BIELEFELD GEOMETRY & TOPOLOGY DAYS

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FOREWORD

GIOVANNI BAZZONI

This document could be considered as the proceedings of the conference Bielefeld Geometry & Topology Days which I organized, on July 2nd and July 3rd, 2015, at Bielefeld University. This would be however not completely true. In fact, the results presented here either have appeared or will appear elsewhere and we refer to each paper for the relevant bibliographical information.

Albeit the content of this document has no presumption of originality, the way in which it is written - the way in which I asked to speakers to compose it - does claim some novelty. It is conceived as a quick-and-dirty overview of the topics touched during the conference, so that the interested reader gets an immediate taste of what motivates the research, what are the main results in the area, who and how proved them and how it would be like to work in the field. Indeed, the goal of the conference was to bring together young researchers in Geometry and Topology, presenting a panoramic view of some of the actual lines of research. Each talk was conceived as a gentle introduction to the speaker’s field of expertise, so that original results could be placed in the adequate context.

Many thanks go to the speakers, first of all for coming to Bielefeld and giving very interesting lectures, but also for writing a report on it, thus contributing to this project and making their work accessible. I would also like to thank SFB 701 “Spectral Structures and Topological Methods in Mathematics”, especially the SFB Webteam and Frau Cole, for supporting the organization of the conference and helping me with the preparation of these notes.
POSITIVE CURVATURE AND TOPOLOGY

MANUEL AMANN

ABSTRACT. Manifolds admitting metrics of positive sectional curvature are conjectured to have a very rigid topological structure. However, this structure is still highly speculative and the strongest results in this direction are known under the assumption of Lie group actions.

In this talk I shall try to illustrate this interplay between geometry, symmetry and topology. In particular, using such symmetry assumptions, I will speak about generalisations of a conjecture by Hopf which states that $\mathbb{S}^2 \times \mathbb{S}^2$ cannot carry a metric of positive curvature.

Presented results come from joint work with Lee Kennard.

The question of whether a given smooth manifold admits a Riemannian metric with positive sectional curvature is nearly as old as the subject of Riemannian geometry itself. The classical Gauss–Bonnet theorem

$$\int_M K \, d\text{vol} = 2\pi \chi(M)$$

relating the sectional curvature $K$ of a compact surface $M$ to its Euler characteristic, $\chi(M)$, can be considered a first classification theorem of that kind.

So it seems surprising that until today only very few examples of simply-connected closed manifolds admitting positively curved metrics are known; from dimension 25 on this list only comprises the compact rank one symmetric spaces $\mathbb{S}^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

At the same time there are not many obstructions to the existence of such a metric: The vanishing of the $\alpha$-invariant on spin manifolds obstructs positive scalar curvature, the Gromov Betti number estimate states that—in the case of a non-negatively curved manifold—the Betti numbers are bounded from above by a (rather large) integer only depending on the dimension of the manifold.

It was a suggestion by Karsten Grove in the 1990s to study positive curvature in the context of isometric Lie group actions, which seems natural in this context for many reasons. This programme has led to many classification results as for example in [3], [4], [5]—passing from diffeomorphism classifications to homeomorphism, (tangential) homotopy and cohomology classifications whilst reducing the symmetry assumptions.

In this talk we focus on effective isometric actions of a torus on a positively curved manifold $M$. We assume that the dimension of the torus acting is basically a logarithm of $\dim M$. As a next step we apply this—in the tradition of the classical Gauss-Bonnet theorem—to derive results on the Euler characteristic $\chi(M)$ (cf. [1]).

**Theorem** (Amann–Kennard). If $M^n$ is a closed Riemannian manifold with positive sectional curvature and symmetry rank at least $\log_4 3(n)$, then $\chi(M) < 2^{3(\log_2 n)^2}$.

The Euler characteristic plays a crucial role in a conjecture of Hopf.
Conjecture (Hopf).

\[ \chi(M^{2n}) \begin{cases} > 0 & \text{for } M \text{ positively curved} \\ \geq 0 & \text{for } M \text{ non-negatively curved} \end{cases} \]

The second part of this conjecture is implied by

Conjecture (Bott–Grove–Halperin). It holds \( \dim \pi_* \otimes \mathbb{Q} < \infty \), i.e. \( M \) is rationally elliptic, if \( M \) is non-negatively curved.

Also a conjecture by Gromov stating that \( b_k(M^n) \leq b_k(T^n) = \left( \binom{n}{k} \right) \) would follow from this conjecture. There is a second famous conjecture by Hopf.

Conjecture (Hopf). The manifold \( S^2 \times S^2 \) does not admit a positively curved Riemannian metric.

Despite several different approaches to these conjectures they seem to remain widely open in general.

It is our goal to illustrate how to use (generalisations of) the first conjecture by Hopf in order to prove generalised versions of his second conjecture—always assuming logarithmic symmetry as above; see [2].

Theorem (Amann–Kennard). Let \( M^{2n} \) be a simply connected, closed manifold with \( b_4(M) = 0 \). Assume \( M \) admits a Riemannian manifold with positive sectional curvature invariant under the action of a torus \( T \) with \( \dim(T) \geq \log_{4/3}(2n) \). Then we derive that \( \chi(M) = \chi(S^{2n}) = 2 \).

We remark that this result leads to several classification results.

- If \( M \) has vanishing odd-dimensional rational cohomology, e.g., if \( M \) is rationally elliptic, then \( M \) is a rational \( S^{2n} \).
- If \( M \) is \( p \)-elliptic for some prime \( p \geq 2n \), then \( M \) is a mod \( p \) homology \( S^{2n} \).
- If \( M \) has vanishing homology in odd degrees, then \( M \) is homeomorphic to \( S^{2n} \).
- If \( M \) is a biquotient, then \( M \) is diffeomorphic to \( S^{2n} \).
- If \( M \) admits a smooth, effective cohomogeneity one action by a compact, connected Lie group, and if the homology of \( M \) has no two–torsion, then \( M \) is equivariantly diffeomorphic to \( S^{2n} \) equipped with a linear \( G \)-action.
- If \( M \) is a symmetric space, then \( M \) is isometric to \( S^n \).

The theorem implies a generalisation of the Hopf conjecture for \( S^2 \times S^2 \) under symmetry. Consider an arbitrary closed manifold \( N^n \) with \( n > 2 \), and consider the double product \( M^{2n} = N \times N \). Suppose that \( M \) admits a metric with positive curvature and an isometric torus action of rank \( r > \log_{4/3}(2n-3) \). Then \( M \) is simply connected and \( b_2(M) \leq b_4(M) \leq 1 \). Due to Künneth we derive that \( b_4(M) = 0 \). The theorem implies that

\[ 2 = \chi(M) = \chi(N)^2, \]

which is impossible. Hence \( N \times N \) has no such metric. A similar conclusion can be drawn for connected sums.
Corollary (Amann–Kennard). Let $N^n$ be a simply connected, closed manifold. The product $N \times N$ does not admit a Riemannian metric with positive sectional curvature and an isometric torus action of rank at least $\log_{1/3}(2n - 3)$.

Similarly, if $n$ is even and $\chi(N) \neq 2$, the connected sum $N \# N$ does not admit a positively curved metric invariant under a torus action of rank at least $\log_{1/3}(n - 3)$.

As mentioned above, the Bott–Grove–Halperin conjecture states that an (almost) non-negatively curved manifold should be (rationally) elliptic, so that in particular its homotopy Euler characteristic

$$\chi_\pi(M) = \sum_i (-1)^{i+1} \dim \pi_i(M) \otimes \mathbb{Q}$$

is defined. We use this in order to suggest and discuss the following generalisation (to odd dimensions) of the Hopf conjecture on the Euler characteristic.

**Conjecture.** A closed manifold $M^n$ of positive curvature satisfies

$$\chi_\pi(M) = n \mod 2$$

A closed manifold $M^n$ of non-negative curvature satisfies

$$\chi_\pi(M) \equiv n \mod 2$$

The first part of the conjecture asserts that the homotopy Euler characteristic is actually equal to 0 or 1. We noted above that the Bott–Grove–Halperin conjecture implies that a non-negatively curved manifold has non-negative Euler characteristic. Thus in the light of this conjecture the Hopf conjecture for non-negatively curved manifolds is a direct consequence. The same phenomenon occurs here: The second part of the conjecture merely claims rational ellipticity, since the congruence holds in general on a rationally elliptic space due to a formula relating the degrees of the homotopy groups to the dimension of the manifold.

In even dimensions on rationally elliptic spaces the conjecture is equivalent to the classical Hopf conjecture. Indeed, it is well-known that rationally elliptic spaces have positive Euler characteristic if and only if their homotopy Euler characteristic vanishes.

We prove the conjecture for rationally elliptic manifolds of positive curvature and the usual logarithmic symmetry assumption, i.e. we show that their homotopy Euler characteristic equals 0 or 1 depending on the parity of the dimension.

**Theorem** (Amann–Kennard). Let $M$ be a rationally elliptic closed simply-connected manifold $M^n$ admitting a metric of positive curvature and symmetry rank $\text{symrk } M \geq \log_{1/3} n + 1$, then $\chi_\pi(M) = \dim M \mod 2$.

As a first application, we observe that this excludes positive curvature under symmetry on $S^k \times S^l$ with $k, l$ odd, as the simplest example. In fact, the homotopy Euler characteristic of a product of spheres equals the number of odd-dimensional factors.

Motivated by the proof of the theorem above concerning $\chi(M) = 2$—which states that the fixed-point set of a torus has to be a rational sphere—we now apply localization theorems in equivariant rational cohomology and homotopy theory. These methods permit to identify large classes of manifolds which cannot carry positive curvature under logarithmic symmetry.
For this we investigate degeneration properties of the Leray–Serre spectral sequence of the Borel construction in the light of rational homotopy techniques. We rephrase the degeneration of this spectral sequence in our special case via the non-existence of certain derivations of negative degree on the rational cohomology algebra of $M$. More in-depth arguments yield

**Theorem** (Amann–Kennard). Let $M$ be rationally elliptic. Suppose that $b_i(M) = 0$ for $0 < i < k$, $k > 4$ and $b_k(M) \neq 0$ for some odd $k$. Then $M$ does not admit a positively curved metric of symmetry rank at least $\log_{4/3}(n-3) + 1$.

as a corollary of a result concerning fibration splittings. Besides, we obtain results on classes of connected sums without positive curvature metrics invariant under tori.

Let us finally summarise some of these considerations by giving a few examples from the class of manifolds not admitting positively curved manifolds with logarithmic symmetry rank.

**Example.** This class comprises $S^k \times S^l$ if one of the following holds: Either both $k,l$ are even or odd, or $l > k$ and $l$ is even and $k$ is odd.

As a further very concrete example, also $S^k \times S^k \times S^k \# S^{2k} \times S^k$ is comprised.

