \text{SL}(2, \mathbb{R}) \), in order to pass from \( \Pi_{\text{temp}}(G) \) to \( T(G) \), one has to replace the limits of discrete series \( \pi_1, \pi_2 \) by the virtual representations \( \tau_\pm = \pi_1 \pm \pi_2 \). Here \( \tau_+ \) is induced, while \( \tau_- \in T_{\text{ell}}(G) \).

## 4 Weighted characters

While weighted orbital integrals appear on the geometric side of the trace formula, weighted characters are the analogous terms on the spectral side. For simplicity, we discuss them for tempered representations of reductive groups over local fields only. Just like the exponential functions in (2), intertwining operators provide another instance of Arthur’s notion of \( (G, M) \)-families (in the given case, \( (G, L) \)-families, because we reserve the letter \( M \) for the geometric side and the letter \( L \) for the spectral side).

**Definition 3** For a Levi subgroup \( L \) of \( G \) and \( \sigma \in \Pi_{\text{temp}}(L) \), set

\[ R_P(\sigma) := \lim_{\lambda \to 0} \sum_{P' \in \mathcal{P}(L)} \frac{R_{P' \mid P}(\sigma)^{-1} R_{P' \mid P}(\sigma_\lambda)}{\theta_P(\lambda)}, \]

where the intertwining operators are considered in the compact picture (on our fixed maximal compact subgroup \( K \) of \( \text{Ind}^G_P \sigma_\lambda \), which does not depend on \( \lambda \).
If $L$ is maximal, we can write this operator as a logarithmic derivative:

$$
\mathcal{R}_P(\sigma) = -\frac{1}{\theta_p(\lambda)} R_{P|P}(\sigma)^{-1} \left. \frac{d}{dz} R_{P|P}(\sigma z\lambda) \right|_{z=0}.
$$

Although $\mathcal{R}_P(\sigma)$ is a priori only defined on $K$-finite vectors, and the weighted character

$$
\phi_L(f, \sigma) := \text{tr}(\text{Ind}_G^G(\tilde{\sigma}, f) \mathcal{R}_P(\tilde{\sigma}))
$$

only for $f \in \mathcal{H}(G)$, one can show ([10], p. 175) that it extends to $f \in C(G)$ and provides a continuous map

$$
\phi_L : C(G) \to I(L).
$$

It follows from (8) and the intertwining property of $R_{P|P}$ that $\phi_L$ is independent of $P \in \mathcal{P}(L)$, as the notation suggests. By definition, we have

$$
\phi_G(f) = \hat{f}. \tag{10}
$$

In Definition 3 one can replace $R_{P|P}$ by the unnormalized intertwining operators $J_{P|P}$, but with caution, as these may have poles on $\Pi_2(L)$.

**Definition 4** For a Levi subgroup $L$ of $G$ and $\sigma \in \Pi_2(L)$, $\lambda \in a^*_L \mathbb{C}$, set

$$
J_P(\sigma_\lambda) = \lim_{\lambda' \to 0} \sum_{P' \in \mathcal{P}(L)} J_{P'|P}(\sigma_\lambda)^{-1} J_{P'|P}(\sigma_{\lambda + \lambda'}) \theta_P(\lambda').
$$

Moreover, if $f \in \mathcal{H}(G)$, set (in view of $(\sigma_\lambda) = \tilde{\sigma}_{-\lambda}$)

$$
\phi_P(f, \sigma_\lambda) := \text{tr}(\text{Ind}_G^G(\tilde{\sigma}_{-\lambda}, f) J_P(\tilde{\sigma}_{-\lambda}))
$$

5 **Invariant distributions**

Neither weighted orbital integrals nor weighted characters are invariant when their subscripts are proper Levi subgroups, hence they have no Fourier transform in the sense of Definition 2. Now since they appear on the two sides of the trace formula, it does not come as a surprise that they behave similarly under inner automorphisms. Interpreting the members of one of the families as transformations, one can use them to modify the other family to
produce invariant distributions $I_M$. These are the main terms in the invariant trace formula [7]. There are two choices which are reciprocal in a certain sense ([3], p. 7/8, [10], §3). It is analytically easier to modify the weighted orbital integrals.