**References**


ON THE BOTT-CHERN AND AEPPLI COHOMOLOGY

DANIELE ANGELLA

ABSTRACT. This survey summarizes the results discussed in a talk at “Bielefeld Geometry & Topology Days” held at Bielefeld University in July 2015. We are interested in quantitative and qualitative properties of Bott-Chern cohomology. We announce an upper bound of the dimensions of Bott-Chern cohomology in terms of Hodge numbers. We also introduce a notion, here called Schweitzer qualitative property, which encodes the existence of a non-degenerate pairing in Bott-Chern cohomology, like the Poincaré duality for the de Rham cohomology. We prove that this property characterizes the $\partial\bar{\partial}$-Lemma.

Dedicated to the memory of Professor Pierre Dolbeault.

INTRODUCTION

We are aimed at decode some of the informations on the complex geometry of a compact complex manifold from the cohomologies associated to its double complex of differential forms. In particular, we focus on the informations contained in the Bott-Chern and Aeppli cohomologies. They provide, in a sense, a bridge between the holomorphic contents of the Dolbeault cohomology, and the topological contents of the de Rham cohomology. In this sense, it is expected that they both provide a better control on the holomorphic structure, (see, e.g., [4] and Theorem 3.2,) and furnish natural tools for treating geometric aspects, (see, e.g., [23]). A possible link between these two settings would be provided by a proof of the conjecture that compact complex manifolds satisfying the $\partial\bar{\partial}$-Lemma admit balanced metrics in the sense of Michelsohn, see [18, §6].

In this survey, we focus on quantitative properties of Bott-Chern and Aeppli cohomologies towards the study of their qualitative properties.

More precisely, as for the “quantitative” aspects, we recall a result proven by the author and A. Tomassini in [4]. It states that, once fixed the topological structure, there is a lower bound on the dimension of the Bott-Chern cohomology in terms of the Betti numbers. See [4, Theorem A] or Theorem 2.1 for a precise statement. Moreover, the lower bound is attained if and only if the complex manifold satisfies the $\partial\bar{\partial}$-Lemma, (that is, the identity induces natural isomorphisms between Bott-Chern, Aeppli, Dolbeault cohomologies). We announce here an upper bound for the Bott-Chern cohomology in terms of Hodge numbers; further details and complete proofs will eventually be provided in [3]. More precisely, in Theorem 2.2, we prove that, on a compact complex manifold $X$ of complex dimension $n$, 2010 Mathematics Subject Classification. 32Q99, 32C35, 55S30.

Key words and phrases. complex manifold, non-Kähler geometry, Bott-Chern cohomology, $\partial\bar{\partial}$-Lemma.

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for any $k \in \mathbb{Z}$, it holds

$$
\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\mathcal{A}}(X) \leq (n+1) \left( \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\mathcal{B}}(X) + \sum_{p+q=k+1} \dim_{\mathbb{C}} H^{p,q}_{\mathcal{B}}(X) \right).
$$

Note that a topological upper bound is not possible.

As a consequence, we get that the difference $\sum_{p+q=k} \left( \dim_{\mathbb{C}} H^{p,q}_{\mathcal{B}}(X) - \dim_{\mathbb{C}} H^{p,q}_{\mathcal{A}}(X) \right)$ is bounded from both above and below by Hodge numbers. Such a quantity yields another characterization of $\partial \bar{\partial}$-Lemma: in Theorem 2.3, we show that a compact complex manifold $X$ satisfies the $\partial \bar{\partial}$-Lemma if and only if

$$
\sum_{k \in \mathbb{Z}} \left| \sum_{p+q=k} \left( \dim_{\mathbb{C}} H^{p,q}_{\mathcal{B}}(X) - \dim_{\mathbb{C}} H^{p,q}_{\mathcal{A}}(X) \right) \right| = 0.
$$

This has to be compared with the characterization in [4, Theorem B], which uses instead the vanishing of the non-negative degrees $\Delta^k$ in (1):

$$
\sum_{k \in \mathbb{Z}} \left( \sum_{p+q=k} \left( \dim_{\mathbb{C}} H^{p,q}_{\mathcal{B}}(X) + \dim_{\mathbb{C}} H^{p,q}_{\mathcal{A}}(X) \right) - 2b_k \right) = 0,
$$

and with [5, Corollary 4.14], which is deduced from interpretation of complex structures as generalized-complex structures in the sense of N. Hitchin.

Our final aim would be to attempt the study of “qualitative” aspects, namely, of the concrete realization of the algebra structure in Bott-Chern cohomology. Some previous attempts were done in [6, 22], with the final motivation of understanding whether a possible notion of formality à la Sullivan for Bott-Chern cohomology makes sense. Roughly speaking, a manifold would be “formal with respect to Bott-Chern cohomology” when the Bott-Chern cohomology functor can be made “concrete” by means of a zigzag of morphisms of bi-differential bi-graded algebras being quasi-isomorphisms with respect to Bott-Chern cohomology. This is the case when there is a suitable choice of the representatives having by themselves a structure of algebra. (A stronger request would be to ask for harmonic representatives with respect to some Hermitian metric.) We introduce here a “qualitative” notion, motivated by the following observation. From the geometric point of view, the algebra structure is mainly important in connection with also the Poincaré duality. But Hermitian duality does not preserve Bott-Chern cohomology. We will introduce a property, here called Schweitzer qualitative property, from the work in [19], that implies in fact the validity of $\partial \bar{\partial}$-Lemma, thanks to the above quantitative results.

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1. Bott-Chern and Aeppli cohomology

We study the geometry of compact complex manifolds $X$ as encoded in their double complex of forms, namely, $(\wedge^\bullet \bullet X, \partial, \bar{\partial})$. This is an object of $\text{bba}$, that is, the category
of bi-differential $\mathbb{Z}^2$-graded algebras. Note that any compact complex manifold admits two natural structures: a real structure; and a non-degenerate pairing structure. These are encoded also in symmetries of the double complex: conjugation yields a symmetry around the bottom-left/top-right diagonal; the duality yields a symmetry around the bottom-right/top-left diagonal.

We keep in mind the case that the double complex is a direct sum of the following two indecomposable objects:

- **zigzags** of length $\ell + 1$, where $\ell \in \mathbb{N}$ counts the number of arrows;

\[
\begin{array}{c}
\ldots \\
\bullet \\
\downarrow^{\partial} \\
\bullet \\
\downarrow^{\bar{\partial}} \\
\bullet \\
\downarrow^{\partial} \\
\bullet \\
\downarrow^{\bar{\partial}} \\
\ldots
\end{array}
\]

- **squares** of isomorphisms:

\[
\begin{array}{c}
\bullet \\
\downarrow^{\partial} \\
\sim \\
\bullet \\
\downarrow^{\partial} \\
\sim \\
\bullet \\
\downarrow^{\bar{\partial}} \\
\end{array}
\]

In particular, zigzags of length one are called *dots*. By Hodge theory and elliptic PDE theory, the cohomologies have finite dimension: whence the number of zigzag is finite, and the number of squares is infinite.

For example, once removed the infinite squares and the arrows arising from symmetries, the double complex associated to a hypothetical complex structure on the 6-dimensional sphere $S^6$ should be as below, where the labels count the number of respective objects and $\alpha, h^{0,2}, \beta, h^{1,0}, h^{1,1}$ are unknown non-negative integers. This example is constructed by using the results in [24] on the Frölicher spectral sequence of a hypothetical complex structure on the six sphere.

\[^1\] it seems that this assumption can be assumed without loss of generality, see the answer by Greg Kuperberg in the MathOverflow discussion at [http://mathoverflow.net/questions/25723/](http://mathoverflow.net/questions/25723/), where he quotes Mikhail Khovanov
Several cohomologies can be defined associated to a double complex. The first ones that appear are the de Rham cohomology,

$$H_{dR}(X; \mathbb{C}) = \ker(\partial + \overline{\partial}) \circ \text{Tot},$$

(here, $\text{Tot}: \text{bba} \to \text{dga}$ denotes the totalization functor,

$$\text{Tot}(A^{\bullet, \bullet}, \partial, \overline{\partial}) := \left( \bigoplus_{p+q=\bullet} A^{p,q}, \partial + \overline{\partial} \right),$$

the category $\text{dga}$ having differential $\mathbb{Z}$-graded algebras as objects,) and the Dolbeault cohomology and its conjugate,

$$H_{\overline{\partial}}(X) = \frac{\ker \overline{\partial}}{\text{im} \overline{\partial}} \quad \text{and} \quad H_{\partial}(X) = \frac{\ker \partial}{\text{im} \partial}.$$

In the model we keep in mind, constituted by squares and zigzags, the Dolbeault cohomology is easily computed by erasing vertical arrows with their ends from the diagram, and counting the remaining points.

Notice that Dolbeault and de Rham cohomology does not suffice, in general, for detecting the complete structure of the double complex. For example, zigzags of odd length do not contribute to the difference between Dolbeault and de Rham cohomology. In other words, to the higher terms in the Frölicher spectral sequence. This means that symmetric zigzags of odd length cannot be detected. For example, the following diagrams have the same de Rham and Dolbeault cohomologies$^2$:

$^2$This example was suggested by Michela Zedda
The last discussion serves as a motivation for introducing Bott-Chern and Aeppli cohomologies. Indeed, the above diagrams differ as for the number of corners. Whence we get the need for having an invariant that counts the corners. In [7, 1], Bott-Chern cohomology and Aeppli cohomology are defined as

\[ H_{BC}^{\bullet, \bullet}(X) = \frac{\ker \partial \cap \ker \overline{\partial}}{\text{imm} \partial \cap \text{imm} \overline{\partial}} \quad \text{and} \quad H_A^{\bullet, \bullet}(X) = \frac{\ker \partial \overline{\partial}}{\text{imm} \partial + \text{imm} \overline{\partial}}. \]

As for the Bott-Chern cohomology, it is easy to recognize that it counts the corners possibly having ingoing arrows, except for the squares:

Dually, Aeppli cohomology counts the corners possibly having outgoing arrows, except for the squares:

Finally, we have the following diagram, where the maps are morphisms of either \( \mathbb{Z} \)-graded or \( \mathbb{Z}^2 \)-graded algebras naturally induced by the identity:

Here, the link between de Rham and (conjugate) Dolbeault cohomology is the Frölicher spectral sequence, naturally arising from the structure of double-complex.

A compact complex manifold \( X \) is said to satisfy the \( \partial \overline{\partial} \)-Lemma if the natural map \( H_{BC}^{\bullet, \bullet}(X) \to H_A^{\bullet, \bullet}(X) \) induced by the identity is injective. This is equivalent to all the map in the above diagram being isomorphisms, see [9, Lemma 5.15]. This is also equivalent to ask that the double complex associated to \( X \) is direct sum of squares and dots, [9, Proposition 5.17].
2. Quantitative properties of Bott-Chern cohomology

In this section, we are interested in determining quantitative relations between the dimensions of the above cohomologies. This is ultimately related to what sequences of integers can appear as dimensions of cohomologies of double complexes of compact complex manifolds. The known restrictions on such sequences are essentially:

- restrictions arising from dimension, compactness, and connectedness;
- symmetries arising from the real structure and the non-degenerate pairing structure;
- inequalities of algebraic type as in [11, Theorem 2], [4, Theorem A], and Theorem 2.2;
- inequalities of analytic type, as the ones that hold for compact complex surfaces, see [2] and subsequent work.