The ordinary orbital integral is already invariant, so one simply sets

$$I_G(g, f) := J_G(g, f).$$

For Levi subgroups $M$ that are maximal in $G$, one can easily single out a non-invariant part of the weighted orbital integral $J_M(m, f)$ by composing the map $\phi_M$ with the Fourier transform of the invariant orbital integral on $M$: If we write

$$J_M(m, f) = \hat{J}_M^M(m, \phi_M(f)) + I_M(m, f),$$

then the remaining term, denoted $I_M(m, f)$, turns out to be invariant. If, moreover, $M$ is minimal, i.e., $M/A_M$ is compact, the Fourier transform can be made explicit using Example 2:

$$J_M(m, f) = |D^M(m)|^{1/2} \int_{\Pi_{\text{temp}}(M)} \Theta_\sigma(m) \phi_M(f, \sigma) d\sigma + I_M(m, f). \quad (11)$$

Arthur has found a generalisation to arbitrary Levi subgroups.

**Theorem 1** ([3], p. 53, [10], p. 179) For all $G$, $M$ as in Definition 7 and $m \in M \cap G_{\text{reg}}$, there are invariant tempered distributions $I_M(m) = \hat{I}_G^M(m)$ on $G$ such that

$$J_M(m, f) = \sum_{M' \supset M} \hat{I}_M^{M'}(m, \phi_{M'}(f))$$

for all $f \in C(G)$, where the sum is taken over all Levi subgroups $M'$ of $G$ containing $M$.

The term with $M' = G$ in the sum simplifies in view of (10), and the equivalent form

$$J_M(m, f) = I_M(m, f) + \sum_{M' \supset M, M' \neq G} \hat{I}_M^{M'}(m, \phi_{M'}(f))$$

of the equation can be read as a recursive definition of $I_M(m, f)$ in terms of the invariant distributions on groups $M'$ of smaller semisimple rank than $G$. The existence of the Fourier transforms is part of the theorem. This is the hardest part of the proof and relies on the local or global trace formula. The
invariance of $I_M(m, f)$ however is straightforward, as is its independence of the choice of a maximal compact subgroup $K$.

In the case $A_G = \{e\}$, the set $T_{\text{ell}}(G)$ is the discrete part of $T(G)$, and a linear functional on $I(G)$, evaluated on functions with support in $T_{\text{ell}}(G)$, yields a linear combination of delta-distributions on $T_{\text{ell}}(G)$. In the Fourier transform of $I_M(m)$, however, delta-distributions at several other points may occur, giving rise to a larger set $T_{\text{disc}}(G) \supset T_{\text{ell}}(G)$, which can be explicitly described. In general, $T_{\text{disc}}(G)$ is $\mathfrak{a}^*_G$-invariant, and a measure on this set can be defined in analogy with Example 3.

**Theorem 2** ([10], p 183) There exist smooth functions $\Phi_{M,L}(m, \tau)$ of $m \in M \cap G_{\text{reg}}$ and $\tau \in T_{\text{disc}}(L)$ such that, for all $\phi \in I(G)$,

$$\hat{I}_M(m, \phi) = \sum_{[L]} \int_{W_L \setminus T_{\text{disc}}(L)} \Phi_{M,L}(m, \tau) \phi(\tau G) d\tau.$$  

If $\pi \in \Pi_2(G) \subset T_{\text{ell}}(G)$, then ([9], p 106)

$$\Phi_{M,G}(m, \pi) = \begin{cases} (-1)^{\dim \mathfrak{a}_M^G} |D(m)|^{1/2} \Theta_\pi(m) & \text{for } m \in M_{\text{ell}}, \\ 0 & \text{otherwise}, \end{cases}$$

Both statements are proved with the aid of the local trace formula. Actually, $\Phi_{M,G}(m, \tau)$ is computed for any $\tau \in T_{\text{ell}}(G)$ in [9], while the special case stated above had been proved already in [1] using differential equations.

Let us restate Example 3 in the present notation. If $l \in L \cap G_{\text{reg}}$ and $\sigma \in \Pi_2(L)$ with $\sigma^G$ irreducible, then

$$\Phi_{G,L}(l, \sigma) = \begin{cases} |D^L(l)|^{1/2} \sum_{w \in W_L} \Theta_{w \sigma}(g) & \text{for } l \in L_{\text{ell}}, \\ 0 & \text{otherwise}. \end{cases}$$

The full Fourier transform of $J_G(g), g \in G_{\text{reg}}$, for $F = \mathbb{R}$ has been determined by Herb [15] using character identities of Shelstad.

The distributions $I_M(m)$ satisfy descent identities ([10], Theorem 4.1), which reduce them to the case when $m \in M_{\text{ell}}$. This allows one to calculate $\Phi_{M,L}$ from the above results for $F = \mathbb{R}$ and a range of pairs $(M, L)$ including those with $\dim \mathfrak{a}_M^G + \dim \mathfrak{a}_L^G \leq \dim \mathfrak{a}_G^G$. The latter have also been obtained using differential equations [16].

The explicit formulas mentioned so far describe the restriction of the Fourier transform $\hat{I}_M(m)$ to a certain subset of $T(G)$. If $\hat{f}$ is supported
on that subset, then \( I_M(m, f) = J_M(m, f) \). The remaining components of \( \hat{I}_M(m) \), however, do depend on the choice of normalizing factors in \(|M|\).

This suggests that one should define invariant distributions, initially for \( f \in \mathcal{H}(G) \), using the transformations \( \phi_\mathcal{P}(f, \sigma) \), which were built of unnormalized intertwining operators. However, one has to avoid the poles of \( \phi_\mathcal{P}(f, \sigma) \) on \( \Pi_{\text{temp}}(L) \). For a parabolic \( \mathcal{P} = MN \) that is both maximal and minimal, we defined \((17), \text{p. 58}\) the invariant distribution \( I_\mathcal{P}(m, f) \) in analogy to (11) by

\[
J_M(m, f) = |D^M(m)|^{1/2} \int_{\Pi_{\text{temp}}(M)} \Theta_{\sigma_\varepsilon}(m) \phi_\mathcal{P}(f, \sigma_\varepsilon) \, d\sigma + I_\mathcal{P}(m, f),
\]

where \( \varepsilon \in \mathfrak{a}_+^\perp \) such that \( \phi_\mathcal{P}(\sigma_\lambda, f) \) has no poles for \( 0 < \Re \lambda \leq \varepsilon \). The integral does not depend on \( \varepsilon \) by Cauchy’s theorem and the decay of the integrand, and in the limit \( \varepsilon \to 0 \) one gets the principal value plus one half of each residue at the poles on \( \Pi_{\text{temp}}(M) \). Let us write \( \phi_{\mathcal{P},\varepsilon}(f, \sigma) = \phi_\mathcal{P}(f, \sigma_\varepsilon) \).

**Theorem 3** For all \( G, M \) as in Definition \([14]\), all \( m \in M \cap G_{\text{reg}} \) and \( \mathcal{P} \in \mathcal{P}(M) \), there are invariant tempered distributions \( I_\mathcal{P}(m) = I_\mathcal{P}^G(m) \) on \( G \) such that

\[
J_M(m, f) = \sum_{M' \supset M} e^{\varepsilon_{M'}(H_{M'}(m))} \hat{I}^{M'}_{\mathcal{P} \cap M'}(m, \phi_{M', \varepsilon_{M'}}(f))
\]

for all \( f \in \mathcal{H}(G) \) and \( \varepsilon_{M'} \in \mathfrak{a}_{M'}^\perp \) small enough. Moreover, there exist smooth functions \( \Phi_{\mathcal{P},L}(m, \tau) \) of \( m \in M \cap G_{\text{reg}} \) and \( \tau \in T_{\text{disc}}(L) \), such that, for all \( \phi \in \mathcal{I}(G) \),

\[
\hat{I}_\mathcal{P}(m, \phi) = \sum_{[L]} \int_{W_L \backslash T_{\text{disc}}(L)} \Phi_{\mathcal{P},L}(m, \tau) \phi(\tau^G) \, d\tau.
\]