A first result in this direction is the Frölicher inequality in [11, Theorem 2]. It states that the structure of double complex yields a spectral sequence of the form

\[ H^\bullet_\partial \Rightarrow H^\bullet_{dR}(X; \mathbb{C}) , \]

whence the inequality

\[ h^k_\partial - b_k \geq 0 \]  

for any \( k \in \mathbb{Z} \) ,

(As a matter of notation, we write, e.g., \( h^k_\partial = \sum_{p+q=k} \dim \mathbb{C} H^{p,q}_\partial(X) \); moreover, \( b_k \) denotes the \( k \)th Betti number of \( X \).)

The discussion in the previous paragraph motivates a similar inequality for the Bott-Chern cohomology. Indeed, recall that Dolbeault cohomology does not care horizontal arrows, conjugate Dolbeault cohomology does not care vertical arrows, Bott-Chern cohomology counts possibly incoming corners, Aeppli cohomology counts possibly outgoing corners, with the exception, in any case, of squares. Hence, just by combinatorical arguments, one recognizes that the sum of the dimension of the Bott-Chern and Aeppli cohomologies is greater or equal than the sum of the dimension of Dolbeault and conjugate Dolbeault cohomologies, which is greater or equal than twice the Betti number. Moreover, both equalities hold if and only if the double complex is direct sum of squares and dots. That is, if the manifold satisfies the \( \partial\bar{\partial} \)-Lemma. This heuristically explains the results in [4], to which we refer for details. The proof there uses the Varouchas exact sequences in [25].

**Theorem 2.1** ([4, Theorem A, Theorem B]). Let \( X \) be a compact complex manifold. Then, for any \( k \in \mathbb{Z} \), the \( k \)th non-\( \partial\bar{\partial} \)-degree

\[ \Delta^k(X) := h^k_{BC} + h^k_A - 2b_k \]  

is a non-negative integer. Moreover, \( X \) satisfies the \( \partial\bar{\partial} \)-Lemma if and only if

\[ \sum_{k \in \mathbb{Z}} \Delta^k(X) = 0 . \]

In a sense, the previous result states that an un-natural isomorphism (the one given by equality of the dimensions — a quantitative property) ensures a natural isomorphism (the one induced by identity — a qualitative property).

As a special case, consider compact complex surfaces. In [2] and in further discussions with A. Tomassini and M. Verbitsky, we showed that the non-\( \partial\bar{\partial} \)-degree of compact complex surfaces are topological invariants. More precisely, \( \Delta^1 = 0 \) and \( \Delta^2 \in \{0, 2\} \) according to the complex surface admitting Kähler metrics. (Notice that, for compact complex surfaces, the \( \partial\bar{\partial} \)-Lemma is in fact equivalent to the existence of a Kähler metric, by the Lamari and the Buchdahl criterion.)
We announce now an upper bound for the dimension of Bott-Chern cohomology in terms of Hodge numbers. Note that we cannot get a just topological upper bound. That is, an opposite Frölicher inequality cannot be obtained. This is because of the even-length zigzags, which contribute to the Dolbeault cohomology but not to the de Rham cohomology.

**Theorem 2.2** ([3]). Let $X$ be a compact complex manifold of complex dimension $n$. Then, for any $k \in \mathbb{Z}$, it holds
\[ h^k_A \leq \min\{k + 1, (2n - k) + 1\} \left( h^k_{\partial} + h^{k+1}_{\partial} \right), \]
whence also
\[ h^k_{BC} \leq \min\{k + 1, (2n - k) + 1\} \left( h^k_{\partial} + h^{k-1}_{\partial} \right), \]

**Proof.** We give just the idea behind the proof. We refer to a subsequent paper [3] for details. The point is that a contribution to Aeppli cohomology arises from zigzags of positive length $\ell + 1$. Any such zigzag, when placed between total degrees $k$ and $k + 1$, creates exactly two non trivial classes in either Dolbeault or conjugate Dolbeault cohomology at either degree $k$ or degree $k + 1$, and at most $\lfloor \ell/2 \rfloor + 1$ classes in Aeppli cohomology at degree $k$. (In particular, $\lfloor \ell/2 \rfloor + 1 \leq \min\{k + 1, (2n - k) + 1\} \leq n + 1$.) The inequality for Bott-Chern cohomology follows from the Schweitzer duality [19, §2.c] and by the Serre duality. \qed

In particular, for any $k \in \mathbb{Z}$, it holds
\[ -2(n + 1) \left( h^{k+1}_{\partial} + h^{k-1}_{\partial} \right) \leq h^k_A - h^k_{BC} \leq 2(n + 1) \left( h^k_{\partial} + h^{k+1}_{\partial} \right). \]

In the following result, we give a characterization of the $\partial\bar{\partial}$-Lemma in terms of the above inequality.

**Theorem 2.3** ([3]). A compact complex manifold $X$ satisfies the $\partial\bar{\partial}$-Lemma if and only if
\[ \sum_{k \in \mathbb{Z}} \left| h^k_{BC} - h^k_A \right| = 0. \]

**Proof.** We give the idea of the proof, referring to [3] for details. Recall that the Bott-Chern cohomology counts the corners with possible incoming arrows, and the Aeppli cohomology counts the corners with possible outcoming arrows, with the exceptions of squares. Therefore the hypothesis can be restated as: for any anti-diagonal, the number of ingoing arrows equals the number of outgoing arrows, except for squares. Since no ingoing arrow can enter the anti-diagonal of total degree 0, it follows that there is no zigzag of positive length in the whole diagram. That is, the $\partial\bar{\partial}$-Lemma holds. \qed

### 3. Qualitative properties of Bott-Chern cohomology

By investigating the qualitative properties of some cohomology, we mean the study of what and how algebraic structures are induced in cohomology from the space of forms.

For example, let us focus on the differential graded algebra structure on the space of forms given by the wedge product and the exterior differential, and on the de Rham cohomology.
By the Leibniz rule, it induces a structure of algebra in cohomology. We look at $H_{dR}$ as a functor inside the category $dga$ of differential $\mathbb{Z}$-graded algebras:

$$H_{dR} : dga \rightsquigarrow dga.$$ 

We ask for what objects $X$ this functor can be made “concrete”, that is, when it can be realized as a composition of quasi-isomorphisms and formal inverses of quasi-isomorphisms in $dga$: e.g.,

$$X \xrightarrow{\text{qis}} C_1 \xleftarrow{\text{qis}} C_2 \xrightarrow{\text{qis}} \cdots \xrightarrow{\text{qis}} C_{h-1} \xleftarrow{\text{qis}} C_h \xrightarrow{\text{qis}} H_{dR}(X) \xleftarrow{\text{qis}} \cdots$$ 

By the existence of minimal models, see, e.g., [26, Theorem in §II.3 at page 29], [8, Proposition 7.7], this corresponds to the $dga$ of forms and the $dga$ of de Rham cohomology sharing the same model. A compact complex manifold whose double complex of forms satisfies such a property is called formal in the sense of Sullivan [21]. Note that the minimal model contains informations on the rational homotopy groups of the manifold [21]: hence the rational homotopy type of formal manifolds is a formal consequence of their de Rham cohomology. Compact complex manifolds satisfying the $\partial \overline{\partial}$-Lemma (e.g., compact Kähler manifolds) are formal in the sense of Sullivan, [9, Main Theorem]. A theory of Dolbeault formality for complex manifolds has been developed in [17].

Notice that every compact complex manifold is formal in the category of $A_\infty$-algebras\(^3\). This follows from the Homotopy Transfer Principle by Kadeishvili. See [16, 27, 15] for the explicit construction of the Merkulov model. By [15], the induced $A_\infty$-structure in cohomology yields the Massey products, up to sign.

With these notations, a compact complex manifold is formal in the sense of Sullivan if there exists a system of representatives $H^\bullet$ for the cohomology such that the induced $A_\infty$-structure on $H^\bullet$ is actually an algebra structure. A particular case is when the chosen representatives are actually the harmonic representatives with respect to some Hermitian metric. This last situation is referred as geometric formality in the sense of Kotschick [14]. In this case, a possible category to which one can restrict is the category whose objects are $dga$ with a non-degenerate pairing.

Note that the wedge products on forms induces an algebra structure in Bott-Chern cohomology, and just a $H_{BC}$-module structure in Aeppli cohomology. The duality pairing given by any fixed Hermitian metric is internal in de Rham and Dolbeault cohomologies by Poincaré and Serre dualities, and it yields an isomorphism between Bott-Chern and Aeppli cohomologies [19, §2.c].

This is an important issue, for example, in defining Massey products for Bott-Chern cohomology. In [6], triple Aeppli-Bott-Chern Massey products are defined, starting from Bott-Chern class, and yielding a class in Aeppli cohomology, up to indeterminacy.

Hence we propose the following definition, which takes into consideration the structure of non-degenerate pairing.

\(^3\)That is, strongly homotopy associative algebras: the category of $A_\infty$-algebras is equivalent to the category of differential graded co-algebras, by means of the bar construction. We refer to [20, 13, 12, 10] for definitions and details.
Definition 3.1. A compact complex manifold $X$ of complex dimension $n$ is said to satisfy the Schweitzer qualitative property if the natural pairing

$$H_{BC}^{\bullet\bullet}(X) \times H_{BC}^{\bullet\bullet}(X) \to \mathbb{C}, \quad ([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

induced by the wedge product and by the pairing with the fundamental class $[X]$ is non-degenerate.

The above qualitative properties implies a quantitative property which, in turn, characterizes the qualitative property of $\partial\overline{\partial}$-Lemma, thanks to Theorem 2.3.

Theorem 3.2 ([3]). Let $X$ be a compact complex manifold. If it satisfies the Schweitzer qualitative property, then it satisfies the $\partial\overline{\partial}$-Lemma.

References

BOTT-CHERN AND AEPPLI COHOMOLOGY


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THE HK/QK CORRESPONDENCE

MALTE DYCKMANNS

Abstract. The HK/QK correspondence was discovered by Andriy Haydys in 2006. Starting from a (pseudo-)hyper-Kähler manifold endowed with a real-valued function fulfilling certain assumptions, it constructs a quaternionic pseudo-Kähler manifold of the same dimension together with a Killing vector field. I will introduce the correspondence, discuss its compatibility with hyper-Kähler and quaternionic Kähler quotients, and present some examples and applications.