The proof of this result will appear in a separate paper. Using the product formula for \((G, \mathcal{M})\)-families \([13], \text{Lemma 6.3}\), one can express \( \phi_\mathcal{P} \) in terms of (the normalizing factors and) \( \phi_{M'} \) with \( M' \supset M \), hence \( I_M(m) \) in terms of \( \hat{I}^{M'}_{\mathcal{P} \cap M'}(m) \), and finally \( \Phi_{M,L} \) in terms of \( \Phi_{M',L} \).

### 6 Differential equations

Let the ground field \( F \) be \( \mathbb{R} \). Now every \( \pi \in \Pi(G) \) gives rise to a representation of the Lie algebra \( \mathfrak{g} \) as well as its universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \)
on the subspace of smooth vectors. By a version of Schur’s lemma, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by scalars, and the resulting algebra homomorphism $\chi_\pi : Z(\mathfrak{g}) \to \mathbb{C}$ is called the infinitesimal character of $\pi$.

If $L$ is a Levi subgroup and $\sigma \in \Pi(L)$, then the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to Z(t)$, $z \mapsto z_L$, is characterised by the formula $\chi_{\sigma}(z) = \chi_\sigma(z_L)$ for the infinitesimal character of a parabolically induced representation. This homomorphism depends only on the complexifications of $\mathfrak{g}$ and $\mathfrak{l}$ and can also be defined for a maximal torus $T$ in place of $L$. In this case, we get the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to Z(T)$, $z \mapsto z_T$, which characterises the infinitesimal character of a parabolically induced representation.

The global characters $\Theta_\pi$ in (5) are invariant eigendistributions of the algebra $Z(\mathfrak{g})$, whose elements can be viewed as bi-invariant differential operators on $G$. Harish-Chandra determined all tempered invariant eigendistributions explicitly, and as a byproduct he showed that orbital integrals satisfy certain differential equations. Arthur generalized these differential equations to the case of weighted orbital integrals and the associated invariant distributions $I_M(m)$, $m \in M \cap G_{\text{reg}}$.

Since those distributions depend on $m$ up to conjugacy only, we may assume that $m = t$ lies in a fixed maximal torus $T$ and normalise the Haar measure on $G_t = T$ independently of $t \in T \cap G_{\text{reg}}$.

**Theorem 4** ([9], p. 279) For every connected reductive $\mathbb{R}$-group $G$, its Levi subgroup $M$ and a maximal torus $T \subset M$, there exists a smooth map $\partial_M = \partial_M^G : T \cap G_{\text{reg}} \to \text{Hom}_\mathbb{C}(Z(\mathfrak{g}), U(t))$ with the following property. If $L$ is a Levi subgroup of $G$, $\tau \in T_{\text{adm}}(L)$ and $\chi = \chi_{\tau, \sigma}$, then the functions $\Phi_M(t) = \Phi_{M,L}(t, \tau)$ satisfy the differential equations

$$\chi(z)\Phi_M(t) = \sum_{M' \supset M} \partial_M^{M'}(t, z_{M'})\Phi_{M'}(t)$$

on $T \cap G_{\text{reg}}$ for every $z \in Z(\mathfrak{g})$. Moreover, $\deg \partial_M(t, z) \leq \deg z$ and

$$\partial_G(t, z) = z_T.$$
get an algebra homomorphism from $Z(\mathfrak{g})$ into the algebra of matrix-valued differential operators on $T \cap G_{\text{reg}}$. Thus, it suffices to consider $\mathfrak{g}$ for $z$ in a (finite) set of generators of $Z(\mathfrak{g})$.