References


Idea:

 hyper-Kähler quotient (with non-zero level)

Swann bundle construction

Swann bundle: Swann ’91, $\mathbb{R}^{>0} \times SO(3)$-bundle $\tilde{\pi} : \tilde{M} \rightarrow \mathcal{M}$ over a QK manifold, $\tilde{M}$ is endowed with a conical hyper-Kähler structure

- hyper-Kähler quotient: Hitchin, Karlhede Lindström, Roček ’87

$S^1$, $\hat{M}_{\text{HK}}^{4n}$

$\hat{M}_{\text{CHK}}^{4n+4}$, $S^1_{\hat{M}}$, $\bar{S}^1_{\hat{M}}$

$P^{4n+1}$, $S^1_P$, $\bar{S}^1_P$

$\hat{M}_{\text{QK}}, \bar{S}^1$

HK/QK correspondence
Haydys’ HK/QK correspondence (as formulated in [ACDM, D])

Let \((M, g, J_1, J_2, J_3, f)\) be a (pseudo-)hyper-Kähler manifold (with Kähler forms \(\omega_\alpha := g(J_\alpha \cdot, \cdot), \alpha = 1, 2, 3\)), together with a real-valued function \(f \in C^\infty(M)\) such that

\[ Z := -\omega_1^{-1}(df) \]  

is a time- or space-like \(J_1\)-holomorphic Killing vector field satisfying

\[ \mathcal{L}_Z J_2 = -2J_3, \]  

and such that \(\sigma := \text{sgn} \ f\) and \(\sigma_1 := \text{sgn} \ f_1\) are constant and non-zero, where \(f_1 := f - \frac{g(Z,Z)}{2}\).

Let \(\pi : P \to M\) be an \(S^1\)-principal bundle and \(\eta \in \Omega^1(P)\) a principal connection one-form such that

\[ d\eta = \pi^*(\omega_1 - \frac{1}{2} d\beta), \]  

where \(\beta := g(Z, \cdot)\). We define the pseudo-Riemannian metric

\[ g_P := \frac{2}{f_1} \eta^2 + \pi^* g, \]  

the vector field \(Z_P^\alpha := \tilde{Z} + f_1 X_P\), and the one-forms

\[ \theta_0^P := \frac{1}{2} df, \quad \theta_1^P := \eta + \frac{1}{2} \beta, \quad \theta_2^P := \frac{1}{2} \omega_1(Z, \cdot), \quad \theta_3^P := -\frac{1}{2} \omega_2(Z, \cdot) \]  

on \(P\), where \(\tilde{Z} \in \Gamma(\ker \eta)\) denotes the horizontal lift of \(Z\) and \(X_P\) denotes the fundamental vector field\(^3\) of \(P\). Let \(M' \subset P\) be a codimension one submanifold that is transversal to \(Z_1^P\) and let \(X \in \mathfrak{X}(M')\) denote the projection of \(X_P\) to \(TM'\) along \(Z_1^P\).

**Theorem** [H, ACDM, D]

\(M'\) admits a quaternionic pseudo-Kähler structure \((g', Q)\), where

\[ g' = \frac{1}{2|f_1|} (g_P - \frac{2}{f} \sum_{a=0}^{3} (\theta_a^P)^2)|_{M'} \]  

and \(Q = \text{span}_\mathbb{R}\{J_1', J_2', J_3'\}\) is defined by

\[ \omega'_\alpha := g'(J'_\alpha \cdot, \cdot) = \frac{\sigma}{2} (d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma) \]  

for any cyclic permutation \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\), where \(\bar{\theta}_\alpha := \frac{1}{f} \theta_\alpha^P|_{M'}\). \(X\) is Killing with respect to \(g'\) and

\[
\text{sign } g' = \begin{cases} 
(4k - 4, 4\ell + 4) & \text{if } \sigma = +1 \& \sigma_1 = -1, \\
(4k + 4, 4\ell - 4) & \text{if } \sigma = -1 \& \sigma_1 = +1, \\
(4k, 4\ell) & \text{if } \sigma = \sigma_1,
\end{cases}
\]

where \(\text{sign } g = (4k, 4\ell)\).

\(\begin{array}{c}
(M, g, J_1, J_2, J_3, f) \\
\xrightarrow{\text{HK/QK correspondence}} \\
(\text{with the choices } P, \eta, M') \\
\xrightarrow{(M', g', Q, X)}
\end{array}\)
1. The HK/QK correspondence can be inverted via the Swann bundle construction (respectively via conical hyper-Kähler manifolds) together with the hyper-Kähler quotient construction \([H,D]\).

**Examples:**

\[
\begin{array}{c}
\mathbb{H}^n \\
\downarrow \text{HK/QK correspondence} \\
(\mathbb{H}^n P^n)^\circ
\end{array}
\]

The superscript \(^\circ\) always denotes some open subset (mostly due to the removal of the zero level of the respective moment maps), which will differ for different examples.

\[
\begin{array}{c}
S^1_{(\text{diag.})} \\
\downarrow \text{HK/QK correspondence} \\
T^*\mathbb{C}P^n \\
\downarrow \text{HK/QK correspondence} \\
(\mathbb{H}^n P^n)^\circ
\end{array}
\]

HK structure on \(T^*\mathbb{C}P^n\): Calabi ’79

HK structure on cotangent bundles of Hermitian symmetric spaces: Biquard & Gauduchon ’96

2. The HK/QK correspondence is compatible with hyper-Kähler and quaternionic Kähler quotients \([D]\).

**Example:**

\[
\begin{array}{c}
S^1_{(\sigma^2)} \\
\downarrow \text{HK/QK correspondence} \\
(\mathbb{H}^n+1)^\circ
\end{array}
\]

The symmetric QK manifold \(X(n) = \frac{SU(n+2)}{SU(2) \times U(n)}\) is a Wolf space of compact type.

3. If \((M,f)\) fulfills the assumptions of the HK/QK correspondence, then \((M,f+c)\) does as well for any \(c \in \mathbb{R}\) (after possibly restricting to an open subset of \(M\) to fulfill the assumptions on \(\sigma\) and \(\sigma_1\)) \([H]\).

**Applications:**

- Deformations of the symmetric metric on \((\mathbb{H}^n P^n)^\circ\) to that on \((X(n))^\circ\) via a smooth family of quaternionic Kähler metrics \([H,D]\).
- Geometric construction of all quaternionic Kähler manifolds in the image of the one-loop deformed supergravity c-map \([ACDM]\).
- Explicit (!) deformations by complete (!) quaternionic Kähler metrics of all Alekseevsky spaces (including all Wolf spaces of non-compact type), except for quaternionic hyperbolic space \([D]\).
A LOCALIZATION FORMULA FOR RIEMANNIAN FOLIATIONS

OLIVER GOERTSCHES

Abstract. These are notes for a talk given at the Bielefeld Geometry & Topology Days in July 2015. After reviewing the Atiyah-Bott-Berline-Vergne localization formula in equivariant cohomology we explain a generalization for Riemannian foliations in the context of equivariant basic cohomology, obtained in recent joint work with Hiraku Nozawa and Dirk Töben. Our formula allows to compute various geometric quantities, such as the volume of Sasakian manifolds.

1. The classical localization formula

The classical localization formula, proven independently by Atiyah–Bott [2] and Berline–Vergne [3], is formulated in the language of equivariant cohomology. We suppose that we are given a compact smooth manifold $M$, equipped with the action of a real torus $T = S^1 \times \cdots \times S^1$. One of several ways to define the equivariant cohomology of the $T$-action is via the so-called Cartan model [4]: let $t$ be the Lie algebra of $T$, and consider polynomial maps defined on $t$ with values in the $T$-invariant differential forms on $M$, $\Omega^p M^q_T$: $S^p t \otimes \Omega^q M_T$.

Elements of $\Omega^p M^q_T$ are called equivariant differential forms. The Cartan differential $d_T : \Omega^p M^q T \to \Omega^p M^q T$ is defined by

$$(d_T \omega)(X) := \left. d\omega(X) \right|_{t=0} - \iota_X \omega,$$

where $X_p = \left. \frac{d}{dt} \right|_{t=0} \exp(t X) \cdot p$ is the fundamental vector field of $X \in t$. Its cohomology is the equivariant cohomology of the $T$-action, denoted $H^p M^q_T$. It carries the structure of an algebra over the polynomial ring $S(t^*)$, but for this talk it is sufficient to regard it as a vector space.

Example 1.1. (1) If the $T$-action on $M$ is trivial, i.e., if all fundamental vector fields are zero, then the Cartan differential reduced to the ordinary exterior differential: $(d_T \omega)(X) = (d\omega(X))$. Hence, $H^p M^q T = S(t^*) \otimes H(M)^T \cong S(t^*) \otimes H(M)$

(2) If the $T$-action on $M$ is free, then the orbit space $M/T$ carries the structure of a smooth manifold, and one can show that $H^p M^q T \cong H(M/T)$.

(3) Let $(M, \omega)$ be a symplectic manifold, endowed with a Hamiltonian $T$-action. Let $\mu : t \to C^\infty(M); X \mapsto \mu^X$ be a momentum map for the action. It satisfies

$$d\mu^X = -\iota_X \omega.$$

It follows that

$$(d_T (\omega - \mu))(X) = d\omega - d\mu^X - \iota_X \omega + \iota_X \mu^X = 0,$$

i.e., $\omega - \mu$ is an equivariantly closed differential form. Note that for $X \in t$ the evaluation $(\omega - \mu)(X) = \omega - \mu^X$ is a non-homogeneous differential form: it is the sum of a form of degree two and a form of degree zero.
As the last example shows, evaluating a closed $\omega \in \Omega_T(M)$ at some $X \in \mathfrak{t}$ yields a possibly non-homogeneous differential form on $M$. Its top degree part one can try to integrate over $M$. The localization formula expresses such an integral as a sum over certain quantities associated to the components of the fixed point set. For simplicity of the presentation we restrict in this talk to the case where the $T$-action has only finitely many fixed points.

**Theorem 1.2** (ABBV localization formula [2], [3]). Consider an action of a torus $T$ on a $2n$-dimensional compact oriented manifold $M$, with only finitely many fixed points $p_1, \ldots, p_n$. Let $\omega \in H_T(M)$. Then we have an equality

$$
\int_M \omega = (-2\pi)^n \sum_k \frac{\omega(p_k)}{\prod_j \alpha_j^{k_j}},
$$

in the fraction field of $S(\mathfrak{t}^*)$, where $(\alpha_j^{k_j})_{j=1}^n \subseteq \mathfrak{t}^*$ are the weights of the isotropy representation at $p_k$.

Here, the expression $\omega(p_k)$ in the numerator is a polynomial on $\mathfrak{t}$: as explained above, evaluating at $X \in \mathfrak{t}$ results in a possibly non-homogeneous differential form, and $\omega(X)(p_k)$ is the evaluation of its degree-zero part at the point $p_k$.

2. **Equivariant cohomology of transverse actions**

Consider now a smooth compact manifold $M$ with a Riemannian foliation $\mathcal{F}$. The main example for us is the one-dimensional foliation given by the flow lines of a nonsingular Killing vector field $\xi$ on a Riemannian manifold $M$. For instance, one may consider the orbit foliation of the Reeb vector field on a compact Sasakian, or more generally $K$-contact manifold. One important feature to be kept in mind is that the foliation may have leaves that are not closed in the ambient manifold.