There is an algorithm to compute the differential operators, but no general closed formula except for the Casimir element $\omega \in Z(\mathfrak{g})$ (shifted by a constant), which is characterised by $\omega_T(\lambda) = \langle \lambda, \lambda \rangle$ for a nondegenerate symmetric bilinear form on $t^*$ whose extension to $t_C^*$ is $W$-invariant. In fact, $\partial_M(t, \omega)$ vanishes unless either $M = G$, in which case it is given by $[\text{13}]$, or $M$ is maximal, in which case case

$$\partial_M(t, \omega) = \sum_{\alpha} \frac{|\langle \theta_P, \alpha \rangle|}{(t^{\alpha} - 1)(1 - t^{\alpha})},$$

where $P \in P(M)$, the linear function $\theta_P$ is as in $[\text{2}]$, and the sum is taken over all roots $\alpha$ of $t_C$ such that $\mathfrak{g}_{C,\alpha} \not\subset \mathfrak{m}_C$.

For a parabolic subgroup $P = M_P N$ containing $T$, set

$$T_P := \{ t \in T : |t^\alpha| > 1 \forall \text{ roots } \alpha \text{ of } t_C \text{ in } n_C \}$$

and define the subset $T_{C,P}$ of $T_C$ by the same conditions. The limit of a function on $T_{C,P}$ as $t \to \infty$ is meant with respect to the filter of all the translates of the sets $T_{C,P}$.

**Theorem 5** ([\text{13}], pp. 780, 785, 790) The system applied to $(\Phi_M)_{M \supseteq M_P}$ is holonomic on $T_C \cap G_{C,\text{reg}}$ and has a regular singularity at $\infty$ on $T_{C,P}$. For every $\lambda \in t_C^*$ with $\chi_\lambda = \chi$ there is a unique solution $\Psi = \Psi^{P,\lambda}$ on the universal covering $\tilde{T}_{C,P}$ such that

- $\Psi_G(\exp H) = e^{\lambda(H)}$,
- $\Psi_M(t) \to 0$ as $t \to \infty$ if $M \neq G$.

For sufficiently regular $\chi$, every solution is of the form

$$\Phi_M(t) = \sum_{\chi_\lambda = \chi} \sum_{M' \supseteq M} c_{M'}(\lambda) \Psi_M^{P \cap M',\lambda}(t)$$

for suitable functions $c_{M'}$. 


Our holonomic system is similar to the system satisfied by matrix coefficients of admissible irreducible representations studied in [14], but it is more complicated as the torus $T$ need not be $\mathbb{R}$-split. By general results on holonomic systems with regular singularity, the standard solutions $\Psi_M(t)$ have a series expansion in powers of $t^\alpha$, where $\alpha$ runs through the roots of $T_C$ in $p_C/p_C \cap m_C$.

In order to compute the Fourier transforms of the invariant distributions $I_m(t)$ explicitly, it remains to solve the following problems:

1. Find $c_{M'}(\lambda) = c_{P,M',L}(\lambda, \tau)$ for the Fourier transforms $\Phi_M(t) = \Phi_{M,L}(t, \tau)$ on each sector $T_P$.

2. Describe the standard solutions $\Psi_M$ explicitly.

Arthur has determined [12] the asymptotics of the Fourier transforms, which takes a simpler shape if expressed in terms of the modified Fourier transforms that use unnormalized intertwining operators.

**Theorem 6** If $L \subset P \neq G$, then

$$\Phi_{P,L}(t, \tau) \to 0 \quad \text{as } t \to \infty.$$ 

If we combine this result with the Riemann-Lebesgue lemma, we can deduce that in an extreme case the coefficients $c_{P,M',L}$ are equal to zero or one:

**Theorem 7** If $T$ is $\mathbb{R}$-split, $P$ is minimal and $\sigma(t) = t^\lambda$ is sufficiently regular, then

$$\Phi_{P,T}(t, \sigma) = \sum_{\lambda' \in W_\lambda} \Psi^{P,\lambda'}_{P,T}(t).$$

Again, the details will appear in a forthcoming paper.