We first recall the notion of a transverse action on the foliated manifold $(M, \mathcal{F})$, introduced in [1]. We denote by $\Xi(M)$ and $\Xi(\mathcal{F})$ the Lie algebras of vector fields on $M$ and tangent to $\mathcal{F}$, respectively. One calls a vector field $X \in \Xi(M)$ foliated if its flow sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$, or, equivalently, if $[X, \Xi(\mathcal{F})] \subseteq \Xi(\mathcal{F})$. The set of foliated fields, denoted by $L(M, \mathcal{F})$, contains $\Xi(\mathcal{F})$ as an ideal and hence

$$
L(M, \mathcal{F}) := L(M, \mathcal{F})/\Xi(\mathcal{F}),
$$

the set of transverse fields, inherits a Lie algebra structure.

**Definition 2.1** ([1]). Let $\mathfrak{a}$ be a finite-dimensional real Lie algebra. Then a transverse action of $\mathfrak{a}$ on $(M, \mathcal{F})$ is a Lie algebra homomorphism $\mathfrak{a} \to L(M, \mathcal{F})$; $X \mapsto \mathcal{X}$. As for ordinary actions, we call $\mathcal{X}$ the fundamental transverse field of $X \in \mathfrak{a}$.

As all the examples of transverse actions we will encounter are actions of Abelian Lie algebras, we restrict immediately to this subclass.

Note that it makes sense to speak about orbits of transverse actions: an orbit is a minimal $\mathcal{F}$-saturated submanifold $O$ such that for all $p \in O$ we have $T_p \mathcal{F} \oplus a \cdot p \subset T_p O$, where $a \cdot p = \{ \mathcal{X}_p | X \in \mathfrak{a} \}$; note that although $a \cdot p$ is not a subset of $T_p M$, its direct sum with $T_p \mathcal{F}$ is. The analogues of fixed points are leaves of $\mathcal{F}$ in which all fundamental transverse fields of the action vanish.

**Example 2.2.** As mentioned above, our main example is the orbit foliation $\mathcal{F}$ of a non-singular Killing vector field $\xi$ on a compact Riemannian manifold $M$. Its flow defines a one-parameter subgroup of the isometry group $I(M)$, and we may consider its closure $T$. As a compact connected Abelian Lie group it is a torus, and its Lie algebra $\mathfrak{t}$ contains $\xi$. 
The dimension of $T$ is at least two if and only if $\xi$ admits nonclosed flow lines. The $T$-action induces an infinitesimal action
\[ t \mapsto \Xi(M), \]
with image in $L(M, F)$ because $T$ is Abelian, and with $\xi$, by definition, sent into $\Xi(F)$. Hence, the abstract quotient $\mathfrak{a} := t/\mathbb{R}\xi$ acts transversely on $(M, F)$ via the induced homomorphism
\[ \mathfrak{a} = t/\mathbb{R}\xi \longrightarrow L(M, F)/\Xi(F) = l(M, F). \]
For an action of this type, the $\mathfrak{a}$-orbits coincide with the $T$-orbits. A leaf of $F$, i.e., a flow line of $\xi$, is fixed by the $\mathfrak{a}$-action if and only if it is closed in $M$.

For transverse actions one can define an analogue of equivariant cohomology [6], as follows. The idea is that, in case the leaf space $M/F$ is a manifold, a transverse action on $(M, F)$ descends to an ordinary (infinitesimal) action on $M/F$, and one can consider the ordinary equivariant cohomology of this induced action, defined via the Cartan model.

This makes use of the de Rham complex of $M/F$, but it is known that the same information is encoded in the $F$-basic de Rham complex:
\[ \Omega(M, F) = \{ \sigma \in \Omega(M) \mid \iota_X \sigma = L_X \sigma = 0 \text{ for all } X \in \Xi(F) \} \]
is a subcomplex of the de Rham complex of $M$; its cohomology $H(M, F)$ is called the basic cohomology of $(M, F)$. If $M/F$ is a manifold, then we have $H(M, F) \cong H(M/F)$, but the basic objects have the advantage that they make perfect sense also in case $M/F$ is not a well-behaved space.

In [6] we replaced, in the definition of equivariant cohomology above, the ordinary action with a transverse action of a finite-dimensional Abelian Lie algebra $\mathfrak{a}$, and differential forms with $F$-basic differential forms, in order to arrive at a definition of a foliated version of equivariant cohomology. To make this work we just need to observe that because for $X \in \mathfrak{a}$ the fundamental transverse field is a vector field well-defined up to tangential part, contraction with it as well as taking the Lie derivative is well-defined on $\Omega(M, F)$. In particular, we may speak about $\Omega(M, F)^\mathfrak{a}$, the subspace of $\mathfrak{a}$-invariant basic differential forms. (Formally, we just observed that a transverse $\mathfrak{a}$-action induces the structure of an $\mathfrak{a}$-differential graded algebra on $\Omega(M, F)$.) We define
\[ \Omega^\mathfrak{a}(M, F) := S(\mathfrak{a}^*) \otimes \Omega(M, F)^\mathfrak{a}, \]
and the Cartan differential
\[ (d_\mathfrak{a}\sigma)(X) = d(\sigma(X)) - \iota_X(\sigma(X)). \]
Elements of $\Omega^\mathfrak{a}(M, F)$ are called equivariant basic differential forms.

**Definition 2.3** ([6]). Let $\mathfrak{a}$ be a finite-dimensional Abelian Lie algebra. Then the equivariant basic cohomology of a transverse $\mathfrak{a}$-action on $(M, F)$ is defined as
\[ H_\mathfrak{a}(M, F) := H(\Omega^\mathfrak{a}(M, F), d_\mathfrak{a}). \]

For a huge class of Riemannian foliations, namely the Killing foliations (which include all Riemannian foliations on simply-connected manifolds), there always exists a canonical transverse action of an Abelian Lie algebra, whose geometric relevance is due to the fact that its orbits are precisely the closures of the leaves of the foliation – which implies that the fixed points of this transverse action are exactly the closed leaves. This Lie algebra was dubbed structural Killing algebra in [6]. For the construction of this action, as well as for the definition of Killing foliations, we refer to [7], but see also [6] for a short summary. For this talk it is sufficient to mention that for Riemannian foliations as in Example 2.2 the transverse action of the Lie algebra $\mathfrak{a} = t/\mathbb{R}\xi$ described there can be identified with the action of the structural Killing algebra.
3. A foliated localization formula

We already observed that the evaluation of a closed equivariant differential form at a Lie algebra element results in an ordinary, but perhaps non-homogeneous differential form. The same holds true for closed equivariant basic differential forms: any such evaluation is a basic differential form, possibly non-homogeneous. But nonzero basic differential forms only exist in degree up to the codimension of $F$, so in order to obtain an object that is worth being integrated we first have to transform it into a top degree form, e.g. by pairing it with a leafwise volume form.

Let $p$ be the dimension of the leaves of $F$, and $q$ the codimension. A $p$-form $\eta$ on $M$ is called relatively closed if

$$d\eta(v_1, \ldots, v_{p+1}) = 0$$

whenever $p$ of the $p+1$ vectors $v_i$ are tangent to $F$.

**Example 3.1.** Let $M$ be a Sasakian, or more generally, $K$-contact manifold, with Reeb vector field $\xi$ and contact form $\eta$. Then $d\eta$ is basic with respect to the orbit foliation of $\xi$, i.e., $\eta$ is relatively closed.

Let $\eta$ be a relatively closed $p$-form on $(M, F)$ and consider the map

$$\int_{(\mathcal{F}, \eta)} : \Omega^p(M, F) \to \mathbb{R}; \quad \sigma \mapsto \int_M \eta \wedge \sigma.$$

Then one can easily show that $\int_{(\mathcal{F}, \eta)}$ descends to a map $\int_{(\mathcal{F}, \eta)} : H^q(M, F) \to \mathbb{R}$. We also consider the natural equivariant extension of this integration operator,

$$\int_{(\mathcal{F}, \eta)} : \Omega_a(M, F) \to S(\mathfrak{a}^*)$$

as well as its induced map on equivariant cohomology.

For our presentation of the foliated localization formula, we restrict to the case that the foliation $\mathcal{F}$ has only finitely many closed leaves. The general theorem, as well as its proof, can be found in [5].

**Theorem 3.2 (ABBV-type localization formula for Riemannian foliations [5]).** Let $\mathcal{F}$ be a transversely oriented Killing foliation of dimension $p$ and codimension $q$ on a compact oriented Riemannian manifold $M$ with only finitely many closed leaves $i_k : L_k \to M$. Furthermore, let $\eta$ be a relatively closed $p$-form. Then for any $\sigma \in H_a(M, \mathcal{F})$, where $\mathfrak{a}$ is the structural Killing algebra of $\mathcal{F}$, we have an equality

$$\int_{(\mathcal{F}, \eta)} \sigma = \int_M \eta \wedge \sigma = (-2\pi)^{q/2} \sum_k \left(\int_{L_k} \eta\right) \cdot \frac{i_k^* \sigma}{\prod_j \alpha_{j}};$$

in the fraction field of $S(\mathfrak{a}^*)$, where $\{\alpha_{j}^{k}\}_{j=1}^{q/2} \subset \mathfrak{a}^*$ are the weights of the transverse isotropy $\mathfrak{a}$-action at $L_k$.

The weights of the transverse action are defined in terms of the chosen orientations: As $\mathfrak{a}$ acts transversely isometrically, in any $p \in L_k$ we have a transverse isotopy representation $\mathfrak{a} \to \mathfrak{so}(\nu_pN); X \mapsto [X, \cdot]$. Since $\mathfrak{a}$ is abelian, we find an oriented orthonormal frame $\xi : \mathbb{R}^k \to \nu_p L_k$ and linear forms $\alpha_j \in \mathfrak{a}^*$, such that any endomorphism $[X, \cdot]$ is a block diagonal matrix with blocks of the form

$$\begin{pmatrix} 0 & -\alpha_j(X) \\ \alpha_j(X) & 0 \end{pmatrix}.$$

We see that the main difference to the classical localization formula is the appearence of the relatively closed $p$-form $\eta$. Only the basic part of the expression $\eta \wedge \sigma$ is localized.
4. An example: The volume of K-contact manifolds

Let $M$ be a compact $K$-contact manifold of dimension $2n + 1$, with contact form $\eta$, Reeb vector field $\xi$, transverse almost complex structure $J$, and Riemannian metric given by

$$g = \frac{1}{2} d\eta \circ (1 \otimes J) + \eta \otimes \eta.$$ 

The $K$-contact condition is that $\xi$ is a Killing vector field with respect to $g$. Any Sasakian manifold is $K$-contact.

The top form $\eta \wedge (d\eta)^n$ is nowhere vanishing and thus defines an orientation on $M$. With this orientation fixed, the Riemannian volume form of $g$ is

$$\frac{1}{2^n n!} \eta \wedge (d\eta)^n.$$ 

We observe that this expression is perfectly suitable for our localization formula: considering on $M$ the foliation $\mathcal{F}$ given by the flow lines of $\xi$, we see that $\eta$ is a relatively closed 1-form and $(d\eta)^n$ is an $\mathcal{F}$-basic $(2n)$-form. The volume of $M$ can thus be written as

$$\text{Vol}(M, g) = \frac{1}{2^n n!} \int_M \eta \wedge (d\eta)^n = \frac{1}{2^n n!} \int_{(\mathcal{F}, \eta)} (d\eta)^n.$$ 

The action of the structural Killing algebra can be identified as the one described in Example 2.2: let $t_\mathcal{F}$ be a compact $K$-contact manifold with only finitely many closed Reeb orbits we can thus apply Theorem 3.2 and obtain:

$$\text{Vol}(M, g) = \frac{1}{2^n n!} \int_{(\mathcal{F}, \eta)} \omega^n = (-1)^n \frac{n!}{n!} \sum_{k=1}^{N} \left( \int_{L_k} \eta \right) \cdot \frac{\eta|_{L_k}(v^\#)^n}{\prod_j \alpha_j^k(v + Rb)},$$ 

where the fractions on the right hand side are considered as rational functions in the variable $v \in \mathfrak{t}$. The total expression is independent of $v \in \mathfrak{t}$.

For the (short) proof see [5]. Among the examples to which one can apply Theorem 4.1 are toric and certain deformations of homogeneous Sasakian manifolds. For a detailed discussion of these examples, including explicit formulas, we also refer to [5].
References


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DIFFERENT GEOMETRIES ON THE SPACE OF KÄHLER AND SASAKIAN METRICS

DAVID PETRECCA
(JOINT WORK WITH S. CALAMAI AND K. ZHENG)

Abstract. On a closed Kähler manifold, the space of all Kähler metrics in a fixed cohomology class has a natural structure of infinite dimensional manifold. On it, several (weak) Riemannian metrics can be assigned and the most studied ones are called $L^2$, Calabi and Gradient (or Dirichlet) metric. I will recall known results about their different geometries and write down and compare the relative geodesic equations as PDEs on the manifold. Finally I will discuss my contribution, joint work with S. Calamai and K. Zheng, about the geodesic equation of the gradient metric and of the Ebin metric (from the space of all Riemannian metrics) restricted to the (similarly defined) space of Sasakian metrics.

1. Outline of the talk

The talk will develop as follows.

(1) Definition of the space $\mathcal{H}$ of Kähler metrics;
(2) Metrics on $\mathcal{H}$:
   (a) $L^2$ metric and its geometry;
   (b) Calabi metric and its geometry;
   (c) Gradient metric and its geometry;
(3) Space of Sasakian metric and the induced geometry from the space of Riemannian metrics;
(4) Applications and questions.

We will discuss our contribution [5] about items 2c and 3 above.

2. The space of Kähler metrics

Let $(M, \omega)$ be a closed Kähler manifold with $n = \dim \mathbb{C} M$. We consider the cohomology class $[\omega] \in \Omega^{1,1}(M) \cap H^2(M, \mathbb{R})$. We are interested in the forms in $[\omega]$ which define a Kähler metric, namely

$$\mathcal{K} = \{ \alpha \in \Omega^{1,1}(M) \cap H^2(M, \mathbb{R}) : \alpha = \omega + d\beta > 0 \text{ for some } \beta \}.$$ 

Applying the $\bar{\partial}\partial$-Lemma we find a real function $\varphi$ such that $d\beta = i\partial\partial\varphi$. So we can write

$$\mathcal{K} = \{ \omega_\varphi \in \Omega^{1,1}(M) \cap H^2(M, \mathbb{R}) : \omega_\varphi := \omega + i\partial\partial\varphi > 0 \}$$

$$\mathcal{\tilde{H}} = \{ \varphi \in C^\infty(M, \mathbb{R}) : \omega_\varphi \geq 0 \}.$$ 

We are interested in the space $\mathcal{\tilde{H}}$ up to constants so we consider “normalized” potentials, namely

$$\mathcal{\tilde{H}} = \{ \varphi \in \mathcal{\tilde{H}} : I(\varphi) = 0 \}$$

(1)

where $I : \mathcal{\tilde{H}} \to \mathbb{R}$ is a functional such that $dI_\varphi\psi = \int_M \psi\omega_\varphi^n$ and $I(0) = 0$, see [11].

With this we can identify $\mathcal{\tilde{H}} = \mathcal{H} \oplus \mathbb{R}$ by $\varphi \mapsto (\varphi - I(\varphi), I(\varphi))$. 

The tangent space of our spaces are

\[ T_{\varphi} \mathcal{H} = C^\infty(M, \mathbb{R}) \]

and

\[ T_{\varphi} \mathcal{H} = \left\{ \psi \in C^\infty(M, \mathbb{R}) : \int_M \psi \omega^n_\varphi = 0 \right\}. \]

We will use another description of the space \( \mathcal{H} \) given as follows. Let

\[ \mathcal{C} = \left\{ u \in C^\infty(M) : \int_M e^n \frac{\omega^n}{n!} = \text{vol} \right\} \]

where \( \text{vol} \) is the volume of \( M \), invariant in the Kähler class.

There is a map \( \mathcal{H} \to \mathcal{C} \) given by

\[ \varphi \mapsto \log \frac{\omega^n_\varphi}{\omega^n} \] (2)

where \( \omega^n_\varphi \) is the unique function \( f_\varphi \) such that \( \omega^n_\varphi = f_\varphi \omega^n \).

The injectivity of this map was proved by Calabi in the 1950s (see e.g. [1]) while the surjectivity follows from the Calabi-Yau theorem that ensure the existence, in a Kähler class, of a Kähler metric with prescribed Ricci form.

By deriving, the tangent space of \( \mathcal{C} \) is

\[ T_u \mathcal{C} = \left\{ v \in C^\infty(M) : \int_M v e^n \frac{\omega^n}{n!} = 0 \right\}. \]

3. Metrics on \( \mathcal{H} \)

3.1. \( L^2 \) (Donaldson, Mabuchi, Semmes metric). Let \( \varphi \in \mathcal{H} \) and \( \psi_1, \psi_2 \in T_{\varphi} \mathcal{H} \). Define

\[ g^M(\psi_1, \psi_2)_\varphi = \int_M \psi_1 \psi_2 \omega^n_\varphi. \] (3)

If \( \varphi(t) \in \mathcal{H} \) is a curve and \( \psi(t) \) is a section along it (i.e. if \( \psi(t) \in T_{\varphi(t)} \mathcal{H} \) for all \( t \)) we have an existence result for the Levi-Civita covariant derivative.

**Proposition 3.1** (Donaldson-Mabuchi-Semmes). Given a curve \( \psi \) and a section \( \psi \) along it, there exist a unique section \( D_t \psi \) along \( \varphi \) such that

1. \( \frac{d}{dt} g^M(\psi, \psi) = 2 g^M(D_t \psi, \psi) \);
2. \( D_t \psi \) is torsion free;
3. The Leibniz rule holds: \( D_t(f \psi) = f D_t \psi + f' \psi \).

About the geometry of the \( L^2 \) metric we have the following

**Proposition 3.2.** The \( L^2 \) metric has non positive sectional curvature and \( (\mathcal{H}, g^M) \) is a locally symmetric space, i.e. its curvature tensor is parallel.

The geodesic equation for this metric is

\[ 0 = D_t \varphi' = \varphi'' - \frac{1}{2} |d\varphi'|_{\varphi}^2. \] (4)
Associated to it there are two problems. The first one is the Cauchy problem and consists in finding a geodesic with prescribed initial point and speed

\[
\begin{align*}
D_t \varphi' &= 0 \\
\varphi(0) &= \varphi_0 \\
\varphi'(0) &= \psi_0
\end{align*}
\] (CP)

and the Dirichlet problem to find a geodesic joining two different points

\[
\begin{align*}
D_t \varphi' &= 0 \\
\varphi(0) &= \varphi_0 \\
\varphi(1) &= \psi_1
\end{align*}
\] (DP)

For the \(L^2\) metric, Chen [7] provided a weak solution for (DP).

**Theorem 3.3** (X. X. Chen [7]). \textit{Given} \(\varphi_0 , \varphi_1 \in \mathcal{H}\) \textit{there exist a} \(C^{1,1}\) \textit{curve}¹ \textit{that solves} (DP).

This allows to prove that the Riemannian pseudo distance on \(\mathcal{H}\) is a distance that gives \(\mathcal{H}\) the structure of a metric space. Its completion was later found by Clarke [9].

Moreover, Donaldson [11] proved that the problem (CP) is ill-posed, i.e. that there exist “forbidden” directions that cannot be initial speeds for a geodesic.

### 3.2. The Calabi metric.

Let us now work in \(\mathcal{C}\) and define the Calabi metric to be

\[
g^C(v_1, v_2)_u = \int_M v_1 v_2 e^u \omega^n. \tag{5}
\]

Pulled back to \(\mathcal{H}\) via the map (2) it looks like

\[
g^C(\psi_1, \psi_2)_\varphi = \int_M \Delta_\varphi \psi_1 \Delta_\varphi \psi_2 \omega^n
\]

where \(\Delta_\varphi\) is the \(\partial\bar{\partial}\)-Laplacian of the metric \(\omega_\varphi\).

This metric was introduced by Calabi in the 1950s and its study was completed by Calamai [4] in the 2010s. Namely he proved the existence of the Levi-Civita covariant derivative,

**Proposition 3.4** (Calamai, 2012). \textit{The Levi-Civita covariant derivative for the Calabi metric on} \(\mathcal{C}\) \textit{is} \(D_tv = v' + 1/2 v u' + 1/4 \omega^n g^C(u', v)\).

the following existence results for the geodesic equations

**Theorem 3.5** (Calamai, 2012). \textit{The geodesic equation is equivalent to the ODE} \((\exp(u/2))_t \!+ \! \exp(u/2) = 0\) \textit{and hence both} (CP) \textit{and} (DP) \textit{admit smooth explicit solutions.}

and the following about the curvature

**Proposition 3.6** (Calamai, 2012). \textit{The sectional curvature of} \((\mathcal{C}, g^C)\) \textit{is constant equal to} \(1/4 \omega^n\).

The solvability of (DP) allows to prove that \(\mathcal{C}\) has a metric space structure, but incomplete. The completion was found by Clarke and Rubinstein [10].

¹Here \(C^{1,1}\) means the space of functions \(f\) for which \(iB_sB\) has bounded components.
3.3. The gradient (Dirichlet) metric. We go back to $H$ and define
\[ g^G(\psi_1, \psi_2)_\varphi = \int_M (d\psi_1, d\psi_2)_\varphi \omega^n. \] (6)

This metric was studied in [4] where the following was proved that the Levi-Civita co-
variant derivative is given implicitly by
\[ \Delta \varphi D_t \psi = \Delta \varphi \psi' + (\Delta \varphi \psi)' + \Delta \varphi \psi \Delta \varphi \varphi' \]
and

**Proposition 3.7** (Calamai, 2012). On a Riemann surface (complex dimension 1) the Gradi-
ent metric is flat.

In higher dimension Calamai and Zheng [6] provide an explicit expression of the sectional
curvature $K_G$.

**Conjecture 3.8** (Calabi). The curvatures of the three metrics are related by
\[ K_M \leq K_G \leq \frac{1}{4 \text{vol}} = K_C. \]

The geodesic equation then is
\[ \Delta \varphi \varphi'' + (\Delta \varphi \varphi')' + (\Delta \varphi \varphi')^2 = 0. \] (7)

Except for Riemann surfaces, both (CP) and (DP) were not solved.