In general, however, the limit formula does not determine the solution of the differential equation uniquely. On needs, in addition, the jump relations satisfied by the the Fourier transforms $\Phi_{M,L}(t, \tau)$ as $t$ approaches singular values where $G_t \neq T$. To determine $c_{P,M',L}(\lambda, \tau)$, we need to find the analogous jump relations for the standard solutions $\Psi_M$. This has been done for $\text{rk}_\mathbb{R} G = 1$ in [18] and for $G = \text{GL}(3, \mathbb{R})$ (unpublished).
7 Explicit Fourier transforms

Now we turn to Problem 2, i.e., the explicit determination of the standard solutions $\Psi_M$ of our holonomic system. For sufficiently regular $\chi$, the coefficients of the series $\Psi_M$ can be determined using the differential equation for the Casimir element alone, and one can show ([18], Theorem 5.8) that those coefficients are rational functions in $\lambda$. Moreover, unlike the case of matrix coefficients, the recursive equations can be solved explicitly at least for groups of low rank, and an unexpected cancellation keeps the degree of the denominators bounded.

**Theorem 8** ([17], p. 71, 75) If $P$ is a maximal parabolic subgroup and $T$ a maximal torus of its Levi component $M$, then

$$\Psi_{P,\lambda}^M(t) = t^\lambda \sum_{\alpha} \eta(\tilde{\alpha}) b(\lambda(\tilde{\alpha}), t^{-\alpha}),$$

where the sum is taken over the roots $\alpha$ of $T_C$ such that $g_\alpha \subset n_C$ but $g_\alpha \not\subset m_C$, and where

$$b(s, z) = \sum_{n=1}^{\infty} \frac{z^n}{n + s} = z \int_0^1 \frac{x^s}{1 - zx} \, dx \quad (|z| < 1, \text{Re}\, s > -1).$$

The function $b$ is a special case of the confluent hypergeometric equation and closely related to the incomplete Beta function.

For nonmaximal Levi subgroups $M$, the coefficients are rational functions in several complex variables, for which no unique decomposition in partial fractions exists. However, experience with groups up to rank three suggests that there should exist a canonical expression in partial fractions associated to certain combinatorial data in the root system, which were called root cones in [15], p. 794. E.g., in the case of the Borel subgroup of GL(3), there are four such root cones, which give rise to the four terms in the following formula.

**Theorem 9** If $G = \text{GL}(3, \mathbb{R})$, $P$ is the subgroup of upper triangular matrices and $M = T$ the subgroup of diagonal matrices $t = \text{diag}(t_1, t_2, t_3)$, then

$$\Psi_{P,\lambda}^T(t) = t^\lambda \left( b(\lambda_{23}, \lambda_{13}, t_{32}, t_{21}) + b(\lambda_{12}, \lambda_{13}, t_{21}, t_{32}) + b(\lambda_{12}, t_{31}) b(\lambda_{23}, t_{32}) + b(\lambda_{12}, t_{21}) b(\lambda_{23}, t_{31}) \right),$$

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where $t^\lambda = t_1^{\lambda_1} t_2^{\lambda_2} t_3^{\lambda_3}$, $t_{ij} = t_i/t_j$, $\lambda_{ij} = \lambda_i - \lambda_j$, and

$$\tilde{b}(s_1, s_2, z_1, z_2) = \sum_{n_2=1}^{\infty} \sum_{n_1=1}^{\infty} \frac{z_1^{n_1} z_2^{n_2}}{(s_1 + n_1)(s_2 + n_2)} (s_1 + n_1)(s_2 + n_2)$$

$$= z_1 z_2 \int_0^1 \int_0^1 \frac{x_1^{s_1} x_2^{s_2}}{(1 - z_1 x_1)(1 - z_2 x_1 x_2)} \, dx_1 \, dx_2$$

($|z| < 1$, Re $s_i > -1$).

References