4. Our contribution

We provide a smooth solution for (CP) for the gradient metric.

**Theorem 4.1** (Calamai, –, Zheng 2014). Given $\varphi_0 \in H$ and $\psi_0 \in T_{\varphi}H$, there exists $\varepsilon > 0$ and a smooth curve $\varphi : (-\varepsilon, \varepsilon) \to H$ that satisfies (CP).

Let us sketch its proof. For an integer $k \geq 1$ and real $\alpha \in (0, 1)$ and $\delta > 0$ define the following spaces
\[ H^{k,\alpha}_\delta = \{ \varphi \in C^{k,\alpha}(g) : \omega + i \partial \bar{\partial} \varphi \geq \delta \} \]
where $C^{k,\alpha}(g)$ is the Hölder space with respect to the reference metric $g$ and $\delta$ depends on
the initial data $\varphi_0, \psi_0$.

Our geodesic equation splits into the system
\[
\begin{align*}
\varphi' &= \psi \\
\psi' &= L_\varphi(\psi)
\end{align*}
\] (8)

where $L_\varphi$ is a pseudo-differential operator.

Consider the metric space
\[ X = C^2([-\varepsilon, \varepsilon], H^{k,\alpha}_\delta) \times C^2([-\varepsilon, \varepsilon], C^{k,\alpha}(g)) \]
as the function space where we are going to look for solutions of our system. The norm that
we consider is defined for $\psi \in C^2([-\varepsilon, \varepsilon], C^{k,\alpha}(g))$ as
\[ ||\psi||_{k,\alpha} := \sup_{t \in [-\varepsilon, \varepsilon]} ||\psi(t, \cdot)||_{C^{k,\alpha}(g)}, \]
and in the product space, the norm of any element $(\varphi, \psi) \in X$ is
\[ ||(\varphi, \psi)||_{k,\alpha} := ||\varphi||_{k,\alpha} + ||\psi||_{k,\alpha}. \]
We work in an appropriate metric ball in $X$ obtained by the following lemma.

**Lemma 4.2.** There exists $r > 0$ such that if $\varphi \in C^{k,\alpha}(g)$ is such that $\|\varphi - \varphi_0\|_{C^{k,\alpha}(g)} < r$ then $\varphi \in H_{\delta}^{k,\alpha}$.

We consider the operator

$$T(\varphi, \psi) = \left( \varphi_0 + \int_0^t \psi(s)ds, \psi_0 + \int_0^t (L_\varphi(\psi))(s)ds \right).$$

(10)

**Proposition 4.3.** If $B_r$ is a metric ball of radius $r$ centered in $(\varphi_0, \psi_0) \in X$ then we have

(i) $T(B_r) \subseteq B_r$;

(ii) $T$ on $B_r$ is a contraction.

The former item boils down to the estimate $\|L_\varphi \psi\|_{k,\alpha}$ using Schauder estimates (see e.g. [1]) while the latter to the estimate $\|L_\varphi \psi - L_\varphi \psi'\|_{k,\alpha}$.

At this point the Banach fixed point theorem provides the existence of a fixed point $(\varphi, \psi) \in B_r$ for the operator $T$, which by construction is a $C^{k,\alpha}(g)$ solution to our problem.

### 4.1. Smoothness

To prove the smoothness of our solution, pick $(k, \alpha)$ and a solution $\varphi \in C^{k+1,\alpha}(g)$ which exists for $|t| < \varepsilon(k+1, \alpha)$.

If $\partial_A = \frac{\partial}{\partial x^A}$ then derive the system and obtain

$$\begin{cases} \left( \partial_A \varphi \right)' = \partial_A \psi \\ \left( \partial_A \psi \right)' = \partial_A (L_\varphi \psi) \end{cases}.$$

This is a linear system in $\varphi_A = \partial_A \varphi$ and $\psi_A = \partial_A \psi$ with $C^{k-1,\alpha}(g)$ coefficients that exist for $|t| < \varepsilon(k, \alpha)$. So also the solutions $\varphi_A, \psi_A$ exist for $|t| < \varepsilon(k, \alpha)$. This means that $\varphi$ is $C^{k+1,\alpha}(g)$ for $|t| < \varepsilon(k, \alpha)$ proving the inequality $\varepsilon(k, \alpha) \leq \varepsilon(k+1, \alpha)$.

Since in general we have $\varepsilon(k, \alpha) \geq \varepsilon(k+1, \alpha)$ we obtain the equality. This means that our solution is $C^{k,\alpha}(g)$ for all $(k, \alpha)$, hence smooth.

### 5. Sasakian manifolds

A Sasakian manifold is a $(2n + 1)$-dimensional $M$ together with a contact form $\eta$, its Reeb field $\xi$, a $(1, 1)$-tensor field $\Phi$ and a Riemannian metric $g$ that makes $\xi$ Killing, such that

$$\begin{align*}
\eta(\xi) &= 1, \ i_\xi d\eta = 0 \\
\Phi^2 &= -\text{id} + \xi \otimes \eta \\
g(\Phi \cdot, \Phi \cdot) &= g + \eta \otimes \eta \\
d\eta &= g(\Phi \cdot, \cdot) \\
N_\Phi + \xi \otimes d\eta &= 0
\end{align*}$$

where $N_\Phi$ is the torsion of $\Phi$. The first four mean that $M$ is a contact metric manifold and the last one means it is normal, see [2, Chap. 6].

The foliation defined by $\xi$ is called characteristic foliation. Let $D = \ker \eta$. It is known that $(d\eta, J = \Phi |_D)$ is a transversally Kähler structure, as the second, third and fourth equation above say.

A form $\alpha$ is said to be basic if $i_\xi \alpha = 0$ and $i_\xi d\alpha = 0$. A function $f \in C^\infty(M)$ is basic if $\xi \cdot f = 0$. The space of smooth basic functions on $M$ is denoted by $C^\infty_B(M)$. The transverse
Kähler structure defines the transverse operators $\bar{\partial}$, $\partial$ and $d^c = \frac{i}{2}(\bar{\partial} - \partial)$ acting on basic forms, analogously as in complex geometry. The form $d\eta$ is basic and its basic class is called transverse Kähler class.

Given an initial Sasakian manifold $(M, \eta_0, \xi_0, \Phi_0, g_0)$, basic functions parameterize a family of other Sasakian structures on $M$ which share the same characteristic foliation and are in the same transverse Kähler class, in the following way. We follow the notation of [2, p. 238].

Let $\varphi \in C^\infty_B(M)$ and define $\eta_\varphi = \eta_0 + d^c \varphi$. The space of all $\varphi$'s is

$$\tilde{\mathcal{H}}_S = \{ \varphi \in C^\infty_B(M) : \eta_\varphi \wedge d\eta_\varphi \neq 0 \}$$

and, in analogy of the Kähler case, we consider normalized “potentials”

$$\mathcal{H}_S = \{ \varphi \in \tilde{\mathcal{H}}_S : I(\varphi) = 0 \}.$$

The equation $I = 0$ is a normalization condition, similar to the one in (1). We refer to [14] for the definition of $I$ in our case, which is such that

$$T_\varphi \mathcal{H}_S = \left\{ \psi \in C^\infty_B(M) : \int_M \psi \frac{1}{n!} \eta_\varphi \wedge d\eta^n_\varphi = 0 \right\}.$$

These deformations are called of type II and it is easy to check that they leave the Reeb foliation and the transverse holomorphic structure fixed, since $\xi$ is still the Reeb field for $\eta_\varphi$.

Every $\varphi \in \mathcal{H}_S$ defines a new Sasakian structure where the Reeb field and the transverse holomorphic structure are the same and the metric transforms as follows

$$g_\varphi = d\eta_\varphi \circ (\text{id} \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi. \quad (11)$$

As in the Kähler case, these deformations keep the volume of $M$ fixed, which will be denoted by $\text{vol}$.

The $L^2$ metric was generalized to $\mathcal{H}_S$ in [14, 15], where Guan and Zhang solved the Dirichlet problem for the geodesic equation and He provided a Sasakian analogue of Donaldson’s picture about extremal metrics.

On the space $\mathcal{H}_S$ one can define the Calabi metric and the gradient metric in the same ways as in formulae (5) and (6) by using the so called basic Laplacian which acts on basic functions in the same way as in the Kähler case and by using the volume form $\frac{1}{n!} \eta_\varphi \wedge d\eta^n_\varphi$ in the integrals.

In this setting, it is easy to see that the map

$$\mathcal{H}_S \ni \varphi \mapsto \log \frac{\eta_\varphi \wedge d\eta^n_\varphi}{\eta_0 \wedge d\eta_0^n}$$

maps basic functions to basic functions. The transverse Calabi-Yau theorem of [3] allows to prove the surjectivity of this map as in the Kähler case, more precisely between $\mathcal{H}_S$ and the space of basic conformal volume forms

$$C_B = \left\{ u \in C^\infty_B(M) : \int_M e^u \frac{1}{n!} \eta_0 \wedge d\eta^n_0 = \text{vol} \right\}.$$

---

2This definition with the $\frac{1}{2}$ is classical in Sasakian geometry and differs from the convention usually used in complex geometry $d^c = i(\bar{\partial} - \partial)$. With this convention it holds the relation $dd^c = i\bar{\partial} \partial$ on basic forms.
5.1. **The Ebin metric.** The space of the Riemannian metrics $\mathcal{M}$ is identified with the space $S^2_+(T^*\mathcal{M})$ of all symmetric positive $(0,2)$-tensors on $\mathcal{M}$. The formal tangent space at a metric $g \in \mathcal{M}$ is then given by all symmetric $(0,2)$-tensors $S^2(T^*\mathcal{M})$. For $a,b \in T_g\mathcal{M}$, the Ebin [12] metric is defined as the pairing

$$g_E(a,b)_g = \int_M g(a,b)dv_g$$

where $g(a,b)$ is the metric $g$ extended to $(0,2)$-tensors and $dv_g$ is the volume form of $g$. In [13] the explicit expression of the Cauchy geodesics is given.

We prove the following fact

**Proposition 5.1** (Calamai, –, Zheng 2014; indep. Chhay 2015). The pullback of the Ebin metric to the space of Sasakian metric is $2(g^C + g^G)$ which will be referred to as the sum metric.

We prove the existence of the Levi-Civita covariant derivative for the sum metric in function of the ones of the Calabi and gradient metric.

The geodesic equation for the sum metric is

$$\left(\Delta_\varphi - 1\right)\left(\varphi'(\varphi'')' + \frac{1}{2}(\Delta\varphi ')^2\right) - \frac{1}{2}\left|i\partial\bar{\partial}\varphi'\right|^2 = 0. \tag{12}$$

The sum metric belongs to the bigger family of combination metrics $\alpha g^M + \beta g^G + \gamma g^C$.

We prove with the same technique explained above

**Theorem 5.2.** The combination metric admits a Levi-Civita connection and its Cauchy problem (CP) has a unique smooth solution.

6. **Applications**

Let us now recall briefly the applications in Kähler geometry of the different geometries on $\mathcal{M}$.

- **The $L^2$ metric:** is used to prove the uniqueness of constant scalar curvature Kähler metrics in each Kähler class;
- **The Calabi metric:** belongs to a bigger family of weighted metrics ([6]) and one of them is used to reprove the uniqueness of Kähler-Einstein metrics when the first Chern class is nonpositive.
- **The gradient metric:** is the metric with respect to which the pseudo-Calabi flow [8] is a gradient flow.

7. **Questions**

We conclude stating some open questions.

1. Does the combination metric $g_\beta = g^M + \beta g^G$ approximate geodesic rays of the $L^2$ metric?
2. Clarke and Rubinstein [10] prove that $\mathcal{H} \subset \mathcal{M}$ is not totally geodesic. What about $\mathcal{H}_S$?
3. Is (DP) for the gradient metric solvable on special classes of manifolds (e.g. ruled surfaces, toric manifolds, ...)?
4. Study the geometry of the finite dimensional space of type I deformations of a Sasakian metric (see [2, Chap. 8])
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THE J-FLOW ON SASAKIAN MANIFOLDS

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Abstract. The J-flow is a gradient geometric flow of Kaehler structures firstly studied by Donaldson from the point of view of moment maps and by Chen in relation to the Mabuchi energy. The aim of this talk is to introduce an odd-dimensional analogue of this flow in the Sasakian context.

This talk is based on the joint work with Luigi Vezzoni [16].

1. J-flow on Kâhler manifolds

Let $(M, \omega_0)$ be a Kâhler manifold and consider the space of Kâhler potentials:

$$\mathcal{H} = \{ \varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi = \omega_0 + i\bar{\partial}\partial^* \varphi > 0 \}.$$  

The geometry of $\mathcal{H}$ has been deeply studied, in particular T. Mabuchi [11] and S. K. Donaldson [5] independently, proved that $\mathcal{H}$ endowed with the Riemannian metric:

$$R_{\omega_\varphi}(\psi_1, \psi_2, \psi_3) = \frac{1}{4} \{\{\psi_1, \psi_2\}_\varphi, \psi_3\}_\varphi,$$

is an infinite dimensional local symmetric space, with curvature

where $\{\ , \}_\varphi$ are the Poisson brackets on $C^\infty(M)$ with respect to $\omega_\varphi$.

Fix a second Kâhler form $\chi$ on $M$ and let $\varphi(t)$ be a smooth path in $\mathcal{H}$. Define the functional $J_\chi : \mathcal{H} \to \mathbb{R}$ by:

$$\partial_t J_\chi(\varphi) = \frac{1}{(n-1)!} \int_M \varphi_\chi \wedge \omega_\varphi^{n-1} = \frac{1}{n!} \int_M \varphi \sigma_\varphi \omega_\varphi^n, \quad J_\chi(0) = 0,$$

where $\sigma_\varphi$ is the trace of $\chi$ with respect to $\omega_\varphi$. Let $\mathcal{H}_0 = \{ h \in \mathcal{H} \mid I(h) = 0 \}$ where $I : \mathcal{H} \to \mathbb{R}$ is the functional defined by:

$$\partial_t I(\varphi) = \int_M \varphi \omega_\varphi^n = 0, \quad I(0) = 0.$$  

Observe that $h \in \mathcal{H}_0$ is a critical point for $J_\chi$ iff $\int_M k \omega_\varphi = 0$, for all $k \in T_h \mathcal{H}_0$, i.e. iff $\sigma_\varphi = c$ is constant. The $J$-flow is the gradient flow of $J_\chi$ and it is defined by:

$$\dot{\varphi} = c - \sigma_\varphi, \quad \varphi(0) = \varphi_0.$$  

In [6, pages 10-11] Donaldson introduced the $J$-flow pointing out the importance of its critical points from the point of view of moment maps. Donaldson observed that a necessary condition to the existence of such critical metrics is that $c \omega_\varphi - \chi > 0$ and he asked if it was also sufficient. In [3] X.X. Chen proved that for complex surfaces it is. Although, in the recent paper [10], M. Lejmi and G. Székelhidi found a counterexample on the blow up of $\mathbb{P}^3$ at one point. The existence of critical points has been studied by B. Weinkove, V. Tosatti and J. Song in [13, 15, 17, 18] in terms of the positivity of some $(n-1, n-1)$ form
and by X.X. Chen in [4] in terms of the sign of the holomorphic sectional curvature of $\chi$. In particular in [16] we extend Chen’s results to the Sasakian context.

2. SASAKIAN MANIFOLDS

Here you find a very brief and not exhaustive introduction to Sasakian manifolds, a more detailed and better exposition can be found e.g. in [1, 14].

Sasaki manifolds are often viewed as the odd-dimensional counterpart of Kähler manifolds. As Kähler geometry lies in the intersection of complex, Riemannian and symplectic geometry, so Sasaki geometry lies in the intersection of CR, contact and Riemannian geometry. A Riemannian manifold $(M, g)$ is Sasaki iff the Riemannian cone $(M \times \mathbb{R}^+, \tilde{g} = r^2 g + dr^2)$ is Kähler. The integrable complex structure $J$ and the Kähler form $\omega$ on $(M \times \mathbb{R}^+, \tilde{g})$ induce in a natural way on $(M, g)$:

1. a killing vector field $\xi$ and its dual 1-form $\eta$, $\eta(\xi) = 1$, $\iota_\xi d\eta = 0$, which is a contact form, i.e. $\eta \wedge (d\eta)^n \neq 0$. The tangent bundle $TM$ splits into $TM = D \oplus L_\xi$, where $D = \ker \eta$ and $L_\xi$ is the line tangent to $\xi$.
2. an endomorphism $\Phi$ defined by $\Phi|_D = J|_D$, $\Phi L_\xi = 0$, which satisfies $\Phi^2 = -\text{Id} + \eta \otimes \xi$.

The triple $(\eta, \xi, \Phi)$ realises a contact structure on $M$, while $(D, \Phi|_D)$ realises a CR structure.

According to $TM = D \oplus L_\xi$, the metric $g$ splits into:

$$g(X, Y) = g^T(X, Y) + \eta(X)\eta(Y), \quad X, Y \in TM,$$

where $g^T(X, Y) = \frac{1}{2} d\eta(X, \Phi Y)$, which is zero along the direction of $\xi$ and it is Kähler with respect to $D$, is called the transverse Kähler metric of $M$.

A $p$-form $\alpha$ on $M$ is basic if it satisfies

$$\iota_\xi \alpha = 0, \quad \iota_\xi d\alpha = 0.$$

In particular, a function is basic if its derivative in the direction of $\xi$ vanishes. If $\{z_1, \ldots, z_n, z\}$ are local coordinates ($z_j$ complex, $z$ real) such that

$$\partial_z = \xi, \quad \Phi(dz^j) = i dz^j, \quad \Phi(d\bar{z}^j) = -i d\bar{z}^j,$$

then a function is basis iff it does not depend on $z$. We denote the space of smooth basic functions on $M$ by $C^B_\Omega(M, \mathbb{R})$.

3. HOW TO GENERALIZE THE $J$-FLOW IN THE SASAKIAN CONTEXT

To define a Sasakian $J$-flow we first need a Sasakian version of the space of Kähler potential $\mathcal{K}$. Given a Sasakian manifold $(M, g, \xi, \Phi, \eta)$, define

$$\mathcal{K}_S = \{ f \in C^B_\Omega (M, \mathbb{R}) | \eta_f = \eta + d^c f \text{ is a contact form} \},$$

where $(d^c f)(X) = -\frac{1}{2} df(\Phi(X))$ for any vector field $X$ on $M$. Observe that any $f \in \mathcal{K}_S$ induces a Sasakian structure $(\xi, \Phi_f, \eta_f)$ on $M$ with the same Reeb vector field $\xi$. The geometry of $\mathcal{K}_S$ has been studied by P. Guan and X. Zhang in [7, 8] from the point of view of geodesics, by W. He in [9] from the point of view of curvature and by S. Calamai, D. Petrrecca and K. Zheng in [2] in relation to the Ebin metric.

Fix a transverse Kähler metric $\chi$ on $M$ and let $f(t)$ be a smooth path on $\mathcal{K}_S$. Define the functional $J_\chi : \mathcal{K}_S \to \mathbb{R}$ by:

$$(\partial_t J_\chi)(f) = \frac{1}{2^{n-1}(n-1)!} \int_M f \chi \wedge \eta \wedge (d\eta_f)^{n-1} = \frac{1}{2^{n-1}(n-1)!} \int_M f \sigma_f \eta \wedge (d\eta_f)^n, \quad J_\chi(0) = 0,$$
where $\sigma_f$ is the trace of $\chi$ with respect to $d\eta_f$. Further, let $\mathcal{H}_S^0 = \{ h \in \mathcal{H}_S | I(h) = 0 \}$, where $I: \mathcal{H}_S \to \mathbb{R}$ is defined by:

$$(\tilde{c}_t I)(f) = \int_M \hat{f} \eta \wedge (d\eta)^n, \quad I(0) = 0.$$

As for the Kähler setting, it is easy to see that $h \in \mathcal{H}_S^0$ is a critical point for $J_\chi$ iff $\sigma_h = c$ is constant. The Sasaki $J$-flow is given by:

$$\dot{f} = c - \sigma_f, \quad f(0) = f_0.$$

Chen’s work [4] can be summarized in the following steps:

1. there exists at most one critical point;
2. the flow is parabolic and thus well defined (short time existence);
3. second order estimates give long time existence;
4. under the additional hypothesis on the sign of the holomorphic sectional curvature of $\chi$, the flow converges at $t \to \infty$ to a critical point.

In the generalization to the Sasakian context, the main differences with the Kähler setting are the following:

1. it is not clear if two points in $\mathcal{H}_S$ can be connected by a geodesic. Although, Guan and Zhang proved in [8] that this can be always done in a weak sense. Using their result we are able to prove the uniqueness of critical points for the Sasakian $J$-flow.

2. The Sasakian $J$-flow is parabolic only along transverse directions. To prove that it is well defined, we use a trick introduced in [12], completing the flow to a parabolic one:

$$\dot{f} = c - \sigma_f - \xi^2(f), \quad f(0) = f_0, \quad (1)$$

and proving that if $f_0$ is basic then a solution to (1) remains basic to all $t$.

3. From the local point of view, a solution to the Sasaki $J$-flow can be seen as a collection of solutions to the Kähler $J$-flow on open sets in $\mathbb{C}^n$. This fact allows us to use all the local estimates about the Kähler $J$-flow provided in [4]. For the global estimates we need to prove a transverse version of the maximal principle.

Remark 3.1. Notice that one can define a Kähler $J$-flow on the Kähler cone $(M \times \mathbb{R}^+, \tilde{g})$ associated to the Sasakian manifold $(M, \xi, \Phi, \eta, g)$. However, there are no reasons for the Kähler structure obtained by deforming the cone Kähler structure under the flow to be conic. Thus, there is no hope to define a Sasakian $J$-flow using the Kähler cone instead of the transverse Kähler metric of $M$.

**References**


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